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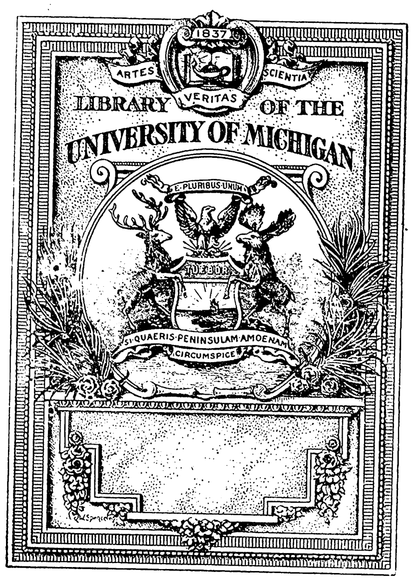
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FURTHER RESEARCHES IN THE THEORY OF DIVERGENT SERIES AND INTEGRALS.

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I. *Further researches in the Theory of Divergent Series and Integrals.*

By G. H. Hardy, M.A.

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§ 1. This paper is a continuation of one published in the *Quarterly Journal of Mathematics* in 1904*.

In § 16 of the paper referred to I said:

'The definitions of the previous sections are perhaps of most use in connection with double limit problems, such as differentiation under the integral sign. Their employment in such problems raises questions which demand a detailed treatment which I must reserve for the present.'

In the present paper I propose to consider some of these questions in greater detail.

A. *Generalised limits and integrals and infinite series.*

§ 2. Two of the most important among the double limit problems of ordinary analysis are the following:

(i) when is the limit of the sum of an infinite series equal to the sum of the limits of the terms of the series?

(ii) when is the integral of the sum of an infinite series equal to the sum of the integrals of the terms?

Or in symbols,

(i) when is $$\lim_{x\to a} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \lim_{x\to a} f_n(x)?$$

(ii) when is $$\int_a^A dx \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \int_a^A f_n(x)\,dx?$$

The case which is of especial interest to us now is that in which in (i) $a=\infty$ and in (ii) $a=0$, $A=\infty$†. The two problems may then be regarded as substantially the same. For if we suppose that the series $\Sigma f_n(x)$ may be integrated term by term over any finite interval $(0, X)$, and write

$$F_n(x) = \int_0^x f_n(t)\,dt,$$

* 'Researches in the Theory of Divergent Series and Divergent Integrals,' *Q. J.*, vol. xxxv. pp. 22—66.

† Throughout this paper I suppose the lower limit of the integrals discussed to be zero: the limitation is of course apparent only.

the second problem takes the form—

(ii)′ when is $$\lim_{X\to\infty} \sum_{n=0}^{\infty} F_n(X) = \sum_{n=0}^{\infty} \lim_{X\to\infty} F_n(X)\,?$$

—which is substantially the same problem as (i).

The problems which we have now to consider are—

(1) when is $$\underset{x\to\infty}{L} \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \underset{x\to\infty}{L} f_n(x)\,?$$

(2) when is $$G\int_0^{\infty} dx \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} G\int_0^{\infty} f_n(x)\,dx\,?$$

—the symbols $\underset{x\to\infty}{L}$ and $G\int_0^{\infty}$ denoting the generalised limit and generalised integral according to the definitions of my former paper. In that paper I in the first instance defined $\underset{x\to\infty}{L} F(x)$ as being

$$\lim_{t\to\infty} \int_0^{\infty} e^{-x} F(tx)\,dx,$$

and $G\int_0^{\infty} f(x)\,dx$ as being

$$\lim_{t\to\infty} \int_0^{\infty} e^{-x/t} f(x)\,dx,$$

or (what is, at any rate in all cases of interest, the same thing)

$$\int_0^{\infty} dt \int_0^{\infty} xe^{-tx} f(x)\,dx\,;$$

and I showed that if, for all positive values of τ,

$$\lim_{x\to\infty} e^{-\tau x} f(x) = 0$$

and if $$F(x) = \int_0^{x} f(t)\,dt,$$

then $$\underset{x\to\infty}{L} F(x) = G\int_0^{\infty} f(x)\,dx.$$

In these circumstances the problems (1) and (2) are equivalent in the same sense as were (i) and (ii). I shall in what follows adopt (2) as the standard form of the problem, as it takes this form in the most interesting applications; and I shall for the present confine myself to the simple definitions recalled above. As I explained in my former paper, more powerful definitions may be given; but those just stated are easy to work with and are sufficient to deal with the most interesting and obvious cases. I shall, moreover, concern myself solely with the difficulties proper to the particular problems under consideration, ignoring those which affect equally the ordinary double limit problems of the Integral Calculus, such as those which arise from discontinuities of the subject of integration.

§ 3. The transformation expressed by the equation (2) is valid if

$$\lim_{t\to 0} \int_0^{\infty} e^{-\tau x}\,dx\, \Sigma f_n(x) = \lim_{t\to 0} \Sigma \int_0^{\infty} e^{-\tau x} f_n(x)\,dx$$
$$= \Sigma \lim_{t\to 0} \int_0^{\infty} e^{-\tau x} f_n(x)\,dx.$$

Now the first of these equations asserts that the series

$$\Sigma\, e^{-\tau x} f_n(x)$$

may be integrated term by term from $x=0$ to $x=\infty$. The second transformation asserts that the series

$$\sum_0^\infty \int_0^\infty e^{-\tau x} f_n(x)\, dx$$

is a continuous function of τ for $\tau = 0$. Thus we obtain

THEOREM I. *The equation*

$$G\int_0^\infty \left\{\sum_0^\infty f_n(x)\right\} dx = \sum_0^\infty G\int_0^\infty f_n(x)\, dx$$

will certainly be true, if

(i) *the series* $\quad \Sigma\, e^{-\tau x} f_n(x) \quad (\tau > 0)$

can be integrated term by term over the interval $(0, \infty)$,

(ii) *the series* $\quad \Sigma \int_0^\infty e^{-\tau x} f_n(x)\, dx$

is a continuous function of τ *for* $\tau = 0$.

Our problem is therefore reduced to the investigation of (1) the legitimacy of a certain ordinary term by term integration, and (2) the continuity of a certain infinite series.

It is useful to notice one case in which the first of the two conditions stated above is certainly fulfilled. This case is that in which

(a) $\Sigma f_n(x)$ is uniformly convergent over any finite range $(0, X)$,

(b) the integral $\quad \int_0^\infty e^{-\tau x} \Sigma\, |f_n(x)|\, dx$

is convergent. For then, as I have proved in a note in the *Messenger of Mathematics* *, the integration term by term, from 0 to ∞, of the series $\Sigma e^{-\tau x} f_n(x)$ is certainly legitimate.

§ 4. By far the most interesting case is that in which

$$f_n(x) = a_n x^n \phi(x),$$

the series $\Sigma\, a_n x^n$ being convergent for all values of x. We have then to consider

(a) whether the series $\quad \Sigma\, e^{-\tau x} \phi(x)\, a_n x^n$

may be integrated term by term from 0 to ∞,

(b) whether the series $\quad \Sigma\, a_n \int_0^\infty e^{-\tau x} \phi(x)\, x^n\, dx$

is continuous for $\tau = 0$.

Let us first notice certain cases in which the first of these questions can certainly be answered in the affirmative.

* Vol. XXXV. p. 126. See also Bromwich, *ibid.*, vol. XXXVI. p. 1, and *Infinite Series*, pp. 448—455.

(1) It can certainly be answered in the affirmative if the integration is legitimate over any finite interval $(0, X)$, and

$$e^{-\tau x}\phi(x), \quad e^{-\tau x}\Sigma |a_n| x^n$$

each tend to zero as x tends to ∞, for any positive value of τ. For then the conditions stated at the end of the last section are satisfied. In particular it is fulfilled if $e^{-\tau x}\phi(x)$ tends to zero, for any positive value of τ, and $\Sigma a_n x^n$ is an integral function of order less than 1.

(2) It can certainly be answered in the affirmative if the integration is permissible over any finite interval $(0, X)$, and $e^{-\tau x}\phi(x)$ tends to zero for any positive value of τ, and $\Sigma n!\, a_n x^n$ is convergent for all values of x. For then, if X is large enough, $|\phi(x)| < e^{\frac{1}{2}\tau x}$ for $x \geqq X$, and

$$\left|\int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right| < \int_X^\infty e^{-\frac{1}{2}\tau x} x^n dx$$
$$= n!\left(\frac{2}{\tau}\right)^{n+1} e^{-\frac{1}{2}\tau X}\left\{1 + \tfrac{1}{2}\tau X + \frac{(\frac{1}{2}\tau X)^2}{2!} + \ldots + \frac{(\frac{1}{2}\tau X)^n}{n!}\right\},$$

which is always less than $\quad n!\left(\dfrac{2}{\tau}\right)^{n+1},$

and, for any assigned values of τ and n, tends to zero as $X \to \infty$. Hence

$$\left|\sum_0^\infty a_n \int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right| < \left(\sum_0^N + \sum_{N+1}^\infty\right)|a_n|\left|\int_X^\infty e^{-\tau x} x^n \phi(x)\, dx\right|$$
$$< \sum_0^N |a_n| \int_X^\infty e^{-\frac{1}{2}\tau x} x^n dx + \sum_{N+1}^\infty n!\,|a_n|\left(\frac{2}{\tau}\right)^{n+1}.$$

We can now choose, first N so that the second sum is less than $\frac{1}{2}\epsilon$, and then X so that the first sum is less than $\frac{1}{2}\epsilon$; and hence we see that

$$\lim_{X\to\infty} \sum_0^\infty a_n \int_X^\infty e^{-\tau x} x^n \phi(x)\, dx = 0,$$

and this is precisely the condition that the integration over the whole range $(0, \infty)$ should be legitimate.

It should be remarked that the results just proved are by no means sufficient for the applications that we have in view. There are many interesting cases in which the result holds for all positive values of τ, but its correctness does not follow from anything that has yet been proved. If, e.g., $\phi(x) = e^{-mix}$, where $m > 0$, and $a_n = (-\sigma i)^n/n!$, where $\sigma > 0$, so that

$$\Sigma a_n x^n = e^{-\sigma x i},$$

the equation states that $\quad \displaystyle\int_0^\infty e^{-\{\tau + (m+\sigma) i\}}\, dx = \Sigma \frac{(-\sigma i)^n}{(\tau + mi)^{n+1}},$

a result which is true for all positive values of τ, if $\sigma < m$. But the conditions (1) are not satisfied, since

$$\Sigma |a_n| x^n = e^{\sigma x};$$

and the conditions (2) are not satisfied, since $\Sigma n!\, a_n y^n$ is not an integral function of y.

§ 5. Before passing on I may make a few further remarks. In a former paper in these *Transactions** I proved that

$$\int_0^\infty e^{-x}(\Sigma a_n x^n)\,dx = \Sigma\, n!\, a_n$$

whenever the series on the right is convergent. It follows that

$$\int_0^\infty e^{-\tau x}(\Sigma a_n x^n)\,dx = \Sigma\, n!\, a_n \tau^{-n-1},$$

if $\tau > 0$ and the series on the right-hand side is convergent.

A more general result is the following.

If $\tau > 0$, $\mu > -1$, *the equations*

$$\int_0^\infty e^{-\tau x}(\Sigma a_n x^{\mu+n})\,dx = \Sigma a_n \int_0^\infty e^{-\tau x} x^{\mu+n}\,dx$$

$$= \Sigma a_n \frac{\Gamma(\mu+n+1)}{\tau^{\mu+n+1}}$$

are certainly true whenever the last series is convergent.

Let $$u_n = a_n \Gamma(\mu+n+1)\,\tau^{-\mu-n-1}.$$

Then, for any positive value of X,

$$\int_0^X e^{-\tau x}\left(\Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n x^{\mu+n}\right) dx = \Sigma \frac{\tau^{\mu+n+1} u_n}{\Gamma(\mu+n+1)} \int_0^X e^{-\tau x} x^{\mu+n}\,dx,$$

and what we have to prove is that

$$\lim_{X\to\infty} \Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx = 0.$$

Now

$$\frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx$$

$$= e^{-\tau X}\left\{\frac{(\tau X)^\mu}{\Gamma(\mu+1)} + \frac{(\tau X)^{\mu+1}}{\Gamma(\mu+2)} + \dots + \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}\right\} + \frac{\tau^\mu}{\Gamma(\mu)} \int_X^\infty e^{-\tau x} x^{\mu-1}\,dx,$$

and so $$\Sigma \frac{\tau^{\mu+n+1}}{\Gamma(\mu+n+1)} u_n \int_X^\infty e^{-\tau x} x^{\mu+n}\,dx = S_1 + S_2,$$

where $$S_1 = e^{-\tau X} \sum_{n=0}^\infty u_n \sum_{\lambda=0}^n \frac{(\tau X)^{\mu+\lambda}}{\Gamma(\mu+\lambda+1)},$$

$$S_2 = \frac{\tau^\mu}{\Gamma(\mu)} \left(\int_X^\infty e^{-\tau x} x^{\mu-1}\,dx\right)(\Sigma u_n).$$

Obviously $S_2 \to 0$ as $X \to \infty$. Also, as in my former proof, we have

$$S_1 = e^{-\tau X} \sum_{n=0}^\infty \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sum_{\lambda=n}^\infty u_\lambda = e^{-\tau X} \sum_{n=0}^\infty \frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)} \sigma_n,$$

say.

* Vol. XIX. p. 299.

Now if
$$\chi(x)=\sum_{n=0}^{\infty}\frac{x^{\mu+n}}{\Gamma(\mu+n+1)}$$
it is known* that
$$\chi(x)=e^x(1+\epsilon_x),$$
where $\epsilon_x \to 0$ as $x\to\infty$, and so, for all positive values of X,
$$\chi(\tau X)<Ke^{\tau X},$$
and, *a fortiori*,
$$\sum_{N+1}^{\infty}\frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}<Ke^{\tau X}.$$
But
$$S_1=e^{-\tau X}\sum_{0}^{N}\frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}\sigma_n+e^{-\tau X}\sum_{N+1}^{\infty}\frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}\sigma_n.$$
Choose N so that
$$|\sigma_n|<\epsilon/K \qquad (n>N).$$
Then
$$|S_1|<e^{-\tau X}\sum_{0}^{N}\frac{(\tau X)^{\mu+n}}{\Gamma(\mu+n+1)}+\epsilon,$$
and from this it follows immediately that $S_1\to 0$ as $X\to\infty$. Thus our theorem is established.

The question is naturally suggested as to whether it is not always true that
$$\int_0^{\infty}e^{-\tau x}(\Sigma\, a_n x^n)\,\phi(x)\,dx=\Sigma\, a_n\int_0^{\infty}e^{-\tau x}x^n\phi(x)\,dx,$$
when $\tau>0$, $e^{-\tau x}\phi(x)\to 0$ for any positive value of τ, and the series on the right-hand side is convergent. But it is easy to show by an example that this is not the case. Suppose
$$\phi(x)=e^{-mix} \qquad (m>0)$$
(a case with which we shall be much concerned in the sequel). Then the question takes the form: *is*
$$\int_0^{\infty}e^{-(\tau+mi)x}(\Sigma\, a_n x^n)\,dx=\Sigma\frac{n!\,a_n}{(\tau+mi)^{n+1}}$$
whenever the last series is convergent? Now let
$$\tau+mi=y,\ \ a_n=\frac{z^n}{n!}.$$
The result would be
$$\int_0^{\infty}e^{-(y-z)x}\,dx=\Sigma\frac{z^n}{y^{n+1}}.$$

The right-hand side is convergent if $|z|<|y|$, the left-hand side when $R(z)<R(y)$, and it is obvious that the first of these conditions does not imply the second.

§ 6. (3) As the sets of conditions (1) and (2) are not sufficiently general for the applications I have in view, I shall indicate a set of conditions of a different character, under which it is always possible to give an affirmative answer to the question (a) of § 4.

Let us suppose first that $e^{-\tau x}\phi(x)\to 0$ for any positive value of τ, and write
$$\psi(\tau)=\int_0^{\infty}e^{-\tau x}\phi(x)\,dx.$$
And further let us suppose that $\psi(\tau)$ is *an analytic function of* τ, *regular in the neighbourhood of the origin*: (*a fortiori* regular in any region throughout which the real part of τ is always positive).

* See e.g. *Proc. Lond. Math. Soc.*, N.S., vol. II. p. 405.

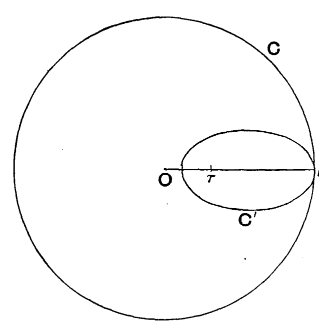
The integrals
$$\int_0^\infty e^{-\tau x} x^n \phi(x)\, dx$$
are all uniformly convergent in any interval (τ_0, τ_1), where $0 < \tau_0 < \tau_1$. Hence, for any positive value of τ, we have
$$\left(\frac{d}{d\tau}\right)^n \psi(\tau) = (-1)^n \int_0^\infty e^{-\tau x} x^n \phi(x)\, dx.$$

Also the integral on the right converges to a limit as $\tau \to 0$, and this limit is equal to $\psi^{(n)}(0)$: or in other words
$$G\int_0^\infty x^n \phi(x)\, dx = (-1)^n \psi^{(n)}(0).$$

Now let δ denote the distance from the origin of the nearest singularity of $\psi(\tau)$. Then if $\rho < \delta$ and the contour of integration is the circle C defined by $|u| = \rho$, we have
$$\left(\frac{d}{d\tau}\right)^n \psi(\tau) = \frac{n!}{2\pi i}\int \frac{\psi(u)\, du}{(u-\tau)^{n+1}}.$$

Finally let us suppose that *the series* $\Sigma n!\, a_n y^n$ *has a radius of convergence greater than* δ. Then, for sufficiently small values of τ, the series
$$\chi(u, \tau) = \Sigma \frac{(-1)^n n!\, a_n}{(u-\tau)^{n+1}},$$
is uniformly convergent along C. We have therefore
$$\begin{aligned}\frac{1}{2\pi i}\int \psi(u)\chi(u, \tau)\, du &= \Sigma \frac{(-1)^n n!\, a_n}{2\pi i}\int \frac{\psi(u)\, du}{(u-\tau)^{n+1}} \\ &= \Sigma(-1)^n a_n \left(\frac{d}{d\tau}\right)^n \psi(\tau) \\ &= \Sigma a_n \int_0^\infty e^{-\tau x}\phi(x)\, x^n\, dx.\end{aligned}$$

Now the only singularity of the subject of integration, within C, is $u = \tau$. We may therefore replace C by a contour C' such as is shown in the figure, cutting the positive real axis between $u = 0$ and $u = \tau$, say at $u = \gamma$. On C', u has its real part positive and greater than γ. Hence



$$\begin{aligned}&\frac{1}{2\pi i}\int_{C'} \psi(u)\chi(u, \tau)\, du \\ &\qquad = \frac{1}{2\pi i}\int_{C'} \chi(u, \tau)\, du \int_0^\infty e^{-u\xi}\phi(\xi)\, d\xi.\end{aligned}$$

In this repeated integral we may invert the order of integration. For, in the first place, this inversion is obviously justifiable when the upper limit ∞ is replaced by any positive number X. And, in the second place,
$$\left|\frac{1}{2\pi i}\int_{C'} \chi(u, \tau)\, du \int_X^\infty e^{-u\xi}\phi(\xi)\, d\xi\right| < K\int_X^\infty e^{-\gamma\xi}|\phi(\xi)|\, d\xi,$$
which may obviously be made as small as we please by sufficiently increasing X.

Hence $$\frac{1}{2\pi i}\int_{C'}\psi(u)\,\chi(u,\ \tau)\,du = \frac{1}{2\pi i}\int_0^\infty \phi(\xi)\,d\xi\int_{C'} e^{-u\xi}\,\chi(u,\ \tau)\,du.$$

But $$\frac{1}{2\pi i}\int_{C'} e^{-u\xi}\,\chi(u,\ \tau)\,du = \frac{1}{2\pi i}\int_C e^{-u\xi}\,\chi(u,\ \tau)\,du$$
$$= \frac{1}{2\pi i}\int_C e^{-u\xi}\left\{\Sigma\frac{(-1)^n n!\,a_n}{(u-\tau)^{n+1}}\right\}du$$
$$= \Sigma\frac{(-1)^n n!\,a_n}{2\pi i}\int_C\frac{e^{-u\xi}}{(u-\tau)^{n+1}}\,du$$
$$= \Sigma(-1)^n a_n\left(\frac{d}{d\tau}\right)^n e^{-\tau\xi}$$
$$= e^{-\tau\xi}\,\Sigma a_n\xi^n.$$

Hence finally $$\Sigma a_n\int_0^\infty e^{-\tau x}\,\phi(x)\,x^n\,dx = \int_0^\infty e^{-\tau\xi}\,\phi(\xi)\,\Sigma a_n\xi^n\,d\xi.$$

Thus the question (a) of § 4 may be answered affirmatively.

§ 7. I shall now pass on to the question (b), and show that if the conditions of § 6 are satisfied it also can be answered affirmatively. In order to prove this let us go back to the equation

$$\Sigma(-1)^n a_n\left(\frac{d}{d\tau}\right)^n\psi(\tau) = \frac{1}{2\pi i}\int_C\psi(u)\,du\,\Sigma\frac{(-1)^n n!\,a_n}{(u-\tau)^{n+1}}.$$

The series under the integral sign converges uniformly for all values of u on C and all values of τ such that $0 \leqq \tau \leqq \tau_0$. Hence each side of the equation is a continuous function of τ for $\tau = 0$, and the series

$$\Sigma(-1)^n a_n\psi^{(n)}(0) = \Sigma a_n G\int_0^\infty x^n\phi(x)\,dx,$$

is convergent and equal to

$$\lim_{\tau\to 0}\Sigma(-1)^n a_n\psi^{(n)}(\tau),$$

or to $$\lim_{\tau\to 0}\Sigma a_n\int_0^\infty e^{-\tau x}x^n\phi(x)\,dx.$$

Thus the question (b) may also be answered in the affirmative.

§ 8. Thus we arrive at:

THEOREM II. *We may evaluate the generalised integral*

$$G\int_0^\infty\phi(x)\,F(x)\,dx,$$

by expanding $F(x)$ *as a power series* $\Sigma a_n x^n$ *and taking the generalised integral term by term, provided*

(1) *the function* $$\psi(\tau) = \int_0^\infty e^{-\tau x}\phi(x)\,dx$$
is regular at the origin,

(ii) *the series* $\Sigma n!\,a_n y^n$ *has a radius of convergence greater than* $1/\delta$, *where* δ *is the distance from the origin of the nearest singularity of* $\psi(\tau)$.

§ 9. Let us consider in particular the case in which

$$\phi(x) = e^{-mix} x^{\mu},$$

where $m > 0$, $\mu > -1$. Then

$$\psi(\tau) = \int_0^{\infty} e^{-(\tau+mi)x} x^{\mu} dx = \frac{\Gamma(\mu+1)}{(\tau+mi)^{\mu+1}},$$

where that branch of $(\tau + mi)^{-\mu-1}$ is chosen which reduces to

$$m^{-\mu-1} e^{-\frac{1}{2}(\mu+1)\pi i},$$

for $\tau = 0$. In this case $\delta = m$, and we shall certainly have

$$G\int_0^{\infty} e^{-mix} (\Sigma a_n x^n) x^{\mu} d\mu = \Sigma a_n G \int_0^{\infty} e^{-mix} x^{n+\mu} dx$$

$$= \Sigma a_n \Gamma(n+\mu+1) m^{-n-\mu-1} e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

if the series $\Sigma n!\, a_n y^n$ has a radius of convergence greater than $1/m$. This condition, however, may be reduced to a simpler form. For the radius of convergence of

$$\Sigma a_n \Gamma(n+\mu+1) e^{-\frac{1}{2}(n+\mu+1)\pi i} y^n$$

is the same as that of $\Sigma n!\, a_n y^n$. The integration is therefore certainly legitimate if its radius of convergence is greater than $1/m$. But we can go further and say that the integration is certainly legitimate if the radius of convergence is *as great as* $1/m$.

For let
$$\psi_1(\tau) = \int_0^{\infty} e^{-(\tau+\kappa+mi)x} x^{\mu} dx = \frac{\Gamma(\mu+1)}{(\tau+\kappa+mi)^{\mu+1}}.$$

The distance of the nearest singularity of $\psi_1(\tau)$ from the origin is

$$\sqrt{(m^2+\kappa^2)} > m,$$

and therefore
$$\int_0^{\infty} e^{-(\kappa+mi)x} (\Sigma a_n x^n) x^{\mu} dx = \Sigma a_n G \int_0^{\infty} e^{-(\kappa+mi)x} x^{n+\mu} dx$$

$$= \Sigma a_n \frac{\Gamma(n+\mu+1)}{(\kappa+mi)^{n+\mu+1}},$$

provided the radius of convergence of $\Sigma n!\, a_n y^n$ is greater than

$$1/\sqrt{(m^2+\kappa^2)},$$

and therefore certainly if it is equal to $1/m$.

But, by a well-known extension of Abel's theorem on the continuity of power-series

$$\lim_{\kappa \to 0} \Sigma a_n \frac{\Gamma(n+\mu+1)}{(\kappa+mi)^{n+\mu+1}} = \Sigma a_n \Gamma(n+\mu+1) m^{-n-\mu-1} e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

*provided only the series on the right is convergent, or even if it is oscillatory, but summable by Cesàro's mean value process or one of its extensions**.

We have thus proved:

THEOREM III. *We may evaluate the generalised integral*

$$G\int_0^{\infty} e^{-mix} F(x) x^{\mu} dx,$$

* Bromwich, *Infinite Series*, pp. 210 *et seq.* and pp. 310 *et seq.*

by expanding $F(x)$ in a power-series $\Sigma a_n x^n$, and taking the generalised integral term by term, whenever this method of procedure leads to a series either convergent, or oscillatory and summable by mean values.

And we may add that *if $F(x)$ is an integral function of order less than 1 the second condition of Theorem II and the solitary condition of Theorem III will certainly be satisfied.*

§ 10. I shall now give some examples of the use of the Theorems II, III, starting with the latter, from which we conclude that if $F(x) = \Sigma a_n x^n$ then

$$G\int_0^\infty e^{-mix} F(x)\, x^\mu\, dx = \sum_0^\infty a_n \Gamma(n+\mu+1)\, m^{-n-\mu-1}\, e^{-\frac{1}{2}(n+\mu+1)\pi i},$$

provided only the series on the right is convergent.

Supposing a_n real, and separating the real and imaginary parts, we obtain

$$G\int_0^\infty \cos mx F(x)\, x^\mu dx = -C \sin \tfrac{1}{2}\mu\pi - S\cos \tfrac{1}{2}\mu\pi,$$

$$G\int_0^\infty \sin mx F(x)\, x^\mu dx = -C \cos \tfrac{1}{2}\mu\pi + S\sin \tfrac{1}{2}\mu\pi,$$

where
$$C = \Sigma a_n \frac{\Gamma(n+\mu+1)}{m^{n+\mu+1}} \cos \tfrac{1}{2} n\pi,$$

$$S = \Sigma a_n \frac{\Gamma(n+\mu+1)}{m^{n+\mu+1}} \sin \tfrac{1}{2} n\pi,$$

or
$$C = \frac{\Gamma(\mu+1)}{m^{\mu+1}}\left\{a_0 - \frac{(\mu+1)(\mu+2)}{m^2} a_2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{m^4} a_4 - \dots\right\},$$

$$S = \frac{\Gamma(\mu+1)}{m^{\mu+1}}\left\{-\frac{(\mu+1)}{m} a_1 + \frac{(\mu+1)(\mu+2)(\mu+3)}{m^3} a_3 - \dots\right\}.$$

In particular, if $\mu = 0$,

$$G\int_0^\infty \cos mx\, F(x)\, dx = -\frac{a_1}{m^2} + \frac{3!\, a_3}{m^4} - \frac{5!\, a_5}{m^6} + \dots,$$

$$G\int_0^\infty \sin mx\, F(x)\, dx = \frac{a_0}{m} - \frac{2!\, a_2}{m^3} + \frac{4!\, a_4}{m^5} - \dots$$

By taking $\mu = 0$ and $F(x) = J_0(\sqrt{x})$ we obtain

$$\int_0^\infty J_0(\sqrt{x}) \begin{matrix}\cos\\ \sin\end{matrix}\, mx\, dx = \frac{1}{m} \begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{1}{4m}\right) \quad \dots\dots(1);$$

by taking $F(x) = \cos\sqrt{x}$ and $\mu = -\frac{1}{2}$ we obtain

$$\int_0^\infty \frac{\cos\sqrt{x}}{\sqrt{x}} \cos mx\, dx = \sqrt{\left(\frac{\pi}{2m}\right)}\left(\cos\frac{1}{4m} + \sin\frac{1}{4m}\right) \quad \dots\dots(2),$$

or
$$\int_0^\infty \cos x^2 \cos 2\mu x\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\cos\mu^2 + \sin\mu^2) \quad \dots\dots(3);$$

and similarly
$$\int_0^\infty \sin x^2 \cos 2\mu x\, dx = \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}\,(\cos\mu^2 - \sin\mu^2) \quad \dots\dots(4).$$

These are cases in which $F(x)$ is an integral function of order < 1, and the integrals on the left are all convergent in the ordinary sense.

More general results may be obtained by taking

$$x^{\mu} F(x) = x^k J_0(\sqrt{x}),\ x^k \frac{\cos\sqrt{x}}{\sqrt{x}}$$

(where k is a positive integer); e.g.

$$G\int_0^\infty J_0(\sqrt{x})\, x^{2k} \cos mx\, dx = (-)^k \left(\frac{d}{dm}\right)^{2k} \left\{\frac{1}{m} \sin \frac{1}{4m}\right\} \qquad (5).$$

§ 11. Taking $\mu = 0$ and

$$F(x) = J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \dots,$$

we obtain

$$\int_0^\infty \cos mx\, J_0(x)\, dx = 0 \qquad (6),$$

$$\int_0^\infty \sin mx\, J_0(x)\, dx = \frac{1}{m} + \frac{1}{2}\frac{1}{m^3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{1}{m^5} + \dots$$
$$= \frac{1}{\sqrt{(m^2-1)}} \qquad (7),$$

provided $m > 1$. The results agree with Weber's well-known formulae*.

If we take $\mu = k$, and $F(x) = J_0(x)$, we obtain formulae for

$$G\int_0^\infty x^k \frac{\cos}{\sin} mx\, J_0(x)\, dx,$$

which agree with the results of formal differentiation of (6) and (7).

More generally we may take

$$x^{\mu} F(x) = x^{\rho-1} J_\alpha(x),$$

where $\rho + \alpha > 0$, and express

$$G\int_0^\infty x^{\rho-1} \frac{\cos}{\sin} mx\, J_\alpha(x)\, dx$$

as a hypergeometric series. When $-\alpha < \rho < \frac{3}{2}$ we obtain a known expression of an ordinary integral. An interesting special case is that in which $\rho - 1 = \alpha$. In this case we find

$$G\int_0^\infty x^\alpha J_\alpha(x)\, e^{-mix}\, dx = \Sigma \frac{(-)^n}{2^{\alpha+2n}\, n!\, \Gamma(n+\alpha+1)} G\int_0^\infty e^{-mix} x^{2n+2\alpha}\, dx$$
$$= \Sigma \frac{(-)^n}{2^{\alpha+2n}\, n!\, \Gamma(n+\alpha+1)} \frac{\Gamma(2n+2\alpha+1)}{m^{2n+2\alpha+1}} e^{-\frac{1}{2}(2n+2\alpha+1)\pi i}.$$

Using the formula

$$\Gamma(\alpha)\, \Gamma(\alpha + \tfrac{1}{2}) = \Gamma(2\alpha)\, 2^{\frac{1}{2} - 2\alpha} \sqrt{2\pi},$$

we can reduce this series to

$$\frac{2^\alpha\, \Gamma(\alpha+\frac{1}{2})\, e^{(-\alpha+\frac{1}{2})\pi i}}{m^{2\alpha+1}\sqrt{\pi}} \Sigma \frac{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})\dots(\alpha+n-\frac{1}{2})}{1 \cdot 2 \dots n} \left(\frac{1}{m^2}\right)^n$$
$$= \frac{2^\alpha e^{(-\alpha+\frac{1}{2})\pi i}\, \Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\,(m^2-1)^{\alpha+\frac{1}{2}}}.$$

* Gray and Mathews, *Bessel Functions*, p. 73.

Equating real and imaginary parts we obtain

$$G\int_0^\infty x^\alpha J_\alpha(x)\cos mx\,dx = -\frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}(m^2-1)^{\alpha+\frac{1}{2}}}\sin\alpha\pi \quad\text{.....................(8)},$$

$$G\int_0^\infty x^\alpha J_\alpha(x)\sin mx\,dx = \frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}(m^2-1)^{\alpha+\frac{1}{2}}}\cos\alpha\pi \quad\text{.....................(9)}.$$

Since $$\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-\alpha)=\frac{\pi}{\cos\alpha\pi}$$

the second of these two formulae agrees with the formula

$$\int_0^\infty x^\alpha J_\alpha(x)\sin mx\,dx = \frac{2^\alpha\sqrt{\pi}}{\Gamma(\frac{1}{2}-\alpha)(m^2-1)^{\alpha+\frac{1}{2}}}, \qquad (n>1,\ -\tfrac{1}{2}<n<\tfrac{1}{2})$$

given by Sonine*. Our formulae are valid provided only $m>1$ and the integral is convergent at the lower limit, which requires $\alpha>-\frac{1}{2}$ for (8) and $\alpha>-1$ for (9). If $\alpha=-\frac{1}{2}$ the formula (9) becomes illusory and reduces to the well-known result

$$\int_0^\infty \frac{\cos x\sin mx}{x}\,dx=\tfrac{1}{2}\pi \qquad (m>1) \quad\text{.....................(10)}.$$

Another interesting pair of results is

$$\int_0^\infty J^\alpha(x)\,{\cos \atop \sin}\,(x\cosh\omega)\,\frac{dx}{x}={\cos \atop \sin}\,(\tfrac{1}{2}\alpha\pi)\,\frac{e^{-\alpha\omega}}{\alpha} \quad\text{.......................(11)}$$

(valid for $\alpha>1$)†.

To prove these formulae we observe that if $m=\cosh\omega$

$$\int_0^\infty J^\alpha(x)\,e^{-mix}\frac{dx}{x}=\sum_{n=0}^\infty\frac{(-)^n\,2^{-\alpha-2n}}{n!\,\Gamma(\alpha+n+1)}\,G\int_0^\infty x^{\alpha-1+2n}e^{-mix}\,dx$$

$$=(2m)^{-\alpha}e^{-\frac{1}{2}\alpha\pi i}\sum_{n=0}^\infty\frac{\Gamma(\alpha+2n)}{n!\,\Gamma(\alpha+n+1)}(2m)^{-2n}$$

$$=\frac{(2m)^{-\alpha}e^{-\frac{1}{2}\alpha\pi i}}{\alpha}\left\{1+\alpha\left(\frac{1}{2m}\right)^2+\frac{\alpha(\alpha+3)}{1.2}\left(\frac{1}{2m}\right)^4+\frac{\alpha(\alpha+4)(\alpha+5)}{1.2.3}\left(\frac{1}{2m}\right)^6+\dots\right\}$$

$$=\frac{m^{-\alpha}e^{-\frac{1}{2}\alpha\pi i}}{\alpha}\left\{1+\sqrt{\left(1-\frac{1}{m^2}\right)}\right\}^{-\alpha}$$

$$=e^{-\frac{1}{2}\alpha\pi i}\,\frac{e^{-\alpha\omega}}{\alpha},$$

the only condition required being $m>1$, which is satisfied.

§ 12. The following three examples are also instructive:

(i) $$G\int_0^\infty\cos mx\cos\lambda x\,dx=0+0+0+\dots=0,$$

$$G\int_0^\infty\sin mx\cos\lambda x\,dx=\frac{1}{m}+\frac{\lambda^2}{m^3}+\frac{\lambda^4}{m^5}+\dots=\frac{m}{m^2-\lambda^2},$$

provided $m>\lambda$—of course in this case the formulae are also true for $m<\lambda$;

* *Math. Annalen*, Bd. XVI. p. 39.

† For $\alpha>-2$ if the *sine* be taken: cf. Nielsen, *Handbuch der Cylinderfunktionen*, p. 197, and the analogous results of Schafheitlin, *Math. Annalen*, Bd. XXX. p. 171.

(ii) $$\int_0^\infty \frac{\sin mx \sin x}{x} dx = \Sigma \frac{1}{(2n+1)\, m^{2n+1}} = \tfrac{1}{2}\log\left(\frac{m+1}{m-1}\right);$$

(iii) $$\int_0^\infty \frac{\sin mx \sin x}{x^2} dx = \Sigma \frac{(-)^n}{(2n+1)!} G \int_0^\infty \sin mx\, x^{2n-1}\, dx$$
$$= \tfrac{1}{2}\pi,$$

all the integrals in this case vanishing save that for which $n = 0$*. In these formulae we must have $m > 1$. It might appear that the last formula should also hold for $m < 1$, since the series

$$\tfrac{1}{2}\pi + 0 + 0 + 0 + \ldots$$

is always convergent. But our condition was that the series obtained by integrating

$$\int_0^\infty e^{-mix} F(x)\, dx$$

should converge, i.e. that the two series obtained by integrating

$$\int_0^\infty {\cos \atop \sin} mx\, F(x)\, dx$$

should *both* converge: and it is easy to see that in this case the series obtained by taking the cosine diverges for $m < 1$. This must always be borne in mind. Otherwise we should be tempted to infer that

$$G\int_0^\infty \cos mx\, \phi(x^2)\, dx$$

(where ϕ is an integral function) is always zero, which is evidently not the case, as, e.g.,

$$\int_0^\infty \cos mx\, e^{-x^2}\, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-\frac{1}{4}m^2}.$$

In this case it will be found that the series for $\int_0^\infty \sin mx e^{-x}\, dx$ is divergent for all values of m.

§ 13. Let us consider next some applications of the more general theorem II. It is easy to prove that

$$\int_0^\infty J^\alpha(mx)\, e^{-\tau x} x^\rho\, dx = \frac{m^\alpha}{\tau^{\alpha+\rho+1}} \frac{\Gamma(\alpha+\rho+1)}{2^\alpha\, \Gamma(\alpha+1)} F\left(\frac{\alpha+\rho+1}{2}, \frac{\alpha+\rho+2}{2}, \alpha+1, -\frac{m^2}{\tau^2}\right) \ldots (12),$$

provided $\alpha + \rho > -1$, $\tau > m$. We have only to replace $J^\alpha(mx)$ by its expression as a series and integrate term by term. This formula fails us for small values of τ, but, by the help of a formula of Euler's connecting two hypergeometric functions, we can deduce from it †

$$\int_0^\infty J^\alpha(mx)\, e^{-\tau x} x^\rho\, dx$$
$$= \frac{\Gamma(\alpha+\rho+1)}{\Gamma(\alpha+1)} \frac{(\frac{1}{2}m)^\alpha}{(m^2+\tau^2)^{\frac{1}{2}(\alpha+\rho+1)}} F\left(\frac{\alpha+\rho+1}{2}, \frac{\alpha-\rho}{2}, \alpha+1, \frac{m^2}{m^2+\tau^2}\right) \ldots (13).$$

* We note that $$G\int_0^\infty x^{2n} \cos mx\, dx = G\int_0^\infty x^{2n+1} \sin mx\, dx = 0,$$
$$G\int_0^\infty x^{2n+1} \cos mx\, dx = (-)^{n+1} \frac{(2n+1)!}{m^{2n+2}}, \quad G\int_0^\infty x^{2n} \sin mx\, dx = (-)^n \frac{2n!}{m^{2n+1}}.$$

† Hankel, *Math. Annalen*, Bd. VIII. p. 467. Nielsen, *Handbuch der Cylinderfunktionen*, pp. 185, 188.

In the limit for $\tau = 0$ this equation becomes

$$G\int_0^\infty J^\alpha(mx)\, x^\rho\, dx = \frac{2^\rho}{m^{\rho+1}} \frac{\Gamma\{\frac{1}{2}(\alpha+\rho+1)\}}{\Gamma\{\frac{1}{2}(\alpha-\rho+1)\}} \quad\text{.......................(14).}$$

This formula holds for $\alpha+\rho>-1$. If also $\rho<\frac{1}{2}$, the integral is convergent in the ordinary sense *.

§ 14. Now let us, in the general integral

$$G\int_0^\infty \phi(x)\, F(x)\, dx,$$

suppose

$$F = \Sigma\, a_n x^n,$$

and

$$\phi(x) = J^\alpha(mx)\, x^\rho.$$

Then

$$\psi(\tau) = \int_0^\infty e^{-\tau x} J^\alpha(mx)\, x^\rho\, dx$$

is, as we can see from the equation (13), a function of τ regular within the circle whose centre is the origin and whose radius is m.

From Theorem II we deduce that if the series $\Sigma\, n!\, a_n y^n$ has a radius of convergence greater than $1/m$, then

$$G\int_0^\infty J^\alpha(mx) x^\rho F(x)\, dx = \Sigma a_n\, G\int_0^\infty J^\alpha(mx)\, x^{n+\rho} dx$$

$$= \Sigma\, \frac{2^{\rho+n}}{m^{\rho+n+1}}\, a_n \frac{\Gamma\{\frac{1}{2}(\alpha+1+\rho+n)\}}{\Gamma\{\frac{1}{2}(\alpha+1-\rho-n)\}}.$$

Writing $\alpha-\beta$ for α and putting $m=1$ we obtain

$$G\int_0^\infty J^{\alpha-\beta}(x)\, x^\rho F(x)\, dx = 2^\rho\, \Sigma\, 2^n a_n \frac{\Gamma\{\frac{1}{2}(\alpha-\beta+1+\rho+n)\}}{\Gamma\{\frac{1}{2}(\alpha-\beta+1-\rho-n)\}}.$$

Now suppose

$$x^\rho F(x) = x^{-\gamma+\alpha+\beta} J^{\gamma-1}(xz) \qquad (0<z<1)$$

$$= x^{\alpha+\beta-1}\left(\frac{z}{2}\right)^{\gamma-1} \sum_{\nu=0}^{\infty} \frac{(-)^\nu (\frac{1}{2}xz)^{2\nu}}{\nu!\,\Gamma(\gamma+\nu)}.$$

Then we must take

$$\rho = \alpha+\beta-1,$$

$$a_{2\nu+1} = 0, \quad a_{2\nu} = \left(\frac{z}{2}\right)^{2\nu+\gamma-1} \frac{(-)^\nu}{\nu!\,\Gamma(\gamma+\nu)},$$

and we find

$$G\int_0^\infty J^{\alpha-\beta}(x)\, J^{\gamma-1}(xz)\, x^{-\gamma+\alpha+\beta}\, dx$$

$$= 2^{\alpha+\beta-1}\left(\frac{z}{2}\right)^{\gamma-1} \Sigma\, (-)^\nu\, 2^{2\nu} \left(\frac{z}{2}\right)^{2\nu} \frac{\Gamma(\alpha+\nu)}{\nu!\,\Gamma(\gamma+\nu)\,\Gamma(1-\beta-\nu)}$$

$$= \frac{z^{\gamma-1}}{2^{\gamma-\alpha-\beta}} \frac{\sin\beta\pi}{\pi} \Sigma\, \frac{\Gamma(\alpha+\nu)\,\Gamma(\beta+\nu)}{\nu!\,\Gamma(\gamma+\nu)} z^{2\nu}$$

$$= \frac{z^{\gamma-1}}{2^{\gamma-\alpha-\beta}} \frac{\Gamma(\alpha)}{\Gamma(1-\beta)\,\Gamma(\gamma)} F(\alpha, \beta, \gamma, z^2) \quad\text{.......................................(15);}$$

* Nielsen, *loc. cit.*, p. 189.

a formula which contains a very large number of interesting particular cases. Our proof involves only that the integral should be convergent at the lower limit, *i.e.* that $\alpha > 0$. If $\gamma - \alpha - \beta > -1$ the integral is convergent in the ordinary sense*.

§ 14. As a final example of the use of Theorem II let us consider the integrals

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx,$$

where m and a are positive, λ a positive integer, and $f(x)$ is an even function of x defined by a series $f(x) = \Sigma\, a_n x^{2n}$, convergent for all values of x.

Since $$\frac{x^{2k}}{c^2+x^2} = (-1)^k c^{2k} \left\{\frac{1}{c^2+x^2} - \frac{1}{c^2} + \frac{x^2}{c^4} - \ldots + (-1)^k \frac{x^{2k-2}}{c^{2k-2}}\right\},$$

we have $$G\int_0^\infty \frac{x^{2k} \cos mx}{c^2+x^2}\, dx = (-1)^k c^{2k} \int_0^\infty \frac{\cos mx}{c^2+x^2}\, dx = (-1)^k \tfrac{1}{2}\pi c^{2k-1} e^{-mc},$$

$$G\int_0^\infty \frac{x^{2k+1} \sin mx}{c^2+x^2}\, dx = (-1)^k c^{2k} \int_0^\infty \frac{x \sin mx}{c^2+x^2}\, dx = (-1)^k \tfrac{1}{2}\pi c^{2k} e^{-mc}.$$

Hence, taking the divergent integral term by term, we obtain

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = \Sigma a_n\, G\int_0^\infty \frac{x^{2(n+\lambda)} \cos mx}{c^2+x^2}\, dx$$
$$= (-1)^\lambda \tfrac{1}{2}\pi c^{2\lambda-1} e^{-mc}\, \Sigma\, (-1)^n a_n c^{2n},$$

$$G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = \Sigma a_n\, G\int_0^\infty \frac{x^{2(n+\lambda)+1} \sin mx}{c^2+x^2}\, dx$$
$$= (-1)^\lambda \tfrac{1}{2}\pi c^{2\lambda} e^{-mc}\, \Sigma\, (-1)^\lambda a_n c^{2n}.$$

These results may be stated in the form

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = (-1)^\lambda \tfrac{1}{4}\pi c^{2\lambda-1} e^{-mc} \{f(ci) + f(-ci)\} \quad \text{............(16)},$$

$$G\int_0^\infty \frac{x \sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx = (-1)^\lambda \tfrac{1}{4}\pi c^{2\lambda} e^{-mc} \{f(ci) + f(-ci)\} \text{...............(17)}.$$

We have now to consider under what conditions this procedure is legitimate. In the first place

$$\theta(z) = \int_0^\infty \frac{e^{-zx}}{c^2+x^2} x^{2\lambda}\, dx$$

is known to be an analytic function of z, regular save at the origin and at infinity. It follows that

$$\psi(\tau) = \int_0^\infty \frac{e^{-(\tau+mi)x}}{c^2+x^2} x^{2\lambda}\, dx$$

has as its singularity nearest to the origin the point $\tau = -mi$, so that $\delta = m$.

* For special examples see Nielsen, *loc. cit.* pp. 191 *et seq.* If $\gamma - \alpha - \beta > 0$ the hypergeometric series converges for $z = 1$, and the result may be extended to this case (Nielsen, *loc. cit.* p. 194).

Thus we are justified in evaluating the integrals

$$G\int_0^\infty \frac{\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{\sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx,$$

in the manner adopted above, if the series

$$\Sigma\, 2n!\; a_n y^{2n}$$

has a radius of convergence greater than $1/m$. A similar argument can of course be applied to the integrals

$$G\int_0^\infty \frac{x\cos mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx, \quad G\int_0^\infty \frac{x\sin mx}{c^2+x^2} f(x)\, x^{2\lambda}\, dx.$$

It is however only in the two cases considered above that the result can be calculated in finite terms.

§ 15. So far we have considered only the particular case of the general theorem in which the integral is of the form

$$G\int_0^\infty \phi(x) f(x)\, dx,$$

where $f(x) = \Sigma a_n x^n$. Another interesting case is that in which $f(x)$ is a periodic function representable by a Fourier's series

$$\Sigma a_n e^{-2n\pi i x}.$$

If we suppose $f(x)$ *continuous*, it is known that

$$|a_n| < \frac{K}{n}.$$

If $f(x)$ is continuous except for a finite number of points x_ν in the interval (0, 1), at which it has infinities of the type $A/(x-x_\nu)^\beta$, where $0<\beta<1$, it is known that

$$|a_n| < \frac{K}{n^{1-\beta}}.$$

We have to consider,

(1) whether $\displaystyle\int_0^\infty e^{-\tau x}\phi(x)\, \Sigma a_n e^{-2n\pi i x}\, dx = \Sigma a_n \int_0^\infty e^{-(\tau+2n\pi i)x}\phi(x)\, dx,$

(2) whether the last series is a continuous function of τ for $\tau = 0$.

I shall consider only the particular case in which

$$\phi(x) = x^{\alpha-1}. \qquad (\alpha > 0)$$

It is in this case not hard to show that the question (1) can be answered affirmatively. We have to prove that

$$\lim_{X\to\infty} \Sigma a_n \int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx = 0.$$

Now
$$\begin{aligned}\int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx &= X^{\alpha-1}\frac{e^{-(\tau+2n\pi i)X}}{\tau+2n\pi i} \\ &\quad + \frac{\alpha-1}{\tau+2n\pi i}\int_X^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-2}\, dx \\ &= \lambda_n + \mu_n,\end{aligned}$$

say. But
$$|\lambda_n| < \frac{K}{n} X^{\alpha-1} e^{-\tau X}$$
and
$$|\mu_n| < \frac{K}{n} e^{-\frac{1}{2}\tau X} \int_X^\infty e^{-\frac{1}{2}\tau x} x^{\alpha-2}\, dx < \frac{K}{n} e^{-\frac{1}{2}\tau X},$$
and so
$$|\Sigma a_n \lambda_n| < K X^{\alpha-1} e^{-\tau X} \Sigma n^{\beta-2},$$
$$|\Sigma a_n \mu_n| < K e^{-\frac{1}{2}\tau X} \Sigma n^{\beta-2},$$
each of which tends to zero as $X \to \infty$.

The second question can also be answered in the affirmative if $\alpha \geqq 1$. For then the series
$$\Sigma a_n \int_0^\infty e^{-(\tau+2n\pi i)x} x^{\alpha-1}\, dx = \Gamma(\alpha)\, \Sigma \frac{a_n}{(\tau+2n\pi i)^\alpha},$$
where that value of $(\tau+2n\pi i)^{-\alpha}$ is chosen which reduces to
$$(2n\pi)^{-\alpha} e^{-\frac{1}{2}\alpha\pi i}$$
for $\tau = 0$, has its terms numerically less than those of the series
$$K \Sigma n^{-\alpha-1+\beta},$$
and so is uniformly convergent for any interval $0 \leqq \tau \leqq \tau_0$.

Thus the equations
$$G\int_0^\infty x^{\alpha-1} f(x)\, dx = \sum_1^\infty a_n G \int_0^\infty x^{\alpha-1} e^{-2n\pi i x}\, dx$$
$$= \Gamma(\alpha)(2\pi)^{-\alpha} e^{-\frac{1}{2}\alpha\pi i} \sum_1^\infty \frac{a_n}{n^\alpha},$$
$$G\int_0^\infty x^{\alpha-1} \sum_1^\infty a_n \begin{matrix}\cos\\ \sin\end{matrix} 2n\pi x\, dx = \Gamma(\alpha)(2\pi)^{-\alpha} \begin{matrix}\cos\\ \sin\end{matrix} \tfrac{1}{2}\alpha\pi \sum_1^\infty \frac{a_n}{n^\alpha} \quad \text{..............(18)}$$
are certainly valid if $\alpha > 1$. On the other hand they are not necessarily valid if $0 < \alpha < 1$. Thus if $\alpha = \frac{1}{2}$ and $a_n = 1/\sqrt{n}$ we are led to the series
$$\Sigma \frac{1}{n},$$
which is divergent. In this case the integral also is divergent at the lower limit, since
$$e^{-2\pi i x} + \frac{e^{-4\pi i x}}{\sqrt{2}} + \frac{e^{-6\pi i x}}{\sqrt{3}} + \dots$$
has an infinity of order $1/2$ for $x = 0$.

§ 16. Sufficient will have been said by now to show that, however difficult it may be in some cases to justify our procedure, the method of expansion and taking the generalised integrals of the separate terms is, in such cases as naturally occur in analysis, generally defensible; and *as a rule leads to correct results.* The reader will have no difficulty in constructing any number of further examples for himself, there being a large variety of integrals, of very different types, whose values are most easily determined in this way.

The process may be combined with Borel's method for the summation of a divergent series. This will probably be illustrated best by an example.

Consider the integral $$G\int_0^\infty \frac{x^{a-1}\,e^{-mix}}{1+x}\,dx,$$

where a and m are positive. Expand $1/(1+x)$ in the series

$$1 - x + x^2 - \dots$$

convergent if $0 \leqq x < 1$, summable if $x \geqq 1$. Taking the generalised integral term by term we obtain

$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\,m^{-a}\left\{1 + a\left(\frac{i}{m}\right) + a(a+1)\left(\frac{i}{m}\right)^2 + \dots\right\}.$$

The series in brackets is divergent for all values of m. Its sum, according to Borel's definition, is

$$\int_0^\infty e^{-v}\,dv\left\{1 + a\left(\frac{iv}{m}\right) + \frac{a(a+1)}{1.2}\left(\frac{iv}{m}\right)^2 + \dots\right\}.$$

The series under the sign of integration is itself divergent if $v > m$; but it is summable for all positive values of v, and its sum is known to be

$$\left(1 - \frac{iv}{m}\right)^{-a}*.$$

Hence we are led to the result

$$G\int_0^\infty \frac{x^{a-1}\,e^{-mix}}{1+x}\,dx = \Gamma(a)\,e^{-\frac{1}{2}a\pi i}\int_0^\infty \frac{e^{-v}\,dv}{(m-iv)^a}.$$

Whether our work can be justified is another matter. We shall see in a moment that to attempt to do so would involve considerable difficulties. The point is that the work *leads us to the result*, which is as a matter of fact correct and includes a number of interesting special cases. In particular if $m = 0$, $a < 1$, we obtain

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin a\pi},$$

and if $a = 1$ we obtain

$$\int_0^\infty \frac{e^{-mix}}{1+x}\,dx = \frac{1}{i}\int_0^\infty \frac{e^{-v}\,dv}{m - iv},$$

$$\int_0^\infty \frac{\cos mx}{1+x}\,dx = \int_0^\infty \frac{ve^{-v}\,dv}{m^2+v^2},\quad \int_0^\infty \frac{\sin mx}{1+x}\,dx = \int_0^\infty \frac{me^{-v}\,dv}{m^2+v^2},$$

a pair of formulae due originally to Cauchy†.

Let us consider what our transformations really involve. In summing the series

$$1 + a\left(\frac{i}{m}\right) + a(a+1)\left(\frac{i}{m}\right)^2 + \dots$$

we had to use two repetitions of Borel's process: hence

$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\int_0^\infty \frac{e^{-v}\,dv}{(m-iv)^a}$$

is in reality the equivalent of

$$\Gamma(a)\,e^{-\frac{1}{2}a\pi i}\,m^{-a}\int_0^\infty e^{-v}\,dv\int_0^\infty e^{-w}\,dw\,\Sigma\,a(a+1)\dots(a+n-1)\,\frac{(ivw/m)^n}{(n!)^2},$$

* See, *e.g.*, Bromwich, *Infinite Series*, p. 302.

† 'Mémoire sur les Intégrales Définies,' *Oeuvres*, t. I. p. 377.

or of
$$\int_0^\infty e^{-v}\,dv \int_0^\infty e^{-w}\,dw\, \Sigma\,\Gamma(a+n)\, e^{-\frac{1}{2}(a+n)\pi i}\, m^{-a-n}\frac{(-vw)^n}{(n!)^2},$$

or of
$$\int_0^\infty e^{-v}\,dv \int_0^\infty e^{-w}\,dw \sum_{n=0}^{\infty}\frac{(-vw)^n}{(n!)^2}\lim_{\tau=0}\int_0^\infty e^{-(\tau+im)x}\,x^{a+n-1}\,dx\,;$$

and what we assert is that this is equal to

$$\lim_{\tau=0}\int_0^\infty e^{-(\tau+im)x}\,x^{a-1}\,dx\int_0^\infty e^{-v}\,dv\int_0^\infty e^{-w}\,dw\sum_{n=0}^{\infty}\frac{(-vwx)^n}{(n!)^2}.$$

Now each of these expressions is *an 8-ple repeated limit*, for an infinite integral

$$\int_0^\infty f(x)\,dx$$

is itself a repeated limit. Hence our work is in reality a shorthand representation of a multiple limit permutation of extreme complexity.

§ 17. As a further application of Borel's method let us notice the following. We obtained in § 6 the two formulae

$$G\int_0^\infty F(x)\cos mx\,dx = -\frac{a_1}{m^2}+\frac{3!\,a_3}{m^4}-\frac{5!\,a_5}{m^6}+\ldots,$$

$$G\int_0^\infty F(x)\sin mx\,dx = \frac{a_0}{m}-\frac{2!\,a_2}{m^3}+\frac{4!\,a_4}{m^5}-\ldots,$$

where $F(x)=\Sigma a_n x^n$. If we sum the series on the right by Borel's method we obtain

$$\int_0^\infty e^{-v}\psi_1(v)\,dv,\quad \int_0^\infty e^{-v}\psi_2(v)\,dv,$$

where
$$m\psi_1(v) = -a_1\left(\frac{v}{m}\right)+a_3\left(\frac{v}{m}\right)^3-\ldots = \tfrac{1}{2}i\left\{F\left(\frac{iv}{m}\right)-F\left(-\frac{iv}{m}\right)\right\},$$

$$m\psi_2(v) = a_1 - a_2\left(\frac{v}{m}\right)^2+\ldots = \tfrac{1}{2}\left\{F\left(\frac{iv}{m}\right)+F\left(-\frac{iv}{m}\right)\right\}.$$

We thus obtain the formulae

$$G\int_0^\infty F(x)\cos mx\,dx = \frac{i}{2m}\int_0^\infty e^{-v}\left\{F\left(\frac{iv}{m}\right)-F\left(-\frac{iv}{m}\right)\right\}dv,$$

$$G\int_0^\infty F(x)\sin mx\,dx = \frac{1}{2m}\int_0^\infty e^{-v}\left\{F\left(\frac{iv}{m}\right)+F\left(-\frac{iv}{m}\right)\right\}dv,$$

or
$$G\int_0^\infty F(x)\,e^{mix}\,dx = \frac{i}{m}\int_0^\infty e^{-v}F\left(\frac{iv}{m}\right)dv = i\int_0^\infty e^{-m\eta}F(i\eta)\,d\eta.$$

This is exactly the formula which we obtain by integrating

$$\int F(x)\,e^{mix}dx \qquad (n>0)$$

round the contour formed by the positive parts of the real and imaginary axes and a very large quadrant of a circle, supposing the integrals along the axes convergent in the ordinary sense, and the curvilinear part of the integral evanescent in the limit.

This fact suggests that we must expect errors if $\Sigma a_n x^n$ has only a finite radius of convergence (though summable for all positive values of x), and $F(x)$ has poles situated within or on this contour. It also suggests that when $F(x)$ is integral we may be liable to error when the order of $F(x)$ is not less than unity or at any rate when it is not true that $|e^{mix} F(x)| \to 0$ as $|x| \to \infty$, for any value of $am.x$ between 0 and $\frac{1}{2}\pi$. It is instructive to consider from this general point of view, no less than from that of the precise theorems of the earlier sections, the results

$$\int_0^\infty \cos mx\, J_0(\sqrt{x})\, dx = \frac{1}{4m^2} - \frac{1}{3!}\frac{1}{4^3 m^4} + \frac{1}{5!}\frac{1}{4^5 m^6} - \dots$$

(valid for all positive values of m),

$$\int_0^\infty \cos mx\, J_0(x)\, dx = 0 + 0 + 0 + \dots,$$

$$\int_0^\infty \sin mx\, J_0(x)\, dx = \frac{1}{m} + \frac{1}{2}\frac{1}{m^3} + \frac{1.3}{2.4}\frac{1}{m^5} + \dots,$$

$$\int_0^\infty \cos mx\, e^{-x}\, dx = \frac{1}{m^2} - \frac{1}{m^4} + \frac{1}{m^6} - \dots,$$

$$\int_0^\infty \sin mx\, e^{-x}\, dx = \frac{1}{m} - \frac{1}{m^3} + \frac{1}{m^5} - \dots$$

(valid for $m > 1$ only),

and

$$\int_0^\infty \cos mx\, e^{-x^2}\, dx = 0 + 0 + 0 + \dots,$$

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = 0 + 0 + 0 + \dots$$

(valid for no value of m).

B. *Continuity of generalised integrals which contain a continuous parameter.*

§ 18. I shall now consider the generalised integral

$$G\int_0^\infty f(x, \alpha)\, dx \quad \dots\dots\dots\dots(1),$$

and the question of its continuity for a particular value of α, which we may suppose to be zero.

The integral (1) will be continuous for $\alpha = 0$ if

$$\left.\begin{aligned} \lim_{\alpha\to 0} G\int_0^\infty f(x,\alpha)\, dx &= \lim_{\alpha\to 0}\lim_{\tau\to 0}\int_0^\infty e^{-\tau x} f(x,\alpha)\, dx \\ &= \lim_{\tau\to 0}\lim_{\alpha\to 0}\int_0^\infty e^{-\tau x} f(x,\alpha)\, dx \\ &= \lim_{\tau\to 0}\int_0^\infty e^{-\tau x} f(x,0)\, dx \\ &= G\int_0^\infty f(x,0)\, dx \end{aligned}\right\} \quad \dots\dots\dots\dots(2).$$

The applications of this transformation do not appear to be so interesting as those of the transformation of § 2. I shall therefore not discuss its legitimacy in great detail.

The most interesting case appears to be the following. Suppose that $f(x, \alpha)$ is a continuous function of both variables throughout the rectangle

$$(0, \alpha_0;\ 0, X),$$

where α_0 is some positive value of α, and X any positive value of X. Suppose also that for $0 \leqq \alpha \leqq \alpha_0$, and all positive values of x,

$$|f(x, \alpha)| < Hx^K,$$

where H and K are constants.

Further suppose that
$$\int_0^\infty f(x, \alpha)\, dx$$
is convergent for $\alpha > 0$, and that
$$G \int_0^\infty f(x, 0)\, dx$$
is summable*.

Then
$$\lim_{\tau \to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \int_0^\infty f(x, \alpha)\, dx,$$
$$\lim_{\tau \to 0} \int_0^\infty e^{-\tau x} f(x, 0)\, dx = G \int_0^\infty f(x, 0)\, dx,$$
so that the first and last steps of the argument expressed by the equations (2) are justified. Further, it is easy to see that, for any particular positive value of τ, the integral
$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx$$
is uniformly convergent throughout the interval $(0, \alpha_0)$. Hence
$$\lim_{\alpha \to 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \int_0^\infty e^{-\tau x} f(x, 0)\, dx.$$
Thus the last step but one of the argument is justified. Thus if we write
$$\int_0^\infty e^{-\tau x} f(x, \alpha)\, dx = \phi(\tau, \alpha)$$
the whole question reduces to the question whether
$$\lim_{\alpha \to 0} \lim_{\tau \to 0} \phi(\tau, \alpha) = \lim_{\tau \to 0} \lim_{\alpha \to 0} \phi(\tau, \alpha) \quad \text{..............................(3).}$$
We may notice that we are already assured of the existence of the second repeated limit and of the inner limit on the left-hand side.

Now Mr Bromwich† has enunciated the following necessary and sufficient conditions for the truth of (3):—

(i) the simple limits
$$\phi(\tau) = \lim_{\alpha \to 0} \phi(\tau, \alpha),$$
$$\phi(\alpha) = \lim_{\tau \to 0} \phi(\tau, \alpha)$$
exist,

* It seems better, on account of the ambiguity of the uses of the term *divergent*, to call a generalised integral *summable* than *convergent*.

† *Proc. Lond. Math. Soc.*, N.S., vol. I. p. 184, and vol. VI. p. 119. See also Hobson, *Proc. Lond. Math. Soc.*, N.S., vol. V. p. 225, and *Theory of Functions of a Real Variable*, pp. 303—311 and 464—467.

(ii) the repeated limit

$$\lim_{\tau\to 0} \phi(\tau)$$

exists,

(iii) given ϵ and τ_0 (each positive but as small as we please) we can choose a positive value of τ less than τ_0, and a positive value of α_0, so that

$$|\phi(\alpha) - \phi(\tau, \alpha)| < \epsilon$$

for the one value τ and every positive α less than α_0.

In the present case the first two conditions are certainly satisfied, and the third is equivalent to

$$\left|\int_0^\infty f(x, \alpha)(1 - e^{-\tau x})\,dx\right| < \epsilon$$

for one positive τ less than τ_0 and every positive α less than α_0.

We have therefore:

THEOREM IV. *If $f(x, \alpha)$ is a function of x and α continuous for $0 \leqq x \leqq X$, $0 \leqq \alpha \leqq \alpha_0$, however great X may be, and numerically less than Hx^K for all these values of x and α; if moreover $\int_0^\infty f(x, \alpha)\,dx$ $(\alpha > 0)$, is convergent, and $G\int_0^\infty f(x, 0)\,dx$ summable; and if finally, however small be ϵ and τ_0, we can find τ so that $0 < \tau < \tau_0$ and*

$$\left|\int_0^\infty f(x, \alpha)(1 - e^{-\tau x})\,dx\right| < \epsilon, \qquad (0 < \alpha \leqq \alpha_0)$$

then

$$\lim_{\alpha\to 0}\int_0^\infty f(x, \alpha)\,dx = G\int_0^\infty f(x, 0)\,dx.$$

§ 19. The most interesting case of this theorem is the following:

Let

$$f(x, \alpha) = \phi(\alpha x)\, e^{-mix} x^\mu,$$

where $m > 0$, $\mu > -1$, and $\phi(u)$ is a function of u which possesses the following properties:

(1) as $u \to \infty$, $\phi(u) \to 0$, and that more rapidly than any power of u;

(2) $\phi(u)$ has continuous derivatives $\phi'(u), \phi''(u), \ldots \phi^{(n+1)}(u)$, where $n > \mu$;

(3) the integral

$$\int_0^\infty \phi^{(n+1)}(u)\, u^\mu\, du$$

is absolutely convergent.

Then it can be shown that the conditions of Theorem IV are satisfied: but the proof rests on a number of preliminary results.

§ 20. Let

$$\chi_0(x, \tau) = e^{-(\tau + mi)x} x^\mu,$$

where $\tau > 0$. Further let

$$\chi_1(x, \tau) = \int_x^\infty \chi_0(t, \tau)\,dt,$$

$$\chi_2(x, \tau) = \int_x^\infty \chi_1(t, \tau)\,dt,$$

....

Then it is easy to prove, by a repeated integration by parts, that, if $a = \tau + mi$, we have

$$\chi_1(x,\tau) = \frac{e^{-ax}x^\mu}{a}\left\{1 + \frac{\mu}{ax} + \frac{\mu(\mu-1)}{(ax)^2} + \dots + \frac{\mu(\mu-1)\dots(\mu-r+1)}{(ax)^r}\right\} + \frac{\mu(\mu-1)\dots(\mu-r)}{a^r}\int_x^\infty e^{-at}t^{\mu-r-1}dt \dots (4),$$

where r is any positive integer (we may set aside the case in which μ is a positive integer, when $\chi_1(x,\tau)$ can be found in finite terms). In fact $\chi_1(x,\tau)$ has the asymptotic expansion

$$\frac{e^{-ax}x^\mu}{a}\left\{1 + \frac{\mu}{ax} + \frac{\mu(\mu-1)}{(ax)^2} + \dots\right\} \dots\dots (5).$$

We can find an expression of the same kind for $\chi_k(x,\tau)$, k being any positive integer. For it is easy to prove* that

$$\begin{aligned}\chi_k(x,\tau) &= \frac{1}{(k-1)!}\int_x^\infty e^{-at}t^\mu(t-x)^{k-1}dt \\ &= \sum_{\lambda=0}^{k-1}\frac{(-x)^\lambda}{\lambda!(k-1-\lambda)!}\int_x^\infty e^{-at}t^{\mu+k-1-\lambda}dt \\ &= \sum_{\lambda=0}^{k-1}\frac{(-x)^\lambda}{\lambda!(k-1-\lambda)!}\left[\frac{e^{-ax}x^{\mu+k-1-\lambda}}{a}\left\{1 + \frac{\mu+k-1-\lambda}{ax} + \dots \right.\right. \\ &\qquad \left. + \frac{(\mu+k-1-\lambda)(\mu+k-2-\lambda)\dots(\mu+k-r-\lambda)}{(ax)^r}\right\} \\ &\qquad \left. + \frac{(\mu+k-1-\lambda)\dots(\mu+k-r-1-\lambda)}{a^r}\int_x^\infty e^{-at}t^{\mu+k-r-2-\lambda}dt\right] \dots (6).\end{aligned}$$

This furnishes for $\chi_k(x,\tau)$ the asymptotic expansion

$$\frac{e^{-ax}x^{\mu+k-1}}{a}\sum\frac{A_\nu}{(ax)^\nu},$$

where $$A_\nu = \sum_{\lambda=0}^{k-1}\frac{(-1)^\lambda}{\lambda!(k-1-\lambda)!}(\mu+k-1-\lambda)(\mu+k-2-\lambda)\dots(\mu+k-\nu-\lambda).$$

It is, however, easy to prove that

$$A_0 = A_1 = \dots = A_{k-2} = 0,\ A_{k-1} = 1.$$

For $$\begin{aligned}A_\nu &= \frac{1}{(k-1)!}\sum_{\lambda=0}^{k-1}(-1)^\lambda\binom{k-1}{\lambda}\left[\left(\frac{d}{dx}\right)^\nu x^{\mu+k-1-\lambda}\right]_{x=1} \\ &= \frac{1}{(k-1)!}\left[\left(\frac{d}{dx}\right)^\nu x^{\mu+k-1}\left(1-\frac{1}{x}\right)^{k-1}\right]_{x=1} \\ &= \frac{1}{(k-1)!}\left[\left(\frac{d}{dx}\right)^\nu x^\mu(x-1)^{k-1}\right]_{x=1},\end{aligned}$$

from which the result follows at once. Thus for large values of x

$$\chi_k(x,\tau) \sim \frac{e^{-(\tau+mi)x}x^\mu}{(\tau+mi)^k},$$

and $$|\chi_k(x,\tau)| < Kx^\mu$$

for all positive values of x and τ.

Now let $$\chi_0(x,0) = \lim_{\tau\to 0}\chi_0(x,\tau) = e^{-mix}x^\mu.$$

* Jordan, *Cours d'Analyse*, t. III. p. 59.

Then (see p. 59 of my paper in the *Quarterly Journal* already quoted) we know that

$$\chi_1(x, 0) = G\int_x^\infty \chi_0(t, 0)\,dt$$

is summable, and that

$$\chi_1(x, 0) = \lim_{\tau \to 0} \chi_1(x, \tau).$$

Now suppose, in the equation (4), that $r > \mu$, and make τ tend to zero. Clearly we obtain in the limit

$$\chi_1(x, 0) = \frac{e^{-mix}x^\mu}{mi}\left\{1 + \frac{\mu}{mix} + \frac{\mu(\mu-1)}{(mix)^2} + \dots + \frac{\mu(\mu-1)\dots(\mu-r+1)}{(mix)^r}\right\} + \frac{\mu(\mu-1)\dots(\mu-r)}{(mi)^r}\int_x^\infty e^{-mit}t^{\mu-r-1}dt \dots (4)'.$$

Since $r > \mu$, we can determine a positive value of ϵ such that

$$\mu - r - 1 + \epsilon < -1,$$

and

$$\left|\int_x^\infty e^{-mit}t^{\mu-r-1}dt\right| = \left|x^{\mu-r-1+\epsilon}\int_x^\xi e^{-mit}t^{-\epsilon}dt\right| < Kx^{-1-\alpha}. \qquad (\xi > x,\ \alpha > 0)$$

It follows that the last term in the equation (4)′ possesses an absolutely convergent integral up to ∞. The other terms on the right-hand side possess generalised integrals up to ∞. Hence $\chi_1(x, 0)$ possesses a generalised integral up to ∞, and we may write

$$\chi_2(x, 0) = G\int_x^\infty \chi_1(t, 0)\,dt.$$

It is, moreover, not difficult to prove that

$$\lim_{t \to 0} \chi_2(x, \tau) = \chi_2(x, 0).$$

This point, however, does require proof, for

$$\lim_{\tau \to 0} \chi_2(x, \tau) = \lim_{\tau \to 0} \int_x^\infty \chi_1(t, \tau)\,dt$$

and

$$\chi_2(x, 0) = \lim_{\tau \to 0} \int_x^\infty e^{-\tau t}\chi_1(t, 0)\,dt$$

are not defined in the same manner. But it is easy to show that the two limits are the same. For

$$\begin{aligned}\lim_{\tau \to 0} \chi_2(x, \tau) &= \lim_{\tau \to 0} \int_x^\infty \chi_1(t, \tau)\,dt\\ &= \lim_{\tau \to 0}\left[\int_x^\infty \left\{\sum_{s=0}^{r} \frac{\mu(\mu-1)\dots(\mu-s+1)}{a^{s+1}} e^{-at}t^{\mu-s} + \frac{\mu(\mu-1)\dots(\mu-r)}{a^{r+1}}\int_t^\infty e^{-au}u^{\mu-r-1}du\right\}\right]\end{aligned}$$

and

$$\begin{aligned}\chi_2(x, 0) &= \lim_{\tau \to 0} \int_x^\infty e^{-\tau t}\chi_1(t, 0)\,dt\\ &= \lim_{\tau \to 0}\left[\int_x^\infty \left\{\sum_{s=0}^{r} \frac{\mu(\mu-1)\dots(\mu-s+1)}{(mi)^{s+1}} e^{-at}t^{\mu-s} + \frac{\mu(\mu-1)\dots(\mu-r)}{(mi)^{r+1}} e^{-\tau t}\int_t^\infty e^{-miu}u^{\mu-r-1}du\right\}\right].\end{aligned}$$

If we suppose $r > \mu$, the integral in the last line is absolutely convergent in the ordinary sense. And all that we have to show is that

$$\lim_{\tau \to 0} \int_x^\infty dt \int_t^\infty e^{-(\tau+mi)u} u^{\mu-r-1} du = \lim_{\tau \to 0} \int_x^\infty e^{-\tau t} dt \int_t^\infty e^{-miu} u^{\mu-r-1} du;$$

and since r may be as large as we like, this is very easily proved. For if $r > \mu + 1$ it follows, by comparison with the integral $\int_x^\infty dt \int_t^\infty u^{\mu-r-1} du$, that each of the integrals in question is uniformly convergent in an interval including $\tau = 0$, so that they have the common limit

$$\int_x^\infty dt \int_t^\infty e^{-miu} u^{\mu-r-1} du.$$

Thus $\chi_2(x, \tau) \to \chi_2(x, 0)$ as $\tau \to 0$; and we can now prove, precisely on the lines of the deduction of (1′) from 1, that $\chi_2(x, 0)$ possesses an asymptotic expansion precisely similar to that of $\chi_2(x, \tau)$, τ being replaced by 0, i.e. mi written for $\tau + mi$. It is clear that we can proceed indefinitely in this way, and so establish the existence of a series of functions

$$\chi_0(x, 0), \quad \chi_1(x, 0) = G \int_x^\infty \chi_0(t, 0)\, dt, \ldots, \quad \chi_k(x, 0) = G \int_x^\infty \chi_{k-1}(t, 0)\, dt, \ldots,$$

such that

$$\lim_{\tau \to 0} \chi_k(x, \tau) = \chi_k(x, 0),$$

and $\chi_k(x, 0)$ possesses an asymptotic expansion deducible from that of $\chi_k(x, \tau)$ by merely replacing τ in it by zero. In particular $\chi_k(x, 0)$ satisfies the same inequality

$$|\chi_k(x, 0)| < K x^\mu$$

that we found to be satisfied by $\chi_k(x, \tau)$.

It may be observed that $\qquad \chi_k(0, \tau), \ \chi_k(0, 0)$

can be found in finite terms. For if we integrate by parts and observe that

$$\lim_{x \to \infty} x^\alpha \chi_\nu(x, \tau) = 0 \qquad (\tau > 0)$$

for all values of α and ν, we see that

$$\begin{aligned}\chi_k(0, \tau) &= \int_0^\infty \chi_{k-1}(x, \tau)\, dx = \int_0^\infty x \chi_{k-2}(x, \tau)\, dx \\ &= \int_0^\infty \frac{x^2}{2!} \chi_{k-3}(x, \tau)\, dx = \ldots \\ &= \int_0^\infty \frac{x^{k-1}}{(k-1)!} \chi_0(x, \tau)\, dx = \frac{1}{(k-1)!} \int_0^\infty e^{-(\tau+mi)x} x^{\mu+k-1} dx \\ &= \frac{\Gamma(\mu+k)}{\Gamma(k)} (\tau + mi)^{-\mu-k}.\end{aligned}$$

Hence also $$\chi_k(0, 0) = \frac{\Gamma(\mu+k)}{\Gamma(k)} m^{-\mu-k} e^{-\frac{1}{2}(\mu+k)\pi i}.$$

§ 21. We are now in a position to establish the result enunciated in § 19. For

$$\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx = \int_0^\infty \phi(\alpha x)(1-e^{-\tau x})\,e^{-mix}x^\mu\,dx$$

$$= \int_0^\infty \phi(\alpha x)\{\chi_0(x, 0)-\chi_0(x, \tau)\}\,dx$$

$$= \phi(0)\{\chi_1(0, 0)-\chi_1(0, \tau)\} + \int_0^\infty \alpha\phi'(\alpha x)\{\chi_1(x, 0)-\chi_1(x, \tau)\}\,dx$$

$$= \sum_{r=0}^{n} \alpha^r \phi^{(r)}(0)\{\chi_{r+1}(0, 0)-\chi_{r+1}(0, \tau)\}$$

$$+ \int_0^\infty \alpha^{n+1}\phi^{(n+1)}(\alpha x)\{\chi_{n+1}(x, 0)-\chi_{n+1}(x, \tau)\}\,dx.$$

But the last integral is in absolute value less than

$$K\int_0^\infty \alpha^{n+1}|\phi^{(n+1)}(\alpha x)|\,x^\mu\,dx = K\alpha^{n-\mu}\int_0^\infty |\phi^{(n+1)}(u)|\,u^\mu\,du,$$

so that

$$\left|\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx\right| < \sum_{r=0}^{n}\alpha^r|\phi^{(r)}(0)|\,|\chi_{r+1}(0, 0)-\chi_{r+1}(0, \tau)| + K\alpha^{n-\mu};$$

and so the condition of Theorem IV is clearly satisfied.

Hence we obtain

THEOREM V. *If $\phi(u)$ is a function of u which tends to zero, as $u \to \infty$, more rapidly than any power of u, and has continuous derivatives $\phi'(u), \phi''(u), \dots \phi^{(n+1)}(u)$, and*

$$\int_0^\infty \phi^{n(+1)}(u)\,u^\mu\,du$$

is absolutely convergent, then

$$\lim_{\alpha\to 0}\int_0^\infty e^{-mix}x^\mu\phi(\alpha x)\,dx = \phi(0)\,G\int_0^\infty e^{-mix}x^\mu\,dx = \phi(0)\,\Gamma(\mu+1)\,m^{-\mu-1}e^{-\frac{1}{2}(\mu+1)\pi i},$$

provided $m > 0$, $\mu > -1$.

Examples of the preceding theorem are given by supposing $\phi(u) = e^{-u}$ (in which case the result is obvious), $\phi(u) = e^{-u^2}$, $\phi(u) = \operatorname{sech} u$, etc.

§ 22. The case in which $\mu = 0$ is of especial interest. In this case we have

$$\int_0^\infty f(x, \alpha)(1-e^{-\tau x})\,dx = \int_0^\infty \phi(\alpha x)\{e^{-mix}-e^{-(\tau+mi)x}\}\,dx$$

$$= \left(\frac{1}{mi}-\frac{1}{\tau+mi}\right)\phi(0) + \int_0^\infty \alpha\phi'(\alpha x)\left\{\frac{e^{-mix}}{mi}-\frac{e^{-(\tau+mi)x}}{\tau+mi}\right\}dx$$

$$= \left(\frac{1}{mi}-\frac{1}{\tau+mi}\right)\phi(0) + \left\{\frac{1}{(mi)^2}-\frac{1}{(\tau+mi)^2}\right\}\alpha\phi'(0)$$

$$+ \int_0^\infty \alpha^2\phi''(\alpha x)\left\{\frac{e^{-mix}}{(mi)^2}-\frac{e^{-(\tau+mi)x}}{(\tau+mi)^2}\right\}dx,$$

all that we have assumed so far being that $\phi'(u)$ and $\phi''(u)$ are continuous and tend to zero as $u \to \infty$, and that the original integral is convergent. If in addition

$$\int_0^\infty \phi''(u)\, du$$

is absolutely convergent, it follows as in the general case that the conditions of Theorem IV are satisfied, and that

$$\lim_{\alpha \to 0} \int_0^\infty \cos mx \phi(\alpha x)\, dx = 0 \quad \text{.................................(7)},$$

$$\lim_{\alpha \to 0} \int_0^\infty \sin mx \phi(\alpha x)\, dx = \frac{1}{m}\phi(0) \quad \text{........................(8)}.$$

A particular case in which $\int_0^\infty \phi''(u)\, du$ is certainly absolutely convergent is that in which $\phi''(u)$ changes sign only a finite number of times.

As examples we have

$$\lim_{\alpha \to 0} \int_0^\infty \cos mx e^{-(\alpha x)^2} dx = \lim_{\alpha \to 0} \frac{\sqrt{\pi}}{2\alpha} e^{-(m/2\alpha)^2} = 0,$$

$$\lim_{\alpha \to 0} \int_0^\infty \sin mx e^{-(\alpha x)^2} dx = \lim_{\alpha \to 0} \frac{1}{\alpha} e^{-(m/2\alpha)^2} \int_0^{m/2\alpha} e^{t^2} dt = \frac{1}{m},$$

$$\lim_{\alpha \to 0} \int_0^\infty \frac{\cos mx}{1 + \alpha^2 x^2}\, dx = \lim_{\alpha \to 0} \frac{\pi}{2\alpha} e^{-m/\alpha} = 0,$$

$$\lim_{\alpha \to 0} \int_0^\infty \frac{\sin mx}{1 + \alpha x}\, dx = \lim_{\alpha \to 0} \left\{ \frac{1}{\alpha} \cos\left(\frac{m}{\alpha}\right) \int_{m/\alpha}^\infty \frac{\sin u}{u}\, du - \frac{1}{\alpha} \sin\left(\frac{m}{\alpha}\right) \int_{m/\alpha}^\infty \frac{\cos u}{u}\, du \right\}$$

$$= \lim_{\alpha \to 0} \left\{ \frac{1}{\alpha} \cos^2\left(\frac{m}{\alpha}\right) \cdot \frac{\alpha}{m} + \frac{1}{\alpha} \sin^2\left(\frac{m}{\alpha}\right) \cdot \frac{\alpha}{m} \right\} = \frac{1}{m}.$$

It is also instructive to notice that the result of the theorem is true when

$$\phi(u) = \frac{\sin u}{u}, \quad J_0(\sqrt{u}), \quad J_0(u),$$

since

$$\int_0^\infty \frac{\sin \alpha x}{\alpha x} \cos mx\, dx = 0, \qquad \int_0^\infty \frac{\sin \alpha x}{\alpha x} \sin mx\, dx = \frac{1}{2\alpha} \log\left(\frac{m+\alpha}{m-\alpha}\right),$$

$$\int_0^\infty J_0(\sqrt{\alpha x}) \cos mx = \frac{1}{m} \sin\left(\frac{\alpha}{4m}\right), \qquad \int_0^\infty J_0(\sqrt{\alpha x}) \sin mx\, dx = \frac{1}{m} \cos\left(\frac{\alpha}{4m}\right),$$

$$\int_0^\infty J_0(\alpha x) \cos mx\, dx = 0, \qquad \int_0^\infty J_0(\alpha x) \sin mx\, dx = \frac{1}{\sqrt{(m^2 - \alpha^2)}},$$

which tend to the prescribed limits as $\alpha \to 0$. But in these cases the conditions which we have laid down are not satisfied, the integral $\int_0^\infty \phi''(u)\, du$ not being absolutely convergent.

In the case of the integrals

$$\int_0^\infty \cos(\alpha x)^2 \frac{\cos mx}{\sin mx}\, dx = \frac{1}{2\alpha} \sqrt{(\tfrac{1}{2}\pi)} \left\{ \cos\left(\frac{m}{2\alpha}\right)^2 \pm \sin\left(\frac{m}{2\alpha}\right)^2 \right\}$$

the conditions are not satisfied and the result does not hold.

These examples naturally suggest that the conditions of this section may be generalised. Indeed a variety of generalisations of Theorems IV and V are naturally suggested: but I shall be content with investigating the simplest and most obvious cases.

Uniform summability and continuity.

§ 23. We have not, so far, used the idea of *uniform* summability of

$$G\int_0^\infty f(x, \alpha)\,dx \quad \text{...(9).}$$

We shall naturally say that the integral is uniformly summable if

$$\int_0^\infty e^{-\tau x} f(x, \alpha)\,dx \text{ ....(10)}$$

tends to a limit as $\tau \rightarrow 0$, uniformly for all values of α in question.

If, as in Theorem IV, $f(x, \alpha)$ is a continuous function of both variables, and

$$|f(x, \alpha)| < H_x^K,$$

the integral (10) is for any positive τ, uniformly convergent and continuous: and so (9) is also continuous.

This test is, however, less general than that of Theorem IV, at any rate in the only case to which that theorem applies, viz. that in which

$$\int_0^\infty f(x, \alpha)\,dx$$

is convergent for $\alpha > 0$. For if

$$\lim_{\tau \rightarrow 0} \int_0^\infty e^{-\tau x} f(x, \alpha)\,dx = \int_0^\infty f(x, \alpha)\,dx$$

uniformly for all positive values of α, we can, given τ, find τ_0 so that

$$\left| \int_0^\infty f(x, \alpha)\,(1 - e^{-\tau x})\,dx \right| < \epsilon,$$

for *all positive values of* τ *less than* τ_0 *and all positive values of* α, and this is more than the test of Theorem IV demands. At the same time this more stringent test is satisfied in the cases considered in the preceding paragraphs, and such an integral as

$$G\int_0^\infty \phi(\alpha x)\, e^{-mix} x^\mu\,dx$$

is uniformly summable throughout the interval $0 \leqq \tau \leqq \tau_0$.

§ 24. Before passing on to other questions I may point out the simplest example of a *discontinuous* generalised integral, viz.

$$\begin{aligned} G\int_0^\infty \alpha \sin \alpha x\,dx &= 1 \qquad (\alpha \neq 0) \\ &= 0. \qquad (\alpha = 0) \end{aligned}$$

It is easy to see that in this case the condition for uniform summability is not satisfied, since

$$\int_0^\infty \alpha e^{-\tau x} \sin \alpha x\,dx = \frac{\alpha^2}{\alpha^2 + \tau^2}$$

has the limit 1 if $\alpha \neq 0$ and the limit 0 if $\alpha = 0$. Also to make

$$1 - \frac{\alpha^2}{\alpha^2 + \tau^2} = \frac{\tau^2}{\alpha^2 + \tau^2} < \epsilon$$

we must take $\tau < \alpha\sqrt{\{\epsilon/(1-\epsilon)\}}$, which cannot be effected by a choice of τ independent of α. Similarly

$$G\int_0^\infty \alpha^{\mu+1}\cos\alpha x\, x^\mu\, dx = \Gamma(\mu+1)\cos\tfrac{1}{2}\mu\pi,\quad G\int_0^\infty \alpha^{\mu+1}\sin\alpha x\, x^\mu\, dx = \Gamma(\mu+1)\sin\tfrac{1}{2}\mu\pi$$

are discontinuous for $\alpha = 0$.

The integral

$$G\int_0^\infty \alpha\cos\alpha x\, dx = 0$$

is continuous, but not uniformly convergent. For

$$\int_0^\infty \alpha e^{-\tau x}\cos\alpha x\, dx = \frac{\alpha\tau}{\alpha^2+\tau^2},$$

which has the limit zero for all values of α, but does not approach its limit uniformly.

C. *Differentiation with respect to a parameter.*

§ 25. Let us next consider the equation

$$\frac{d}{d\alpha}G\int_0^\infty f(x,\alpha)\, dx = G\int_0^\infty \frac{\partial f}{\partial\alpha}\, dx.$$

This rests upon the equations

$$\begin{aligned}\frac{d}{d\alpha}G\int_0^\infty f(x,\alpha)\, dx &= \frac{d}{d\alpha}\lim_{\tau\to 0}\int_0^\infty e^{-\tau x}f(x,\alpha)\, dx\\ &= \lim_{\tau\to 0}\frac{d}{d\alpha}\int_0^\infty e^{-\tau x}f(x,\alpha)\, dx\\ &= \lim_{\tau\to 0}\int_0^\infty e^{-\tau x}\frac{\partial f}{\partial\alpha}\, dx\\ &= G\int_0^\infty \frac{\partial f}{\partial\alpha}\, dx.\end{aligned}$$

It is however, in dealing with the question of differentiation, more convenient to adopt the alternative form of the definition of the generalised integral, viz.*

$$G\int_0^\infty f(x)\, dx = \int_0^\infty dt\int_0^\infty xe^{-tx}f(x)\, dx,$$

which is equivalent to the definition hitherto followed in all cases of any practical interest. For if, as we shall throughout suppose,

$$\lim_{x\to\infty} e^{-\tau x}f(x) = 0$$

for any positive value of τ, it is easy to see that

$$\begin{aligned}\int_\tau^T dt\int_0^\infty xe^{-tx}f(x)\, dx &= \int_0^\infty xf(x)\, dx\int_\tau^T e^{-tx}\, dx\\ &= \int_0^\infty (e^{-\tau x}-e^{-Tx})\, f(x)\, dx,\end{aligned}$$

and

$$\int_0^\infty dt\int_0^\infty xe^{-tx}f(x)\, dx = \lim_{\tau\to 0}\int_0^\infty e^{-\tau x}f(x)\, dx,$$

if, and only if, the latter limit exists.

* *Quarterly Journal, loc. cit.* p. 50.

The transformation which we have to justify is then

$$\begin{aligned}\frac{d}{d\alpha} G\int_0^\infty f(x, \alpha)\, dx &= \frac{d}{d\alpha}\int_0^\infty dt \int_0^\infty xe^{-tx} f(x, \alpha)\, dx \\ &= \int_0^\infty dt \frac{\partial}{\partial\alpha}\int_0^\infty xe^{-tx} f(x, \alpha)\, dx \\ &= \int_0^\infty dt \int_0^\infty xe^{-tx}\frac{\partial f}{\partial\alpha}\, dx \\ &= G\int_0^\infty \frac{\partial f}{\partial\alpha}\, dx,\end{aligned}$$

which rests upon a double application of Leibniz's theorem.

Let us suppose now that $\dfrac{\partial f}{\partial\alpha}$ is a continuous function of x and α and that

$$e^{-\tau x}\frac{\partial f}{\partial\alpha},$$

where τ has any positive value, tends to zero, as $x \to \infty$, uniformly throughout an interval

$$(\alpha_0 - H,\ \alpha_0 + H).$$

Then
$$\int_0^\infty xe^{-tx}\frac{\partial f}{\partial\alpha}\, dx$$

is uniformly convergent throughout the region

$$0 < \tau_0 \leqq \tau, \quad \alpha_0 - H \leqq \alpha \leqq \alpha_0 + H.$$

For
$$\left|\frac{\partial f}{\partial\alpha}\right| < Ke^{\frac{1}{2}\tau_0 x},$$

and so
$$\left|\int_X^{X'} xe^{-tx}\frac{\partial f}{\partial\alpha}\, dx\right| < \int_X^{X'} xe^{-\frac{1}{2}\tau_0 x}\, dx.$$

From this it follows (1) that, for any positive value of t,

$$\frac{\partial}{\partial\alpha}\int_0^\infty xe^{-tx} f(x, \alpha)\, dx = \int_0^\infty xe^{-tx}\frac{\partial f}{\partial\alpha}\, dx,$$

(2) that each side of this equation is a continuous function of t and α throughout the region of values just defined. Now let

$$F(t, \alpha) = \int_0^\infty xe^{-tx} f(x, \alpha)\, dx.$$

Then $\dfrac{\partial F}{\partial\alpha}$ is a continuous function of both variables throughout the region; and so a sufficient further condition for the truth of the equation

$$\frac{d}{d\alpha}\int_0^\infty F(t, \alpha)\, dt = \int_0^\infty \frac{\partial F}{\partial\alpha}\, dt,$$

is that the latter integral should be uniformly convergent, i.e. that it should be possible to make

$$\left|\int_\tau^{\tau_1}\frac{\partial F}{\partial\alpha}\, dt\right| < \epsilon, \quad \left|\int_{T_1}^{T}\frac{\partial F}{\partial\alpha}\, dt\right| < \epsilon,$$

for $0 < \tau < \tau_1$ and $T > T_1$ respectively, by a choice of τ_1 and T_1 independent of α.

Now if t_1 and t_2 are any positive values of t

$$\int_{t_1}^{t_2} \frac{\partial F}{\partial \alpha}\, dt = \int_{t_1}^{t_2} dt \int_0^\infty x e^{-tx} \frac{\partial f}{\partial \alpha}\, dx$$
$$= \int_0^\infty (e^{-t_1 x} - e^{-t_2 x}) \frac{\partial f}{\partial \alpha}\, dx,$$

the inversion of integrations being easily justified on the hypotheses that were made above concerning $\frac{\partial f}{\partial \alpha}$. Hence our conditions take the form

$$\left| \int_0^\infty (e^{-\tau x} - e^{-\tau_0 x}) \frac{\partial f}{\partial \alpha}\, dx \right| < \epsilon, \quad \left| \int_0^\infty (e^{-T_0 x} - e^{-Tx}) \frac{\partial f}{\partial \alpha}\, dx \right| < \epsilon.$$

The second condition can obviously be satisfied: the first can be satisfied if

$$\int_0^\infty e^{-tx} \frac{\partial f}{\partial \alpha}\, dx,$$

tends to a limit, as $t \to 0$, uniformly for $\alpha_0 - H \leqq \alpha \leqq \alpha_0 + H$. Hence we obtain

THEOREM VI. *If f and $\frac{\partial f}{\partial \alpha}$ are continuous functions of x and α, for $\alpha_0 - H \leqq \alpha \leqq \alpha_0 + H$ and all positive values of x; if further $e^{-\tau x}\frac{\partial f}{\partial \alpha}$ tends uniformly to zero as $x \to \infty$, for any positive value of τ; if finally $G\int_0^\infty \frac{\partial f}{\partial \alpha}\, dx$ is uniformly summable, then*

$$\frac{d}{d\alpha} G \int_0^\infty f(x, \alpha)\, dx = G \int_0^\infty \frac{\partial f}{\partial \alpha}\, dx,$$

for $\alpha = \alpha_0$.

§ 26. *Examples of differentiation.* (i) The integrals

$$G\int_0^\infty x^{a-1} e^{-mix}\, dx, \quad G\int_0^\infty x^\alpha J_\alpha(x) e^{-mix}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

are summable if $a > 0$, $\alpha > -\frac{1}{2}$. Moreover, if n is any positive integer, the integrals

$$G\int_0^\infty x^{a+n-1} e^{-mix}\, dx, \quad G\int_0^\infty x^{\alpha+n} J_\alpha(x) e^{-mix}\, dx$$

are uniformly summable throughout any interval of values of m which does not include $m = 0$, as appears directly from the analysis by which they are evaluated. Hence the integrals (1) can be differentiated any number of times with respect to m, as may be immediately verified.

(ii) If
$$I(\alpha) = \int_0^\infty \frac{\cos \beta x - \cos \alpha x}{x} f(\sin^2 x)\, dx,$$

where α and β are positive, and f is continuous, we find that

$$\frac{dI}{d\alpha} = G\int_0^\infty \sin \alpha x f(\sin^2 x)\, dx$$
$$= \frac{1}{\sin \frac{1}{2}\alpha\pi} \int_0^\pi \cos \alpha x f(\sin^2 x)\, dx,$$

the uniform summability of the derived integral following from the analysis by which its value is found*.

In particular we find

$$\frac{d}{d\alpha}\int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\,dx = G\int_0^\infty \sin\alpha x\,dx = \frac{1}{\alpha},$$

$$\int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\,dx = \log\left(\frac{\alpha}{\beta}\right).$$

The result may be extended to cover the case in which f becomes infinite for certain values of x in such a way that

$$\int_0^\pi f(\sin^2 x)\,dx$$

is convergent.

Similarly, if $$I(\alpha) = \int_0^\infty \frac{\cos\beta x - \cos\alpha x}{x}\log x\,dx,$$

we find $$\frac{dI}{d\alpha} = G\int_0^\infty \sin\alpha x\log x\,dx = -\frac{1}{\alpha}(\log\alpha + \gamma),$$

where γ is Euler's constant, and so

$$I = \gamma\log\left(\frac{\alpha}{\beta}\right) + \tfrac{1}{2}\{(\log\alpha)^2 - (\log\beta)^2\}.$$

(iii) If $$I(\alpha) = \int_0^\infty \frac{\sin\alpha x}{x} f(\sin^2 x)\,dx,$$

$$\frac{dI}{d\alpha} = G\int_0^\infty \cos\alpha x\, f(\sin^2 x)\,dx = 0,$$

unless α is an even integer $2n$†. Hence $I(\alpha)$ is constant for $2n < \alpha < 2(n+1)$, and so

$$I(\alpha) = \int_0^\infty \frac{\sin(2n+1)x}{x} f(\sin^2 x)\,dx,$$

which is easily found to be equal to

$$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)x}{\sin x} f(\sin^2 x)\,dx.$$

In particular, if $0 < \alpha < 2$,

$$\int_0^\infty \frac{\sin\alpha x}{x} f(\sin^2 x)\,dx = \tfrac{1}{2}\pi\int_0^\pi f(\sin^2 x)\,dx.$$

(iv) Suppose (as in § 20) that

$$\chi_0(x, \tau) = e^{-(\tau+mi)x}x^\mu,$$

and consider the integral

$$I(\tau, m, \beta) = \int_0^\infty \chi_0(x, \tau)\,\phi(\beta x)\,dx \quad \text{.............................(2)},$$

where β is positive, and $\phi(u)$ is a function of u which has continuous derivatives of all orders, while

$$e^{-\tau u}\phi(u) \to 0,$$

* *Quarterly Journal, loc. cit.* pp. 55—58.

† *Quarterly Journal, loc. cit.*

as $u \to \infty$, for any positive value of τ. Then, integrating by parts, as in § 20, we find

$$I(\tau, m, \beta) = \sum_{s=0}^{\nu-1} \beta^s \phi^{(s)}(0) \chi_{s+1}(0, \tau) + \int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, \tau)\, dx \quad \text{.........(3).}$$

Now suppose that, for some value of ν, the integral

$$\int_0^\infty \phi^{(\nu)}(u)\, u^\mu\, du$$

is absolutely convergent. Then the integral

$$\int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, \tau)\, dx$$

is absolutely and uniformly convergent throughout any region of values of τ, m, and β defined by inequalities such as

$$0 \leqq \tau \leqq \tau_0, \quad 0 < m_0 < m, \quad 0 < \beta_0 < \beta,$$

and tends uniformly to the limit

$$\int_0^\infty \beta^\nu \phi^{(\nu)}(\beta x) \chi_\nu(x, 0)\, dx$$

as $\tau \to 0$, for all such values of m and β. Also

$$\chi_{s+1}(0, \tau) \to \chi_{s+1}(0, 0),$$

uniformly for all such values of m. Hence, under the conditions stated, the integral

$$G \int_0^\infty \chi_0(x, 0)\, \phi(\beta x)\, dx$$

is uniformly summable.

Thus, e.g., the integrals

$$G \int_0^\infty \frac{e^{-mix} x^\mu\, dx}{1 + \beta x}, \quad \int_0^\infty \frac{e^{-mix} x^\mu\, dx}{1 + \beta^2 x^2},$$

where $\mu > -1$, are uniformly summable for $m_0 < m$, $\beta_0 < \beta$, and the reader will easily write down any number of such examples. This result enables us to justify the processes used in the following examples.

(v) If
$$I(m) = G \int_0^\infty \frac{e^{-mix} x^{\lambda-1}\, dx}{1 + x},$$

where $\lambda > 0$, then
$$\frac{dI}{dm} = -i\, G \int_0^\infty \frac{e^{-mix} x^\lambda\, dx}{1 + x},$$

and
$$\frac{dI}{dm} - iI = -i\, G \int_0^\infty e^{-mix} x^{\lambda-1}\, dx$$
$$= -i\, \Gamma(\lambda)\, m^{-\lambda}\, e^{-\frac{1}{2}\lambda\pi i},$$

an equation whose solution is

$$I = i\, \Gamma(\lambda)\, e^{(m - \frac{1}{2}\lambda\pi) i} \left(\int_m^\infty e^{-it} t^{-\lambda}\, dt + C \right).$$

It is easy to prove that $I \to 0$ as $m \to \infty$. Hence

$$G\int_0^\infty \frac{e^{-mix}\, x^{\lambda-1}\, dx}{1+x} = i\,\Gamma(\lambda)\, e^{(m-\frac{1}{2}\lambda\pi)i} \int_m^\infty e^{-it}\, t^{-\lambda}\, dt,$$

a result which is easily verified in special cases.

(vi) If
$$I(\alpha) = \int_0^\infty \frac{\cos \alpha x}{1+x^2} f(\sin^2 x)\, dx,$$

we find that
$$\frac{d^2 I}{d\alpha^2} - I = -G\int_0^\infty \cos \alpha x f(\sin^2 x)\, dx = 0,$$

unless α is an even integer.

Thus if
$$2n < \alpha < 2(n+1),$$

we have
$$I(\alpha) = A_n e^{\alpha} + B_n e^{-\alpha},$$

where A_n, B_n are given by the equations

$$I_{2n} = A_n e^{2n} + B_n e^{-2n},$$

$$I_{2n+2} = A_n e^{2n+2} + B_n e^{-2n-2}.$$

If $f(\sin^2 x) \equiv 1$, $\dfrac{d^2 I}{d\alpha^2} - I = 0$ for all values of α, so that

$$I = Ae^{\alpha} + Be^{-\alpha},$$

from which we deduce the well-known formula

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2}\, dx = \tfrac{1}{2}\pi e^{-\alpha}.$$

The same method may be applied to obtain the formula

$$J(\alpha) = \int_0^\infty \frac{\sin \alpha x}{1+x^2}\, dx = \tfrac{1}{2}\{e^{-\alpha}\, li\,(e^{\alpha}) - e^{\alpha}\, li\,(e^{-\alpha})\},$$

by means of the differential equation

$$\frac{d^2 I}{d\alpha^2} - I = -\frac{1}{\alpha}.$$

And it is evident that this method is capable of very general application to integrals of the form

$$\int_0^\infty {\cos \atop \sin} \alpha x\, R(x)\, dx.$$

(vii)* If
$$I(\alpha) = \int_0^\infty \tanh \tfrac{1}{2}\pi x \sin \alpha x \frac{dx}{1+x^2},$$

we find
$$\frac{d^2 I}{d\alpha^2} - I = -G\int_0^\infty \tanh \tfrac{1}{2}\pi x \sin \alpha x\, dx$$

$$= -G\int_0^\infty \sin \alpha x\, dx + 2\int_0^\infty \frac{\sin \alpha x}{e^{\pi x}+1}\, dx$$

$$= -\operatorname{cosech} \alpha,$$

and hence can deduce that

$$I(\alpha) = \tfrac{1}{2}\{e^{-\alpha} \log(e^{2\alpha}-1) - e^{\alpha} \log(1-e^{-2\alpha})\}.$$

Many other integrals of a similar type may be calculated in the same way.

* For the next two examples cf. Bromwich, *Infinite Series*, pp. 496—7.

(viii) Let $$P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n:$$

then the integral $$G\int_0^\infty x^{\mu-1} e^{iP(x)}\, dx \qquad (\mu > 0)$$

is uniformly convergent throughout the region defined by

$$0 < a_0^0 < a_0 < a_0^1, \quad a_r^0 < a_r < a_r^1.$$

This will be proved if we can show that, if X is any positive number, the integral

$$\int_X^\infty x^{\mu-1} e^{-\tau x + iP(x)}\, dx$$

tends uniformly to a limit as $\tau \to 0$.

Let $y = P(x)$. Then, if X is large enough, and $z = y^{1/n}$, we have, for $y \geqq Y = P(X)$, expansions of the forms

$$x = Az\left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots\right),$$

$$x^{\mu-1}\frac{dx}{dy} = Bz^{\mu-n}\left(1 + \frac{B_1}{z} + \frac{B_2}{z^2} + \dots\right),$$

$$e^{-\tau x} = e^{-\tau Az\left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots\right)}$$

$$= e^{-\tau Az}\left(1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \dots\right),$$

$$x^{\mu-1} e^{-\tau x}\frac{dx}{dy} = Dz^{\mu-n} e^{-\tau Az}\left(1 + \frac{D_1}{z} + \frac{D_2}{z^2} + \dots\right),$$

where $A > 0$, these series being absolutely and uniformly convergent for $y \geqq Y$ and for all values of the coefficients in question.

Now, if m is large enough, the integral

$$\int_Y^\infty e^{-\tau Az + iy} z^{\mu-n-m}\left(D_m + \frac{D_{m+1}}{z} + \dots\right) dy$$

is uniformly convergent and tends uniformly to a limit as $\tau \to 0$. Hence the problem is reduced to that of showing that

$$\int_Y^\infty y^\lambda e^{-\tau y^{1/n} + iy}\, dy$$

tends uniformly to a limit as $\tau \to 0$. But if

$$\psi_0(y) = y^\lambda e^{iy},$$

$$\psi_1(y) = G\int_y^\infty \psi_0(t)\, dt,$$

$$\psi_2(y) = G\int_y^\infty \psi_1(t)\, dt,$$

$$\dots\dots\dots\dots\dots\dots$$

we have $$\int_Y^\infty \psi_0(y) e^{-\tau y^{1/n}}\, dy = \sum_{s=0}^{\nu-1} \psi_{s+1}(Y)\left(\frac{d}{dY}\right)^s (e^{-\tau Y^{1/n}})$$

$$+ \int_Y^\infty \psi_\nu(y)\left(\frac{d}{dy}\right)^\nu (e^{-\tau y^{1/n}})\, dy.$$

Now (§ 20)
$$|\psi_\nu(y)| < Ky^\lambda$$
and
$$\left|\left(\frac{d}{dy}\right)^\nu e^{-\tau y^{1/n}}\right| < Ky^{-\nu\sigma},$$
where $\sigma = 1 - \frac{1}{n}$. By supposing ν large enough we can make $\lambda - \nu\left(1 - \frac{1}{n}\right)$ negative and as large as we like, and so ensure that
$$\int_Y^\infty \psi_\nu(y)\left(\frac{d}{dy}\right)^\nu (e^{-\tau y^{1/n}})\,dy$$
is uniformly convergent, for $0 \leqq \tau \leqq \tau_0$, and tends uniformly to a limit as $\tau \to 0$. And so
$$\int_Y^\infty y^\lambda e^{-\tau y^{1/n} + iy}\,dy$$
tends uniformly to the limit
$$\psi_1(Y) = G\int_Y^\infty y^\lambda e^{iy}\,dy.$$

The result enunciated originally is thus established.

Now let
$$I(\alpha) = \int_0^\infty e^{iP(x)}\,dx,$$
where
$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + \alpha x.$$
The integral is convergent if $x > 1$. Also
$$\frac{d^kI}{d\alpha^k} = i^kG\int_0^\infty x^k e^{iP(x)}\,dx,$$
these integrals being, as is easily proved, convergent in the ordinary sense if $k \leqq n-2$. Hence
$$na_0 i^{-(n-1)}\frac{d^{n-1}I}{d\alpha^{n-1}} + (n-1)a_1 i^{-(n-2)}\frac{d^{n-2}I}{d\alpha^{n-2}} + \ldots$$
$$+ 2a_{n-2}i^{-1}\frac{dI}{d\alpha} + \alpha I = G\int_0^\infty P'(x)\,e^{iP(x)}_1\,dx = i;$$
since
$$\underset{x\to\infty}{L}\,e^{iP(x)} = \lim_{t\to\infty}\int_0^\infty e^{-x+iP(tx)}\,dx = \lim_{t\to\infty}\frac{1}{t}\int_0^\infty e^{-(x/t)+iP(x)}\,dx = \lim_{t\to\infty}\frac{1}{t}\int_0^\xi e^{iP(x)}\,dx,$$
and
$$\left|\int_0^\xi e^{iP(x)}\,dx\right| < K,$$
so that $\underset{x\to\infty}{L}\,e^{iP(x)} = 0$. Suppose in particular that $a_0 = 1$, $a_1 = a_2 = \ldots = a_{n-2} = 0$. Then
$$ni^{-(n-1)}\frac{d^{n-1}I}{d\alpha^{n-1}} + \alpha I = i.$$

Thus, if $n = 3$, we see that
$$\int_0^\infty e^{i(x^3+\alpha x)}\,dx$$
satisfies the equation
$$-3\frac{d^2I}{d\alpha^2} + \alpha I = i,$$

or that $$\int_0^\infty \cos(x^3+\alpha x)\,dx, \quad \int_0^\infty \sin(x^3+\alpha x)\,dx,$$

satisfy the equations $$\frac{d^2I}{d\alpha^2} = \tfrac{1}{3}\alpha I, \quad \frac{d^2I}{d\alpha^2} - \tfrac{1}{3}\alpha I = -\tfrac{1}{3},$$

a result originally due to Stokes*.

D. *Integration of a generalised integral with respect to a parameter.*

§ 27. I shall consider finally the question of the integration of a generalised integral with respect to a parameter, as expressed by the equation

$$\int_\beta^\gamma d\alpha\, G\int_0^\infty f(x,\alpha)\,dx = G\int_0^\infty dx\int_\beta^\gamma f(x,\alpha)\,d\alpha \qquad\text{.......................(1).}$$

This formula rests upon the transformations expressed by the equations

$$\left.\begin{aligned}\int_\beta^\gamma d\alpha\, G\int_0^\infty f(x,\alpha)\,dx &= \int_\beta^\gamma d\alpha\int_0^\infty dt\int_0^\infty xe^{-tx}f(x,\alpha)\,dx\\ &= \int_0^\infty dt\int_\beta^\gamma d\alpha\int_0^\infty xe^{-tx}f(x,\alpha)\,dx\\ &= \int_0^\infty dt\int_0^\infty xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,d\alpha\\ &= G\int_0^\infty dx\int_\beta^\gamma f(x,\alpha)\,dx\end{aligned}\right\}\qquad\text{..............(2),}$$

i.e. on a repeated inversion of integrations.

I shall suppose (a) that $f(x,\alpha)$ is continuous throughout the region $0 \leqq x \leqq X$, $\beta \leqq \alpha \leqq \gamma$, for any value of X however large, (b) that, for any positive value of τ,

$$e^{-\tau x}f(x,\alpha) \to 0$$

as $x\to\infty$, uniformly for all values of α in the interval (β,γ), and (c) that

$$G\int_0^\infty f(x,\alpha)\,dx$$

is uniformly summable throughout the same interval.

In the first place, the conditions (a) and (b) are sufficient to ensure that, for any positive value of t,

$$\int_\beta^\gamma d\alpha\int_0^\infty xe^{-tx}f(x,\alpha)\,dx = \int_0^\infty xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,d\alpha \qquad\text{.................(3).}$$

For, however large X may be, we have

$$\int_\beta^\gamma d\alpha\int_0^X xe^{-tx}f(x,\alpha)\,dx = \int_0^X xe^{-tx}\,dx\int_\beta^\gamma f(x,\alpha)\,dx \qquad\text{.................(4).}$$

But we can choose K so that

$$|f(x,\alpha)| < Ke^{\frac{1}{2}tx}$$

* Stokes, *Math. Papers*, vol. II. p. 329 and vol. IV. pp. 77, 283; Stolz, *Grundzüge*, bd. III. p. 30; Bromwich, *Infinite Series*, p. 497. Since this paper has been in type Mr Bromwich has devised a shorter and simpler method of arriving at the results of this section, which will be printed shortly in the *Messenger of Mathematics*.

for all values of x and α in question, and then

$$\left|\int_\beta^\gamma d\alpha \int_X^{X'} xe^{-tx} f(x, \alpha)\, dx\right| < K(\gamma-\beta)\int_X^\infty xe^{-\frac{1}{2}tx}\, dx$$

$$= K(\gamma-\beta)\frac{e^{-\frac{1}{2}tX}}{(\frac{1}{2}t)^2}(1+\tfrac{1}{2}tX)$$

for all values of X' greater than X. The last expression has the limit zero as $X \to \infty$. From this it follows that

$$\lim_{X\to\infty}\int_\beta^\gamma d\alpha\int_0^X xe^{-tx}f(x, \alpha)\, dx = \int_\beta^\gamma d\alpha\int_0^\infty xe^{-tx}f(x, \alpha)\, dx,$$

and so that (3) follows from (4).

Now let
$$F(t, \alpha) = \int_0^\infty xe^{-tx}f(x, \alpha)\, dx.$$

This integral, as is easily seen, converges uniformly with respect to t and α throughout any domain bounded by inequalities

$$0 < \tau \leqq t \leqq T,\ \beta \leqq \alpha \leqq \gamma,$$

and so is a continuous function of t and α throughout any such domain. Hence

$$\int_\beta^\gamma d\alpha\int_\tau^T F(t, \alpha)\, dt = \int_\tau^T dt\int_\beta^\gamma F(t, \alpha)\, d\alpha.$$

But
$$|f(x, \alpha)| < Ke^x,$$
and so, if $t > 1$,

$$\left|F(t, \alpha)\right| < K\int_0^\infty xe^{-(t-1)x}dx = \frac{K}{(t-1)^2}.$$

Thus, if $T' > T > 1$,

$$\left|\int_\beta^\gamma d\alpha\int_T^{T'} F(t, \alpha)\, dt\right| < (\gamma-\beta)K\int_T^{T'}\frac{dt}{(t-1)^2} < \frac{K}{T},$$

which tends to zero as $T \to \infty$. Hence

$$\int_\beta^\gamma d\alpha\int_\tau^\infty F(t, \alpha)\, dt = \lim_{T\to\infty}\int_\beta^\gamma\int_\tau^T = \lim_{T\to\infty}\int_\tau^T\int_\beta^\gamma = \int_\tau^\infty dt\int_\beta^\gamma F(t, \alpha)\, d\alpha.$$

I shall now prove that if the condition (c) is satisfied we may replace τ by 0 in this equation. To see this we observe that if $0 < \tau' < \tau$

$$\int_\beta^\gamma d\alpha\int_{\tau'}^\tau F(t, \alpha)\, dt = \int_{\beta'}^\gamma d\alpha\int_{\tau'}^\tau dt\int_0^\infty xe^{-tx}f(x, \alpha)\, dx$$

$$= \int_\beta^\gamma d\alpha\int_0^\infty (e^{-\tau' x} - e^{-\tau x})f(x, \alpha)\, dx.$$

But if
$$\int_0^\infty e^{-\tau x}f(x, \alpha)\, dx$$

converges uniformly to a limit, as $\tau \to 0$, we can, given ϵ, so choose τ_0 that

$$\left|\int_0^\infty (e^{-\tau' x} - e^{-\tau x}) f(x, \alpha)\, dx\right| < \epsilon$$

for

$$0 < \tau' < \tau \leqq \tau_0, \quad \beta \leqq \alpha \leqq \gamma,$$

and so

$$\left|\int_\beta^\gamma d\alpha \int_{\tau'}^{\tau} F(t, \alpha)\, dt\right| < \epsilon$$

for all such values of τ', τ and α.

We can now state

THEOREM VII. *If (a) $f(x, \alpha)$ is continuous for $0 \leqq x \leqq X$, $\beta \leqq \alpha \leqq \gamma$, however large X may be; (b) for any positive value of τ*

$$e^{-\tau x} f(x, \alpha) \to 0$$

uniformly for $\beta \leqq \alpha \leqq \gamma$; and (c) the integral

$$G\int_0^\infty f(x, \alpha)\, dx$$

is uniformly summable for $\beta \leqq \alpha \leqq \gamma$, then

$$\int_\beta^\gamma d\alpha\, G\int_0^\infty f(x, \alpha)\, dx = G\int_0^\infty dx \int_\beta^\gamma f(x, \alpha)\, d\alpha.$$

§ 28. A particularly interesting special case of this theorem is one which leads us to certain extensions of Dirichlet's integral and Fourier's double integral, which are due to Sommerfeld*.

Suppose that $$f(x, \alpha) = f(\alpha)\, e^{-imx\alpha},$$

where $m > 0$. Then $G\int_0^\infty e^{-imx\alpha} dx$ is uniformly summable in the interval $\beta \leqq \alpha \leqq \gamma$ if β and γ have the same sign, say the positive. On this hypothesis we obtain the equations

$$G\int_0^\infty dx \int_\beta^\gamma f(\alpha)\, e^{-imx\alpha}\, d\alpha = \frac{1}{im}\int_\beta^\gamma \frac{f(\alpha)}{\alpha}\, d\alpha,$$

$$G\int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx\alpha\, d\alpha = 0,$$

$$G\int_0^\infty dx \int_\beta^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \frac{1}{m}\int_\beta^\gamma \frac{f(\alpha)}{\alpha}\, d\alpha.$$

It is of course well known that if $f(\alpha)$ is monotonic as well as continuous (or, more generally, is a *fonction à variation bornée*) the integrals on the left-hand side of these equations are convergent in the ordinary sense. For

$$\int_0^X dx \int_\beta^\gamma f(\alpha) \cos mx\alpha\, d\alpha = \frac{1}{m}\int_\beta^\gamma f(\alpha)\frac{\sin mX\alpha}{\alpha}\, d\alpha,$$

$$\int_0^X dx \int_\beta^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \frac{1}{m}\int_\beta^\gamma f(\alpha)\frac{1 - \cos mX\alpha}{\alpha}\, d\alpha,$$

* *Die willkürlichen Funktionen in der Math. Physik*, Inaug. Diss., Königsberg, 1901. See also Carslaw, *Fourier's Series and Integrals*, p. 186. I owe these references to Mr Bromwich. The idea which is the base of Sommerfeld's work appears to go back to Cauchy; see *e.g.* his *Mémoire sur la Théorie des Ondes*, Note VI. (*Œuvres*, t. i, p. 133).

and the integrals on the right-hand side are known to have the limits

$$0, \quad \frac{1}{m}\int_\beta^\gamma \frac{f(\alpha)}{\alpha}\, d\alpha$$

as $X \to \infty$.

If $\beta = 0$ the formulae cease to be true, and in fact

$$G\int_0^\infty dx \int_0^\gamma f(\alpha)\cos m x\alpha\, d\alpha = \frac{\pi}{2m} f(0) \quad\text{..........................(5)},$$

whereas the corresponding sine integral is in general divergent.

It is very easy to establish the formula (5) on the assumption that $f(\alpha)$ is continuous, the only case contemplated in the general theorem. As however the result is one of considerable interest in itself, I shall adopt less restrictive hypotheses*. I shall suppose only that

(i) $f(\alpha)$ is integrable in any interval throughout which it is limited,

(ii) $f(+0)$ is determinate,

(iii) $\int_0^\gamma |f(\alpha)|\, d\alpha$ is convergent.

Then it is easy to see that

$$\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha)\cos m x\alpha\, d\alpha = \int_0^\gamma f(\alpha)\, d\alpha \int_0^\infty e^{-\tau x}\cos m x\alpha\, dx$$

$$= \int_0^\gamma \frac{\tau f(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2},$$

for any positive value of τ. For $e^{-\tau x}\cos m x\alpha f(\alpha)$ is an integrable function of the two variables x and α throughout any rectangle $(0, X; 0, \gamma)$, so that the equation certainly holds when ∞ is replaced by any positive number X, however large. And

$$\left|\int_0^\gamma f(\alpha)\, d\alpha \int_X^\infty e^{-\tau x}\cos m x\alpha\, dx\right| < \frac{1}{\tau} e^{-\tau X}\int_0^\gamma |f(\alpha)|\, d\alpha,$$

which tends to zero as $X \to \infty$.

Moreover, on the hypotheses which we have adopted,

$$\lim_{\tau\to 0}\int_0^\gamma \frac{\tau f(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2} = \frac{\pi}{2m} f(+0).$$

For let

$$f(\alpha) - f(+0) = \phi(\alpha)$$

so that $\phi(\alpha) \to 0$ with α. Then

$$\left|\int_0^\gamma \frac{\tau\phi(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2}\right| = \left|\left(\int_0^\delta + \int_\delta^\gamma\right)\frac{\tau\phi(\alpha)\, d\alpha}{\tau^2 + m^2\alpha^2}\right|$$

$$< \omega\int_0^\delta \frac{\tau\, d\alpha}{\tau^2 + m^2\alpha^2} + \frac{\tau}{\tau^2 + m^2\delta^2}\int_\delta^\gamma |\phi(\alpha)|\, dx$$

$$< \frac{\omega\pi}{2m} + \frac{\tau}{\tau^2 + m^2\delta^2}\int_0^\gamma |\phi(\alpha)|\, d\alpha,$$

where ω is the upper limit of $|\phi(\alpha)|$ in the interval $(0, \delta)$.

* The succeeding analysis is not essentially different from Sommerfeld's, but rather more general and direct.

Let $\delta = \tau^s$, where $0 < s < \frac{1}{2}$. Then ω and $\tau/(\tau^2 + m^2\delta^2)$ each tend to zero with τ, and so

$$\int_0^\gamma \frac{\tau\phi(\alpha)\,d\alpha}{\tau^2 + m^2\alpha^2} \to 0.$$

But $$f(+0)\int_0^\gamma \frac{\tau\,d\alpha}{\tau^2 + m^2\alpha^2} = \frac{1}{m} f(+0)\int_0^{m\gamma/\tau} \frac{du}{1+u^2} \to \frac{\pi}{2m} f(0),$$

which establishes the result desired.

Similarly we can show that

$$\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha) \sin mx\alpha\, d\alpha = \int_0^\gamma \frac{m\alpha f(\alpha)\,d\alpha}{\tau^2 + m^2\alpha^2}$$

in general tends to $+\infty$ or to $-\infty$ (according to the sign of $f(+0)$) as $\tau \to 0$.

If however $f(\alpha) = \alpha F(\alpha)$, and $F(\alpha) \to F(+0)$ as $\alpha \to 0$,

$$\begin{aligned}\int_0^\infty e^{-\tau x} dx \int_0^\gamma f(\alpha) \sin mx\alpha\, d\alpha &= m\int_0^\gamma \frac{\alpha^2 F(\alpha)\,d\alpha}{\tau^2 + m^2\alpha^2} \\ &= \frac{1}{m}\left\{\int_0^\gamma F(\alpha)\,d\alpha - \tau\int_0^\gamma \frac{\tau F(\alpha)\,d\alpha}{\tau^2 + m^2\alpha^2}\right\} \to \frac{1}{m}\int_0^\gamma F(\alpha)\,d\alpha.\end{aligned}$$

§ 29. The equation (5) expresses a generalisation of Dirichlet's integral much on the lines of Fejér's generalisation of Fourier's theorem, in which the 'conditions of Dirichlet' are removed and mere continuity (or integrability) assumed, and the Fourier's series, while possibly oscillatory, summable by Cesàro's method of mean values.

It is easy to obtain other generalisations on similar lines. For example, if $f(\alpha)$ satisfies conditions similar to those imposed in it in § 28, we have

$$\begin{aligned}\lim_{\tau\to 0}\int_0^\infty e^{-(\tau x)^2} dx \int_0^\gamma f(\alpha) \cos mx\alpha\, d\alpha &= \lim_{\tau\to 0}\int_0^\gamma f(\alpha)\,d\alpha \int_0^\infty e^{-(\tau x)^2} \cos mx\alpha\, dx \\ &= \frac{1}{2\tau}\sqrt{\pi}\int_0^\gamma f(\alpha)\, e^{-(m\alpha/2\tau)^2} d\alpha \\ &= \frac{\pi}{2m} f(+0),\end{aligned}$$

by a well-known theorem of Weierstrass*.

But a generalisation more precisely on Fejér's lines can be obtained by using the definition†

$$G\int_0^\infty \phi(x)\,dx = \lim_{x\to\infty} \frac{1}{x}\int_0^x dt \int_0^t \phi(u)\,du.$$

We have then to state conditions under which

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dt \int_0^t du \int_0^\gamma f(\alpha) \cos mu\alpha\, d\alpha = \frac{\pi}{2m} f(+0).$$

* This formula also is given by Sommerfeld (*loc. cit.*).

† *Quarterly Journal*, *loc. cit.*, p. 53. For a proof that, under very general conditions, this definition is included in our previous definition, see the same paper, p. 54, and C. N. Moore, *Trans. Amer. Math. Soc.*, vol. VIII. p. 299.

If $f(\alpha)$ satisfies the conditions employed above, we can invert the integrations, and obtain

$$\int_0^x dt \int_0^t du \int_0^\gamma f(\alpha)\cos mu\alpha\, d\alpha = \int_0^\gamma f(\alpha)\, d\alpha \int_0^x dt \int_0^t \cos mu\alpha\, du$$
$$= \int_0^\gamma f(\alpha)\frac{1-\cos mx\alpha}{(m\alpha)^2}\, d\alpha,$$

so that
$$G\int_0^\infty dx \int_0^\gamma f(\alpha)\cos mx\alpha\, d\alpha = \lim_{x\to\infty}\int_0^\gamma \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}mx\alpha}\right)^2 f(\alpha)\,\tfrac{1}{2}x\,d\alpha,$$

if the latter limit exists. Since

$$\lim_{x\to\infty}\int_0^\gamma \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}mx\alpha}\right)^2 \tfrac{1}{2}x\,d\alpha = \lim_{x\to\infty}\frac{1}{m}\int_0^{\frac{1}{2}mx\gamma}\left(\frac{\sin u}{u}\right)^2 du$$
$$= \frac{1}{m}\int_0^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\pi}{2m},$$

it will be seen that what we have to prove is that

$$\frac{1}{x}\int_0^\gamma \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}m\alpha}\right)^2 \phi(\alpha)\,d\alpha \to 0,$$

where $\phi(\alpha) = f(\alpha) - f(+0)$ tends to zero with α.

Let ρ be a positive number less than γ, and let ω be the upper limit of $|\phi(x)|$ in the interval $(0, \rho)$. Then

$$\frac{1}{x}\left|\int_0^\rho \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}m\alpha}\right)^2 \phi(x)\,d\alpha\right| < \frac{\omega}{x}\int_0^\infty \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}m\alpha}\right)^2 d\alpha$$
$$< \frac{2\omega}{m}\int_0^\infty \left(\frac{\sin u}{u}\right)^2 du = \frac{\omega\pi}{m},$$

and
$$\frac{1}{x}\left|\int_\rho^\gamma \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}m\alpha}\right)^2 \phi(\alpha)\,d\alpha\right| < \frac{4}{m^2\rho^2 x}\int_0^\gamma |\phi(\alpha)|\,d\alpha.$$

Thus
$$\frac{1}{x}\left|\int_0^\gamma \left(\frac{\sin\frac{1}{2}mx\alpha}{\frac{1}{2}m\alpha}\right)^2 \phi(\alpha)\,d\alpha\right| < \frac{\omega\pi}{m} + \frac{4}{m^2\rho^2 x}\int_0^\gamma |\phi(\alpha)|\,d\alpha,$$

and if we choose ρ so that

$$\rho \to 0, \quad \rho^2 x \to \infty,$$

as by taking $\rho = x^{-s}$, where $0 < s < \frac{1}{2}$, we see that the limit of the right-hand side is zero. Thus the result is established.

It is of course well known that there are continuous functions $f(\alpha)$ for which the equations

$$\lim_{X\to\infty}\int_0^\gamma f(\alpha)\frac{\sin mX\alpha}{\alpha}\,d\alpha = \tfrac{1}{2}\pi f(+0), \quad \int_0^\infty dx\int_0^\gamma f(\alpha)\cos(mx\alpha)\,d\alpha = \frac{\pi}{2m}f(+0)$$

do not hold. An example of such a function was given by Du Bois Reymond, and a simpler one by Schwarz*. The functions given by these writers are of a very complicated type and defined by an enumerable sequence of different formulae, in a corresponding

* See Hobson, *Theory of Functions of a real Variable*, pp. 701 *et seq.*, for references and further discussion.

sequence of intervals of values of x approaching the origin. In all such cases any of the generalised forms of Dirichlet's Theorem hold.

§ 30. (ii) Suppose

$$f(x, \alpha) = f(\alpha) \cos mx (\alpha - a).$$

Then
$$G \int_0^\infty \cos mx (\alpha - a)\, dx = 0$$

is uniformly convergent in any interval which does not include $x = \alpha$; and so

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = 0 \quad \text{..........................(6)},$$

if (β, γ) does not include $\alpha = a$.

On the other hand, if $\beta < \alpha < \gamma$,

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = \frac{\pi}{m} f(\alpha) \quad \text{........................(7)},$$

as may be shown by arguments precisely similar to those of § 28. When $f(\alpha)$, besides being continuous, satisfies Dirichlet's conditions, the sign of the generalised integral may be omitted, and the formulae reduce to Fourier's double integral formulae.

These formulae of course hold under wider conditions, and so do (6) and (7). It may be proved, precisely on the lines of § 28, that if $f(\alpha)$ satisfies the conditions there laid down,

$$G \int_0^\infty dx \int_\beta^\gamma f(\alpha) \cos mx (\alpha - a)\, d\alpha = \begin{cases} \tfrac{1}{2}\pi f(\beta + 0) & (a = \beta) \\ \tfrac{1}{2}\pi f(\gamma - 0) & (a = \gamma) \\ \tfrac{1}{2}\pi \{f(a - 0) + f(a + 0)\} & (\beta < \alpha < \gamma) \\ 0 \text{ (otherwise)}, & \end{cases}$$

a formula equivalent to Sommerfeld's principal result.

Again, precisely on the lines of § 29, we can show that the above equations remain valid when either of the definitions

$$G \int_0^\infty f(x)\, dx = \lim_{\tau \to 0} \int_0^\infty e^{-(\tau x)^2} f(x)\, dx,$$

$$G \int_0^\infty f(x)\, dx = \lim_{x \to \infty} \frac{1}{x} \int_0^x dt \int_0^t f(u)\, du,$$

is adopted.

§ 31. (iii) As a final illustration I shall consider Hankel's generalisation of Fourier's double integral theorem by means of Bessel functions. Hankel* first gave the formula

$$\int_0^\infty x J^\nu (ax)\, dx \int_\beta^\gamma f(\alpha) J^\nu (\alpha x)\, d\alpha = \begin{matrix} 0 \\ f(a) \end{matrix} \quad \text{....................(8)},$$

* *Math. Annalen*, bd. VIII. p. 482.

where β, γ, and a are positive, according as a does not or does fall inside the interval (β, γ) A rigorous proof that the value of the integral is

$$\tfrac{1}{2}\{f(a-0)+f(a+0)\} \qquad (\beta < a < \gamma),$$
$$\tfrac{1}{2}f(\beta+0) \qquad (\beta = a), \qquad \tfrac{1}{2}f(\gamma-0) \qquad (a=\gamma),$$
$$0 \qquad (a < \beta \text{ or } \gamma < a),$$

has been given by Nielsen *, it being assumed that

$$R(\nu) > -1,$$

and that $f(\alpha)$ satisfies Dirichlet's conditions.

A generalisation of Hankel's formula on the lines of §§ 29, 30 has been attempted and partly achieved by Sommerfeld in his dissertation already quoted. Sommerfeld shews that the formula holds for any integrable function in the form

$$\lim_{\tau\to 0}\int^{\infty} e^{-(\tau x)^2} x J^{\nu}(ax)\,dx \int_{\beta}^{\gamma} f(\alpha) J^{\nu}(\alpha x)\,d\alpha = \left.\begin{matrix}\tfrac{1}{2}\{f(a-0)+f(a+0)\}\\ \tfrac{1}{2}f(\beta+0),\ \tfrac{1}{2}f(\gamma-0)\\ 0\end{matrix}\right\} \quad \ldots\ldots\ldots(9),$$

i.e. that

$$G\int_0^{\infty} xJ^{\nu}(ax)\,dx \int_{\beta}^{\gamma} f(\alpha) J^{\nu}(ax)\,d\alpha = \tfrac{1}{2}\{f(a-0)+f(a+0)\}, \text{ etc.}\ldots\ldots\ldots(10),$$

if the definition of the generalised integral by means of the convergence factor $e^{-(\tau x)^2}$ is adopted.

But when the convergence factor $e^{-\tau x}$ is used Sommerfeld only succeeded in establishing the result for *integral* values of ν. I shall now prove that the formula holds for all values of ν whose real part is greater than -1.

§ 32. For this purpose we require the value of the integral

$$\int_0^{\infty} xe^{-\tau x} J^{\nu}(ax)\, J^{\nu}(\alpha x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where $R(\nu) > -1$ and τ, a, and α are positive. For this purpose we start from the formula†

$$\sum_{s=0}^{\infty}(\nu+s)\,P_s^{\nu}(\cos\theta)\,J^{\nu+s}(ax)\,J^{\nu+s}(\alpha x)$$
$$= \frac{(\frac{1}{2}a\alpha x)^{\nu}}{\Gamma(\nu)}(a^2-2a\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}\nu} J^{\nu}\{x\sqrt{(a^2-2a\alpha\cos\theta+\alpha^2)}\}.$$

It is easy to prove that we are justified in multiplying this series by $xe^{-\tau x}$ and integrating term by term from 0 to ∞. We thus obtain

$$\sum_{s=0}^{\infty}(\nu+s)\,P_s^{\nu}(\cos\theta)\,I_s$$
$$= \frac{(\frac{1}{2}a\alpha)^{\nu}}{\Gamma(\nu)}(a^2-2a\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}\nu}\int_0^{\infty} x^{\nu+1}e^{-\tau x} J^{\nu}\{x\sqrt{(a^2-2a\alpha\cos\theta+\alpha^2)}\}\,dx$$
$$= \frac{2\,(a\alpha)^{\nu}\,\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu)\sqrt{\pi}}\,\frac{\tau}{(\tau^2+a^2+\alpha^2-2a\alpha\cos\theta)^{\nu+\frac{3}{2}}},$$

* *Cylinderfunktionen*, pp. 366—370. † *ibid.* p. 280.

where $$I_s = \int_0^\infty x e^{-\tau x} J^{\nu+s}(ax) J^{\nu+s}(\alpha x)\, dx.$$

Let us suppose for the present that $R(\nu) > -\frac{1}{2}$. Then it is easy to see that we can multiply this equation by

$$(\sin\theta)^{2\nu},$$

and integrate term by term from $\theta = 0$ to $\theta = \pi$. Since it is known that

$$\int_0^\pi (\sin\theta)^{2\nu} P_s^{\nu}(\cos\theta)\, d\theta = 0 \qquad (s > 0)$$

and $$\int_0^\pi (\sin\theta)^{2\nu} d\theta = \frac{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}{\Gamma(\nu+1)},$$

we obtain

$$I_0 = \frac{(2\nu+1)(a\alpha)^\nu}{\pi} \int_0^\pi \frac{\tau(\sin\theta)^{2\nu} d\theta}{(\tau^2 + a^2 + \alpha^2 - 2a\alpha\cos\theta)^{\nu+\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Let us, in Sommerfeld's notation, write

$$A^2 = (\tau^2 + a^2 + \alpha^2)^2 - (2a\alpha)^2$$
$$= \{\tau^2 + (a+\alpha)^2\}\{\tau^2 + (a-\alpha)^2\},$$
$$\tau^2 + a^2 + \alpha^2 = A\xi, \quad 2a\alpha = A\sqrt{(\xi^2 - 1)}.$$

We then obtain

$$\int_0^\infty x e^{-\tau x} J^\nu(ax) J^\nu(\alpha x)\, dx = \frac{2^{-\nu}(2\nu+1)(\xi^2-1)^{\frac{1}{2}\nu}}{\pi A^{\frac{3}{2}}} \int_0^\pi \frac{\tau(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu+\frac{3}{2}}} \ldots\ldots (13).$$

Now Hobson* has given the formula

$$(\xi^2-1)^{\frac{1}{2}\nu} \int_0^\pi \frac{(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu-n}}$$
$$= \frac{\Gamma(n-\nu+1)}{\Gamma(n+\nu+1)} 2^\nu \sqrt{\pi}\Gamma(\nu+\tfrac{1}{2}) \left\{P_n^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_n^\nu(\xi)\right\},$$

and also the formulae†

$$\frac{\Gamma(n-\nu+1)}{\Gamma(n+\nu+1)} \left\{P_n^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_n^\nu(\xi)\right\} = P_n^{-\nu}(\xi),$$
$$P_n^{-\nu}(\xi) = P_{-n-1}^{-\nu}(\xi).$$

If in these formulae we put $n = -\frac{3}{2}$, we see that

$$(\xi^2-1)^{\frac{1}{2}\nu} \int_0^\pi \frac{(\sin\theta)^{2\nu} d\theta}{\{\xi - \sqrt{(\xi^2-1)}\cos\theta\}^{\nu+\frac{3}{2}}}$$
$$= \frac{\Gamma(-\nu-\frac{1}{2})}{\Gamma(\nu-\frac{1}{2})} 2^\nu \sqrt{\pi}\, \Gamma(\nu+\tfrac{1}{2}) \left\{P_{-\frac{3}{2}}{}^\nu(\xi) - \frac{2}{\pi} e^{-\nu\pi i} \sin\nu\pi\, Q_{-\frac{3}{2}}{}^\nu(\xi)\right\}$$
$$= 2^\nu \sqrt{\pi}\, \Gamma(\nu+\tfrac{1}{2}) P_{-\frac{3}{2}}{}^{-\nu}(\xi)$$
$$= 2^\nu \sqrt{\pi}\, \Gamma(\nu+\tfrac{1}{2}) P_{\frac{1}{2}}{}^{-\nu}(\xi).$$

* *Phil. Trans. Roy. Soc.* (A), vol. CLXXXVII. p. 493. † *ibid.* pp. 462, 452.

Hence we deduce

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx = \frac{2\Gamma(\nu+\frac{3}{2})}{A^{\frac{3}{2}}\sqrt{\pi}}\, \tau\, P_{\frac{1}{2}}{}^{-\nu}(\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

If however ν is integral we have also

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx$$

$$= \frac{2^{-\nu}(2\nu+1)}{\pi A^{\frac{3}{2}}} \cdot \frac{\Gamma(-\nu-\frac{1}{2})}{\Gamma(\nu-\frac{1}{2})}\, 2^\nu \sqrt{\pi}\, \Gamma(\nu+\tfrac{1}{2})\, \tau\, P_{-\frac{3}{2}}{}^{\nu}(\xi)$$

$$= \frac{2\Gamma(\frac{3}{2}-\nu)}{A^{\frac{3}{2}}\sqrt{\pi}}\, \tau P_{-\frac{3}{2}}{}^{\nu}(\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

The function which Hobson denotes by $P_{-\frac{3}{2}}{}^{\nu}(\xi)$ would be denoted by Heine or Sommerfeld by

$$\sqrt{\left(\frac{2}{\pi}\right)} \frac{P_\nu{}^{\frac{1}{2}}(\xi)}{\Gamma(\frac{3}{2}-\nu)},$$

so that in this notation we obtain

$$\frac{\tau}{\pi}\left(\frac{2}{A}\right)^{\frac{3}{2}} P_\nu{}^{\frac{1}{2}}(\xi);$$

and this is the result obtained by Sommerfeld for integral values of ν.

The formula (14) has been proved on the assumption that $R(\nu) > -\frac{1}{2}$. Each side of the equation represents an analytic function of ν regular for all values of ν for which $R(\nu) > -1$, and the equation therefore holds for all such values.

§ 33. We have thus the formula

$$\int_0^\infty xe^{-\tau x} J^\nu(ax)\, J^\nu(\alpha x)\, dx$$

$$= \frac{2\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \frac{\tau}{\{\tau^2+(a+\alpha)^2\}^{\frac{3}{4}}\{\tau^2+(a-\alpha)^2\}^{\frac{3}{4}}}\, P_{\frac{1}{2}}{}^{-\nu}\left\{\frac{\tau^2+a^2+\alpha^2}{[\tau^2+(a+\alpha)^2]^{\frac{1}{2}}\,[\tau^2+(a-\alpha)^2]^{\frac{1}{2}}}\right\} \ldots(16),$$

and it is clear that, unless $a=\alpha$, this expression tends to zero with τ, and moreover does so uniformly in any interval of values of α which does not include $\alpha=a$. Hence

$$G\int_0^\infty x\, J^\nu(ax)\, J^\nu(\alpha x)\, dx = 0 \qquad (\alpha \neq a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17);$$

moreover the integral is uniformly summable in (β, γ) if that interval does not include a. In this case, therefore, by Theorem VII

$$G\int_0^\infty x\, J^\nu(ax)dx \int_\beta^\gamma f(\alpha)\, J^\nu(\alpha x)\, dx = 0.$$

If, however, $a=\alpha$, $A \to 0$ and $\xi \to \infty$ as $\tau \to 0$. In this case we require an asymptotic formula for $P_{\frac{1}{2}}{}^{-\nu}(\xi)$.

Now it is known* that

$$P_{\frac{1}{2}}{}^{-\nu}(\xi) = \frac{(2\xi)^{\frac{1}{2}}}{\Gamma(\nu+\frac{3}{2})\sqrt{\pi}} + \epsilon_\xi,$$

where

$$|\epsilon_\xi| < K\xi^{-\frac{3}{2}},$$

* Hobson, *loc. cit.*, p. 463. When $n=\frac{1}{2}$ the expansions there given become illusory, but the appropriate expansion can easily be deduced by a passage to the limit.

and, if $\beta > 0$, K is independent of ξ or α. Hence it follows that, if $\beta < a < \gamma$,

$$\lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,d\alpha\int_0^\infty xe^{-\tau x}J^\nu(ax)\,J^\nu(\alpha x)\,dx$$

$$= \lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,d\alpha\left\{\frac{2\Gamma(\nu+\frac{3}{2})}{A^{\frac{3}{2}}\sqrt{\pi}}\,\tau P_{\frac{1}{2}}{}^{-\nu}(\xi)\right\}$$

$$= \frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^\gamma f(\alpha)\,\tau\xi^{\frac{1}{2}}A^{-\frac{3}{2}}\,d\alpha$$

$$= \frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^\gamma \frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,f(\alpha)\,d\alpha,$$

provided that the last limit exists.

We divide the range of integration into the two parts (β, a), (a, γ). Let us first evaluate the limit

$$\lim_{\tau\to 0}\int_\beta^a \frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,d\alpha.$$

The integral is equal to

$$\frac{(\tau^2+a^2+\beta^2)^{\frac{1}{2}}}{\tau^2+(a+\beta)^2}\arctan\left(\frac{a-\beta}{\tau}\right)+\int_\beta^a \arctan\left(\frac{a-\alpha}{\tau}\right)\frac{d}{d\alpha}\left\{\frac{(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\tau^2+(a+\alpha)^2}\right\}d\alpha.$$

It is easy to see that the last integral is continuous for $\tau = 0$, so that the expression tends, as $\tau\to 0$, to the limit

$$\tfrac{1}{2}\pi\left[\frac{(a^2+\beta^2)^{\frac{1}{2}}}{(a+\beta)^2}+\int_\beta^a\frac{d}{d\alpha}\left\{\frac{(a^2+\alpha^2)^{\frac{1}{2}}}{(a+\alpha)^2}\right\}d\alpha\right] = \frac{\pi}{4\sqrt{2}}.$$

Hence, if $f(a-0)$ is determinate,

$$\frac{2^{\frac{3}{2}}}{\pi}\lim_{\tau\to 0}\int_\beta^a \frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,f(a-0)\,d\alpha = \tfrac{1}{2}f(a-0).$$

Now let
$$f(\alpha)-f(a-0)=\phi(\alpha),$$
so that $\phi(\alpha)\to 0$ as $\alpha\to a-0$. Then

$$\left|\int_\beta^a \frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,\phi(\alpha)\,d\alpha\right| \leqq \left|\int_\beta^{a-\delta}\right|+\left|\int_{a-\delta}^a\right|$$

$$< \frac{\tau(\tau^2+2a^2)^{\frac{1}{2}}}{(\tau^2+a^2)(\tau^2+\delta^2)}\int_\beta^a|\phi(\alpha)|\,d\alpha$$

$$+\ \omega\int_{a-\delta}^a \frac{\tau(\tau^2+a^2+\alpha^2)^{\frac{1}{2}}}{\{\tau^2+(a+\alpha)^2\}\{\tau^2+(a-\alpha)^2\}}\,d\alpha,$$

where ω is the upper limit of $|\phi(\alpha)|$ in the interval $(a-\delta, a)$.

The first term is less than
$$K\tau/(\tau^2+\delta^2),$$
and the second is less than

$$\frac{\omega\sqrt{3}}{a}\int_{a-\delta}^a\frac{\tau\,d\alpha}{\tau^2+(a-\alpha)^2} < \frac{\pi\omega\sqrt{3}}{2a}.$$

If $\delta = \tau^s$, where $0 < s < \frac{1}{2}$, each of these expressions tends to zero with τ. Hence

$$\frac{2^{\frac{3}{2}}}{\pi} \lim_{\tau \to 0} \int_\beta^a \frac{\tau(\tau^2 + a^2 + \alpha^2)^{\frac{1}{2}}}{\{\tau^2 + (a+\alpha)^2\}\{\tau^2 + (a-\alpha)^2\}} f(\alpha)\, d\alpha = \tfrac{1}{2} f(a-0).$$

The integral from a to γ can be treated in a precisely similar manner, and so we arrive at the equation (10) of § 31. Thus the result proved by Sommerfeld for integral values of ν is extended to all values of ν whose real part is greater than -1.

§ 34. When $e^{-(\tau x)^2}$ is used as the factor of convergence the work is easier since, by a well-known formula,

$$\int_0^\infty e^{-(\tau x)^2} x\, J^\nu(ax)\, J^\nu(\alpha x)\, dx = \frac{i^\nu}{2\tau^2} e^{-\frac{a^2+\alpha^2}{4\tau^2}} J^\nu\left(\frac{-ia\alpha}{2\tau^2}\right).$$

* Sommerfeld, *loc. cit.* p. 31; Nielsen, *Cylinderfunktionen*, p. 184.

TRANSACTIONS

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CONTENTS OF VOLUME XX.

GENERAL INDEX

CORRIGENDA AND ADDENDUM.

p. 36, line 13, *for* "k", *read* "k_0".

p. 42, l. 9, *for* "as unity", *read* "as of the order unity".

p. 43, l. 8, *for* "function for y_0 as the first", *read* "function of x for y_0, the first".

p. 51, last line, *for* "$\mathfrak{R}\Delta x$", *read* "$\mathfrak{R}$".

p. 52, l. 7, *for* "S", *read* "$S\phi(\xi)$".

p. 53, l. 6 from bottom, *for* "$\int$", *read* "$\int_0^\infty$."

p. 60, l. 9, put brackets outside the right member of the equation and outside the bracket on the right, "y_0".

p. 60, l. 10. Make similar correction.

p. 61, l. 8, *for* "x^3", *read* "$x^3 + \ldots$".

p. 118, put at the beginning of the last line, also of the line fourth from the bottom, and the line seventh from the bottom, "$-$" (the minus symbol).

p. 119. Make similar correction on lines 3 and 5.

p. 123, l. 10, *for* "in general", *read* "at first".

p. 123, l. 11, *for* "will be found necessary", *read* "is convenient".

p. 141 *Add* 'The writer desires to refer to his paper on "The Generalised Law of Error" in the *Journal of the Statistical Society* for September 1906; where a condition which is mentioned only incidentally in the paper on the Law of Error in the *Camb. Phil. Trans.* (at pp. 114, 115), viz., the case in which the series of coefficients $k_1, k_2, k_3 \ldots$ descends less rapidly than by powers of $1/\sqrt{m}$ (m being the number of elements), is shown to be generally admissible, and to permit the extension of the generalised formula to a large class of concrete statistical groups.'

p. 285 *for* $\phi(s) = f(s) - \lambda \int_0^1 k(s, t)\,\phi(t)\,dt.$

read $f(s) = \phi(s) - \lambda \int_0^1 k(s, t)\,\phi(t)\,dt.$

p. 286 *for* $\phi(s) = f(s) + \lambda \int_0^1 K(s, t)\,\phi(t)\,dt,$

read $\phi(s) = f(s) + \lambda \int_0^1 K(s, t)\,f(t)\,dt.$

TRANSACTIONS

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VOLUME XXI. No. II. pp. 49—105.

ON THE LONGITUDINAL IMPACT OF METAL RODS WITH ROUNDED ENDS.

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CAMBRIDGE:
AT THE UNIVERSITY PRESS.

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II. *On the Longitudinal Impact of Metal Rods with Rounded Ends.*

By J. E. SEARS, B.A., St John's College.

[*Read* 28 October, 1907.]

PREAMBLE.

In a former paper under the above heading*, a description has been given of some experiments undertaken with the object of comparing the values of Young's modulus in metal rods as determined by static and dynamic methods. Previous experiments on the subject† had given rise to no conclusive results, though a higher value of E seemed to be generally obtained in the dynamic tests. Such an effect might be accounted for by assuming a time-lag between stress and strain, so that in the case of extremely rapid applications of stress (such as are involved in the propagation of elastic waves in the metal), the full value of the strain would not have time to be developed. This would give rise to an *apparently higher* value of E, and consequently of the velocity of wave propagation, which, according to St Venant's‡ theory, is $\sqrt{\frac{Eg}{\rho}}$. It was with the object of finally settling this point that the experiments were originally undertaken.

The method employed was to determine the velocities of propagation of elastic waves in long rods of the metals by means of observations on the duration of longitudinal impact between them. According to St Venant, for pairs of rods of equal length, this should be simply the time required by a wave to travel twice the length of either rod. The experiments of Voigt § and others had, however, shewn that this is influenced by the nature of the ends of the rods where contact takes place, and that the actual duration of the impact is always larger than the above by an amount which we shall call the "end-effect."

For convenience in working, the ends of the rods in these experiments were rounded off, but even with the most carefully polished plane ends, the end-effect still persists. It was, however, found to be independent of the length of the rods (using them always in pairs of equal length) provided the velocity and other circumstances of the impact were kept the same. By plotting duration of impact against length of rods a straight line was thus obtained whose slope gave the required value of the wave velocity.

* *Proc. Camb. Phil. Soc.* vol. XIV. pt. III.

† Wertheim, *Ann. de Chim. et de Phys.* 3e sér. tome XII.; Hopkinson, *Proc. Roy. Soc.* vol. LXXIV.

‡ Liouville, sér. 2, t. XII. 1867.

§ *Wied. Ann.* XIX. 1883.

The durations of the impacts were determined by the method first introduced by Pouillet*, and subsequently applied to the same problem by Schneebeli†, Hausmaninger‡, and Hamburger§. The particular form of the method used in these experiments is simply represented by Fig. 1. An electric circuit is completed by the contact between the rods r_1, r_2, the total quantity of electricity which passes, from the moment they first meet, until they again separate, being measured on the ballistic galvanometer G.

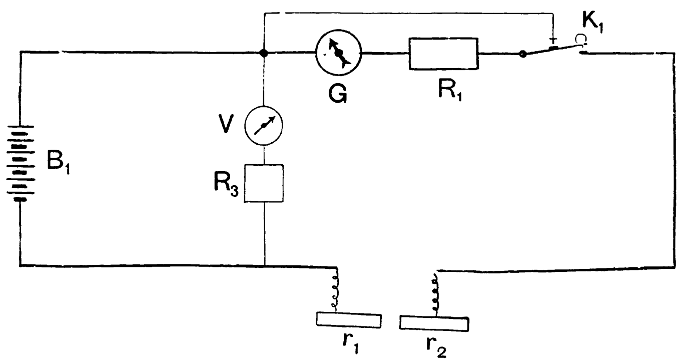


FIG. 1.

The resistance R_1 is made so great that the contact resistance between the rods may at all times be neglected, so that, ignoring for the moment the effects of self-induction, we should have

$$q = \frac{V}{R} T, \text{ or } T = \frac{R}{V} q \quad \text{.................................}(1),$$

where q is the quantity of electricity; V the voltage; R the total resistance of the circuit; and T the duration of the impact.

The galvanometer was calibrated *in situ* by placing a standard condenser in the circuit, and charging to a known voltage. In this way the galvanometer works during calibration under as nearly as possible the same conditions as in the actual experiments—viz., making its flings on open circuit‖.

Some difficulty was experienced in determining the necessary corrections for self-induction, the whole effect being very small, and the coil of the galvanometer, which accounted for nearly all of it, having a very high resistance. The ultimate correction was found to be simply an addition of $2{\cdot}5 \times 10^{-6}$ seconds to the duration of impact¶.

The rods were suspended from two parallel wooden beams by 'V's of fine cord (fishing line), Fig. 2, being thus compelled to swing always in the same vertical plane, and with their axes always horizontal. They were connected to the electric circuit by long light flexible wires w_1 and w_2. In their lowest position the rods were collinear, and just in contact at their ends. The radius of the arc described was 5 feet.

* *Pogg. Ann.* 64, 1845.

† *Ibid.* 143–145, 1871–2.

‡ *Wiener Sitzungsberichte*, 88, 2te Abth.

§ *Wied. Ann.* 28, 1886.

‖ This is a matter of some importance. See also p. 73.

¶ For the method by which this was finally determined, as well as for more detailed descriptions of the apparatus in general, the reader is referred to the paper (*Proc. Camb. Phil. Soc.* vol. XIV. pt. III.) already mentioned.

The cords, C, were used as guides in adjusting the rods for collinearity, and were removed when this was attained.

The rod, r_2, was then withdrawn by means of the cord, c, to a distance which could be measured by means of a travelling telescope. This cord passes under a small pulley, P, to a dropper, D, which is supported at its upper end by a fuse wire, F. The rod is released by blowing the fuse. The diameter of all the rods used was $\frac{1}{2}''$, and the radius of the spherical ends $1''$.

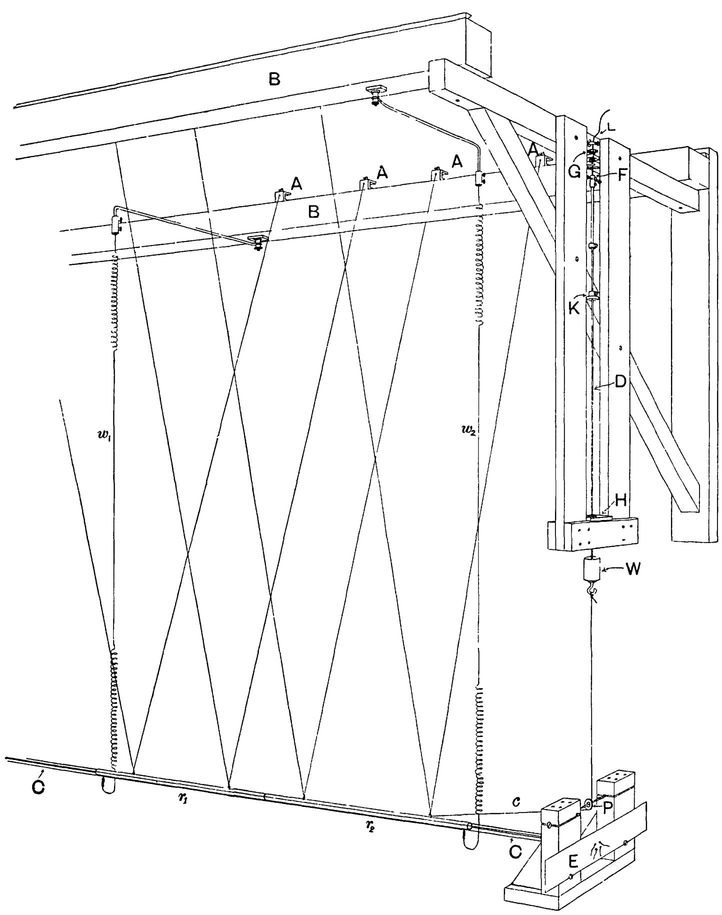


FIG. 2.

The withdrawal finally adopted was $2''$, giving rise to a velocity of impact of about $5''$ per sec. It was found, during the preliminary experiments on steel rods, that any higher velocity than this led to overstraining at the centres of the ends of rods, the slight flattening of the ends (which was easily detected by observing the reflexion of a straight line in their highly polished surfaces) giving rise to an appreciable diminution in the end-effect. Even such a small velocity as $5''$ per sec. produces a mean pressure over the area of contact far higher than the elastic limit of the steel under static compression, and it appears probable that very high pressures may be applied *instantaneously* without producing any permanent effects*.

Any reduction in the withdrawal below about 2″ increased considerably the difficulty of obtaining accurate results, so that, in the cases of copper and aluminium, this overstraining effect could not be avoided. The end-effect thus diminished gradually with successive impacts until a steady state was reached. Referring to Fig. 3, in which the results of the experiments are plotted, it will be seen that for these metals two parallel straight lines are obtained, the points on the dotted lines each representing the first observation of a set, and those on the full lines representing the means of several readings taken after the steady state was established.

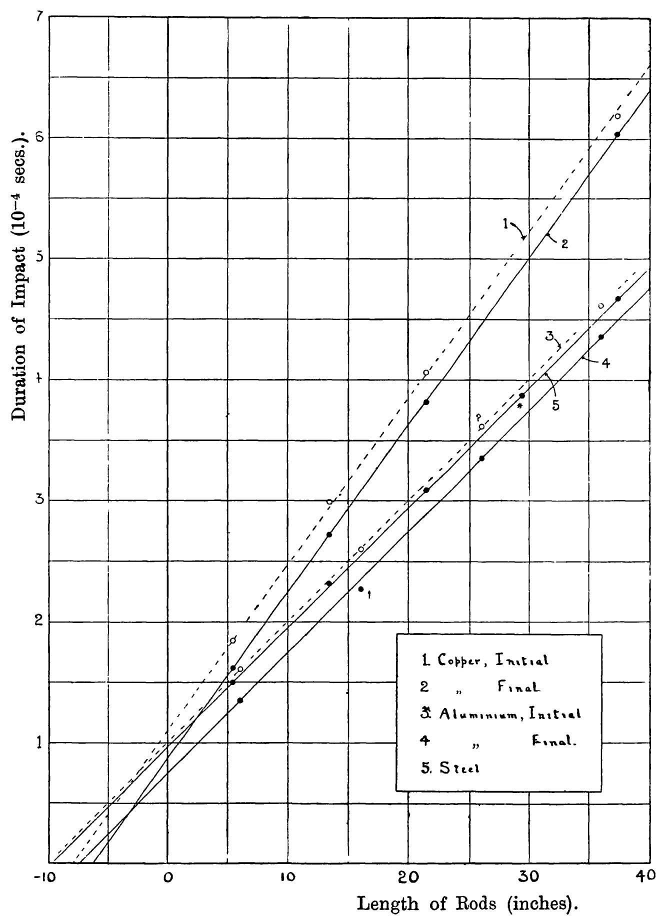


Fig. 3.

* This conclusion is in agreement with Prof. Hopkinson's observations, *loc. cit.* See also p. 78.

The figures are as follows:

	Steel rods, 2″ withdrawal				
Length of rods (inches)	$5\frac{3}{8}$	$13\frac{3}{8}$	$21\frac{3}{8}$	$29\frac{3}{8}$	$37\frac{3}{8}$
Time of impact (10^{-4} secs.)	1·501	2·319	3·089	3·868	4·675

		Copper rods, 2″ withdrawal			
Length of rods (inches)...		$5\frac{3}{8}$	$13\frac{3}{8}$	$21\frac{3}{8}$	$37\frac{3}{8}$
Time of impact (10^{-4} secs.)	Initial	1·845	2·993	4·064	6·193
	Steady	1·617	2·721	3·817	6·041

		Aluminium rods, 2″ withdrawal			
Length of rods (inches)...		6	16	26	36
Time of impact (10^{-4} secs.)	Initial	1·611	2·602	? 3·621	4·619
	Steady	1·349	2·272	3·354	4·359

From the slope of the lines in Fig. 3 we then get the following values for the velocity of wave-propagation:

Steel ... ... 16,820 ft. per sec.
Copper ... ... 12,060 „ „ „
Aluminium ... ... 16,620 „ „ „

The values of $\sqrt{\frac{Eg}{\rho}}$ (St Venant's theoretical wave-velocity), as calculated from static tests on the rods, were as follows:

Steel ... ... 16,750 ft. per sec.
Copper ... ... 12,010 „ „ „
Aluminium ... ... 16,580 „ „ „

These values have to be multiplied by the following factors to correct from isothermal to adiabatic propagation,

Steel ... ... 1·0010.
Copper ... ... 1·0015.
Aluminium ... ... 1·0026.

So that we ultimately get the following table:

Velocity......	Observed	Calculated	
		Isothermal	Adiabatic
Steel	16,820	16,750	16,770
Copper	12,060	12,010	12,030
Aluminium ...	16,620	16,580	16,620

The agreement between the first and last columns here is very close, the difference in the worst case (that of steel) being only $\frac{1}{3}$ per cent. This is well within the limits of experimental error, and we are consequently justified in assuming that the value of Young's modulus is the same for instantaneous as for steady stresses.

A slight modification of the above statement is, however, necessary when the material employed displays an appreciable amount of elastic after-working. In the static test of the aluminium rods this effect was found to be present to such an extent that the final extension produced by the application of a load was rather more than 1 per cent. greater than that obtained by reading the extensometer as soon as possible after the load was applied. This elastic after-working requires time, and consequently does not occur with stresses of the very short duration involved in wave-propagation.

The static value of E for aluminium was therefore found, not by increasing the load step by step and plotting a straight line (a process requiring considerable time for its completion), but by running the whole load quickly on (or off), and reading the extensometer at once. This was repeated several times, and the figure quoted above represents the mean of the readings so obtained. It will be seen that it agrees exactly with that obtained by the impact method.

The time occupied in taking a reading by the above method was not checked, but it would probably be something like 15 seconds. It follows therefore that elastic after-working represents, not the completion of an extension growing—rapidly at first and afterwards more slowly—from zero, but a distinct effect; the greater part of the extension being produced *instantaneously* on the application of the load*.

* A similar result has been recorded by Hopkinson and Rogers, *Proc. Roy. Soc.* vol. A. LXXVI. 1905, p. 424. See also Appendix III.

MATHEMATICAL THEORY.

In the present paper it is our object to give a mathematical investigation of the effects of the rounded ends of the rods on the nature and duration of impact, together with the results of some further experiments performed with the object of fully checking the theory developed.

We have at our disposal two distinct theories of impact, due respectively to St Venant* and Hertz†. Neither of these theories in itself represents adequately the whole of the facts; but it will be shewn how, by a suitable modification and combination of the two, a solution may be obtained which agrees well, in every respect, with experiment.

St Venant's theory, which refers particularly to the case of long rods, treats the problem by considerations of wave-propagation. Thus, if two rods impinge longitudinally, waves will be set up in each, travelling out from the point of contact with the velocity $\sqrt{\frac{Eg}{\rho}}$, and these waves, by their reflexions in the free ends of the rods, determine the whole course of the impact. If, for example, the longer rod be at rest before impact, then, after impact, the shorter rod will be left at rest and inert, the whole of its momentum being transferred to the longer, which rebounds vibrating. The duration of the impact is the time taken by the wave to travel up and down the longer rod. It is assumed throughout this theory that the ends of the rods are mathematical planes, so that, during contact, the rods may be treated as a continuous whole. This of course, owing to the granular structure of metals, can never be attained, even with the most carefully polished ends, and Voigt‡ has given a modification of the theory in which he attempts to overcome this difficulty by postulating an indefinitely thin region of separation between the rods, having a mean elastic modulus different from that of the rods themselves. This form of the theory is also not very satisfactory, owing to the indefinite character of the constant thus introduced. In either form, moreover, the theory is applicable only to plane-ended rods.

To get any information on the impact of bodies with curved surfaces, we have to turn to Hertz' theory. In this theory it is assumed that *no wave motion* is set up, so that the masses of the bodies may be treated as concentrated at their centres of gravity, and the pressure between them taken to be the same as would be produced statically by the same relative displacement. This limits the application of the theory to cases in which the duration of impact is very long compared with the gravest mode of vibration of the bodies concerned; and for this to be the case either the velocity of impact must be extremely small, or the dimensions of the bodies very closely the same in all directions. This theory then is also not directly applicable to the case under consideration.

It is only necessary, however, in order to get a satisfactory theory, to combine with the wave-theory of St Venant a condition representing the law of compression for that

* *Liouville*, sér. 2, t. XII. 1867. † *Crelle*, Bd. XCII. 1882. ‡ *Wied. Ann.* Bd. XIX. 1883.

region of the rods which includes the point of contact*. This condition is obtainable in a simple and convenient form by a modification of Hertz' theory which we now proceed to discuss.

Consider first the case of statical compression.

Hertz gives, as the law of compression between two infinite elastic bodies in contact,

$$P = K\alpha^{\frac{3}{2}},$$

where P = total pressure between the bodies,

α = relative displacement of infinitely distant points in the two bodies,

K = a constant depending only on the curvatures at the point of contact and on the elastic constants of the materials.

(For a pair of equal spherical surfaces of radius r, K has the value

$$\frac{\sqrt{2}E}{3(1-\sigma^2)} r^{\frac{1}{2}},$$

where E is Young's Modulus, and σ Poisson's ratio.)

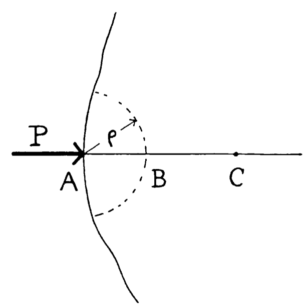


FIG. 4.

We may treat either body separately as consisting of two parts. Within a hemisphere of radius ρ, large compared with the area of contact, but small compared with the dimensions of the body, the displacements will be dependent only on the distribution of the pressure over the area of contact, and the shape of the body in the immediate neighbourhood; while in the rest of the body they will be simply proportional to the whole pressure, P, independent of its distribution, but dependent on the general shape of the body.

We may thus divide α', the displacement of the infinitely distant parts of the body relative to the point of contact, into two parts, α_1' and α_2', of which α_1', the displacement of A relative to B, is independent of the shape of the body at a distance from A, but depends on the distribution of pressure at A, while α_2', the displacement of B relative to infinity, is simply proportional to the total pressure P.

If, now, the body, instead of being infinite, is finite, the value of α_1' will remain unaltered provided ρ is small compared with the dimensions of the body. The displace-

* This, in fact, is what Voigt attempts to do for plane-ended rods; but, the law of compression in the ends being unknown, he is obliged to represent it by a mean elastic modulus whose value remains quite indeterminate. See also pp. 79–82.

ment of B relative to any point of the body external to the hemisphere will still be proportional to P, and it will therefore be possible to find some point, C, such that the displacement of B relative to $C=\alpha_2'$. The displacement of A relative to C will then be $\alpha_1'+\alpha_2'=\alpha'$, so that by taking α as the relative displacement of two such points C_1 and C_2 suitably chosen in the bodies, the formula, $P=K\alpha^{\frac{3}{2}}$, is made applicable to bodies of finite size.

In dealing with the impact problem, we now apply Hertz' principle to that part of either rod lying between A and C, and calculate the *total compression* of this 'end element' as though the pressure in it were statical*. In other words we neglect the effect, on this total compression, of the pressure-gradient set up in the end-element by the inertia of its parts. We then have the law

$$P=K\alpha^{\frac{3}{2}}=\frac{\sqrt{2}\,E}{3\,(1-\sigma^2)}\,r^{\frac{1}{2}}\alpha^{\frac{3}{2}}$$

connecting the pressure at A with the compression simultaneously existing between C_1 and C_2. (Fig. 5.)

Apart from this one step, however, we shall suppose that waves are propagated along the *whole length* of the rods (including the end-elements) in accordance with St Venant's theory†.

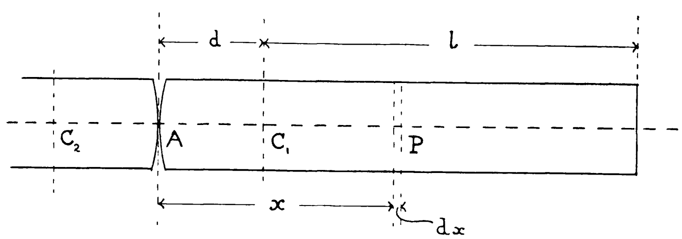


FIG. 5.

To determine the circumstances of the impact we then proceed as follows:

Consider any element, dx, of the rod.

(In particular we shall use the element situated at C.)

The pressure, at a distance x along the rod, will be that which existed at A at a time $\frac{x}{v}$ previously, where v is the velocity of propagation of waves along the rod $\left(=\sqrt{\frac{Eg}{\rho}}\right)$.

* For the determination of the appropriate length of AC in this case see Appendix I.

† There appears, at first sight, to be an inconsistency in thus treating the end-element on a *static* basis in calculating its total compression, and on a *dynamic* basis for the rest of the problem. A moment's reflection will shew that this inconsistency is only apparent, and that the end-element is really being treated on a dynamic basis throughout. The static calculation is used in one step merely as a good approximation to the dynamic facts. The greater part of the compression takes place in the immediate neighbourhood of the point of contact, and is, consequently determined solely by the pressure actually existing at that point at the instant. With moderate velocities of impact, moreover, the pressure-gradient in the end-element is always small, so that, altogether, the variation of pressure which does occur in the parts of the end-element remote from the point of contact can have but little influence on the total compression. A numerical estimate of the error involved will be found in Appendix I, p. 88.

Hence the pressure difference between the two faces of the element at P

$$=\frac{\partial p}{\partial x}\,dx=\frac{1}{v}\frac{\partial}{\partial t}\{p_{\left(t-\frac{x}{v}\right)}\}\,dx,$$

where p_t denotes the pressure existing at A at time t.

Thus, if ξ be the displacement of P towards A, and $S\,(=\pi a^2)$, the cross-section of the rod, the equation of motion for the element dx is

$$\rho S dx\ddot{\xi}+\frac{1}{v}\cdot\frac{d}{dt}(p_{\left(t-\frac{x}{v}\right)})\,dx=0,$$

or

$$\rho S\dot{\xi}+\frac{1}{v}\cdot p_{\left(t-\frac{x}{v}\right)}=\text{const.}$$

(ρ now standing for the density of the material).

Applying this to the motion of C_1; if β be the displacement of C_1 we get

$$\rho S\dot{\beta}+\frac{1}{v}\,p_{\left(t-\frac{d}{v}\right)}=\rho S V_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

since, at all times previous to $t=0$, $p=0$ and $\dot{\beta}_0=V_1$, the velocity of the rod before impact.

Similarly for the other rod, if γ be the displacement of C_2 in the same direction, and V_2 the initial velocity of the rod in that direction,

$$\rho S\dot{\gamma}-\frac{1}{v}\,p_{\left(t-\frac{d}{v}\right)}=\rho S V_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Subtracting,

$$\rho S\,(\dot{\beta}-\dot{\gamma})+\frac{2}{v}\,p_{\left(t-\frac{d}{v}\right)}=\rho S\,(V_1-V_2)$$

or,

$$\rho S\dot{\alpha}+\frac{2}{v}\,p_{\left(t-\frac{d}{v}\right)}=\rho S\,(V_1-V_2)\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Taking this equation with $p=K\alpha^{\frac{3}{2}}$, we can get an approximate mathematical solution up to the time when the first reflected wave reaches either C_1 or C_2. Another term has then to be added to the equation, which involves the previous part of the solution in the form $p=\phi(t)$. But as the original solution appears in the form $t=f(p)$, where f is the sum of transcendental functions, it is impossible to get the form of ϕ, and the solution becomes impracticable. It appears better therefore to plot the pressure-time curve step by step from the beginning, as this process can be carried straight on to the end with any desired degree of accuracy, and without any further assumptions. The process is as follows:

Integrating (i), and using the conditions there stated, we get

$$\rho S\beta+\frac{1}{v}\int_0^t p_{\left(t-\frac{d}{v}\right)}\,dt=\rho S V_1 t\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Let us plot the pressures at intervals of time, τ, chosen sufficiently small to ensure the degree of accuracy required, and let $\beta_1\beta_2\beta_3\ldots$ etc. be the values of β at times τ, 2τ, 3τ, ... etc.

Then, from (iv),

$$\left.\begin{aligned}\rho S\beta_1 + \frac{1}{v}\int_0^{\tau} p_{\left(t-\frac{d}{v}\right)}\,dt &= \rho S V_1 \tau \\ \rho S\beta_2 + \frac{1}{v}\int_0^{2\tau} p_{\left(t-\frac{d}{v}\right)}\,dt &= 2\rho S V_1 \tau \\ \rho S\beta_3 + \frac{1}{v}\int_0^{3\tau} p_{\left(t-\frac{d}{v}\right)}\,dt &= 3\rho S V_1 \tau \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \end{aligned}\right\},$$

or, subtracting these in turn and dividing by ρS,

$$\left.\begin{aligned}\beta_1 &= \phantom{\beta_1 +{}} V_1\tau - \frac{1}{v\rho S}\int_0^{\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \beta_2 &= \beta_1 + V_1\tau - \frac{1}{v\rho S}\int_{\tau}^{2\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \beta_3 &= \beta_2 + V_1\tau - \frac{1}{v\rho S}\int_{2\tau}^{3\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\text{v}).$$

Similarly we have

$$\left.\begin{aligned}\gamma_1 &= \phantom{\gamma_1 +{}} V_2\tau + \frac{1}{v\rho S}\int_0^{\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \gamma_2 &= \gamma_1 + V_2\tau + \frac{1}{v\rho S}\int_{\tau}^{2\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \gamma_3 &= \gamma_2 + V_2\tau + \frac{1}{v\rho S}\int_{2\tau}^{3\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\text{vi}).$$

If $\alpha_1\alpha_2\alpha_3 \ldots$ be the values of α at times $\tau, 2\tau, 3\tau, \ldots$ etc., $\alpha_n = \beta_n - \gamma_n$, so that, subtracting equations (v) and (vi) we get

$$\left.\begin{aligned}\alpha_1 &= \phantom{\alpha_1 +{}} (V_1 - V_2)\,\tau - \frac{2}{v\rho S}\int_0^{\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \alpha_2 &= \alpha_1 + (V_1 - V_2)\,\tau - \frac{2}{v\rho S}\int_{\tau}^{2\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \alpha_3 &= \alpha_2 + (V_1 - V_2)\,\tau - \frac{2}{v\rho S}\int_{2\tau}^{3\tau} p_{\left(t-\frac{d}{v}\right)}\,dt \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\text{vii}).$$

The equations (i) and (ii) hold until the pressure waves, being reflected from the free ends of the rods, arrive again at C_1 and C_2 respectively, when new terms have to be added. Equation (i) then takes the form

$$\rho S\dot{\beta} + \frac{1}{v}\,p_{\left(t-\frac{d}{v}\right)} + \frac{1}{v}\,p_{\left(t-\frac{d+2l}{v}\right)} = \rho S V_1.$$

This holds until the wave, being once more reflected at A, again returns to C_1, when a further term $\frac{1}{v}\,p_{\left(t-\frac{d+2l+2d}{v}\right)}$ has to be added, and so on. Similar additions have to be made to equation (ii), and the corresponding terms inserted in equations (v), (vi), and (vii).

For purposes of calculation we may take any value for $\tau \ngtr \frac{d}{v}$. It is convenient to take $\tau = \frac{d}{v}$ or some sub-multiple of $\frac{d}{v}$*. Suppose $\tau = \frac{d}{nv}$. Then, since, at all times previous to $t = 0$, $p = 0$, we get

$$\alpha_r = r(V_1 - V_2)\tau = r(V_1 - V_2)\frac{d}{nv},$$

for all values of $r \ngtr n$, and

$$\left.\begin{aligned} \alpha_{n+1} &= \alpha_n + (V_1 - V_2)\frac{d}{nv} - \frac{2}{v\rho S}\int_{n\frac{d}{nv}}^{(n+1)\frac{d}{nv}} p_{\left(t-\frac{d}{v}\right)}\,dt \\ \alpha_{n+2} &= \alpha_{n+1} + (V_1 - V_2)\frac{d}{nv} - \frac{2}{r\rho S}\int_{(n+1)\frac{d}{nv}}^{(n+2)\frac{d}{nv}} p_{\left(t-\frac{d}{v}\right)}\,dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\},$$

or,

$$\left.\begin{aligned} \alpha_r &= r(V_1 - V_2)\frac{d}{nv} \text{ up to } r = n, \\ \alpha_{n+1} &= (V_1 - V_2)\frac{d}{nv} + \alpha_n - \frac{2}{v\rho S}\int_0^{\frac{d}{nv}} p_t\,dt \\ \alpha_{n+2} &= (V_1 - V_2)\frac{d}{nv} + \alpha_{n+1} - \frac{2}{v\rho S}\int_{\frac{d}{nv}}^{\frac{2d}{nv}} p_t\,dt \\ \alpha_{n+3} &= (V_1 - V_2)\frac{d}{nv} + \alpha_{n+2} - \frac{2}{r\rho S}\int_{\frac{2d}{nv}}^{\frac{3d}{nv}} p_t\,dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(\text{viii}),$$

where extra terms have to be inserted for the reflected waves when they appear.

Now $p = K\alpha^{\frac{3}{2}}$, so that, up to time $\frac{d}{v}$, we can plot $p = K(V_1 - V_2)^{\frac{3}{2}} t^{\frac{3}{2}}$ at intervals $\frac{d}{nv}$. Then in the second of equations (viii), α_n is known $\left\{= (V_1 - V_2)\frac{d}{v}\right\}$ and $\int_0^{\frac{d}{nv}} p_t\,dt =$ area under curve up to point (1) (Fig. 6) (equals impulse on end of rod during first time interval). Hence α_{n+1} is known and thence p_{n+1}. The point $(n+1)$ can now be plotted, and so on.

It is convenient, for purposes of actual calculation, to have $\frac{d}{n}$ some submultiple of the whole length of the rod. In this way, when the terms due to the reflected waves appear, the areas which have previously been calculated may be used over again. The method will be quite clear from the consideration of the worked out examples which are given in Appendix II.

The general course of the impact is shewn in the accompanying sketches (Fig. 7). To understand the shading of these sketches, they may be regarded as representing a piece

* In making my numerical calculations I found that sufficient accuracy was attained by taking $\tau = \frac{d}{v}$ simply.

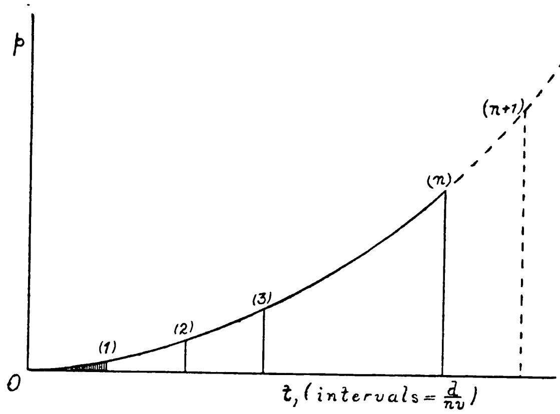

p
O
(1)
(2)
(3)
(n)
(n+1)
t, (intervals = $\frac{d}{nv}$)

FIG. 6.

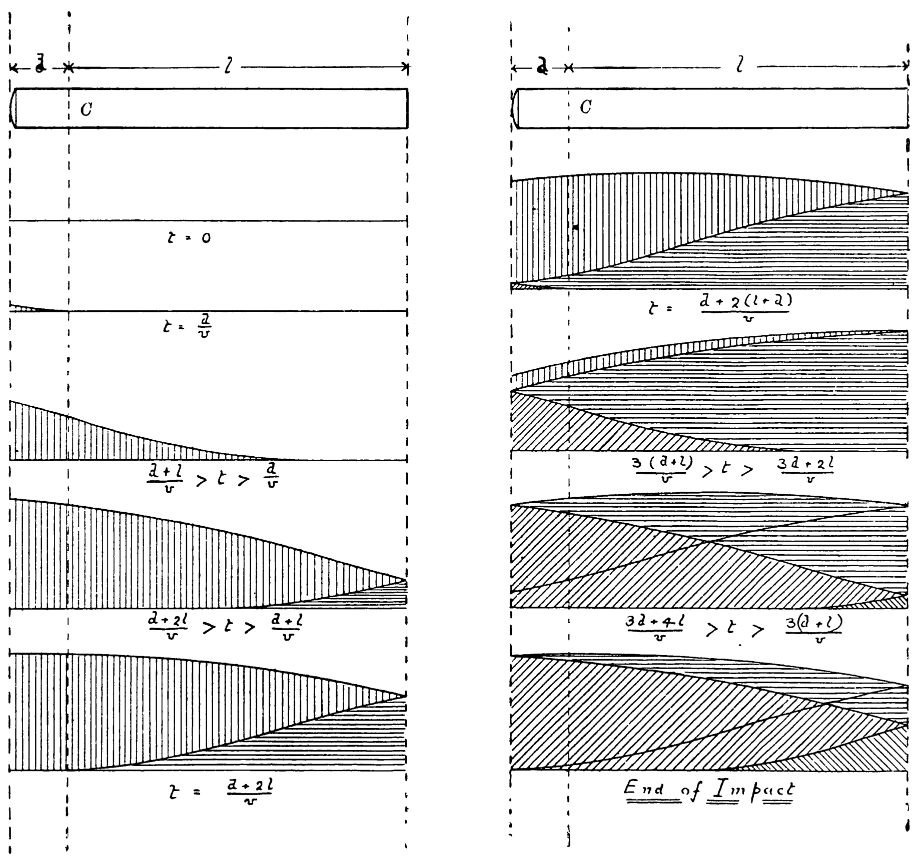

d
l
C
$t = 0$
$t = \frac{d}{v}$
$\frac{d+l}{v} > t > \frac{d}{v}$
$\frac{d+2l}{v} > t > \frac{d+l}{v}$
$t = \frac{d+2l}{v}$
$t = \frac{d+2(l+d)}{v}$
$\frac{3(d+l)}{v} > t > \frac{3d+2l}{v}$
$\frac{3d+4l}{v} > t > \frac{3(d+l)}{v}$
End of Impact

FIG. 7.

of paper, cut to the shape of the pressure-time curve, and folded backwards and forwards within the length of the rod. For clearness, only those portions of the paper actually visible at any stage have been shaded, but it is understood that the stress actually existing at any point is the *sum of all the whole ordinates* of the paper at that point, *with their appropriate signs*. In this way we can determine the state of pressure in the rod at any time during the impact. Fresh terms are to be inserted in the equations every time the head of the wave returns to the point C. Shading thus ||||| and thus ///// indicates compression; shading thus ≡ and thus ≋ indicates tension.

It is necessary to distinguish carefully between the step by step method of solution here employed, and the attempt to solve an ordinary differential equation by producing the curve. The successive points in the present method are determinable with any desired degree of accuracy from those already known, and the errors are not cumulative. The accuracy of the final result depends only on the accuracy of the assumptions made in using the statical compression of the end-element.

EXPERIMENTAL TESTS.

Duration of Impact and Velocity of Rebound.

It seemed best to check this theory in the first place by the use of rods of unequal length. In this way we get a check, not only from the duration of the impact, but from the changes in velocity as well (the total impulse being simply the area of the pressure-time curve from end to end). In the case of equal rods, the relative velocities before and after impact are practically equal, so that this latter check is of little value. I therefore made some experiments on unequal rods with a view to determining their velocities. These I calculated from observations on the amplitude of swing, which were made as follows*:—The travelling telescope was fitted with a micrometer eye-piece, and so adjusted that the end of the rod, in its lowest position, just coincided with one of the principal graduations. The divided head was then turned through such a number of complete revolutions that the rod, at the end of its swing, nearly returned to the original graduation. The outstanding difference was then measured by the small divisions of the micrometer eye-piece, and the whole amplitude thus determined. Small corrections had to be made for air-damping. Taking the amplitude as A, the velocity at the lowest point of the swing (where the impact takes place) will then be $\frac{2\pi A}{P}$, where P is the time of a complete swing.

Great difficulty was at first experienced in getting self-consistent observations, the total momentum after impact being apparently different from (generally greater than) that before. This was attributed to transverse waves set up in the suspensions at the moment of impact, which, returning to the rods after reflexion at the upper ends, so affected the amplitude as to make it unreliable as a means of measuring the velocities

* See also p. 73.

immediately after impact. It was some time before a suspension was found sufficiently light and inextensible for this error not to be serious. Finally, a suspension of fine gut fishing line was obtained. This was much lighter and less elastic than cotton, and, as will be seen from the figures, worked fairly satisfactorily.

The following* are the figures for this series of experiments:

Length of striking rod $5\frac{1}{3}''$.

Velocity of striking rod before impact $= 5\cdot030''$ per sec.

The struck rod was at rest previous to the impact.

Length of struck rod	Velocity of struck rod after impact. Ins. per sec.		Velocity of striker after impact. Ins. per sec.		Duration of impact. 10^{-4} secs.		Coefficient of restitution. 'e'	
	Calculated	Observed	Calculated	Observed	Calculated	Observed	Calculated	Observed
$5\frac{1}{3}''$	5·030	5·041	0	0	1·414	1·445	1·000	1·002
$9\frac{1}{3}''$†	3·648	3·653	− 1·354	− 1·336	1·607	1·604	·994	·992
$13\frac{1}{3}''$	2·841	2·815	− 2·074	− 2·019	1·790	1·795	·977	·969
$17\frac{1}{3}''$†	2·247	2·247	− 2·273	− 2·219	1·937	1·960	·904	·888
$21\frac{1}{3}''$	1·834	1·825	− 2·305	− 2·256	1·999	2·046	·823	·811
$29\frac{1}{3}''$	1·334	1·324	− 2·305	− 2·256	1·999	2·047	·723	·712
$37\frac{1}{3}''$	1·048	1·042	− 2·305	− 2·261	1·999	2·037	·667	·657

Fig. 8 exhibits these readings graphically. In it the dotted curves represent the calculated, and the black dots and full curves the experimental, values. In Case B the agreement is so close that only the theoretical curve could be shewn, and in all cases the agreement in general character, between corresponding curves, is remarkably good.

The first point which calls for remark is the impact of the equal $5\frac{1}{3}''$ rods. It will be seen that the calculated value of 'e' is unity, and the velocity of rebound of the striking rod consequently zero, and that the experimental readings agree well with this. This was by no means the case in the earlier experiments, the striking rod then following up the struck rod after impact through a small distance which Voigt‡, in his experiments, used to provide a correction for other cases. When the gut suspension was employed, however, the change produced was remarkable, the striking rod, when observed through the telescope, appearing absolutely at rest after the impact. At the same time it will be seen that the duration of the impact was less than that previously recorded for the corresponding pair of rods§. It appeared desirable, therefore, to repeat the experiments there given, and this was accordingly done, care being taken to adjust the apparatus so that the

* See also footnote, Appendix II, p. 89.

† The cases marked † were calculated for $9\cdot48''$ and $17\cdot48''$ rods, respectively, the values given being found, as explained in Appendix I (footnote, p. 87), by graphic interpolation.

‡ Voigt, *loc. cit.*

§ See p. 53.

actual velocity of impact was the same in each case, instead of, as formerly, relying for this purpose on the constancy of the withdrawal. The results were as follows:

Steel rods. Velocity 5·030″ per sec.					
Length of rods (inches) ...	$5\frac{1}{3}$	$13\frac{1}{3}$	$21\frac{1}{3}$	$29\frac{1}{3}$	$37\frac{1}{3}$
Time of impact (10^{-4} secs.)	1·445	2·306	3·089	3·895	4·683

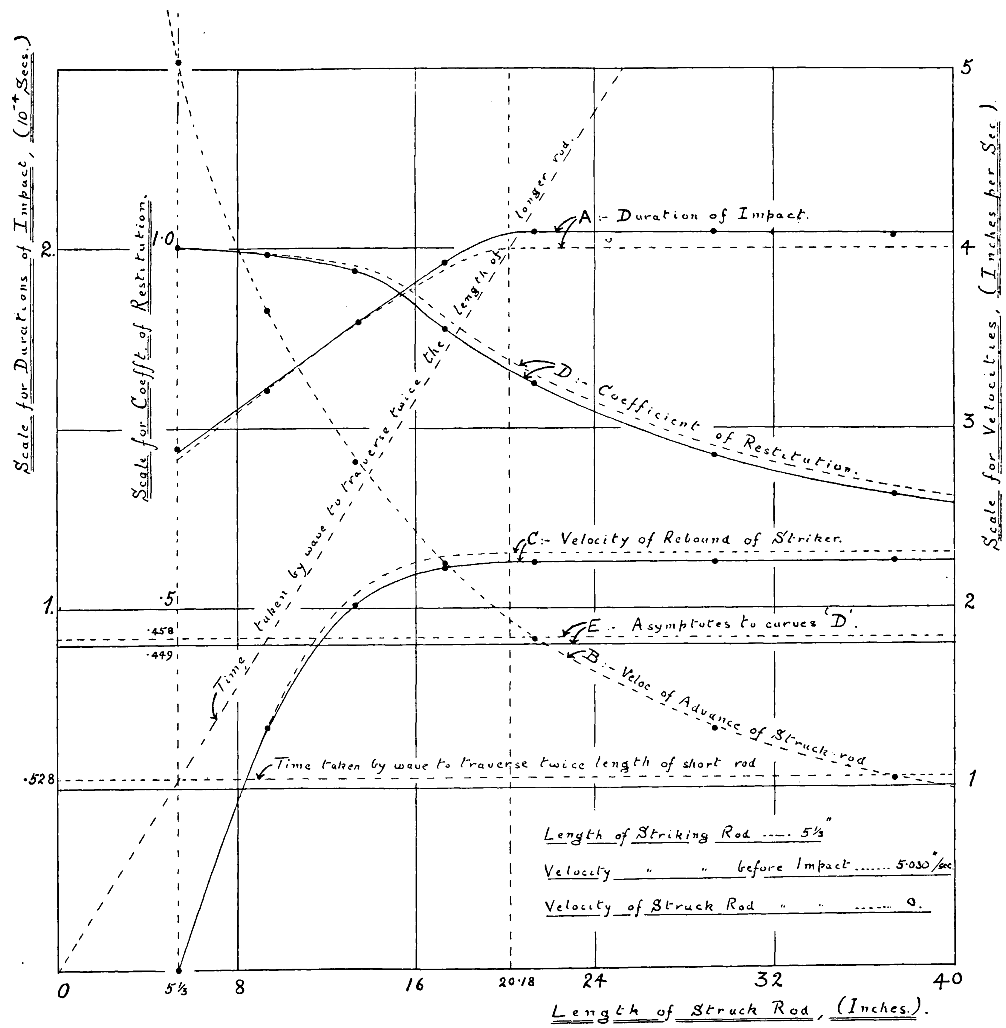


FIG. 8.

It will be seen that, but for the shortest pair of rods, there is very little difference from the earlier series of results (the last four readings lie, in fact, more accurately on a straight line than before). The reduction in the end-effect for the shortest pair is just what the theory would lead us to expect. Referring to Fig. 9*, which gives the calculated pressure-time curves for pairs of equal rods of different lengths, we see that, until the first reflected wave appears, the pressure rises asymptotically to the value 28,020, which corresponds to the formula $P = \frac{1}{2}\pi a^2 E \frac{V}{v}$ given by St Venant's theory. Provided, then, the rods are sufficiently long for this value to be practically attained before the arrival of the first reflected wave, the end-effect will be independent of the length of the rods. In the case of $5\frac{1}{3}''$ rods, however, not only does the first reflected wave return much sooner than this, but, before the end of the impact, a second reflected wave comes into play. The consequence is that for these rods the end-effect is only $\cdot 886 \times 10^{-4}$ secs., while for the $13\frac{1}{3}''$ rods it is $\cdot 990 \times 10^{-4}$ secs. Beyond this it only increases to $\cdot 999 \times 10^{-4}$ secs. for indefinitely long rods.

Taking the last four readings in the above table, we get the same value for the velocity of wave propagation (16,820 ft. per sec.) as before, so that the conclusions of the previous article are not impaired. It is rather curious, however, but perhaps not very surprising, that this effect should have been so exactly masked by the use of the heavier suspension in all three cases then tried. Unfortunately I have not had time to repeat the experiments on copper and aluminium. The calculated figures for the steel rods are as follows:

Steel rods. Velocity 5·030″ per sec. (calculated)					
Length of rods (inches)	$5\frac{1}{3}$	$13\frac{1}{3}$	$21\frac{1}{3}$	$29\frac{1}{3}$	$37\frac{1}{3}$
Duration of impact (10^{-4} secs.)	1·414	2·311	3·113	3·905	4·698

Returning now to the consideration of Fig. 8, a most striking feature is the close agreement shewn by the theoretical and experimental values of the coefficient of restitution. The curves D do not really adequately represent this agreement. The values of 'e' are calculated from the velocities of the rods after impact, and these again from the impulse, which is equal to the change of momentum of either rod. In the case of the short striking rod, the greater part of this is taken up in the fixed known momentum before impact, so that any error in the impulse will make a considerably greater error in the velocity of the striking rod after impact, and, consequently, in the coefficient of restitution. It is fairer, therefore, to compare the impulses direct. These are, of course, given at once by the velocities of the struck rod after impact, and a reference to the curve B, shews how remarkable the agreement really is.

It will be seen that, as the length of the struck rod increases, the duration of impact also increases; but slowly, so that we finally reach a length (whose calculated value is 20·18″) at which the time taken by a wave to travel up and down the longer rod is equal to the whole duration of the impact. Beyond this point the impact ceases before the reflected wave in the struck rod gets back to the point of contact, and the

* p. 96.

nature of the impact is independent of the length of the longer rod. In fact, to the right of this ordinate the curves A and C become horizontal straight lines. The impulse at impact being constant, it follows that the velocity of the struck rod after impact must be inversely proportional to its mass, so the curve B is in this region an hyperbola. The curve D, whose ordinate is proportional to the sum of the ordinates of B and C, is therefore also an hyperbola, whose asymptote is shewn.

It appears then that the rod of length $5\frac{1}{3}''$, if allowed to impinge on an infinitely long rod, would give rise to a limiting value for 'e' of ·449. This result is surely very remarkable. We have here a set of values for 'e' varying from absolute unity in the case of equal rods, to ·449 for one short and one very long rod; and this is entirely due to the energy taken up in vibration, the rods still behaving perfectly elastically.

In a paper in the *Philosophical Magazine* for February, 1906, Lord Rayleigh concludes that, in the case of spheres, the energy of vibration is extremely small compared with that of translation, so that Hertz' theory would apply. The present case is far different, and no former theory of the impact of long rods has, I believe, shewn an agreement with experimental results, both as regards duration of impact and velocity of rebound, nearly so good as this. It will be seen at once that the actual results are far from agreement with either the Newtonian (or Hertzian) or the St Venant theories*.

The next series of experiments was similar to the one just described, except that the longest rod was chosen for the fixed length instead of the shortest. In these experiments I concerned myself only with the durations of impact. The results, which are plotted in Fig. 11, are as follows:

Steel rods: Relative velocity before impact 5·030″/sec. Length of striking rod $37\frac{1}{3}''$.	Length of struck rod (inches)	Duration of Impact (10^{-4} secs.)	
		Calculated	Observed
	$5\frac{1}{3}$	1·999	2·037
	$9\frac{1}{3}$	2·700	2·807
	$13\frac{1}{3}$	3·335	3·386
	$17\frac{1}{3}$	3·938	3·934
	$21\frac{1}{3}$	4·197	4·213
	$25\frac{1}{3}$	4·278	4·298
	$29\frac{1}{3}$	4·380	4·399
	$33\frac{1}{3}$	4·510	4·516
	$37\frac{1}{3}$	4·698	4·683

In Fig. 11, the curve shewn is the calculated curve, the black dots being the experimental values. With the longer rods it will be seen that the agreement is again remarkably good, while the shorter rods give readings not very different from the theoretical values. The general character of the curve, at any rate, is well established.

* See Love, *Theory of Elasticity*, 2nd ed. pp. 25, 26.

The interesting feature of this curve is its hump. This occurs when the length of the shorter rod is half that of the longer, and may be explained as follows:—When two long unequal rods impinge together, the first reflected wave in the shorter rod, on its arrival at the point of contact, initiates a separation between the rods. The terms in the equations

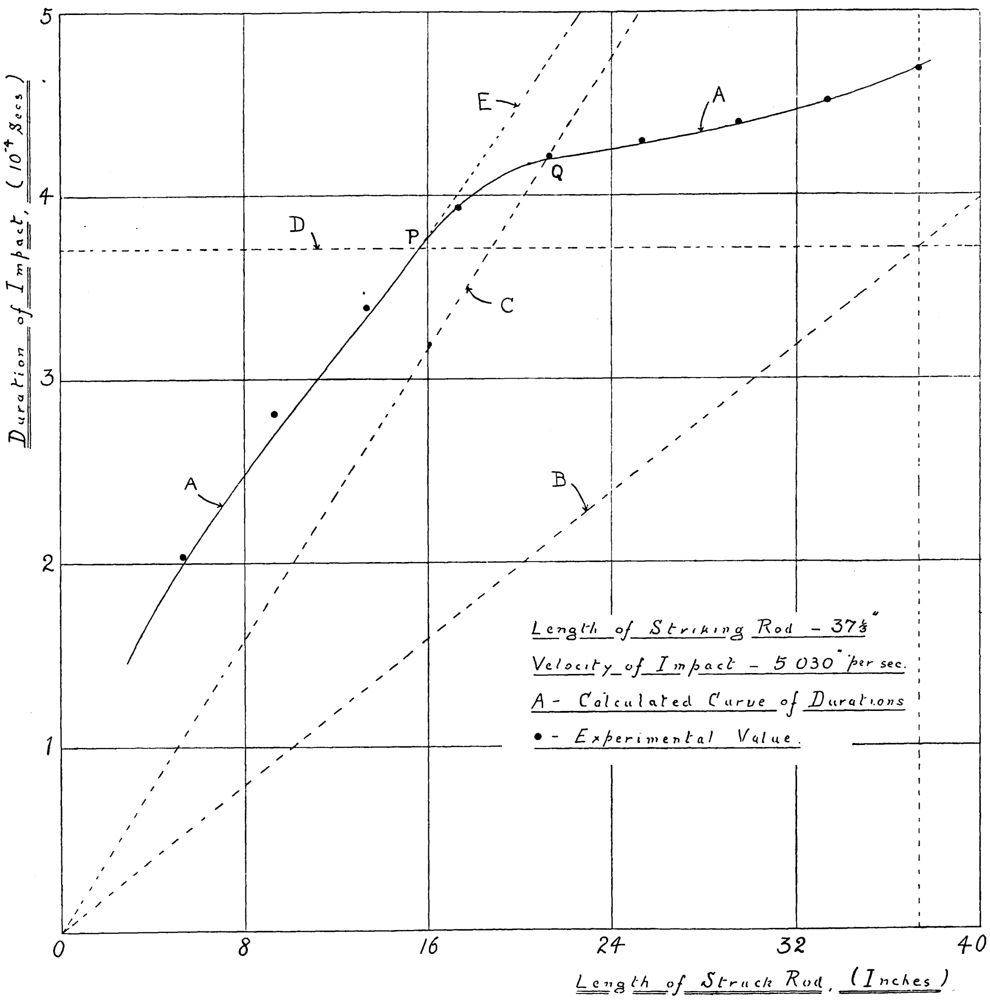


FIG. 11.

arising from the progressing wave consequently begin to diminish, thereby partially counteracting the effect of the reflected wave, with the result that the displacement (and therefore the pressure) between the ends of the rods, instead of approaching zero in a finite manner, tends asymptotically towards this limit, and it is not until a reflected wave in the longer rod, or a second reflected wave in the shorter rod, also reaches the point of contact, that separation finally ensues*.

* To illustrate this point I have given in full (Appendix II B) the calculation of the end-effect for two indefinitely long rods with 16″ difference in length. This is calculated from the arrival of the first reflected wave in the shorter rod. The pressure-time curve is shewn in Fig. 10.

It follows that, if the difference between the two rods is greater than the length of the shorter, the duration of impact will be greater than the time taken by the wave to travel *four times* the length of the shorter rod. The greater this length, however, the nearer will the pressure be to zero when the second reflected wave reaches the point of contact, and the sooner after this time will the impact, consequently, cease. The duration of impact, in fact, in the case of two long rods, one of which is more than twice as long as the other, tends asymptotically to this value. This is indicated by the dotted curve, E, and the straight line, C (Fig. 11).

The above statement, as it stands, is not, however, quite complete. If the length of the longer rod be only slightly greater than twice that of the shorter, a reflected wave in the longer rod will arrive at the point of contact before the impact is finished and help to hasten the end. The curve, A, consequently begins to fall away from the curve, E, (Fig. 11), at the point, P, where the duration of impact attains a value equal to twice the time taken by a wave to traverse the longer rod. When the length of the shorter rod is exactly half that of the longer, the two waves will arrive together, and, as the shorter rod continues to increase in length, the wave in the longer rod acquires more and more importance, until, at the point, Q, it is sufficient to complete the impact before the second wave in the shorter rod arrives. From this point on, the end-effect (measured, as usual, from the arrival of the first reflected wave in the shorter rod) will depend only on the difference between the two rods, becoming less as the rods approach equality.

My appa ıatus was not capable of dealing with rods of greater length than those used in the above series of experiments; but, the theory being so well supported by experiment in all cases tried, it seems quite justifiable at this stage to follow up the result of using still longer rods by the aid of theory alone. This is done in Fig. 12, which represents the result of allowing rods of all lengths up to 120″ to impinge with the same velocity (5·030″ per sec.) on a single rod of length 48″. The black dots on this curve represent, not experimental values, but the points for which the calculation was actually performed.

By taking the fixed rod of this length, the character of the first portion of the curve is rendered more apparent. From P to Q we have the transition stage, as before. To obtain the points on the curve beyond Q, we add the end-effect, calculated for indefinitely long rods of the proper difference in length, to the time required by the first wave to run up and down the shorter rod. The blank circle indicates the duration for a 24″ rod impinging on an indefinitely long one. The effect of a new reflected wave on the relative displacement, α, between C_1 and C_2, being independent of the rod in which it appears, it follows that the end-effect in this case is the same as that for indefinitely long rods which differ by 24″. The prolongations of the portions of the curve on either side of the transition stage therefore meet in this point. The black dot below shews the duration for 24″—48″ rods.

From Q to R, the length of the shorter rod increases uniformly, but at R the two rods become equal, and from this point onward the length of the shorter rod does not alter. In plotting the curve to the right of Q by means of the end-effects, a discontinuity

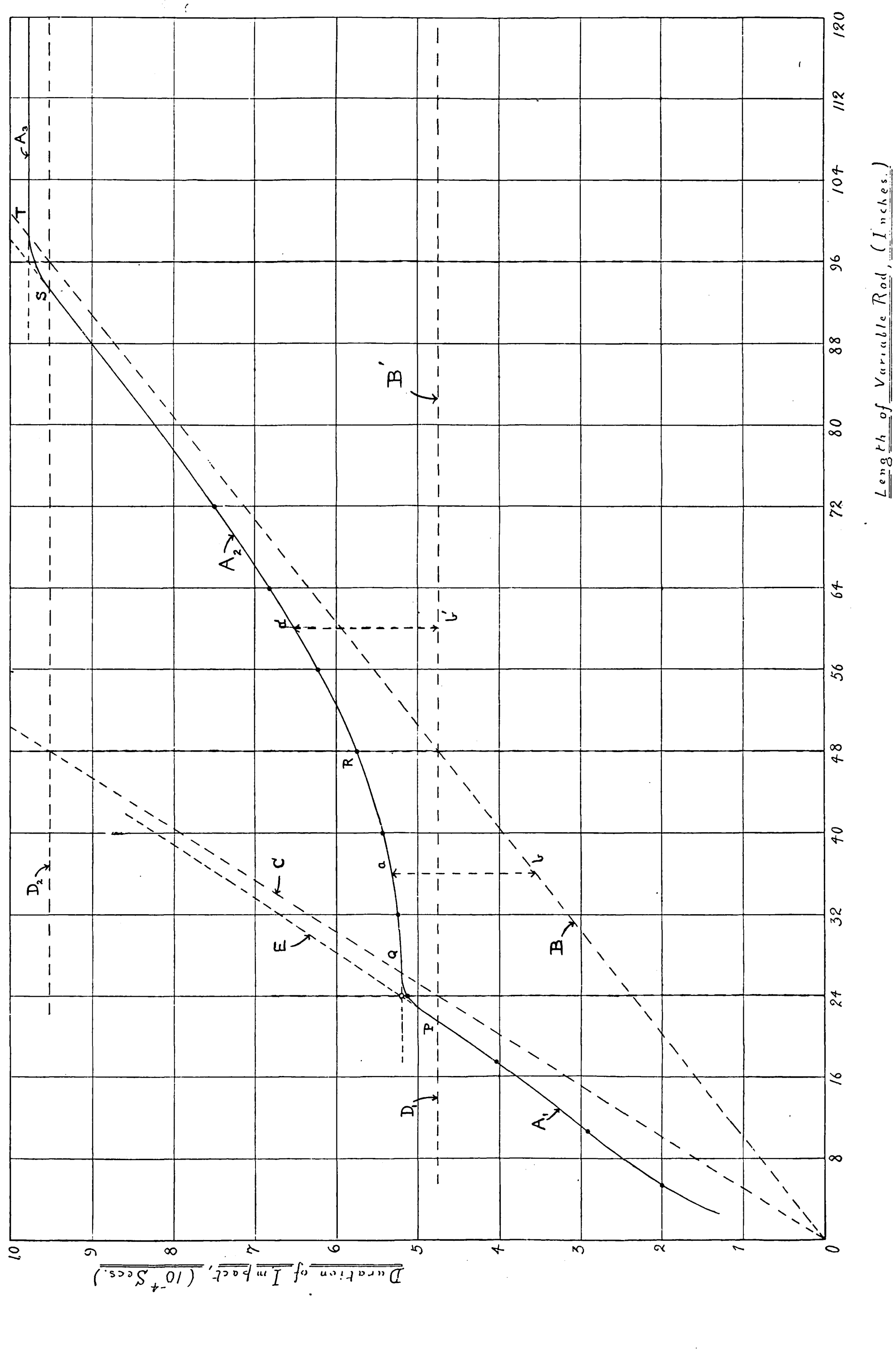


FIG. 12.

is thus introduced at R. This however makes no apparent difference to the continuity of the curve*, which now becomes asymptotic to the straight line, B, which represents the time taken by a wave to travel *once* up and down the variable rod. The ordinates ab, $a'b'$, at equal distances on either side of R, are equal.

At S, where the variable rod attains a length nearly twice that of the fixed rod, the duration becomes equal to the time taken by a wave to travel twice up and down the latter, and a second transition period sets in, of exactly the same character as before. At T, the variable rod attains such a length that the second reflected wave in the fixed shorter rod completes the impact before the return of the first reflected wave in the longer. From this point onwards the duration of impact becomes independent of the length of the longer rod, just as we saw in the first series of experiments (Fig. 8). As before the two portions of the curve on either side of the transition stage, intersect, if produced on the ordinate which gives to one rod twice the length of the other.

It is particularly interesting to find a curve consisting of three (or possibly four) distinct portions like this, as the representation of a process apparently perfectly continuous; and the agreement with experiment which has been established leaves no room to doubt its accuracy.

Variation of End-effect with velocity of Impact.

A last series of experiments was undertaken with a view to determining the variation of the end-effect with the velocity of impact. For this purpose the shortest ($5\frac{1}{3}''$) rods were chosen. The results of this series are tabulated below, and the figures plotted in Fig. 13.

Withdrawal (inches)	Velocity (inches per second)	Duration of impact (10^{-4} secs.)		
		Calculated	Observed (a)	Observed (b)
$\frac{1}{4}$	·6005	2·098	2·184	—
$\frac{1}{2}$	1·229	—	1·891	—
1	2·502	1·607	1·646	—
$1\frac{1}{2}$	3·758	—	1·522	—
2	5·030	1·414	1·445	1·400
3	7·553	—	—	1·311
4	10·05	1·258	—	1·242
6	15·13	—	—	1·175
8	20·22	1·125	1·132	1·119

With velocities up to $5''$ per sec., no overstrain is produced at the ends of the rods. These readings were accordingly taken first, and appear in column (a). The velocity was

* A discontinuity was found at this point by Hamburger in his experiments on plane-ended rods (*Wied. Ann.* XXVIII. 1886). The correction of certain errors made in plotting his figures, renders this discontinuity less conspicuous, but it is, none the less, quite unmistakeable.

then at once increased to 20″ per sec., and a set of readings taken with this velocity. Of these, the first is recorded in column (*a*), while the mean of the last four readings appears in column (*b*). The remainder of the observations (including a repeat of the 2″ withdrawal) were made with the ends thus overstrained, and appear in column (*b*)*.

In the figure, the full curve, *A*, represents the experimental values, points from column (*a*) being shown as black dots, and points from column (*b*) as blank circles. The dotted curve, *B*, gives the calculated values.

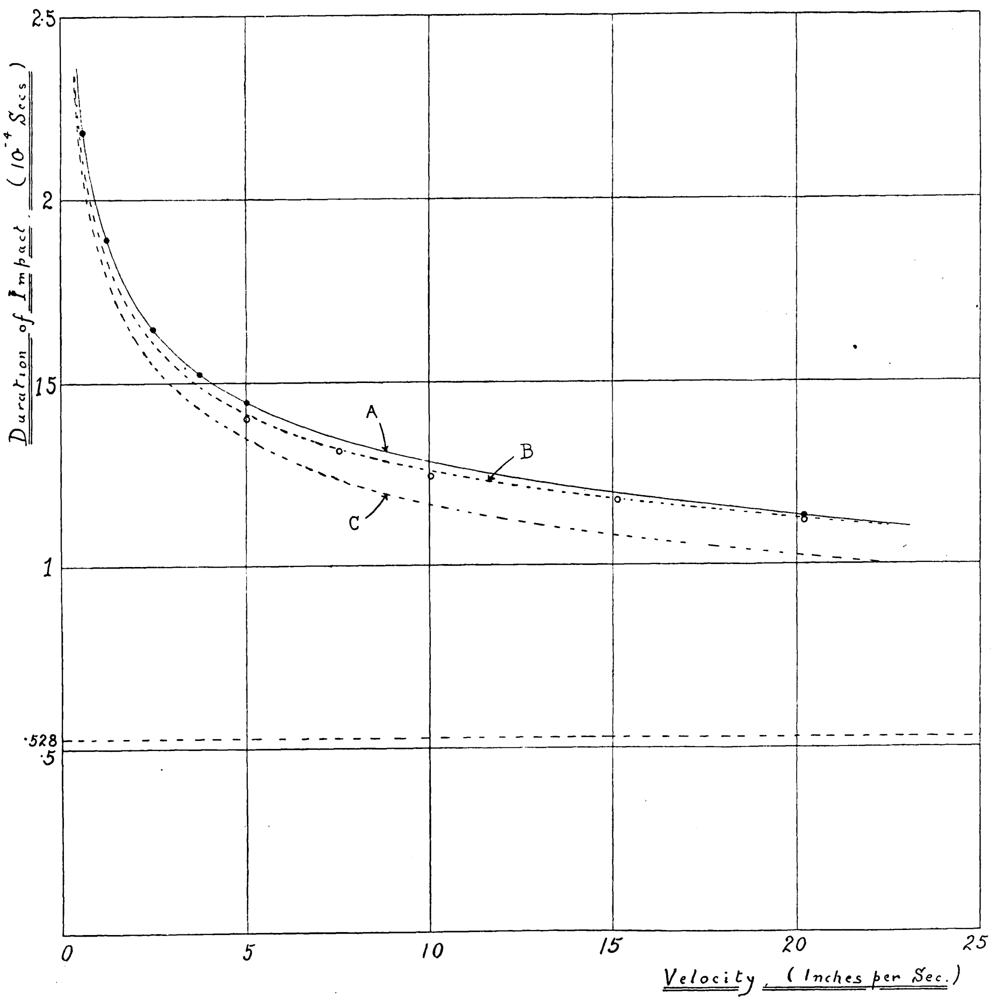


Fig. 13.

This is the case in which (the length of the rods being only ten times their diameter) Hertz' theory may be expected to agree best with experiment. I have therefore also plotted, in curve *C*, the values given by this theory. It will be seen that, with small velocities of

* See p. 51. Also *Camb. Phil. Proc.* xiv. 3, pp. 261, 262.

impact, the three curves run close together, but that, as the velocity increases, Hertz' curve begins to fall away from the other two. The formula given by Hertz' theory is, in fact, $TV^{\frac{1}{5}} = \text{const.}$, so that this curve is asymptotic to the line $T=0$, while the other two are asymptotic to the line $T = \cdot 528 \times 10^{-4}$, which represents the time of a complete vibration of the rods, and corresponds to St Venant's theory.

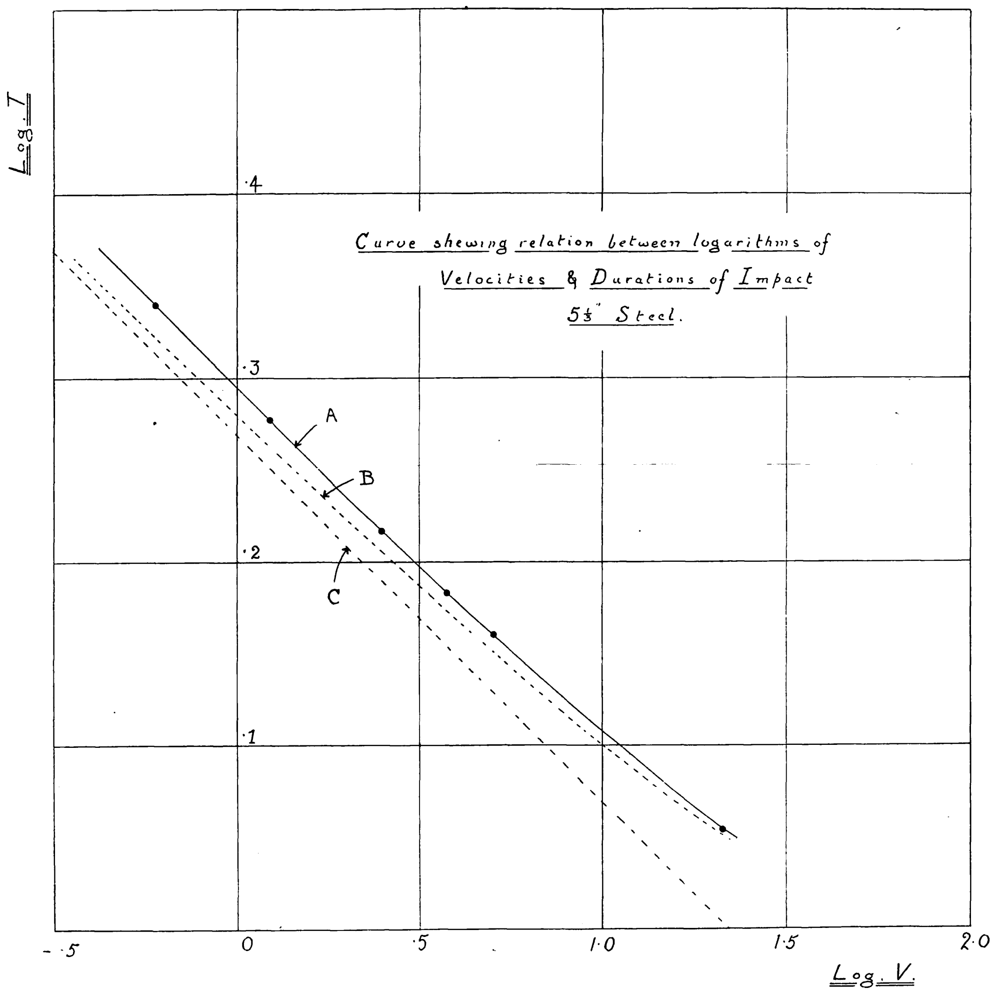


FIG. 14.

With longer rods the deviation becomes much greater: for instance, with $21\frac{1}{3}''$ rods impinging with a velocity of $5\cdot030''$ per sec., Hertz' theory gives $2\cdot340 \times 10^{-4}$ secs. instead of $3\cdot089 \times 10^{-4}$, while for the $37\frac{1}{3}''$ pair we get $2\cdot927 \times 10^{-4}$ as against the true value $4\cdot683 \times 10^{-4}$.

In Fig. 14 are plotted the logarithms of the quantities represented in Fig. 13. The curve C, here becomes simply a straight line with a slope of 1 in 5, and the relationships between the curves are more clearly seen. I suspect that, with small velocities of impact, the experimental values tend to become slightly too high*, and that the curve A should really be asymptotic to this line as well as the curve B.

Details of Method.

I have not thought it necessary to give in full all the experimental data upon which the figures in the foregoing tables are based. The durations of impact were determined by the method described in the previous paper, and the tables of observations are of precisely the same character as those there published.

A question was, however, raised as to the legitimacy of calibrating the galvanometer by means of a standard condenser, and I therefore made some experiments with a view to checking this method against those known as "steady deflexion" and "standard field." Using various galvanometers, discrepancies were always found between the three methods which amounted in some cases to as much as 3 per cent., and it is certain that great care must be taken, in all cases where accurate calibration is required, to ensure that the galvanometer works during the calibration test under conditions closely resembling those obtaining in actual use. In the present case the galvanometer makes its fling on open circuit, a condition which is most nearly attained in the standard condenser method. As explained in the previous paper, fatigue effects in the suspension rendered it difficult, with the galvanometer used in these experiments, to obtain a satisfactory calibration by the steady deflexion method; but after several failures I at last succeeded in doing so, the constant so arrived at differing by only 1 part in 3000 from that obtained with the standard condenser. That given by the standard field method, however (corrected in the usual way for damping), differed from these by nearly 1 per cent. This latter calibration was the least regular of the three, and the method is, moreover, that which differs most from the conditions obtaining in actual practice. There is, I think, no reason to doubt the accuracy of the constant employed, though of course it is not claimed that the experimental figures are correct to 1 part in 3000.

It may perhaps be worth while also to describe here at somewhat greater length the method of determining the velocities†, and to give a sample table. In the first place it is necessary to determine the velocity of the striking rod at impact. To get this, the other rod was first removed, and the striker, after being withdrawn to the correct distance, was released (by blowing the fuse) and allowed to swing freely without any impact, consecutive elongations on the same side of the lowest point being observed by means of the micrometer eye-piece in the telescope. It was found that the elongation, at the end of the first complete swing from release, differed from the original withdrawal by an amount which, though small, was considerably greater than could be accounted for by air-damping (as calculated from the consecutive values of the elongations, neglecting the original withdrawal). This was attributed

* This would be the effect of the slight irregularities of curvature due to the granular structure of the metal, which would be negligible compared with the whole effect of the mean curvature of the ends of the rods except when the pressure was extremely small.

† See p. 62.

to a loss of energy during the release*, and the velocity at the lowest point of the first swing (which in the actual experiments is the velocity of impact) was therefore calculated, not from the original withdrawal, but from the elongation after the first complete oscillation, adding to this the appropriate damping correction.

The following table exhibits the figures for the $5\frac{1}{3}''$ rod with $2''$ withdrawal.

MICROMETER EYE-PIECE READINGS.				
(1) Initial	(2) After 1st complete swing	(3) After 21st complete swing	(2) – (1)	(3) – (2)
8·0	8·35	10·35	·35	2·00
8·0	8·4	10·5	·4	2·10
8·0	8·3 +	10·4	·3 +	2·1 –
8·0	8·4 –	10·5 –	·4 –	2·10
8·0	8·35	10·4 –	·35	2·05 –
8·0	8·3 +	10·4	·3 +	2·1 –
8·0	8·4	10·45	·4	2·05
8·0	8·4	10·45	·4	2·05
8·0	8·3 +	10·45	·3 +	2·15 –
8·0	8·3	10·3	·3	2·00
		Means...	·355	2·06

One division of the micrometer eye-piece corresponds to $\cdot 104''$. (This was determined simply by getting one of the principal graduations of the eye-piece in line with the end of one of the rods (fixed), and then turning the divided head of the travelling telescope until the next graduation came opposite the same point.)

The mean amplitude after the first swing is therefore $2'' - \cdot 355 \times \cdot 104'' = 2'' - \cdot 037'' = 1\cdot 963''$.

The mean amplitude after the 21st swing is less than this by $2\cdot 06 \times \cdot 104''$ or $\cdot 214''$. It has therefore the value $1\cdot 749''$.

To get the equivalent amplitude for the velocity of impact, we have to correct the amplitude ($1\cdot 963''$), at the end of the first swing, for the air-damping during the last three quarters of that swing.

The coefficient of air-damping in 20 complete swings is $\dfrac{1\cdot 963}{1\cdot 749}$ (logarithm $= \cdot 05013$).

The logarithmic coefficient for one complete swing is therefore $\dfrac{\cdot 05013}{20} = \cdot 00251$; and for a three-quarter swing, $\frac{3}{4} \times \cdot 00251 = \cdot 00188$ (antilog. $= 1\cdot 0043$).

* The effect appeared also when the rod was released by burning cotton, instead of in the ordinary manner by blowing the fuse.

Correcting the value 1·963″ by this factor we get the required amplitude = 1·972″. The time of a complete oscillation was found experimentally to be 2·464 secs. We thus get finally the velocity of impact $= \dfrac{2\pi \times 1{\cdot}972}{2{\cdot}464} = 5{\cdot}030''$ per sec.

The velocities after impact were determined directly from the first elongation after impact, the proper correction being applied for air-damping.

THE ELASTIC MODULI.

Referring back to p. 63, and comparing the experimental and theoretical values of the impulses (or, what is the same thing, of the velocities of the struck rod after impact), we get the following table:—

Length of struck rod (inches)	Velocity of struck rod after impact (ins. per sec.)		Difference	°/₀ Error
	Calculated	Observed		
$5\frac{1}{3}$	5·030	5·041	− ·011	− ·22
$9\frac{1}{3}$	3·648	3·653	− ·005	− ·14
$13\frac{1}{3}$	2·841	2·815	·026	·92
$17\frac{1}{3}$	2·247	2·247	·0	·0
$21\frac{1}{3}$	1·834	1·825	·011	·60
$29\frac{1}{3}$	1·334	1·324	·010	·75
$37\frac{1}{3}$	1·048	1·042	·006	·57

The mean absolute error is thus only ·46 °/₀, while the mean algebraic error comes out at ·34 °/₀.

The results with regard to duration of impact, though not quite so close as this, are still very good. Omitting those cases in which the velocity of impact was very small (so that, as mentioned above, the experimental value may be expected to be too high), we may construct the following table:—

Length of Rods (inches)	Velocity of Impact (ins. per sec.)	Duration of Impact (10^{-4} secs.)		Difference	% Error
		Calculated	Observed		
$5\frac{1}{3}$—$5\frac{1}{3}$	5·030	1·414	1·445	·031	2·1
$5\frac{1}{3}$—$9\frac{1}{3}$	,,	1·607	1·604	− ·003	− ·3
$5\frac{1}{3}$—$13\frac{1}{3}$	,,	1·790	1·795	·005	·3
$5\frac{1}{3}$—$17\frac{1}{3}$	,,	1·937	1·960	·023	1·2
$5\frac{1}{3}$—$21\frac{1}{3}$	,,	1·999	2·046	·047	2·3
$5\frac{1}{3}$—$29\frac{1}{3}$	,,	1·999	2·047	·048	2·4
$5\frac{1}{3}$—$37\frac{1}{3}$	,,	1·999	2·037	·038	1·9
$13\frac{1}{3}$—$13\frac{1}{3}$	,,	2·311	2·306	− ·005	− ·2
$21\frac{1}{3}$—$21\frac{1}{3}$	,,	3·113	3·089	− ·024	− ·8
$29\frac{1}{3}$—$29\frac{1}{3}$	,,	3·905	3·895	− ·010	− ·3
$37\frac{1}{3}$—$37\frac{1}{3}$	,,	4·698	4·683	− ·015	− ·3
$37\frac{1}{3}$—$33\frac{1}{3}$	,,	4·510	4·516	·006	·1
$37\frac{1}{3}$—$29\frac{1}{3}$	,,	4·380	4·399	·019	·4
$37\frac{1}{3}$—$25\frac{1}{3}$	,,	4·278	4·298	·020	·5
$37\frac{1}{3}$—$21\frac{1}{3}$	,,	4·197	4·213	·016	·4
$37\frac{1}{3}$—$17\frac{1}{3}$	,,	3·938	3·934	− ·004	− ·1
$37\frac{1}{3}$—$13\frac{1}{3}$	,,	3·335	3·386	·051	1·5
$37\frac{1}{3}$—$9\frac{1}{3}$	,,	2·700	2·807	·107	3·9
$5\frac{1}{3}$—$5\frac{1}{3}$*	7·553	1·317	1·344	·027	2·0
$5\frac{1}{3}$—$5\frac{1}{3}$*	10·05	1·258	1·281	·023	1·8
$5\frac{1}{3}$—$5\frac{1}{3}$*	15·13	1·178	1·193	·015	1·3
$5\frac{1}{3}$—$5\frac{1}{3}$	20·22	1·125	1·132	·007	·6

* Interpolated from curve.

The mean absolute error here is 1·1 %, and the mean algebraic error ·9 %.

Of the 22 cases here considered, however, 11 give rise to errors lying between − ·3 % and ·6 %, which is less than one-fifth of the whole range covered. Considering this group alone, we get, instead of the above figures, ·32 % and ·10 % respectively*.

Such agreement is, of course, extremely good; but, in view of the fact that the value of the wave-velocity used in making these calculations was that actually obtained from the experiments on equal rods, it seems, perhaps, fairer to consider the errors, not in the total times, but in the end-effects alone, the peculiar features of the theory being

* In the early stages of these experiments, spasmodic high readings were very frequent, and proved a great source of difficulty. At a later stage (when the above observations were made) these readings were practically eliminated; but the variations which did occur were almost invariably in the same sense. The last vestiges of this tendency may perhaps be traced in the fact that, in 10 out of the 11 cases in which larger errors occur, these errors are positive.

only manifested after the return of the first reflected wave in the shorter rod. It must, however, be borne in mind that in doing this we are artificially emphasizing the purely experimental errors, which are, of course, errors in the *total* time. The result is that the marked grouping of the errors in the neighbourhood of zero now disappears, and we are compelled once more to take into consideration all the cases with the exception of the two extremes.

The value of the mean absolute error is thus increased to 1·7 °/₀, and that of the mean algebraic error to 1·3 °/₀.

Even so, the discrepancy between the experimental and theoretical results is not great; and, but for a slight mistake in the calculations*, the agreement, both as regards impulse and duration of impact, would be even better.

In any case we are certainly justified in regarding the differences which do exist as the result rather of accident than of any fundamental disagreement.

Now the theory on which the above calculations are based depends essentially on the formula

$$P = \frac{\sqrt{2}E}{3(1-\sigma^2)} r^{\frac{1}{2}} \alpha^{\frac{3}{2}} = K\alpha^{\frac{3}{2}},$$

given by Hertz for the compression between two spheres of radius r.

In this formula σ represents Poisson's ratio, and has the value $\dfrac{E}{2C} - 1$.

Thus $$(1-\sigma^2) = \frac{E}{2C}\left(2 - \frac{E}{2C}\right) = \frac{E}{C}\left(1 - \frac{E}{4C}\right),$$

or $$\log(1-\sigma^2) = \log E - \log C + \log\left(1 - \frac{E}{4C}\right),$$

so that $$\frac{1}{(1-\sigma^2)}\frac{d(1-\sigma^2)}{dC} = -\frac{1}{C} + \frac{E}{4C^2} \times \frac{1}{1-\dfrac{E}{4C}},$$

or $$\frac{d(1-\sigma^2)}{(1-\sigma^2)} = \frac{dC}{C}\left\{\frac{2E-4C}{4C-E}\right\}$$

$$= \frac{2\sigma}{1-\sigma}\frac{dC}{C}.$$

The value of σ for steel is about ·27, so that

$$\frac{d(1-\sigma^2)}{(1-\sigma^2)} = \frac{3}{4}\frac{dC}{C} \text{ very nearly.}$$

An error of 1 °/₀ in the value of C will thus give rise to an error of $\frac{3}{4}$ °/₀ in the value of Hertz' constant K.

Now the value of C used in calculating this constant was taken from a *static* torsion test on the specimen rod; and, as it has already been seen that the value of E is the same for both steady and instantaneous stresses, it follows from the close agreement

* See note, Appendix II, p. 89.

above established between the experimental and theoretical results, that the same remark must apply also to the rigidity modulus, C.

But *all* the elastic constants of the material are expressible in terms of E and C, so that we are now entitled to make the following general statement:—

The values of all the elastic constants of a metal are the same under instantaneous as under steady stresses.

THE ELASTIC LIMIT.

One other point appears worthy of remark. It was mentioned in the previous paper that, with a withdrawal of 2″, the pressure at the centre of the area of contact (as calculated by Hertz' formula) reaches a value of 108 tons per sq. in. (the mean value over the whole area being 72 tons per sq. in.) and that this pressure apparently produces no overstrain. The agreement established in this paper between theory and experiment may be taken to prove that the application of the formula is correct, and that the pressure may therefore rise instantaneously to as much as five times the elastic limit of the material without producing any permanent effects. It would be most interesting to enquire further into this phenomenon.

A REMARK BY HERTZ.

The results obtained in this work lend considerable weight to a paragraph in Hertz' paper* which appears to have been generally overlooked by experimenters and others who have applied themselves to the problem of impact†. Hertz there suggests that, by combining the static compression for the parts of the bodies in the immediate neighbourhood of the point of contact with the general equations of motion for the rest of the bodies, we could probably get the laws of impact for bodies of any shape. It is not easy to see how this could be done, however, nor wherein the process would be likely to be simpler than that of applying the laws of motion to the whole of the bodies direct. Hertz himself offers no suggestion. It will be seen that the solution, in the case of long rods, has been brought about by a combination of the static compression in the end-element, not with the general equations of motion‡, but with certain simple laws which, in this case, may be deduced from them. It is possible that other simple cases may afford similar solutions.

* Hertz, *Miscellaneous Papers*, trans. by Jones, pp. 159, 160.

† I may perhaps remark that the passage in question escaped my own notice till after the completion of this work. It is, I find, reproduced in full by Lord Rayleigh in the paper (*Phil. Mag.* Feb. 1906) above mentioned.

‡ I attempted to do this first.

EXTENSION TO PLANE-ENDED RODS.

It is interesting to try to extend the present work to the case of plane-ended rods, taking account of the irregularities of surface produced by the granular structure of the metal. We cannot in this way get any quantitative information without actually determining the mean curvature of these irregularities, and the manner of their distribution, but we can get some idea, at any rate, of the sort of laws which may be expected to govern the 'Zwischenschicht' of Voigt's theory*.

For the sake of clearness, we shall suppose, for the present, that the end of the rod consists simply of a large number of small convex areas regularly disposed, such as might be formed by fitting together a number of equal hexagonal rods each provided with a rounded end. (Each of these elementary rods will behave independently of its neighbours provided only the diameter of the whole rod is so small that the energy of the radial displacements may be neglected.)

Consider one such elementary rod. Let b be the side of the hexagon, r the radius of the end. Let $r = \lambda b$.

We have then
$$P = \frac{\sqrt{2}E}{3(1-\sigma^2)} r^{\frac{1}{2}} \alpha^{\frac{3}{2}} \dagger$$
$$= \frac{\sqrt{2}E}{3(1-\sigma^2)} \lambda^{\frac{1}{2}} b^{\frac{1}{2}} \alpha^{\frac{3}{2}}.$$

The cross-section of the rod $= \frac{3\sqrt{3}}{2} b^2$, so that, if the pressure per unit cross-section be p,
$$p = \frac{2\sqrt{2}E}{9\sqrt{3}(1-\sigma^2)} \lambda^{\frac{1}{2}} . \frac{\alpha^{\frac{3}{2}}}{b^{\frac{3}{2}}}.$$

Again (Love, 2nd Ed. p. 197, Eqn. 70), if a_1 be the radius of the area of contact,
$$a_1 = \left(\alpha . \frac{r}{2}\right)^{\frac{1}{2}} = \left(\frac{\alpha\lambda b}{2}\right)^{\frac{1}{2}}$$
or,
$$\frac{a_1}{b} = \left(\frac{\lambda}{2} . \frac{\alpha}{b}\right)^{\frac{1}{2}}.$$

Now the maximum pressure per unit cross-section in the composite rod is determined solely by the velocity of impact. It follows that greatest value of the ratio $\frac{\alpha}{b}$, and therefore also of the ratio $\frac{a_1}{b}$, will be independent of the size of the rod provided its proportions are kept the same.

The dimensions of the areas of contact will thus always bear the same relation to the diameters of the elementary rods, no matter how much the latter are reduced, and we are therefore still entitled to employ the methods of the present article.

* Voigt, *loc. cit.* Up to the present the use of the rounded ends has enabled us to replace this Zwischenschicht by two end-elements whose mechanical properties were definitely known.

† See p. 56.

The pressure per unit cross-section in the composite rod will therefore be:—

$$p=\frac{2\sqrt{2}E}{9\sqrt{3}(1-\sigma^2)}\frac{\lambda^{\frac{1}{2}}}{b^{\frac{3}{2}}}\alpha^{\frac{3}{2}}.$$

so that in this case the Zwischenschicht would follow, not a linear law, but (with a different constant) that obtained above for a single spherical end.

Owing to the smallness of b, the constant involved would be extremely large, and the results obtained from the above formula would probably approximate closely to those given by St Venant's theory. This result is, however, due to assuming a *regular* disposition of the elementary convex surfaces. In any actual case, these surfaces would be irregular, both in shape and distribution, and only a few points would come into contact at the beginning of an impact, more and more contacts coming into play as the compression increased. Moreover the normals at the points of contact would not, in general, be parallel to the axis of the rod. This deviation would not, however, be very great, and, in any case, the law connecting *axial* pressure and compression for any particular contact will still be of the form $P=K\alpha^{\frac{3}{2}}$ the only difference being the increase of K by a factor, $\sec^{\frac{5}{2}}\theta$, due to the obliquity.

We shall probably get a much closer idea of the state of things which actually exists, if we assume that the number of points in contact at any stage during the impact is proportional to the compression, α, at that stage, and that the *sum of the constants*, for the group of points which comes into play during any small change in the compression, is proportional to that change*.

Let us denote this by writing kda for the aggregate constant of all the points which come into play between

$$\alpha=a \text{ and } \alpha=a+da.$$

Then we have

$$P=\int_0^\alpha kda\,(\alpha-a)^{\frac{3}{2}}$$
$$=\tfrac{2}{5}k\alpha^{\frac{5}{2}}.$$

The constant here is, of course, much smaller than in the preceding formula, making the end-element, equivalently, much softer, so that its effects once more become appreciable.

The general character of the results would still be similar to those found from the formula, $P=K\alpha^{\frac{3}{2}}$, for round-ended rods. The mean elasticity of the Zwischenchicht would be less with a ground than with a polished surface, and would increase with increasing velocity of impact, thus accounting for all Voigt's results.

There is one check on the formula $P=\frac{2}{5}k\alpha^{\frac{3}{2}}$ which could be performed without any knowledge of k. If two equal rods impinge with a velocity so small that the duration of impact is large compared with their period of vibration, we may neglect the wave-motion, and calculate the duration of impact after the manner of Hertz. The formula $P=\frac{2}{5}k\alpha^{\frac{5}{2}}$ then gives rise to the law $TV^{\frac{1}{7}}=\text{const.}$, connecting the duration and velocity of impact. If, then, we take a pair of short plane-ended rods, and plot the curve connecting log T

* The total compression in the end-elements during impact will be small compared with the difference in level between the hills and valleys on the surfaces of contact, so that this assumption will probably not be very far wrong.

and log V, as in Fig. 14, it should become asymptotic, for small values of V, to a line having a slope of 1 in 7. In fact, the line to which this curve ultimately becomes asymptotic should give us both the constant and the index in the formula $P = K\alpha^n$ by deduction from those in the formula $TV^m = \text{const.}$

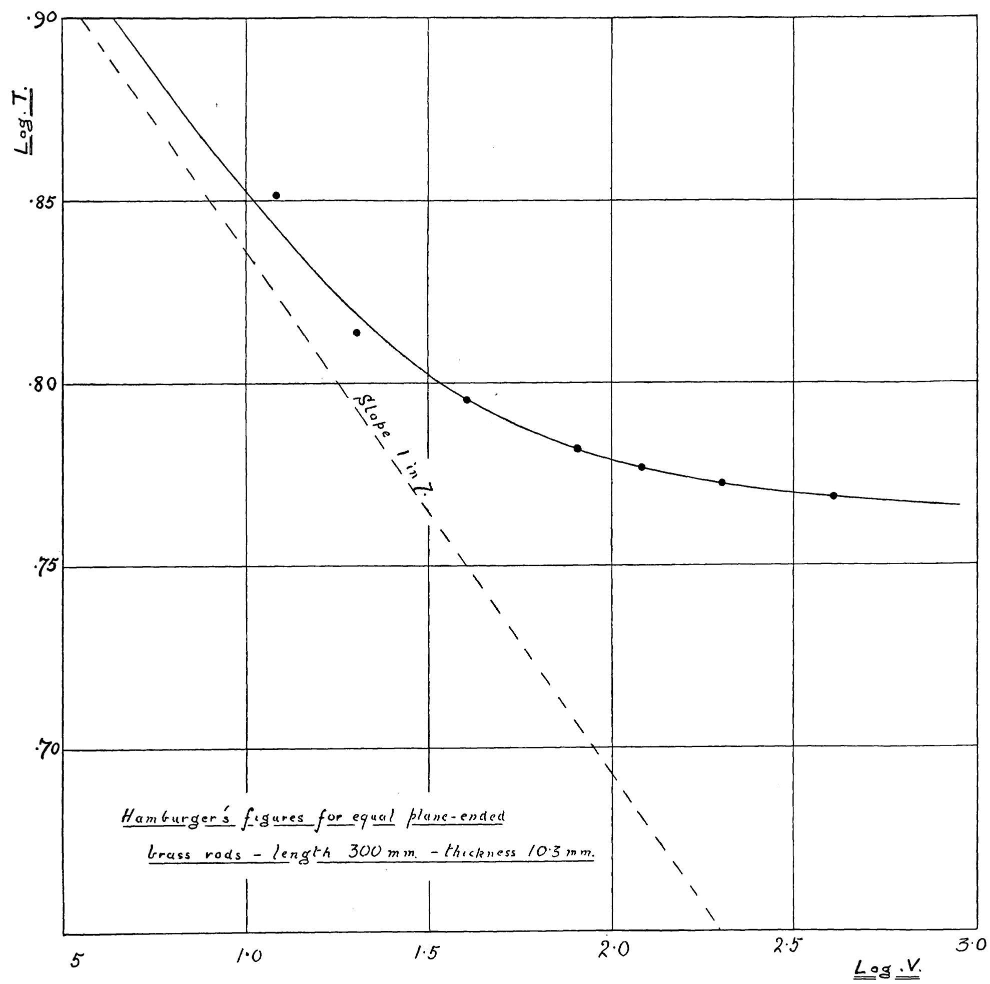


Fig. 15.

Some rough experiments which I made on plane-ended rods, before finally adopting the round-ended form, indicated end-effects of the same general character as are obtained with the latter, and led to the conclusion that, in this case also, the end-effect for unequal rods becomes independent of the length of the longer rod provided the difference in length is sufficiently great. Unfortunately I have not had time to repeat these experiments with the care necessary to provide a check on the foregoing theory.

To obtain this I therefore tried using the figures given by Hamburger* for two plane-ended brass rods of length 300 mm., but without much success. It is possible to draw a smooth curve through the points so obtained which approximates ultimately to a slope of 1 in 7, but the last two points lie one on either side of the curve, and do not really suffice to determine accurately its final direction (Fig. 15). Errors of this sort are unfortunately almost inevitable when dealing with the very small velocities here involved.

CONCLUSION.

In conclusion, I have to express my thanks to Prof. Hopkinson, of the Engineering Laboratory, Cambridge, where the work was carried out, for his unfailing interest and kind advice. When he first suggested that I should undertake experiments on the velocity of wave-propagation in metal rods, the developments he had in view were, I believe, of a far more practical character than those here described. I happened, however, to be interested in the abstract problem of impact, and he has always shewn himself perfectly willing that I should follow up the work on these lines. I have also to thank Mr H. Booth, of Trinity College, who was good enough to relieve me of some part of the arithmetical calculations.

* Hamburger, *Wied. Ann.* XXVIII. 1886.

APPENDIX I.

(A)

Determination of the Length of the End-element.

The determination of the appropriate length of AC (p. 57), for the particular case of a long rod with flattish spherical end, led to unexpected difficulties. I had hoped to be able to use the Bessel solution of the general equations of equilibrium for this purpose, applying over the area of contact a pressure distributed according to the law given by Hertz. Unfortunately I found it impossible by this method to satisfy the necessary condition that there should be no other traction of any sort on the whole of the rest of the surface, and I had, consequently, to adopt the approximate method here described. It will be seen (p. 87 below) that the error involved is probably less than might at first sight be supposed; but the method is far from being satisfactory, and the sufficient accuracy of the value found for AC must be allowed to rest, in the last resort, on the close agreement invariably found between the experimental and theoretical results.

The method in question consists in a consideration of the transition from the case of an infinite plane solid, to that of a long rod under the same applied stress. Take first, for simplicity, the case of a normal pressure applied at a single point of the surface (the centre of the end of the rod) (Fig. 16). We have then (Love, *Theory of Elasticity*, 2nd ed., p. 189) the traction across any plane parallel to the surface

$$=\frac{3P}{2\pi r^2}\cos^2\theta,$$

r being the distance from the point of pressure, and θ the angle between the radius vector and the normal.

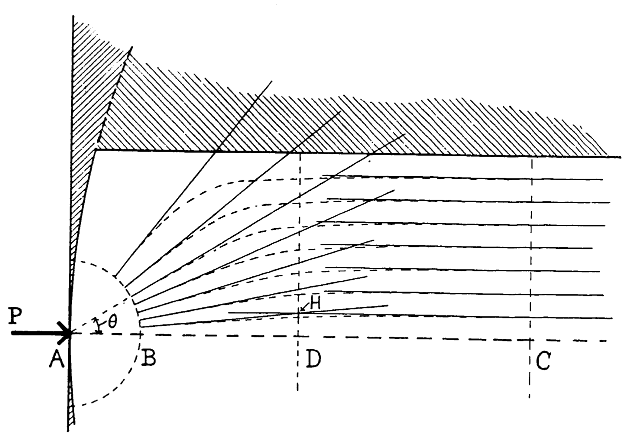


Fig. 16.

The form of this function enables us to represent this system of forces (at all points in the body external to a small hemisphere at A) by a series of radial lines having geometrical properties analogous to those of the lines of force in electrostatics*.

If we now suppose the shaded portions of the infinite solid to be removed, all these lines will have to be crowded into the region occupied by the rod, attaining ultimately a uniform distribution parallel to the axis. Within the small hemisphere at A, the distribution will not be sensibly altered by the change, so that the ultimate distribution must be somewhat as represented by the dotted lines. Up to the point D, at which the pressure in the infinite solid attains the value $\frac{P}{\pi a^2}$ corresponding to a uniform distribution of the pressure over the whole cross-section of the rod, it is clear that the distribution *near the axis* is little different in the two cases, while beyond D the axial pressure *in the rod* never differs greatly from this value, which it soon sensibly attains.

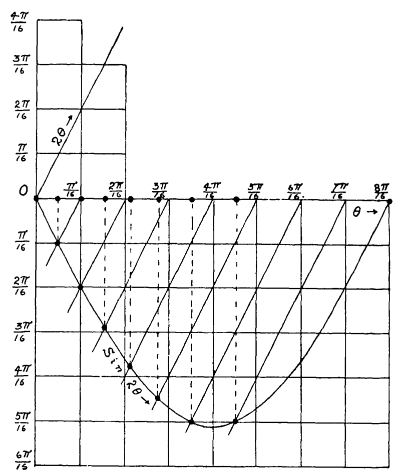


FIG. 17.

* These lines are shewn correctly plotted in Fig. 16.

Suppose n = no. of lines per radian.

Then $n \propto \cos^2\theta = A\cos^2\theta$ (say).

Let N_θ be the no. of lines in the angle θ.

$$N_\theta = \int_0^\theta A\cos^2\theta\, d\theta$$

$$= \frac{A}{4}[2\theta + \sin 2\theta].$$

The total no. of lines within the quadrant is thus

$$N = \frac{A}{4}\,.\,\pi.$$

Suppose we actually plot m lines, then the positions of these lines are got by putting

$$N_\theta = \frac{N}{m},\ \frac{2N}{m},\ \frac{3N}{m},\ \ldots,\ N.$$

or $$2\theta + \sin 2\theta = \frac{\pi}{m},\ \frac{2\pi}{m},\ \frac{3\pi}{m},\ \ldots,\ \pi.$$

The graphic solution of this equation for the case $m=8$ is shewn in the accompanying diagram (Fig. 17).

The extremely open spacing of the lines of force in the neighbourhood of $\theta = \frac{\pi}{2}$ is well brought out by this construction, and serves to illustrate clearly a point to which Hertz, in his memoir, devotes but little attention, viz., the exceedingly slight influence which the shape of the surface outside the area of contact, has (within limits) on the distribution of stress in the interior of the solid.

Now by far the greater part of the compression in AC takes place in the immediate neighbourhood of A (*i.e.*, along AB), and we shall not, therefore, make a very serious error in the *total* value of this compression if we assume that up to D the pressure in the rod is the same as in the infinite solid, and that beyond D it has the uniform value $\frac{P}{\pi a^2}$*.

Exactly similar remarks apply when the pressure at A, instead of being concentrated at a single point, is distributed according to Hertz' formula, and it was on the above assumption that the value to be taken for AC was calculated.

The distance AD is sufficiently great for the stress (and strain) at D to be independent of the distribution at A, and we may therefore calculate these as if the pressure there were concentrated at a point.

Putting $AD = c$, we have then (Love, *loc. cit.*) $p = \frac{3P}{2\pi c^2}$ = pressure at D in the infinite solid. This has to equal $\frac{P}{\pi a^2}$, so that $c = \sqrt{\frac{3}{2}}\, a$ simply.

Again, the displacement of D relative to infinity in the complete solid is

$$\begin{aligned} \alpha_2' &= \frac{P}{4\pi\mu}\left(\frac{\lambda + 2\mu}{\lambda + \mu} + 1\right) . \frac{1}{c} \\ &= \frac{P}{4\pi C}\left\{\frac{6E + 5C}{3E + C} + 1\right\} . \frac{1}{c} = P . \frac{B}{c} \text{ (say)}. \end{aligned}$$

We have to determine the point C so that the compression in DC in the rod $= \alpha_2'$.

Hence, putting $DC = e$,

$$P . \frac{B}{c} = \frac{P}{\pi a^2 E} . e,$$

or

$$e = \frac{\pi a^2 EB}{c}$$

whence

$$\begin{aligned} d = AC &= c + e \\ &= \sqrt{\frac{3}{2}}\, a + \sqrt{\frac{2}{3}}\, a . \pi EB. \end{aligned}$$

* See p. 87.

(B)

Numerical data with a discussion of the errors involved in certain assumptions.

The steel rods which I used had the following constants (taken from previous paper).

Radius of cross-section, $a = \cdot 25''$.

Radius of spherical end, $r = 1''$.

Density, $\rho = 485\cdot 7$ lbs. per cu. ft.

Wave velocity, $v = 16{,}820$ ft. per sec.

From v and ρ we calculate E_ϕ, the *adiabatic* value of Young's modulus, as follows:

$$E_\phi = \frac{\rho v^2}{g} = \frac{485\cdot 7 \times (16820)^2}{144 \times 32\cdot 18}$$
$$= 29{,}640{,}000 \text{ lbs. per sq. in.}$$

A torsion test on the specimen rod gave:

$$C = 11{,}660{,}000 \text{ lbs. per sq. in.}$$

Now $\sigma = \dfrac{E}{2C} - 1$, and, since shear involves no change of volume, $C_\phi = C_\theta = C$, so that

$$\sigma_\phi = \frac{E_\phi}{2C} - 1 = \frac{29\cdot 64}{23\cdot 32} - 1 = \cdot 271.$$

Hence $1 - \sigma_\phi{}^2 = \cdot 9265$

and Hertz' constant, $K_\phi = \dfrac{\sqrt{2} E_\phi r^{\frac{1}{2}}}{3(1 - \sigma_\phi{}^2)}$

$$= \frac{2^{\frac{1}{2}} \times 29\cdot 64 \times 10^6 \times 32\cdot 18 \times 12}{3 \times \cdot 9265}$$

$$= 5\cdot 825 \times 10^9 \text{ absolute inch units.}$$

Lastly $d = \sqrt{\dfrac{3}{2}}\, a + \sqrt{\dfrac{2}{3}}\, \pi E B a$

$$= \sqrt{\frac{3}{2}} \cdot \frac{1}{4} + \sqrt{\frac{2}{3}} \cdot \pi E \cdot \frac{1}{4\pi C}\left(\frac{6E + 5C}{3E + C} + 1\right) \cdot \frac{1}{4}$$
$$= \cdot 3062'' + \cdot 3047''$$
$$= \cdot 6109''$$

For the reason already stated, that the greater part of the compression takes place in the immediate neighbourhood of the point of contact, a considerable latitude is permissible in this figure, and for convenience in working we therefore take the next smaller submultiple of the length of the rod*. The same rods were used as in the earlier experiments,

* See p. 60.

but they were all shortened by $\frac{1}{24}''$, making the shortest of them $5\frac{1}{3}''$ long. The submultiple of this nearest to $\cdot6109''$ is $\cdot5926''$, which is $\frac{1}{18}$th*.

The difference between $\cdot6109''$ and $\cdot5926''$ is $\cdot0183''$, so that the error thus made in d is about 3 per cent. The corresponding error made in α is, however, much smaller. The greatest pressure reached in any case I tried was 96,060 absolute inch units, corresponding to $\alpha = 645 \times 10^{-6''}$. The compression in $\cdot0183''$ due to this pressure is

$$\frac{p \times \cdot0183}{\pi a^2 E} = \frac{16}{\pi} \times \frac{96060}{32\cdot18 \times 12} \times \frac{\cdot0183}{29\cdot64 \times 10^6} = \cdot7375 \times 10^{-6''}.$$

The error in α is thus $\dfrac{2 \times \cdot7375}{645}$ or less than $\frac{1}{4}$ per cent.

Again we have $\alpha = \left(\dfrac{p}{K}\right)^{\frac{2}{3}}$, so that $\dfrac{p}{\alpha} = K^{\frac{2}{3}} p^{\frac{1}{3}}$. The error in α therefore increases only as $p^{\frac{1}{3}}$†. The maximum pressure reached being (for long rods) proportional to the velocity of impact, it follows that this velocity may be considerably increased without serious error from this source. By taking the submultiple next *smaller* than $\cdot6109''$, moreover, the tendency of this error will be to neutralize that made in the assumptions on which the value $\cdot6109''$ is calculated,—viz. that, up to the point D (Fig. 16), the *axial* pressure in the rod is no greater than in the infinite solid, and that beyond this point it has everywhere the value, $\dfrac{P}{\pi a^2}$, which it finally attains.

To get an estimate of the magnitude of this latter error, we may take for the radius AB (within which we suppose the pressure to be sensibly the same in the rod and in the infinite solid) a length of $\frac{1}{10}''$; and let us take, as a first approximation to the dotted curves near the axis (Fig. 16), a circle touching the line AH at G, and the line HF' at F. This is shewn on an exaggerated scale in Fig. 18.

We have $AD = DC = \cdot3''$ approximately.

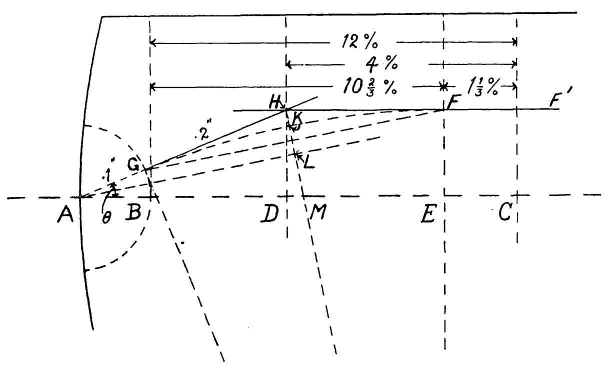


FIG. 18.

* The $13\frac{1}{3}''$, $21\frac{1}{3}''$, $29\frac{1}{3}''$ and $37\frac{1}{3}''$ rods are respectively, 45, 72, 99 and 126 times this length, but the intermediate rods are not exact multiples of it. In these cases the calculation was carried out for the nearest multiple of $\cdot5926''$, and the tabulated values found by graphical interpolation.

† The absolute error in α is proportional to p. The $^0/_0$ error in α is therefore proportional to p/α.

It is at once clear from the figure that in the limit, when θ is indefinitely small,

$$HK = \tfrac{1}{3}HL = \tfrac{1}{6}HM.$$

Now the pressure in the rod at D is to that in the infinite solid as limit $\frac{HD}{KD}$, or $\frac{6}{5}$ so that the error at the point D comes out at 20 per cent. Integrating from B to E, however, the mean error between these points only comes to about 6 per cent.

But, using the value of the maximum pressure given above, the displacement of B relative to C (calculated on the assumption of point pressure at A) is only 12 per cent. of the whole compression in AC (calculated on Hertz' theory), the remaining 88 per cent. occurring within AB. If we take the pressure at A as distributed over a small area, as in the Hertzian theory, the intensity of pressure along the axis will be everywhere slightly less than in the case of point pressure, so that we may take it that the compression between B and E is really not more than say 10 per cent. of α. The final error in α, due to an error of 6 per cent. in this, will thus be only ·6 per cent.

By taking $AB = \frac{1}{20}''$ and proceeding in the same way, we should have arrived at an estimate of the error about three times as great as the above. I do not think, however, that there is any need to make AB so small as this, and I regard the former estimate as probably nearer the truth.

In making this calculation the effect of the transverse pressures has been neglected. It is fairly evident that the changes in this will be considerably less than those in the longitudinal pressure, and their effect on the longitudinal compression will be still further reduced by the action of Poisson's ratio.

On the whole it appears that the *nett error*, due to this and the preceding cause, should certainly be less than 1 per cent., and probably not more than about $\frac{1}{2}$ per cent. even when the pressure is at its greatest. With lower pressures it will, of course, be less still.

Lastly, let us consider the error due to pressure-gradient in the end-element. Taking, for instance, the figures for $5\frac{1}{3}''$ rods impinging with the velocity 5·030″ per second which was usually employed, we see that the greatest difference of pressure ever produced between the two ends of the end-element is about 1080 absolute inch units, this occurring between the points 8 and 9, and 9 and 10, in the tabulated sheet. (Appendix II.)* The value of α_9 is $120{\cdot}96 \times 10^{-6}$ inches. We may suppose the error due to the pressure-gradient to be roughly one half the compression which would be produced in the end-element by a pressure equal to the difference between the pressures at its two ends. This would be

$$\frac{16}{\pi} \times \frac{\frac{1}{2} \times 1080}{32{\cdot}18 \times 12} \times \frac{{\cdot}5926}{29{\cdot}64 \times 10^6} = {\cdot}1425 \times 10^{-6} \text{ ins.}$$

The worst error in α is therefore $\frac{2 \times {\cdot}1425}{120{\cdot}96}$, or again rather less than $\frac{1}{4}$ per cent. During the greater part of the impact the error will, of course, be considerably less than this. Moreover, the pressure-gradients being in opposite senses when the rods are approaching and when they are separating, this error will, to a large extent, be compensating.

* Up to $n=17$, the figures for equal $5\frac{1}{3}''$ rods are identical with those for $5\frac{1}{3}''$—$13\frac{1}{3}''$ rods.

APPENDIX II.

Specimen calculations.

(See pp. 57—60.)

A. $5\frac{1}{3}''$—$13\frac{1}{3}''$ Steel Rods.

Velocity of $5\frac{1}{3}''$ rod before impact...5·030″/sec.

„ „ $13\frac{1}{3}''$ „ „ „ ...0.

Data (see Appendix I):—

	logarithm
$d = \cdot 5926''$	·7727
$v = 16820 \times 12 = 201{,}800$ ins. per sec.	·3050
$K_\phi = 5{\cdot}862 \times 10^9$	·7681*
$\tau = \dfrac{d}{v} = 2{\cdot}936 \times 10^{-6}$ secs.	·4677†
$V = 5{\cdot}030$ ins. per sec.	·7016
$V\tau = 14{\cdot}77 \times 10^{-6}$ ins.	·1693
$\dfrac{1}{v\rho S} = \dfrac{1}{201{,}800} \times \dfrac{1728}{485{\cdot}7} \times \dfrac{16}{\pi}$	·9531
$\dfrac{\tau}{2v\rho S} = 1{\cdot}317 \times 10^{-10}$	·1198

* Throughout my calculations I unfortunately used for the constant K the value $5{\cdot}862 \times 10^9$ instead of the true value $5{\cdot}825 \times 10^9$ (Appendix I, p. 86). This error was introduced by taking the modulus of rigidity C as 11,560,000 instead of 11,660,000. It did not appear worth while to repeat the whole of the rather laborious calculations with the correct value. The difference would be very small (less than the error in K), but would have the general effect of increasing the duration of impact and decreasing the impulse. It would, in fact, tend, on the whole, to improve the agreement already found between theory and experiment.

† See footnote, p. 60.

In the table which follows A_n denotes the area of the pressure-time curve between the ordinates p_{n-1} and p_n $\left(=\int_{(n-1)\tau}^{n\tau} p_t\,dt\right)$. It was found that sufficient accuracy was attained by simply taking this equal to $\frac{\tau}{2}(p_{n-1}+p_n)$. The value of A_1 was obtained direct from the formula, thus

$$A_1=\int_0^\tau p_t\,dt=\int_0^\tau K_\phi\,(Vt)^{\frac{3}{2}}\,dt=\tfrac{2}{5}K_\phi\,V^{\frac{3}{2}}\tau^{\frac{5}{2}},$$

or

$$\frac{A_1}{v\rho S}=\tfrac{2}{5}K_\phi\,\frac{V^{\frac{3}{2}}\tau^{\frac{5}{2}}}{v\rho S}$$
$$=\cdot 035\times 10^{-6}.$$

The successive lines in the columns marked β_n and γ_n represent the calculations of the successive equations ((v) and (vi), p. 59) respectively.

Putting $\tau=\frac{d}{v}$, $V_1=V\,(=5\cdot 030)$, $V_2=0$, and remembering that, at all times previous to zero, $p=0$, these equations become

$$\left.\begin{aligned}\beta_1&=V\tau\\ \beta_2&=\beta_1+V\tau-\frac{1}{v\rho S}\int_0^\tau p_t\,dt\\ \beta_3&=\beta_2+V\tau-\frac{1}{v\rho S}\int_\tau^{2\tau} p_t\,dt\\ &\dots\dots\dots\dots\\ &\dots\dots\dots\dots\end{aligned}\right\},$$

or

$$\left.\begin{aligned}\beta_1&=V\tau\\ \beta_2&=\beta_1+V\tau-\frac{A_1}{v\rho S}\\ \beta_3&=\beta_2+V\tau-\frac{A_2}{v\rho S}\\ &\dots\dots\dots\dots\\ &\dots\dots\dots\dots\end{aligned}\right\},$$

and

$$\left.\begin{aligned}\gamma_1&=0\\ \gamma_2&=\gamma_1+\frac{A_1}{v\rho S}\\ \gamma_3&=\gamma_2+\frac{A_2}{v\rho S}\\ &\dots\dots\dots\dots\\ &\dots\dots\dots\dots\end{aligned}\right\}.$$

The equations in this form hold until the appearance of the reflected waves in the corresponding rods. The terms due to these, when they arise, are simply repetitions of the terms $\frac{A_1}{v\rho S}$, $\frac{A_2}{v\rho S}$, ..., and involve no further calculation *.

* See p. 60.

The process is then as follows:—

From α_n, p_n is calculated, then $p_n + p_{n-1}$, and finally $\frac{A_n}{v\rho S}$. In the first portion of column β we then calculate $\left(V\tau - \frac{A_n}{v\rho S} - \text{reflected terms}\right)$, and adding this to β_n we get β_{n+1}. Similarly we get γ_{n+1}, and thus, by subtraction, α_{n+1}.

Up to $n = 17$, when the first reflected wave appears, the course of this impact is the same as for a pair of equal rods, and may be seen plotted in Fig. 9 (p. 96).

n	$\beta_n = V\tau + \beta_{n-1} - \frac{A_{n-1}}{v\rho S}$ – reflexion terms	$(\times 10^{-6})$	$\gamma_n = \gamma_{n-1} + \frac{A_{n-1}}{v\rho S}$ + reflexion terms	$(\times 10^{-6})$	$\alpha_n = \beta_n - \gamma_n$ $(\times 10^{-6})$	$p_n = K\alpha_n^{\frac{3}{2}}$	$p_n + p_{n-1}$	$\frac{A_n}{v\rho S} = \frac{\tau}{2v\rho S}(p_n + p_{n-1})$ $(\times 10^{-6})$
1	... ...	14·77	... ...	0	14·77	332·9	—	·035
2	14·77 – ·04 = 14·73	29·50	·04	·04	29·46	937·4	1270·3	·167
3	14·77 – ·17 = 14·60	44·10	·17	·21	43·89	1705	2642·4	·348
4	14·77 – ·35 = 14·42	58·52	·35	·56	57·96	2587	4292	·566
5	14·77 – ·57 = 14·20	72·72	·57	1·13	71·59	3550	6137	·809
6	14·77 – ·81 = 13·96	86·68	·81	1·94	84·74	4573	8123	1·070
7	14·77 – 1·07 = 13·70	100 38	1·07	3·01	97·37	5632	10205	1·345
8	14·77 – 1·35 = 13·42	113·80	1·34	4·35	109·45	6715	12347	1·627
9	14·77 – 1·63 = 13·14	126·94	1·63	5 98	120·96	7798	14513	1·913
10	14·77 – 1·91 = 12·86	139·80	1·91	7·89	131·91	8880	16678	2·198
11	14·77 – 2·20 = 12·57	152·37	2·20	10·09	142·28	9948	18828	2·481
12	14·77 – 2·48 = 12·29	164·66	2·48	12·57	152·09	11000	20948	2·761
13	14·77 – 2·76 = 12·01	176·67	2·76	15·33	161·34	12010	23010	3·032
14	14·77 – 3·03 = 11·74	188·41	3·03	18·36	170·05	13000	25010	3·296
15	14·77 – 3·30 = 11·47	199·88	3·30	21·66	178·22	13930	26930	3·549
16	14·77 – 3·55 = 11·22	211·10	3·55	25·21	185·89	14860	28790	3·794
17	14·77 – 3·79 = 10·98	222·08	3·79	29·00	193·08	15730	30590	4·031
18	14·77 – 4·03 – (4·07) ·04 = 10·70*	232·78	4·03	33·03	199·75	16560	32290	4·255

* Arrival of first reflected wave in $5\frac{1}{3}''$ rod.

19	14·77 − 4·26 − (4·43) ·17 = 10·34 ...	..	...	...	...	243·12	4·25	...	...	...	37·28	205·84	17320	33880	4·464
20	14·77 − 4·46 − ·35 − (4·85) ·04 = 9·92 *	...	...	...	...	253·04	4·46	...	...	...	41·74	211·30	18010	35330	4·656
21	14·77 − 4·66 − ·57 − (5·40) ·17 = 9·37	...	...	...	...	262·41	4·66	...	...	...	46·40	216·01	18620	36630	4·827
22	14·77 − 4·83 − ·81 − (5·99) ·35 = 8·78	...	...	...	...	271·19	4·83	...	...	...	51·23	219·96	19130	37750	4·974
23	14·77 − 4·97 − 1·07 − (6·61) ·57 = 8·16	...	...	...	...	279·35	4·97	...	...	...	56·20	223·15	19540	38670	5·095
24	14·77 − 5·09 − 1·35 − (7·25) ·81 = 7·52	...	...	...	...	286·87	5·10	...	...	...	61·30	225·57	19860	39400	5·192
25	14·77 − 5·19 − 1·63 − (7·89) 1·07 = 6·88	...	...	...	...	293·75	5·19	...	...	...	66·49	227·26	20080	39940	5·262
26	14·77 − 5·26 − 1·91 − (8·52) 1·35 = 6·25	...	...	...	...	300 00	5·26	...	...	...	71·75	228·25	20210	40290	5·309
27	14·77 − 5·31 − 2·20 − (9·14) 1·63 = 5·63	...	...	...	...	305·63	5·31	...	...	...	77·06	228·57	20260	40470	5·333
28	14·77 − 5·33 − 2·48 − (9·72) 1·91 = 5·05	...	...	...	...	310 68	5·33	...	...	...	82·39	228·29	20220	40480	5·334
29	14·77 − 5·33 − 2·76 − (10·29) 2·20 = 4·48	...	...	...	...	315·16	5·33	...	...	...	87·72	227·44	20110	40330	5·314
30	14·77 − 5·31 − 3·03 − (10·82) 2·48 = 3·95	...	...	...	...	319·11	5·31	...	...	...	93·03	226·08	19930	40040	5·276
31	14·77 − 5·28 − 3·30 − (11·34) 2·76 = 3·43	...	...	...	...	322·54	5·28	...	...	...	98·31	224·23	19680	39610	5·219
32	14·77 − 5·22 − 3·55 − (11·80) 3·03 = 2·97	...	...	...	...	325·51	5·22	...	...	...	103·53	221·98	19390	39070	5·148
33	14·77 − 5·15 − 3·79 − (12·24) 3·30 = 2·53	...	...	...	...	328·04	5·15	...	...	...	108·68	219·36	19040	38430	5·063

* Return of first reflected wave in $5\frac{1}{3}''$ rod.

n	$\beta_n - V\tau + \beta_{n-1} - \frac{A_{n-1}}{v\rho S}$ – reflexion terms		$\gamma_n = \gamma_{n-1} + \frac{A_{n-1}}{v\rho S}$ + reflexion terms		$\alpha_n = \beta_n - \gamma_n$	$p_n = K\alpha_n^{\frac{3}{2}}$	$p_n + p_{n-1}$	$\frac{A_n}{v\rho S} = \frac{\tau}{2v\rho S}(p_n + p_{n-1})$
34	14·77 – 5·06 – 4·03 – (12·64) 3·55 = 2·13 … … … …	330·17	5·06 .. … …	113·74	216·43	18670	37710	4·968
35	14·77 – 4·97 – 4·26 – (13·02) 3·79 = 1·75 … … … …	331·92	4·97 … … …	118·71	213·21	18250	36920	4·864
36	14·77 – 4·86 – 4·46 – (13·39) 4·03 – ·04 = 1·38* … … …	333·30	4·86 … … …	123·57	209·73	17800	36050	4·750
37	14·77 – 4·75 – 4·66 – 4·26 – (13·83) ·17 = ·94 … … …	334·24	4·75 … … …	128·32	205·92	17330	35130	4·629
38	14·77 – 4·63 – 4·83 – 4·46 – ·35 – (14·31) ·04 = ·46† … …	334·70	4·63 … … …	132·95	201·75	16800	34130	4·498
39	14·77 – 4·50 – 4·97 – 4·66 – ·57 – (14·87) ·17 = – ·10 … …	334·60	4·50 … … …	137·45	197·15	16230	33030	4·352
40	14·77 – 4·35 – 5·09 – 4·83 – ·81 – (15·43) ·35 = – ·66 … …	333·94	4·35 … … …	141·80	192·14	15620	31850	4·197
41	14·77 – 4·20 – 5·19 – 4·97 – 1·07 – (16·00) ·57 = – 1·23 … …	332·71	4·20 … … …	146·00	186·71	14960	30580	4·029
42	14·77 – 4·03 – 5·26 – 5·09 – 1·35 – (16·55) ·81 = – 1·78 … ..	330·93	4·03 … … …	150 03	180·90	14270	29230	3·851
43	14·77 – 3·85 – 5·31 – 5·19 – 1·63 – (17·05) 1·07 = – 2·28 … …	328·65	3·85 … … …	153·88	174·77	13540	27810	3·664
44	14·77 – 3·66 – 5·33 – 5·26 – 1·91 – (17·51) 1·35 = – 2·74 … …	325·91	3·66 … … …	157·54	168·37	12810	26350	3·472
45	14·77 – 3·47 – 5·33 – 5·31 – 2·20 – (17·94) 1·63 = – 3·17 … …	322·74	3·47 + ·04 = 3·51‡ …	161·05	161·69	12050	24860	3·276
46	14·77 – 3·28 – 5·31 – 5·33 – 2·48 – (18·31) 1·91 = – 3·54 … …	319·20	3·28 + ·17 = 3·45 …	164·50	154·70	11280	23330	3·075

* Arrival of second reflected wave in $5\frac{1}{3}''$ rod. † Return of second reflected wave in $5\frac{1}{3}''$ rod. ‡ Arrival of first reflected wave in $13\frac{1}{3}''$ rod.

47	14·77 – 3·07 – 5·28 – 5·33 – 2·76 (18·64) – 2·20 = – 3·87	315·33	3·08 + ·35 + ·04 = 3·47*	167 97	147·36	10490	21770	2·869
48	14·77 – 2·87 – 5·22 – 5·31 – 3·03 (18·89) – 2·48 = – 4·12	311·21	2·87 + ·57 + ·17 = 3·61	171·58	139·63	9674	20164	2·657
49	14·77 – 2·66 – 5·15 – 5·28 – 3·30 (19·15) – 2·76 = – 4·38	306·83	2·66 + ·81 + ·35 = 3·82	175·40	131·43	8834	18508	2·439
50	14·77 – 2·44 – 5·06 – 5·22 – 3·55 (19·30) – 3·03 = – 4·53	302·30	2·44 + 1·07 + ·57 = 4·08	179·48	122·82	7980	16814	2·216
51	14·77 – 2·22 – 4·97 – 5·15 – 3·79 (19·43) – 3·30 = – 4·66	297·64	2·22 + 1·35 + ·81 = 4·38	183·86	113·78	7114	15094	1·988
52	14·77 – 1·99 – 4·86 – 5·06 – 4·03 (19·49) – 3·55 = – 4·72	292·92	1·99 + 1·63 + 1·07 = 4·69	188·55	104·37	6252	13366	1·762
53	14·77 – 1·76 – 4·75 – 4·97 – 4·26 (19·53) – 3·79 = – 4·76	288·16	1·76 + 1·91 + 1·35 = 5·02	193·57	94·59	5393	11645	1·535
54	14·77 – 1·54 – 4·63 – 4·86 – 4·46 (19·56) – 4·03 – ·04 = – 4·79† ...	283·37	1·53 + 2·20 + 1·63 = 5·36	198·93	84·44	4547	9940	1·310
55	14·77 – 1·31 – 4·50 – 4·75 – 4·66 (19·65) – 4·26 – ·17 = – 4·88 ...	278·49	1·31 + 2·48 + 1·91 = 5·70	204·63	73·86	3723	8270	1·090
56	14·77 – 1·09 – 4·35 – 4·63 – 4·83 – 4·46 (19·75) – ·35 – ·04 = – 4·98‡	273·51	1·09 + 2·76 + 2·20 = 6·05	210·68	62·83	2920	6643	·875
57	14·77 – ·86 – 4·20 – 4·50 – 4·97 – 4·66 (19·93) – ·57 – ·17 = – 5·16	268·35	·86 + 3·03 + 2·48 = 6·37	217·05	51·30	2154	5074	·668
58	14·77 – ·67 – 4·03 – 4·35 – 5·09 – 4·83 (20·13) – ·81 – ·35 = – 5·36	262·99	·67 + 3·30 + 2·76 = 6·73	223·78	39·21	1440	3594	·474
59	14·77 – ·47 – 3·85 – 4·20 – 5·19 – 4·97 (20·32) – 1·07 – ·57 = – 5·55	257·44	·47 + 3·55 + 3·03 = 7·05	230·83	26·61	805	2245	·296
60	14·77 – ·30 – 3·66 – 4·03 – 5·26 – 5·09 (20·50) – 1·35 – ·81 = – 5·73	251·71	·30 + 3·79 + 3·30 = 7·39	238·22	13·49	284	1089	·143
61	14·77 – ·14 – 3·47 – 3·85 – 5·31 – 5·19 (20·66) – 1·63 – 1·07 = – 5·89	245·82	·14 + 4·03 + 3·55 = 7·72	245·94	– ·12	—	—	—

* Return of first reflected wave in $13\frac{1}{3}''$ rod. † Arrival of third reflected wave in $5\frac{1}{3}''$ rod. ‡ Return of third reflected wave in $5\frac{1}{3}''$ rod.

Results:—

Impact complete in 60·99 steps.

$$\text{Duration of Impact} = 60{\cdot}99 \times 2{\cdot}936 \times 10^{-6}$$

$$= \underline{1{\cdot}790 \times 10^{-4} \text{ secs.}}$$

$$\Sigma \frac{A_n}{v\rho S} = 187{\cdot}69 \times 10^{-6}.$$

$$\text{Impulse} = \Sigma A_n.$$

$$\text{Mass of rod} = \rho SL.$$

$$\therefore \text{ Change of Velocity of rod } = \frac{\Sigma A_n}{\rho SL} = \frac{v}{L} \Sigma \frac{A_n}{v\rho S}.$$

$$\text{Change of velocity of } 5\tfrac{1}{3}'' \text{ rod} = \frac{201{,}800}{5{\cdot}3333} \times 187{\cdot}69 \times 10^{-6}$$

$$= \underline{7{\cdot}104 \text{ ins. per sec.}}$$

$$\text{Change of velocity of } 13\tfrac{1}{3}'' \text{ rod} = \frac{201{,}800}{13{\cdot}333} \times 187{\cdot}69 \times 10^{-6}$$

$$= \underline{2{\cdot}841 \text{ ins. per sec.}}$$

$$\text{Velocity of } 5\tfrac{1}{3}'' \text{ rod after impact} = 5{\cdot}030'' - 7{\cdot}104'' \text{ per sec.}$$

$$\left.\begin{aligned} &= -\underline{2{\cdot}074'' \text{ per sec.}} \\ \text{Velocity of } 13\tfrac{1}{3}'' \text{ rod after impact} &= \ \ \underline{2{\cdot}841'' \text{ per sec.}} \end{aligned}\right\}$$

When the two rods are of equal length, the reflected waves in both appear at the same time, and there is then no need to keep the columns for β_n and γ_n separately, a single column sufficing for the calculations of α_n. The method is thus considerably simplified.

The results of some of these calculations are seen plotted in Fig. 9.

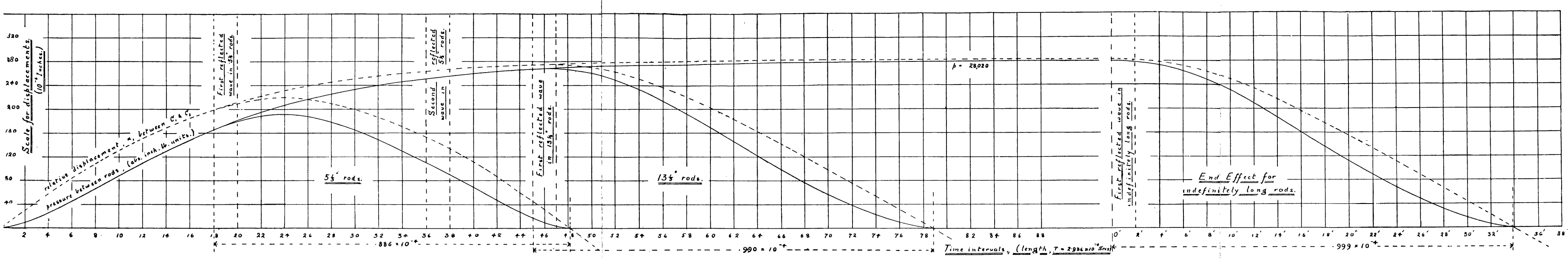
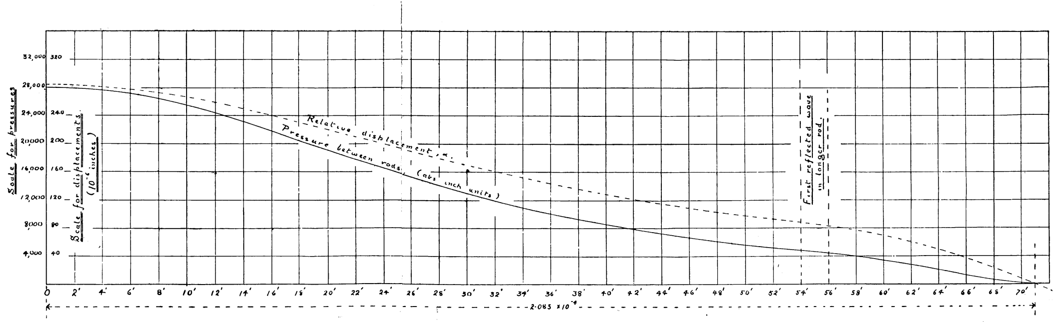


Fig. 9.



End Effect for Indefinitely Long Rods — Difference = 16".

Fig. 10.

B. *End Effect for indefinitely long rods with* 16″ *difference in length.*

The work commences with the arrival of the first reflected wave in the shorter rod, the length of the rods being assumed so great that the pressure has then reached its limiting value 28,020: α has then the value $283{\cdot}7 \times 10^{-6}$. This instant, being indefinite, is denoted by putting $n = 1'$.

The course of this calculation should be considered with reference to the diagram (Fig. 10).

The terms representing the reflected waves are to be found in the $\frac{A_n}{v\rho S}$ column for $5\frac{1}{3}''$—$13\frac{1}{3}''$ rods up to $n = 17$. Beyond this they have the following values (taken from the calculations for pairs of long equal rods):—

18 ...... 4·26	32 ...... 6·30	46 ...... 7·03	60 ...... 7·29
19 ...... 4·47	33 ...... 6·39	47 ...... 7·06	61 ...... 7·29
20 ...... 4·67	34 ...... 6·46	48 ...... 7·08	62 ...... 7·30
21 ...... 4·85	35 ...... 6·54	49 ...... 7·11	63 ...... 7·31
22 ...... 5·04	36 ...... 6·60	50 ...... 7·14	64 ...... 7·32
23 ...... 5·21	37 ...... 6·66	51 ...... 7·16	65 ...... 7·32
24 ...... 5·36	38 ...... 6·72	52 ...... 7·18	66 ...... 7·33
25 ...... 5·51	39 ...... 6·78	53 ...... 7·20	67 ...... 7·33
26 ...... 5·65	40 ...... 6·82	54 ...... 7·22	68 ...... 7·34
27 ...... 5·78	41 ...... 6·87	55 ...... 7·23	69 ...... 7·34
28 ...... 5·90	42 ...... 6·91	56 ...... 7·24	70 ...... 7·34
29 ...... 6·01	43 ...... 6·95	57 ...... 7·26	71 ...... 7·35
30 ...... 6·12	44 ...... 6·98	58 ...... 7·27	72 ..... 7·35
31 ...... 6·21	45 ...... 7·00	59 ...... 7·28	

approaching gradually the asymptotic value $7{\cdot}385$ ($= \frac{1}{2} \times 14{\cdot}77$).

Result:—

End Effect complete in $71{\cdot}95 - 1 = 70{\cdot}95$ steps.

End Effect $= 70{\cdot}95 \times 2{\cdot}936 \times 10^{-6}$

$= \underline{2{\cdot}083 \times 10^{-4} \text{ secs.}}$

n	$\beta_n - x$		$\gamma_n - x$					a_n	p_n	$p_n + p_{n-1}$	$\frac{A_n}{v\rho S}$
0′	Note: $\beta_{0'}$ and $\gamma_{0'}$ are not known, but their difference $a_{0'}$ is. Putting $x = \gamma_{0'}$, we proceed as below.	283·70	...	...	...	...	0	283·70	28020	56040 (= $2p_{0'}$)	7·385
1′	(7·42) 14·77 − 7·38 − ·04 = 7·35	291·05	7·39	...	...	...	7·39	283·66	28010	56030	7·384
2′	(7·55) 14·77 − 7·38 − ·17 = 7·22	298·27	7·38	...	...	...	14·77	283·50	27990	56000	7·379
3′	(7·77) 14·77 − 7·38 − ·35 − ·04 = 7·00	305·27	7·38	...	...	...	22·15	283·12	27940	55930	7·369
4′	(8·11) 14·77 − 7·37 − ·57 − ·17 = 6·66	311·93	7·37	...	...	...	29·52	282·41	27820	55760	7·348
5′	(8·51) 14·77 − 7·35 − ·81 − ·35 = 6·26	318·19	7·35	...	...	...	36·87	281·22	27650	55470	7·308
6′	(8·95) 14·77 − 7·31 − 1·07 − ·57 = 5·82	324·01	7·31	...	...	...	44·18	279·83	27450	55100	7·261
7′	(9·42) 14·77 − 7·26 − 1·35 − ·81 = 5·35	329·36	7·26	...	...	...	51·44	277·92	27160	54610	7·196
8′	(9·90) 14·77 − 7·20 − 1·63 − 1·07 = 4·87	334·23	7·20	...	...	...	58·64	275·59	26810	53970	7·112
9′	(10·37) 14·77 − 7·11 − 1·91 − 1·35 = 4·40	338·63	7·11	...	...	...	65·75	272·88	26430	53240	7·015
10′	(10·84) 14·77 − 7·01 − 2·20 − 1·63 = 3·93	342·56	7·02	...	...	...	72·77	269·79	25980	52410	6·905
11′	(11·30) 14·77 − 6·91 − 2·48 − 1·91 = 3·47	346·03	6·90	...	...	...	79·67	266·36	25490	51470	6·782
12′	(11·74) 14·77 − 6·78 − 2·76 − 2·20 = 3·03	349·06	6·78	...	...	...	86·45	262·61	24950	50440	6·645

13′	(12·15) 14·77 – 6·64 – 3·03 – 2·48 = 2·62	351·68	6·65	...	...	...	93·10	258·58	24380	49330	6·500
14′	(12·56) 14·77 – 6·50 – 3·30 – 2·76 = 2·21	353·89	6·50	...	...	...	99·60	254·29	23780	48160	6·345
15′	(12·92) 14·77 – 6·34 – 3·55 – 3·03 = 1·85	355·74	6·35	...	...	...	105·95	249·79	23150	46930	6·184
16′	(13·27) 14·77 – 6·18 – 3·79 – 3·30 = 1·50	357·24	6·18	...	...	...	112·13	245·11	22500	45650	6·016
17′	(13·60) 14·77 – 6·02 – 4·03 – 3·55 = 1·17	358·41	6·02	...	...	...	118·15	240·26	21830	44330	5·841
18′	(13·89) 14·77 – 5·84 – 4·26 – 3·79 = ·88	359·29	5·84	...	...	...	123·99	235·30	21160	42990	5·665
19′	(14·16) 14·77 – 5·66 – 4·47 – 4·03 = ·61	359·90	5·66	...	...	...	129·65	230·25	20480	41640	5·487
20′	(14·42) 14·77 – 5·49 – 4·67 – 4·26 = ·35	360·25	5·49	...	...	...	135·14	225·11	19800	40280	5·308
21′	(14·63) 14·77 – 5·31 – 4·85 – 4·47 = ·14	360·39	5·31	...	...	...	140·45	219·94	19120	38920	5·128
22′	(14·84) 14·77 – 5·13 – 5·04 – 4·67 = – ·07	360·32	5·13	...	...	...	145·58	214·74	18450	37570	4·950
23′	(15·01) 14·77 – 4·95 – 5·21 – 4·85 = – ·24	360·08	4·95	...	...	...	150·53	209·55	17790	36240	4·775
24′	(15·17) 14·77 – 4·77 – 5·36 – 5·04 = – ·40	359·68	4·78	...	...	...	155·31	204·37	17130	34920	4·602
25′	(15·32) 14·77 – 4·60 – 5·51 – 5·21 = – ·55	359·13	4·60	...	...	...	159·91	199·22	16480	33610	4·428
26′	(15·44) 14·77 – 4·43 – 5·65 – 5·36 = – ·67	358·46	4·43	...	...	...	164·34	194·12	15850	32330	4·260
27′	(15·55) 14·77 – 4·26 – 5·78 – 5·51 = – ·78	357·68	4·26	...	...	...	168·60	189·08	15250	31100	4·099
28′	(15·65) 14·77 – 4·10 – 5·90 – 5·65 = – ·88	356·80	4·10	...	...	...	172·70	184·10	14640	29890	3·939

n	$\beta_n - x$		$\gamma_n - x$					α_n	p_n	$p_n + p_{n-1}$	$\frac{A_n}{v\rho S}$
29′	14·77 − 3·94 − 6·01 − (15·73) 5·78 = − ·96	355·84	3·94	...	...	...	176·64	179·20	14070	28710	3·784
30′	14·77 − 3·78 − 6·12 − (15·80) 5·90 = − 1·03	354·81	3·78	...	...	...	180·42	174·39	13500	27570	3·633
31′	14·77 − 3·63 − 6·21 − (15·85) 6·01 = − 1·08	353·73	3·63	...	...	...	184·05	169·68	12960	26460	3·486
32′	14·77 − 3·49 − 6·30 − (15·91) 6·12 = − 1·14	352·59	3·49	...	...	...	187·54	165·05	12430	25390	3·345
33′	14·77 − 3 34 − 6·39 − (15·94) 6·21 = − 1·17	351·42	3·34	...	...	...	190·88	160·54	11920	24350	3·208
34′	14·77 − 3·21 − 6·46 − (15·97) 6·30 = − 1·20	350·22	3·21	...	...	...	194·09	156·13	11450	23370	3·080
35′	14·77 − 3·08 − 6·54 − (16·01) 6·39 = − 1·24	348·98	3·08	...	...	...	197·17	151·81	10960	22410	2·952
36′	14·77 − 2·95 − 6·60 − (16·01) 6·46 = − 1·24	347·74	2·95	...	...	...	200·12	147·62	10520	21480	2·830
37′	14·77 − 2·83 − 6·66 − (16·03) 6·54 = − 1·26	346·48	2·83	...	...	...	202·95	143·53	10080	20600	2·714
38′	14·77 − 2·71 − 6·72 − (16·03) 6·60 = − 1·26	345·22	2·71	...	...	...	205·66	139·56	9668	19748	2·602
39′	14·77 − 2·60 − 6·78 − (16·04) 6·66 = − 1·27	343·95	2·60	...	...	...	208·26	135·69	9268	18936	2·496
40′	14·77 − 2·50 − 6·82 − (16·04) 6·72 = − 1·27	342·68	2·50	...	...	...	210·76	131·92	8880	18148	2·392
41′	14·77 − 2·39 − 6·87 − (16·04) 6·78 = − 1·27	341·41	2·39	...	...	...	213·15	128·26	8515	17395	2·292
42′	14·77 − 2·29 − 6·91 − (16·02) 6·82 = − 1·25	340·16	2·29	...	...	...	215·44	124·72	8162	16677	2·197

43′	14·77 − 2·20 − 6·95 − 6·87 = − 1·25 (16·02)	338·81	2·20	...	...	...	217·64	121·17	7820	15982	2·106
44′	14·77 − 2·11 − 6·98 − 6·91 = − 1·23 (16·00)	337·58	2·11	...	...	...	219·75	117·83	7500	15320	2·018
45′	14·77 − 2·02 − 7·00 − 6·95 = − 1·20 (15·97)	336·38	2·02	...	...	...	221·77	114·61	7193	14693	1·936
46′	14·77 − 1·94 − 7·03 − 6·98 = − 1·18 (15·95)	335·20	1·94	...	...	...	223·71	111·49	6901	14094	1·857
47′	14·77 − 1·86 − 7·06 − 7·00 = − 1·15 (15·92)	334·05	1·86	...	...	...	225·57	108·48	6625	13526	1·782
48′	14·77 − 1·78 − 7·08 − 7·03 = − 1·12 (15·89)	332·93	1·78	...	...	...	227·35	105·58	6360	12985	1·711
49′	14·77 − 1·71 − 7·11 − 7·08 = − 1·11 (15·88)	331·82	1·71	...	...	...	229·06	102·76	6104	12464	1·643
50′	14·77 − 1·64 − 7·14 − 7·08 = − 1·09 (15·86)	330·73	1·64	...	...	...	230·70	100·03	5865	11969	1·577
51′	14·77 − 1·58 − 7·16 − 7·11 = − 1·08 (15·85)	329·75	1·58	...	...	...	232·28	97·47	5642	11507	1·517
52′	14·77 − 1·52 − 7·18 − 7·14 = − 1·07 (15·84)	328·68	1·52	...	...	...	233·80	94·88	5419	11061	1·457
53′	14·77 − 1·46 − 7·20 − 7·16 = − 1·05 (15·82)	327·63	1·46	...	...	...	235·26	92·37	5205	10624	1·400
54′	14·77 − 1·40 − 7·22 − 7·18 = − 1·03 (15·80)	326·60	1·40	...	...	...	236·66	89·94	5001	10206	1·345
55′	14·77 − 1·35 − 7·23 − 7·20 = − 1·01 (15·78)	325·59	1·34 +	·04 = 1·38 *		...	238·04	87·55	4801	9802	1·291
56′	14·77 − 1·29 − 7·24 − 7·22 = − ·98 (15·75)	324·61	1·29 +	·17 = 1·46		...	239·50	85·11	4604	9405	1·239
57′	14·77 − 1·24 − 7·26 − 7·23 = − ·96 (15·73)	323·65	1·24 +	·35 +	·04 = 1·63		241·13	82·52	4395	8999.	1·186
58′	14·77 − 1·19 − 7·27 − 7·24 = − ·93 (15·70)	322·72	1·19 +	·57 +	·17 = 1·93		243·06	79·66	4168	8563	1·128

* Arrival of first reflected wave in longer rod.

n	$\beta_n - x$		$\gamma_n - x$		α_n	p_n	p_n+p_{n-1}	$\frac{A_n}{v\rho S}$
59′	14·77 − 1·13 − 7·28 − 7·26 (15·67) = − ·90	321·82	1·13 + ·81 + ·35 = 2·29	245·35	76·47	3920	8088	1·065
60′	14·77 − 1·06 − 7·29 − 7·27 (15·62) = − ·85	320·97	1·07 + 1·07 + ·57 = 2·71	248·06	72·91	3651	7571	·998
61′	14·77 − 1·00 − 7·29 − 7·28 (15·57) = − ·80	320·17	1·00 + 1·35 + ·81 = 3·16	251·22	68·95	3357	7008	·923
62′	14·77 − ·92 − 7·30 − 7·29 (15·51) = − ·74	319·43	·92 + 1·63 + 1·07 = 3·62	254·84	64·59	3044	6401	·843
63′	14·77 − ·84 − 7·31 − 7·29 (15·44) = − ·67	318·76	·84 + 1·91 + 1·35 = 4·10	258·94	59·82	2712	5756	·759
64′	14·77 − ·76 − 7·32 − 7·30 (15·38) = − ·61	318·15	·76 + 2·20 + 1·63 = 4·59	263·53	54·62	2367	5079	·669
65′	14·77 − ·67 − 7·32 − 7·31 (15·30) = − ·53	317·62	·67 + 2·48 + 1·91 = 5·06	268·59	49·03	2013	4380	·577
66′	14·77 − ·58 − 7·33 − 7·32 (15·23) = − ·46	317·16	·58 + 2·76 + 2·20 = 5·54	274·13	43·03	1655	3668	·483
67′	14·77 − ·48 − 7·33 − 7·32 (15·13) = − ·36	316·80	·48 + 3·03 + 2·48 = 5·99	280·12	36·68	1303	2958	·390
68′	14·77 − ·39 − 7·34 − 7·33 (15·06) = − ·29	316·51	·39 + 3·30 + 2·76 = 6·45	286·57	29·94	961	2264	·298
69′	14·77 − ·30 − 7·34 − 7·33 (14·97) = − ·20	316·31	·30 + 3·55 + 3·03 = 6·88	293·45	22·86	641	1602	·211
70′	14·77 − ·21 − 7·34 − 7·34 (14·89) = − ·12	316·19	·21 + 3·79 + 3·30 = 7·30	300·75	15·44	356	997	·131
71′	14·77 − ·13 − 7·35 − 7·34 (14·82) = − ·05	316·14	·13 + 4·03 + 3·55 = 7·71	308·46	7·68	125	481	·063
72′	14·77 − ·06 − 7·35 − 7·34 (14·75) = + ·02	316·16	·06 + 4·26 + 3·79 = 8·11	316·57	− ·41	—	—	—

APPENDIX III.

It follows from the remarks on p. 54 (preamble) that the general characteristics of the extension—where elastic after-working is present—must be represented by some such formula as

$$\epsilon = P \,.\, \frac{l}{E}\,(1 - be^{-kt})$$

where $\epsilon =$ extension,

$P =$ load per unit area,

$l =$ length of specimen,

$t =$ time, from application of load,

$E,\ b,\ k =$ constants of the material.

If this formula be correct, we should get for Young's modulus, with $t = \infty$, the ordinary static value, E (which includes elastic after-working); and with $t = 0$, the *instantaneous* value $\frac{E}{1-b}$, or $E\,(1+b)$ since b is small.

For the case of aluminium the difference was about 1 per cent., so that $b = \frac{1}{100}$ roughly.

Suppose we now take two observations, the first at 15 seconds, and the second at 5 minutes from the application of the load *: and suppose that the latter gives the final extension correct to $\frac{1}{10}$ per cent.

We have then $t_1 = 15$, $t_2 = 300$, so that $\frac{t_1}{t_2} = \frac{1}{20}$.

Also by supposition $e^{-kt_2} = e^{-300k} = \frac{1}{10}$,

so that $e^{-kt_1} = (\frac{1}{10})^{\frac{1}{20}} = \frac{9}{10}$ about.

The first observation consequently gives the *instantaneous* value of the extension, also correct to $\frac{1}{10}$ per cent.: which is in accordance with the observed facts.

Formulae of the above character may be deduced from various hypotheses: one of these, which is purely mechanical in its nature, may be simply illustrated as follows:

Suppose that the crystalline grains of which the metal is formed are aeolotropic; and to simplify the argument suppose them to be cubes of side l arranged as in Fig. 19, the value of the modulus in the direction of the shading being E_1, and across it E_2. The behaviour of the metal as a whole will still be isotropic.

* This corresponds roughly to the actual conditions of the experiment, see p. 54.

Suppose also that the *tangential* reaction between adjacent cubes is viscous in character—*i.e.*, proportional to the velocity of relative motion. Let $E_1 < E_2$.

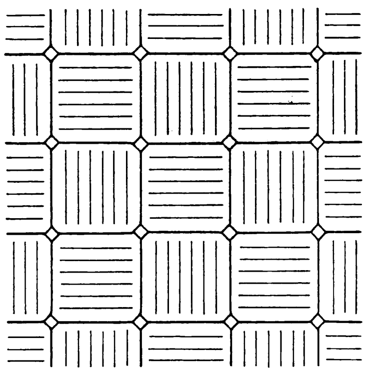


FIG. 19.

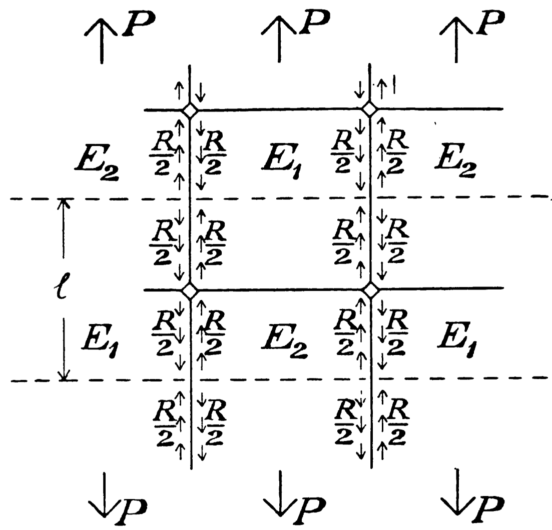


FIG. 20.

Then, on the application of a load, each of the grains will extend, instantaneously, to the same amount, since any *instantaneous* relative motion would involve an *infinite* reaction between the grains. To produce this equality of extension, reactions will be set up between the grains of the character represented in Fig. 20.

Let the equivalent extensive effect of each of these groups of forces be denoted by $R/2$.

The total strain in the length l will then be

$$\frac{P-R}{E_1}\cdot\frac{l}{2}+\frac{P+R}{E_2}\cdot\frac{l}{2}=P\frac{l}{2}\left\{\frac{1}{E_1}+\frac{1}{E_2}\right\}-R\frac{l}{2}\left\{\frac{1}{E_1}-\frac{1}{E_2}\right\}$$

$$=AP-BR \text{ say.}$$

The relative displacement of any two points originally in contact will, at any time, be proportional to the relative displacements of the ends of the adjacent grains, *i.e.* to

$$\frac{P-R}{E_1}\cdot\frac{l}{2}-\frac{P+R}{E_2}\cdot\frac{l}{2} \text{ or to } P\frac{l}{2}\left(\frac{1}{E_1}-\frac{1}{E_2}\right)-R\frac{l}{2}\left(\frac{1}{E_1}+\frac{1}{E_2}\right),$$

$$\text{or to } BP-AR.$$

Now by hypothesis R is proportional to the rate of change of this relative displacement, *i.e.* to $-\dfrac{dR}{dt}$.

That is $$R=-k\frac{dR}{dt},$$

or integrating, $$R=Ce^{-kt}+D.$$

Now when $t=0$, no relative motion has taken place, so that $AP-BR_0=0$, or

$$R_0=\frac{B}{A}\cdot P.$$

Also when $t=\infty$, $R=0$, so that $D=0$, and thus finally

$$R=\frac{B}{A}Pe^{-kt}.$$

Inserting this in the expression for the strain at any time, we get

$$\epsilon=AP-\frac{B^2}{A}Pe^{-kt}$$

$$=AP\left(1-\frac{B^2}{A^2}e^{-kt}\right).$$

Now $A=\frac{l}{2}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)$, so that putting E equal to the harmonic mean of E_1 and E_2, we get $A=\frac{l}{E}$ simply, whence

$$\epsilon=\frac{Pl}{E}\left(1-\frac{B^2}{A^2}e^{-kt}\right),$$

which is exactly the form of equation suggested above.

To get the values for aluminium we should need to have

$$\frac{B^2}{A^2}=b=\frac{1}{100}, \text{ or } \frac{B}{A}=\frac{1}{10}.$$

That is

$$\frac{E_2-E_1}{E_2+E_1}=\frac{1}{10},$$

or

$$\frac{E_2}{E_1}=\frac{11}{9},$$

by no means an unreasonable supposition.

The maximum value of R involved is $\frac{B}{A}P$, or $\frac{1}{10}P$, which is also quite reasonable.

In any ordinary metal the crystalline grains will, of course, not be cubes in regular piling, but of all shapes and sizes mixed indiscriminately together. This would not, however, alter the general character of the formula, which depends solely on the hypothesis of aeolotropic grains with a viscous reaction between them.

In support of such an hypothesis we may remark that as the crystalline grains are built up from their various centres, there will in general be a certain number of molecules to spare between their surfaces; and these molecules will evidently be distributed in the spaces between the grains with relatively small density, *i.e.*, in such a fluid or semi-fluid condition, as would just account for the viscous reaction required. It is, of course, well known from the phenomenon of burnishing that such a fluid condition exists at the surfaces of metals.

It must be understood, however, that up to the present only the general characteristics of the stress-strain-time relation have been proved, and that its exact form still remains a subject for future investigation.

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III. INTEGRAL FORMS AND THEIR CONNEXION WITH PHYSICAL EQUATIONS.

BY

R. HARGREAVES, M.A.

AND

IV. ON THE APPLICATION OF INTEGRAL EQUATIONS TO THE DETERMINATION OF UPPER AND LOWER LIMITS TO THE VALUE OF A DOUBLE INTEGRAL.

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III. *Integral Forms and their connexion with Physical Equations.*

By R. Hargreaves.

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We are concerned here with the variation of integral forms, or more specially with their invariance, under the action of an operator which is an extension of the hydrodynamical operator in Euler's equations. In the integral forms temporal terms are admitted, i.e. terms containing dt as well as the differentials of coordinates, and it appears that the forms have special properties when the temporal terms are associated with the non-temporal in a definite way. These associated terms are significant quantities which include vector and scalar products as particular cases.

The general theory comprises the action of the operator, its conjunction with the process of derivation in Stokes's theorem, and the law of association. The account of the general theory is followed by applications to the equations of hydrodynamics, to those of general dynamics, and to the electromagnetic equations.

It is hoped that this may prove a useful contribution to the unification of the equations of physics on what may briefly be described as the principle of 'the invariance of circuital content.' In connexion with the significance of the circuit Stokes's theorem, in a generalized form, is of fundamental importance as revealing the quantities characteristic of an infinitesimal circuit, and also as furnishing the clue to the treatment of integrals for closed spaces.

§ 1. The operator $\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + \dots$ is denoted by $\frac{D}{Dt}$, the operand by Ω. The latter is a sum of multiple-integral forms $\iint \dots X_{pqr\dots} dx_p \, dx_q \, dx_r \dots$, each term of which contains the same number of differentials, those in one term all distinct; the terms involve various combinations of the letters but not necessarily all the combinations of given order. The letters X and u denote functions of t and of the variables x.

The operation $D\Omega/Dt$ is then understood to mean that

$$\frac{D\Omega}{Dt} dt \text{ is limit of} \int \Sigma X'_{pqr\dots} dx'_p dx'_q dx'_r \dots - \int \Sigma X_{pqr\dots} dx_p dx_q dx_r \dots \quad \dots\dots\dots(1),$$

where $x'_p = x_p + u_p dt$, ... and X' has the value corresponding to the altered values of t, x, viz. $X + \frac{DX}{Dt} dt$. The difference above is the sum of terms

$$\int (X'_{pq...} - X_{pq...})\, dx_p dx_q \,... + \int X_{pq...} (dx'_p dx'_q \,... - dx_p dx_q \,...) \quad \text{..............(2)},$$

and the first term in $\frac{D\Omega}{Dt}$ is therefore $\int \Sigma \frac{DX_{pq...}}{Dt} dx_p dx_q \,...$.

When x'_p, x'_q, ... depend only on x_p, x_q, ..., the variables of their own group, the transformation is made by the Jacobian $\frac{\partial (x'_p, x'_q, x'_r, ...)}{\partial (x_p, x_q, x_r, ...)}$; but in general the group comprises only a part of the variables. Since in any one differential form $dx_p dx_q ...$ the other variables are regarded as constant, the transformation of each element is to be made by applying Jacobi's formula to all the possible combinations of the same order, i.e. we write

$$dx'_{p'} dx'_{q'} \,... = \sum_{p,q,...} \frac{\partial (x'_{p'},\ x'_{q'},\ ...)}{\partial (x_p,\ x_q,\ ...)} dx_p dx_q \,... \quad \text{..........................(3)}.$$

For like subscripts we have $\frac{\partial x'_p}{\partial x_p} = 1 + \frac{\partial u_p}{\partial x_p} dt$, for unlike $\frac{\partial x'_p}{\partial x_q} = \frac{\partial u_p}{\partial x_q} dt$. Thus two types of transformation occur; in the first all the subscripts for x' agree with those for x, in the second all but one agree. If more than one subscript is different, the Jacobian contains the square or a higher power of dt, and the term need not be considered.

When all the subscripts agree the main diagonal gives the only terms of first order in the Jacobian, viz. $1 + \left(\frac{\partial u_p}{\partial x_p} + \frac{\partial u_q}{\partial x_q} + ...\right) dt$; as in (2) we are concerned with

$$dx'_p dx'_q \,... - dx_p dx_q \,..., \text{ this is } \left(\frac{\partial u_p}{\partial x_p} + \frac{\partial u_q}{\partial x_q} + ...\right) dt\, dx_p dx_q \,....$$

Combining this with the term previously given,

$$\int \Sigma \left\{\frac{DX_{pq...}}{Dt} + X_{pq...} \left(\frac{\partial u_p}{\partial x_p} + \frac{\partial u_q}{\partial x_q} + ...\right)\right\} dx_p dx_q \,... \quad \text{.................... (4)}$$

is the section of $\frac{D\Omega}{Dt}$ in which the differentials are all the same as in the term of origin.

If $m-1$ members of the groups $(pq\rho' ...)$, $(pq\rho ...)$ are alike and the remaining one unlike (ρ' and ρ say), then $m-1$ rows of the Jacobian have an element equal to 1, their other elements containing dt; while the remaining row has dt in each element. Only one term therefore of the Jacobian is of the first order in dt, and its magnitude is $\frac{\partial u_{\rho'}}{\partial x_\rho} dt$. Thus the contribution of $X_{p...\rho'...} dx_p \,... dx_{\rho'} \,...$ to the differential $dx_p \,... dx_\rho \,...$ in

$$\frac{D\Omega}{Dt} \text{ is } \pm X_{p...\rho'...} \frac{\partial u_{\rho'}}{\partial x_\rho} dx_p \,... dx_\rho \,... \quad \text{...............................(5)}.$$

We consider now the question of sign. The order of the differentials is only important in the sense that, having chosen an order we must adhere to it, or recognise a change by change of sign in the result: in general a cyclic or symmetrical arrangement is convenient. Thus in transforming $dx'_p dx'_q dx'_r$ to $dx_p dx_q dx_r$ and to $dx_q dx_p dx_r$ respectively, there is a difference of sign corresponding to the single interchange of adjacent members. If the $m-1$ like subscripts in (5) fall in corresponding positions, the only relevant term of the Jacobian is the main diagonal and the sign is positive. Thus if the number of single interchanges of adjacent terms, needed to make the $m-1$ like subscripts fall into corresponding positions is even, the sign of the term is positive; if odd, it is negative.

§ 2. The parts (4) and (5), the latter summed with attention to the rule of signs, constitute the value of $D\Omega/Dt$. But it is not convenient to examine the Jacobian for each sign, and a convention will permit us to state the result of differentiation in a fully defined form. *The convention is that $X_{pq\ldots}$ shall be regarded as a quantity which changes sign with each single interchange of adjacent subscripts, and its actual value got by bringing the subscripts to an original defined order.* The use of this convention replaces the direct consideration of the Jacobian, and the collection of terms becomes simple. If

$$\left.\begin{aligned} \Omega = \int \Sigma\, X_{pq\ldots}\, dx_p\, dx_q \ldots, \quad \text{then} \quad \frac{D\Omega}{Dt} = \int \Sigma\, X'_{pq\ldots}\, dx_p\, dx_q \ldots \\ \text{where} \qquad X'_{pq\ldots} = \frac{DX_{pq\ldots}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} X_{nqr\ldots} + \frac{\partial u_n}{\partial x_q} X_{pnr\ldots} + \ldots \right) \end{aligned}\right\} \ldots\ldots\ldots\ldots(6),$$

and n may have any value present in the subscripts appearing in Ω. A term $X_{nqr\ldots}$ is to be taken as zero if two subscripts agree (e.g. if $n = q$), or if the group $(nqr\ldots)$ does not agree with some group appearing in Ω. But it is not necessary that a group of differentials in $\frac{D\Omega}{Dt}$ should agree with some group in Ω; *one* differential may be different.

One other point remains for consideration, viz. the position as regards dt. In the operator $\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x_1} + \ldots$ the position of dt is exceptional in that the u which corresponds to it, u_τ say, is 1. If this is borne in mind, the general method applies also to t. Two features should be noted: (1) no terms are carried over from temporal to non-temporal sections in the operation D/Dt, since $\frac{\partial u_\tau}{\partial x} = 0$; and (2) all terms carried from a non-temporal to a temporal section contain $\frac{\partial u_{\rho'}}{\partial t}$, ρ' being the one subscript in the source-term which does not appear in the final term, but is replaced by dt.

The notation and result of (6), as furnishing an explicit collection of terms, are effective for the purpose of establishing general theorems. But an equivalent rule may be given which is easy of application in the simpler forms. The terms which arise from a source-term $dx_p dx_q \ldots$ by modification of any differential, say dx_q, are got by writing $\sum_n \frac{\partial u_q}{\partial x_n} dx_n$ in the place of dx_q. The rule of signs is used to bring any term to a standard order, and the summation extends to all values of n (including τ as representing dt) not contained in the other differentials of the group.

§ 3. Some simple examples will now be given. The linear form does not introduce the question of sign, and Ω_1 being $\int \sum_p X_p dx_p + T dt$,

$$\frac{D\Omega_1}{Dt} = \int \Sigma \left(\frac{DX_p}{Dt} + \sum_q X_q \frac{\partial u_q}{\partial x_p} \right) dx_p + \left(\frac{DT}{Dt} + \sum_p X_p \frac{\partial u_p}{\partial t} \right) dt \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

In three dimensions with the more usual notation $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$,

and $$\Omega_1 = \int X\,dx + Y dy + Z dz + T dt,$$

$$\frac{D\Omega_1}{Dt} = \int \left(\frac{DX}{Dt} + X\frac{\partial u}{\partial x} + Y\frac{\partial v}{\partial x} + Z\frac{\partial w}{\partial x}\right) dx + \ldots + \left(\frac{DT}{Dt} + \Sigma\, X\frac{\partial u}{\partial t}\right) dt \quad\ldots\ldots\ldots\ldots(8).$$

A form of the second order is

$$\Omega_2 = \int X\,dy\,dz + Y dz\,dx + Z dx\,dy + U dt\,dx + V dt\,dy + W dt\,dz,$$

for which $$\frac{D\Omega_2}{Dt} = \int \left\{\frac{DX}{Dt} + X\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - Y\frac{\partial u}{\partial y} - Z\frac{\partial u}{\partial z}\right\} dy\,dz + \ldots$$
$$+ \int \left(\frac{DU}{Dt} + U\frac{\partial u}{\partial x} + V\frac{\partial v}{\partial x} + W\frac{\partial w}{\partial x} + Y\frac{\partial w}{\partial t} - Z\frac{\partial v}{\partial t}\right) dt\,dx + \ldots \quad\ldots\ldots(9).$$

The terms $-Y\frac{\partial u}{\partial y}$, $+Y\frac{\partial w}{\partial t}$ are got through the Jacobians $\frac{\partial(z', x')}{\partial(y, z)}$ and $\frac{\partial(z', x')}{\partial(t, x)}$ respectively, i.e. through

$$\begin{vmatrix} \frac{\partial w}{\partial y}dt, & \frac{\partial u}{\partial y}dt \\ 1, & \frac{\partial u}{\partial z}dt \end{vmatrix} \text{ and } \begin{vmatrix} \frac{\partial w}{\partial t}dt, & \frac{\partial u}{\partial t}dt \\ \frac{\partial w}{\partial x}dt, & 1 \end{vmatrix}.$$

The rule at the end of § 2 derives these terms by writing $Y dz\,\frac{\partial u}{\partial y}dy$, and $Y\frac{\partial w}{\partial t}dt\,dx$, and changing the sign in the first for the alteration in order of the differentials.

A form of the third order is

$$\Omega_3 = \int \rho\,dx\,dy\,dz + U dt\,dy\,dz + V dt\,dz\,dx + W dt\,dx\,dy,$$

for which $$\frac{D\Omega_3}{Dt} = \int \left\{\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\} dx\,dy\,dz$$
$$+ \int \left\{\frac{DU}{Dt} + U\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - V\frac{\partial u}{\partial y} - W\frac{\partial u}{\partial z} + \rho\frac{\partial u}{\partial t}\right\} dt\,dy\,dz + \ldots \quad\ldots\ldots\ldots\ldots(10).$$

In (9) and in (10) we note that the temporal terms derived from a temporal source follow the rule of their order in xyz; i.e. in (9) $\frac{DU}{Dt} + U\frac{\partial u}{\partial x} + \ldots$ is a linear form; in (10) $\frac{DU}{Dt} + U\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\ldots$ is a surface form.

Special values may be assigned to the temporal terms in (7) to (10), which have the property that the temporal terms in $D\Omega/Dt$ vanish automatically with the non-temporal terms, i.e. without further condition. These values are for (7)

$$-T = \sum_p u_p X_p\,; \quad \text{for (8)} \; -T = \Sigma\, uX\,;$$

for (9) $$U = vZ - wY, \quad V = wX - uZ, \quad W = uY - vX\,;$$

and for (10) $$-U = \rho u, \quad -V = \rho v, \quad -W = \rho w.$$

Thus by placing the temporal terms in a definite relation to the non-temporal we are led to forms of fundamental importance. The examples given are readily verified; they are particular cases of a general theorem proved below.

§ 4. The operator will be used often in conjunction with a process which is a generalization of the mode of derivation used in Stokes's theorem. The rule of derivation applied to integrals of any order is as follows: In the integral form write for each integrand its complete differential, remove terms which with the existing differentials yield the square of any differential, and as regards order follow the rule given above: the result with one more sign of integration (not here written) is the derived form.

For $\Omega_1 = \int X dx + Y dy + Z dz + T dt$, the derived form is

$$\int \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) dy dz + \ldots + \left(\frac{\partial X}{\partial t} - \frac{\partial T}{\partial x}\right) dt dx + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (11\,a).$$

For $\Omega_2 = \int X dy dz + \ldots + U dt dx + \ldots$, the derived form is

$$\int \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{dZ}{dz}\right) dx dy dz + \left(\frac{\partial X}{\partial t} + \frac{\partial V}{\partial z} - \frac{\partial W}{\partial y}\right) dt dy dz + \ldots \quad \ldots\ldots\ldots\ldots (11\,b).$$

For $\Omega_3 = \int \rho dx dy dz + U dt dy dz + \ldots$, the derived form is

$$\int \left(\frac{\partial \rho}{\partial t} - \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} - \frac{\partial W}{\partial z}\right) dt dx dy dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11\,c).$$

In (11 *b*) for example we have a term $\left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\right) dt dy$, the first and third to be rejected, the second requiring a change of sign to give $-\frac{\partial V}{\partial x} dt dx dy$, the fourth no change of sign because the movement of dz is through two places.

If in (11 *c*) we put $U = -\rho u$, ... we obtain $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho u + \ldots$ the same integrand as in (10), but here the form is of the fourth order.

In a form of the mth order an elementary circuit is made by varying $m + 1$ coordinates so that each coordinate shews a change, and the path of return to the original value is associated with different values of the other coordinates. It is convenient to have a name for the process described above, and we shall call the result the *reticular* form of Ω and denote it by $\mathscr{R}\Omega$; having in mind the figure employed in describing Ampère's view of the addition of circuits. If X_p, X_{pq}, ... are integrands of original forms and ξ_{pq}, ξ_{pqr}, ... integrands of the reticular forms, then

$$\left.\begin{aligned} \xi_{pq} &= \frac{\partial X_q}{\partial x_p} - \frac{\partial X_p}{\partial x_q}, \ \ldots \\ \xi_{pqr} &= \frac{\partial X_{qr}}{\partial x_p} - \frac{\partial X_{pr}}{\partial x_q} + \frac{\partial X_{pq}}{\partial x_r}, \ \ldots \\ \xi_{pqrs} &= \frac{\partial X_{qrs}}{\partial x_p} - \frac{\partial X_{prs}}{\partial x_q} + \frac{\partial X_{pqs}}{\partial x_r} - \frac{\partial X_{pqr}}{\partial x_s}, \ \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

and the operation may be regarded as an extended type of curl.

The process repeated always leads to a result vanishing identically. Thus if we watch the operation on X_{pq} with a view to the term in $dx_p dx_q dx_r dx_s$, the first step gives

$$\left(\frac{\partial X_{pq}}{\partial x_r} dx_r + \frac{\partial X_{pq}}{\partial x_s} dx_s\right) dx_p dx_q,$$

the second

$$\left(\frac{\partial^2 X}{\partial x_s \partial x_r} dx_s dx_r + \frac{\partial^2 X_{pq}}{\partial x_r \partial x_s} dx_r dx_s\right) dx_p dx_q ;$$

and the terms are cancelled because the differentials are introduced in opposite order in the two terms.

A converse theorem is also true, viz. if the integrands in $\mathscr{R}\Omega_m$ vanish, then Ω_m is $\mathscr{R}\Omega_{m-1}$, i.e. a reticular form of an integral form of lower order; for example if $\xi_{pqr}=0, \ldots$, then X_{qr} has the form of ξ_{qr} belonging to a linear integral. The reticular form of zero order of a function f is $\int \Sigma \frac{\partial f}{\partial x_p} dx_p$, and differentiation is a curl of zero order.

§ 5. As regards the conjunction of D/Dt and the process of reticulation, the order of operations is indifferent, i.e. $\frac{D}{Dt}\mathscr{R}\Omega = \mathscr{R}\frac{D\Omega}{Dt}$. It will be sufficient to write the proof for a general form of Ω of the third order. If D/Dt is applied first we quote (6), and curl the result by the third of the relations (12), thus obtaining

$$\begin{aligned}
&\frac{\partial}{\partial x_p}\left[\frac{DX_{qrs}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_q} X_{nrs} + \frac{\partial u_n}{\partial x_r} X_{qns} + \frac{\partial u_n}{\partial x_s} X_{qrn}\right)\right] \\
-&\frac{\partial}{\partial x_q}\left[\frac{DX_{prs}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} X_{nrs} + \frac{\partial u_n}{\partial x_r} X_{pns} + \frac{\partial u_n}{\partial x_s} X_{prn}\right)\right] \\
+&\frac{\partial}{\partial x_r}\left[\frac{DX_{pqs}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} X_{nqs} + \frac{\partial u_n}{\partial x_q} X_{pns} + \frac{\partial u_n}{\partial x_s} X_{pqn}\right)\right] \\
-&\frac{\partial}{\partial x_s}\left[\frac{DX_{pqr}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} X_{nqr} + \frac{\partial u_n}{\partial x_q} X_{pnr} + \frac{\partial u_n}{\partial x_r} X_{pqn}\right)\right],
\end{aligned}$$

which is equal to

$$\begin{aligned}
&\frac{D}{Dt}\left[\frac{\partial X_{qrs}}{\partial x_p} - \frac{\partial X_{prs}}{\partial x_q} + \frac{\partial X_{pqs}}{\partial x_r} - \frac{\partial X_{pqr}}{\partial x_s}\right] \\
&+ \sum_n \frac{\partial u_n}{\partial x_p}\left[\frac{\partial X_{qrs}}{\partial x_n} - \frac{\partial X_{nrs}}{\partial x_q} + \frac{\partial X_{nqs}}{\partial x_r} - \frac{\partial X_{nqr}}{\partial x_s}\right] \\
&+ \sum_n \frac{\partial u_n}{\partial x_p}\left[\frac{\partial X_{nrs}}{\partial x_p} - \frac{\partial X_{prs}}{\partial x_n} + \frac{\partial X_{pns}}{\partial x_r} - \frac{\partial X_{pnr}}{\partial x_s}\right] \\
&+ \sum_n \frac{\partial u_n}{\partial x_r}\left[\frac{\partial X_{qns}}{\partial x_p} - \frac{\partial X_{pns}}{\partial x_q} + \frac{\partial X_{pqs}}{\partial x_n} - \frac{\partial X_{pqn}}{\partial x_s}\right] \\
&+ \sum_n \frac{\partial u_n}{\partial x_s}\left[\frac{\partial X_{qrn}}{\partial x_p} - \frac{\partial X_{prn}}{\partial x_q} + \frac{\partial X_{pqn}}{\partial x_r} - \frac{\partial X_{pqr}}{\partial x_n}\right],
\end{aligned}$$

or to

$$\frac{D}{Dt}\xi_{pqrs} + \sum_n \left[\frac{\partial u_n}{\partial x_p}\xi_{nqrs} + \frac{\partial u_n}{\partial x_q}\xi_{pnrs} + \frac{\partial u_n}{\partial x_r}\xi_{pqns} + \frac{\partial u_n}{\partial x_s}\xi_{pqrn}\right],$$

by (12); and this is $\xi`_{pqrs}$ by (6). The passage from the first block of terms to the second is made by writing

$$\frac{\partial}{\partial x_p}\frac{D}{Dt}X_{qrs} - \frac{D}{Dt}\frac{\partial X_{qrs}}{\partial x_p} = \sum_n \frac{\partial u_n}{\partial x_p}\frac{\partial X_{qrs}}{\partial x_n}, \ \ldots,$$

and collecting terms with reference to $\dfrac{\partial u_n}{\partial x_p}$, ..., as coefficients.

§ 6. We now consider the theorem as to associated temporal forms, of which (7) to (10) were examples. It will be sufficient to state and prove it for an integral form of the fourth order. The statement is that if

$$\left.\begin{aligned} \Omega &= \int \Sigma X_{pqrs} dx_p dx_q dx_r dx_s + \Sigma T_{pqr} dt dx_p dx_q dx_r \\ \text{where} \qquad T_{pqr} &= -\sum_m u_m X_{mpqr} \end{aligned}\right\} \qquad (13),$$

the temporal integrands in $D\Omega/Dt$ vanish automatically when the non-temporal integrands vanish. We set out from (6) as applied to X and T, i.e.

$$X`_{pqrs} = \frac{DX_{pqrs}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} X_{nqrs} + \frac{\partial u_n}{\partial x_q} X_{pnrs} + \frac{\partial u_n}{\partial x_r} X_{pqns} + \frac{\partial u_n}{\partial x_s} X_{pqrn}\right),$$

and

$$T`_{pqr} = \frac{DT_{pqr}}{Dt} + \sum_n \left(\frac{\partial u_n}{\partial x_p} T_{nqr} + \frac{\partial u_n}{\partial x_p} T_{pnr} + \frac{\partial u_n}{\partial x_r} T_{pqn}\right) + \sum_n \frac{\partial u_n}{\partial t} X_{npqr}.$$

When the value of T in (13) is used in the last formula,

$$\begin{aligned} T`_{pqr} = &-\sum_m \frac{\partial u_m}{\partial t} X_{mpqr} - \sum_{m,n} u_n \frac{\partial u_m}{\partial x_n} X_{mpqr} \\ &- \sum_m u_m \left[X`_{mpqr} - \sum_n \left(\frac{\partial u_n}{\partial x_m} X_{npqr} + \frac{\partial u_n}{\partial x_p} X_{mnqr} + \frac{\partial u_n}{\partial x_q} X_{mpnr} + \frac{\partial u_n}{\partial x_r} X_{mpqn}\right)\right] \\ &- \sum_{m,n} \left[\frac{\partial u_n}{\partial x_p} X_{mnqr} + \frac{\partial u_n}{\partial x_q} X_{mpnr} + \frac{\partial u_n}{\partial x_r} X_{mpqn}\right] + \sum_n \frac{\partial u_n}{\partial t} X_{npqr} \\ = &-\sum_m u_m X`_{mpqr} \qquad (14). \end{aligned}$$

Thus the temporal terms $T`$ vanish with $X`$, and moreover the relation between $T`$ and $X`$ is the same as that between T and X, when there is no question of $X`$ vanishing. The form (13) covers the cases on p. 102; thus for (9) write $\int X\,dy\,dz + \ldots$ as

$$\int X_{23} dx_2 dx_3 + X_{31} dx_3 dx_1 + X_{12} dx_1 dx_2,$$

then

$$T_1 = -\sum_m u_m X_{m1} = -u_2 X_{21} - u_3 X_{31} = vZ - wY,$$

is integrand of $dt\,dx_1$ or $dt\,dx$.

It is clear from the form of T in (13) that the associated temporal terms may be derived from the non-temporal by writing for each differential as dx_p in the latter the form $-u_p dt$, and using the rule of signs for each change of order that may be needed. The terms associated with $X\,dy\,dz$ in (9) are $X(-v dt)\,dz + X\,dy\,(-w dt)$ or $-vX\,dt\,dz + wX\,dt\,dy$.

If with these associated terms Ω is called a *complete* integral form, then (14) shows that $\dfrac{D\Omega}{Dt}$ is also a complete form with $X`$ for X, and the operation may be repeated.

§ 7. When Ω is a complete form $\mathscr{R}\Omega$ is not a complete form, but the residual term has a simple form, viz. Ω being

$$\left.\begin{aligned} &\int \Sigma X_{pqr} dx_p dx_q dx_r - \sum_m u_m X_{mpq} dt dx_p dx_q \\ \mathscr{R}\Omega = &\int \Sigma \xi_{pqrs} dx_p dx_q dx_r dx_s + (X'_{pqr} - \sum_m u_m \xi_{mpqr}) dt dx_p dx_q dx_r \end{aligned}\right\} \quad \text{.........(15)},$$

for example. The integrand of $dt dx_p dx_q dx_r$ is

$$\frac{\partial X_{pqr}}{\partial t} + \frac{\partial}{\partial x_p} \sum_m u_m X_{mqr} - \frac{\partial}{\partial x_q} u_m X_{mpr} + \frac{\partial}{\partial x_r} u_m X_{mpq}$$

$$= \frac{\partial X_{pqr}}{\partial t} + \sum_m \left(\frac{\partial u_m}{\partial x_p} X_{mqr} + \frac{\partial u_m}{\partial x_q} X_{pmr} + \frac{\partial u_m}{\partial x_r} X_{pqm} \right) + \sum_m u_m \left(\frac{\partial X_{mqr}}{\partial x_p} - \frac{\partial X_{mpr}}{\partial x_q} + \frac{\partial X_{mpq}}{\partial x_r} \right),$$

of which the first two terms are a part of X'_{pqr}. The last term may, if n is a number not equal to pq or r, be written

$$u_p \frac{\partial X_{pqr}}{\partial x_p} + u_q \frac{\partial X_{pqr}}{\partial x_q} + u_r \frac{\partial X_{pqr}}{\partial x_r} + \sum_n u_n \left(\frac{\partial X_{nqr}}{\partial x_p} - \frac{\partial X_{npr}}{\partial x_q} + \frac{\partial X_{npq}}{\partial x_r} \right):$$

the first part is a further section of X'_{pqr} which now only wants $\sum_n u_n \frac{\partial X_{pqr}}{\partial x_n}$ for its completion. Hence the term is

$$X'_{pqr} - \sum_n u_n \left(\frac{\partial X_{pqr}}{\partial x_n} - \frac{\partial X_{nqr}}{\partial x_p} + \frac{\partial X_{npr}}{\partial x_q} - \frac{\partial X_{npq}}{\partial x_r} \right), \quad \text{or} \quad X'_{pqr} - \sum_n u_n \xi_{npqr},$$

as stated in (15).

If now Ω is a general form with associated terms shown explicitly, viz.

$$\left.\begin{aligned} \Omega &= \int \Sigma X_{pqr} dx_p dx_q dx_r + \Sigma (L_{pq} - \Sigma u_m X_{mpq}) dt dx_p dx_q + \dots \\ &\text{we have} \\ \mathscr{R}\Omega &= \int \Sigma \xi_{pqrs} dx_p dx_q dx_r dx_s + \Sigma (X'_{pqr} - \Sigma u_m \xi_{mpqr} - \lambda_{pqr}) dt dx_p dx_q dx_r \end{aligned}\right\} \text{.........(16 } a\text{)}.$$

Thus $\mathscr{R}\Omega$ can be made a complete form by writing

$$X'_{pqr} = \lambda_{pqr} \equiv \frac{\partial L_{qr}}{\partial x_p} - \frac{\partial L_{pr}}{\partial x_q} + \frac{\partial L_{pq}}{\partial x_r}, \quad \text{.........................(16 } b\text{)}.$$

But λ_{pqr} being a curl of the second order, the third order curl of it vanishes, i.e. ξ'_{pqrs} vanishes and therefore $\frac{D}{Dt}\mathscr{R}\Omega$. Since also $\mathscr{R}\frac{D\Omega}{Dt} = 0$, $\frac{D\Omega}{Dt}$ is a reticular form, and in fact

$$\frac{D\Omega}{Dt} = \mathscr{R} \int L_{pq} dx_p dx_q - \sum_m u_m L_{mp} dt dx_p \dots \quad \text{.................... (16 } c\text{)},$$

each member yielding

$$\int X'_{pqr} dx_p dx_q dx_r + (L'_{pq} - \sum_m u_m X'_{mpq}) dt dx_p dx_q \dots .$$

The conditions (16 b), therefore, make $\mathscr{R}\Omega$ a complete form, $\frac{D\Omega}{Dt}$ a reticular form, and $\frac{D}{Dt}\mathscr{R}\Omega = 0$. The properties are general, though the proof is only written for a form of the third order.

Thus having found the associated terms to be significant quantities, we write Ω with these terms in evidence, and leave the temporal terms general by the use of supplementary terms, which again we expect to be significant. The equations (16 *b*) needed to make $\mathscr{R}\Omega$ a complete form, so that $\frac{D}{Dt}\mathscr{R}\Omega$ may vanish in virtue of the scheme $\xi' = 0$, prove to be integrals of the latter when L's are regarded as arbitrary functions of xt; and the conditions are presented in an alternative form.

If we add to a given Ω a form $\mathscr{R}\Omega_{m-1}$, then since $\mathscr{R}^2\Omega_{m-1} = 0$, $\mathscr{R}\Omega$ is unchanged; hence with the conditions above all satisfied Ω may contain terms of type $\mathscr{R}\Omega_{m-1}$, as well as those leading to the characteristics of $\mathscr{R}\Omega$, if it is our object to determine a general form of Ω corresponding to the conditions.

§ 8. To explain the relation of these forms to integrals for closed spaces we set out from a statement of Stokes's theorem; and we suppose that in the most general case vorticity may be accompanied by Kelvin's circulation dependent on the nature of the space. We have then the general statement (*a*), the difference between the line-integrals over two reconcileable circuits is equal to the vortex-integral over a surface bounded by the two circuits; a particular case (*b*), when the circuit can shrink to a point without passing outside the fluid, the line-integral for such a circuit is the vortex-integral over a surface bounded by the circuit; a case (*c*) when both the circuits shrink to points (distinct poles), and accordingly the vortex-integral over a closed surface within the fluid is seen to vanish; and finally a case (*d*) when vorticity is absent, and the line-integral is the same for all reconcileable circuits, i.e. it is constant for a continuous range of circuits.

To the line-integral in these statements corresponds in cases of higher order the form Ω, to the surface integral the form $\mathscr{R}\Omega$. The latter contains only the part due to integration of the continuous quantities characteristic of the infinitesimal circuit. Its integral over a closed space at any time vanishes if the quantities appearing in Ω are single-valued. Thus as regards (16), $\mathscr{R}\Omega$ is absolutely invariant, all the integrands of $\frac{D}{Dt}\mathscr{R}\Omega$ being zero; and $\frac{D\Omega}{Dt}$ vanishes for a closed space at any time, because it has reticular form expressed in terms of the single-valued quantities L_{pq}, and so Ω is a relative invariant.

The order of procedure in the applications which follow is to write $\frac{D}{Dt}\mathscr{R}\Omega = 0$, i.e. to find the reticular form and make it invariant. This formula is the expression of the principle of invariance of circuital content.

§ 9. Take the linear form $\Omega = \int udx + vdy + wdz + Tdt$, for which

$$\mathscr{R}\Omega = \int\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) dy\,dz + \ldots + \left(\frac{\partial u}{\partial t} - \frac{\partial T}{\partial x}\right) dt\,dx + \ldots;$$

and form the non-temporal integrands of $\frac{D}{Dt}\mathscr{R}\Omega$ as in (9). These equated to zero are the vortex-equations of hydrodynamics, in respect to which we note that they appear in the form

$$\frac{D\xi}{Dt} + \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \eta\frac{\partial u}{\partial y} - \zeta\frac{\partial u}{\partial z} = 0,$$

which is valid whether density is constant or a function of pressure. The integrand of $dtdx$ in the temporal section, cf. (9) U section, is

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial t}-\frac{\partial T}{\partial x}\right)+\left(\frac{\partial u}{\partial t}-\frac{\partial T}{\partial x}\right)\frac{\partial u}{\partial x}+\left(\frac{\partial v}{\partial t}-\frac{\partial T}{\partial y}\right)\frac{\partial v}{\partial x}+\left(\frac{\partial w}{\partial t}-\frac{\partial T}{\partial z}\right)\frac{\partial w}{\partial t}$$

$$-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial t}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial t}=\frac{\partial}{\partial t}\frac{Du}{Dt}-\frac{\partial}{\partial x}\frac{DT}{Dt}.$$

The temporal section therefore vanishes if

$$\frac{DT}{Dt}=-\frac{\partial f}{\partial t},\quad \frac{Du}{Dt}=-\frac{\partial f}{\partial x},\quad \frac{Dv}{Dt}=-\frac{\partial f}{\partial y},\quad \frac{Dw}{Dt}=-\frac{\partial f}{\partial z}\quad \text{..................(17 }a\text{)},$$

where f is some function of $xyzt$. From these equations follows

$$\frac{D}{Dt}(T+\tfrac{1}{2}\Sigma u^2)=-\frac{Df}{Dt},\quad \text{and } \therefore\quad -T=f+\tfrac{1}{2}\Sigma u^2 \quad \text{..................(17 }b\text{)}.$$

This determines the form of T, and the last three equations of (17 a) constitute the equations of hydrodynamics when density is a function of pressure, here derived from the linear form with special integrands. The value of $\frac{\partial u}{\partial t}-\frac{\partial T}{\partial x}$, a temporal integrand in $\mathscr{R}\Omega$, is found by (17 b) and the second of (17 a) to be $v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$, the associated term which makes $\mathscr{R}\Omega$ a complete form. [The equation $\frac{\partial T}{\partial x}=\frac{\partial u}{\partial t}-2\zeta v+2\eta w$ is a form given by Nanson.] We have also by (8)

$$\frac{D\Omega}{Dt}=\int\left(\frac{Du}{Dt}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)dx+\ldots+\left(\frac{DT}{Dt}+u\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}+w\frac{\partial w}{\partial t}\right)dt$$

$$=\int\frac{\partial}{\partial x}(\tfrac{1}{2}\Sigma u^2-f)\,dx+\ldots+\frac{\partial}{\partial t}(\tfrac{1}{2}\Sigma u^2-f)\,dt$$

$$=[\tfrac{1}{2}\Sigma u^2-f],$$

the reticular form of zero order. Here $fu\ldots$ being single-valued, $\frac{D\Omega}{Dt}$ vanishes for a closed circuit. In the particular case $u=\frac{\partial\phi}{\partial x},\ldots$

$$\Omega=\int\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz+\frac{\partial\phi}{\partial t}dt-\left(f+\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\Sigma u^2\right)dt$$

$$=\int d\phi-F'(t)\,dt=[\phi-F(t)].$$

The potential ϕ need not be single-valued, but may possess Kelvin's constants of circulation.

§ 10. The surface scheme in (9) may be connected with the electromagnetic equations. Write Va for $U,\ldots$, V being a constant velocity so that (abc) and (XYZ) have the same dimensions; then

$$\left.\begin{aligned}&\Omega_2(e)\text{ being }\int X\,dy\,dz+\ldots+aV\,dt\,dx+\ldots\\&\mathscr{R}\Omega_2(e)=\int\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)dx\,dy\,dz+\left\{\frac{\partial X}{\partial t}-V\left(\frac{\partial c}{\partial y}-\frac{\partial b}{\partial z}\right)\right\}dt\,dy\,dz+\ldots\end{aligned}\right\}\text{......(18)}.$$

Denote $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$ by ρ, then the non-temporal integrand in $\frac{D}{Dt}\mathscr{R}\Omega_2(e)$ is

$$\frac{D\rho}{Dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

by (10); and this equated to zero is the condition of continuity when ρ is treated as a density. The temporal terms in $\frac{D}{Dt}\mathscr{R}\Omega_2(e)$ will also vanish in virtue of the sole condition of continuity if $\mathscr{R}\Omega_2(e)$ is a complete form, viz. $\int \rho\, dxdydz - \rho u dtdydz \ldots$, cf. p. 102, or if

$$\frac{\partial X}{\partial t}+\rho u = V\left(\frac{\partial c}{\partial y}-\frac{\partial b}{\partial z}\right), \ldots \qquad (19).$$

The analysis therefore furnishes directly a suggestion of the terms introduced by Fitzgerald in amendment of Maxwell's equations.

The equations

$$\frac{\partial a}{\partial t}=V\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right), \ldots \text{ with } \frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}+\frac{\partial c}{\partial z}=0 \qquad (20)$$

have not the same character; they express $\mathscr{R}\Omega_2(m)=0$, or that $\Omega_2(m)$ has no quantities characteristic of the infinitesimal circuit, where

$$\Omega_2(m)=\int a\,dydz+\ldots - XV dtdx \ldots \qquad (21).$$

But if $\mathscr{R}\Omega_2(m)=0$, then $\Omega_2(m)$ is a reticular form of a linear integral form, say of

$$\Omega_1(m)=\int F dx + G dy + H dz - \psi V dt \qquad (22).$$

The identification of $\Omega_2(m)$ with $\mathscr{R}\Omega_1(m)$ or

$$\int\left(\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}\right)dydz+\ldots+\left(\frac{\partial F}{\partial t}+V\frac{\partial \psi}{\partial x}\right)dtdx+\ldots$$

gives

$$a=\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}, \ldots \quad \text{and} \quad X=-\frac{\partial \psi}{\partial x}-\frac{1}{V}\frac{\partial F}{\partial t}, \qquad (23),$$

which are well-known auxiliary forms.

Again $\frac{D}{Dt}\mathscr{R}\Omega_2(e)=0$ implies $\mathscr{R}\frac{D}{Dt}\Omega_2(e)=0$, and $\frac{D}{Dt}\Omega_2(e)$ is a reticular form of a linear integral form. This latter is got at once by (16 a, b) on writing $\Omega_2(e)$ as

$$\int X dydz+\ldots+(vZ-wY+Va)\,dtdx \ldots,$$

i.e. by shewing the associated terms explicitly. We have therefore

$$\left.\begin{aligned}\frac{1}{V}\frac{D\Omega_2(e)}{Dt}&=\mathscr{R}\int \alpha dx+\beta dy+\gamma dz-\Sigma u\alpha dt\\ \text{and similarly}\quad \frac{1}{V}\frac{D\Omega_2(m)}{Dt}&=-\mathscr{R}\int \xi dx+\eta dy+\zeta dz-\Sigma u\xi dt\end{aligned}\right\} \qquad (24),$$

where

$$V\alpha = Va + wY - vZ, \quad V\xi = VX + vc - wb \qquad (25).$$

The new vectors therefore belong to line-integral forms, whereas the original vectors were attached to surface forms.

With respect to the second of (24), we note that as $\Omega_2(m) = \mathscr{R}\Omega_1(m)$ the left-hand member is $\frac{1}{V}\mathscr{R}\frac{D}{Dt}\Omega_1(m)$; and therefore the line-integral forms

$$\frac{1}{V}\frac{D}{Dt}\Omega_1(m) \quad \text{and} \quad -\int \xi dx + \ldots - \Sigma u\xi dt$$

can only differ by an exact integral. This integral is $\psi - \Sigma uF/V$, and in fact

$$\left.\begin{aligned} &V\xi + \frac{\partial}{\partial x}(V\psi - \Sigma uF) = -\left(\frac{DF}{Dt} + F\frac{\partial u}{\partial x} + G\frac{\partial v}{\partial x} + H\frac{\partial w}{\partial x}\right), \text{ i.e. } -F', \ldots \\ \text{and} \quad &-V\Sigma u\xi + \frac{\partial}{\partial t}(V\psi - \Sigma uF) = V\frac{D\psi}{Dt} - \Sigma\frac{\partial u}{\partial t}F = \frac{D}{Dt}(V\psi - \Sigma uF) + \Sigma uF' \end{aligned}\right\} \quad \ldots\ldots(26),$$

results derivable from (23) and the definition of ξ.

§ 11. The integral forms (18) and (20) are brought into relation by the use of (abc) in the temporal terms of $\Omega_2(e)$, and of (XYZ) in those of $\Omega_2(m)$. If we follow the method of § 7 we have the statement that

$$\Omega_2(e) = \int X dy dz + \ldots + (vZ - wY + V\alpha)\, dt dx + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(27\,a)$$

is a form in which the conditions

$$\frac{DX}{Dt} + X\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - Y\frac{\partial u}{\partial y} - Z\frac{\partial u}{\partial z} = V\left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}\right), \ldots \quad \ldots\ldots\ldots\ldots\ldots(27\,b)$$

make

$$\frac{1}{V}\frac{D}{Dt}\Omega_2(e) = \mathscr{R}\int \alpha dx + \beta dy + \gamma dz - \Sigma u\alpha dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27\,c),$$

and $\frac{D}{Dt}\mathscr{R}\Omega_2(e) = 0$, without special assumption as to the nature of $(\alpha\beta\gamma)$. In fact (27 b) is an integrated form of the equation of continuity. The same treatment applied to $\Omega_2(m)$ gives

$$\Omega_2(n) = \int a dy dz + \ldots + (vc - wb - V\xi)\, dt dx + \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28\,a),$$

$$\frac{Da}{Dt} + a\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - b\frac{\partial u}{\partial y} - c\frac{\partial u}{\partial z} = V\left(\frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28\,b),$$

and

$$\frac{1}{V}\frac{D}{Dt}\Omega_2(m) = -\mathscr{R}\int \xi dx + \eta dy + \zeta dz - \Sigma u\xi dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots(28\,c).$$

The forms (18) and (20) being supposed to hold for free aether, $(\alpha\beta\gamma)$ and $(\xi\eta\zeta)$ then have the values given by (25). In view of the difference in the character of 'a' and α alluded to above, it is natural to apply coefficients in the connecting equation. Thus in a general way, for (25) we may write

$$\xi = \frac{X}{K} + \frac{vc - wb}{VL}, \quad \alpha = \frac{a}{M} + \frac{wY - vZ}{VL} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29),$$

and apply these values in (27 b) and (28 b). The assumption $L = KM = \mu^2$, where μ is index of refraction, applied to the case of constant translation gives Fresnel's formula for modified velocity as an exact form.

It may be remarked that in (18) the scalar ρ or $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$ takes the place of the vector of vorticity in hydrodynamics; and in the theory of conductors charges are the analogues of Kelvin's constants of circulation.

§ 12. The general linear form gives the equations of Lagrange in what may be called an Eulerian form. Thus if we write $\Omega=\int \sum_i p_i dq_i+(L-\sum_i p_i v_i)\,dt$, with associated terms shown explicitly, we obtain as representing (16 b)

$$\frac{Dp_i}{Dt}+\sum_j p_j\frac{\partial v_j}{\partial q_i}=\frac{\partial L}{\partial q_i} \qquad\qquad (30),$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+\sum_i v_i\frac{\partial}{\partial q_i}$. In Lagrange's equations T is a function of v and q, and $p_i=\frac{\partial T}{\partial v_i}$. As in (30) v is supposed to be a function of q, we may distinguish the notations by supposing $\frac{d}{dq}$ written in (30). Then

$$\frac{dT}{dq_i}=\frac{\partial T}{\partial q_i}+\sum_j\frac{\partial T}{\partial v_j}\frac{\partial v_j}{\partial q_i}=\frac{\partial T}{\partial q_i}+\sum_j p_j\frac{\partial v_j}{\partial q_i},$$

and (30) becomes

$$\frac{Dp_i}{Dt}-\frac{\partial T}{\partial q_i}=\frac{d}{dq_i}(L-T)=-\frac{dV}{dq_i}=-\frac{\partial V}{\partial q_i},$$

since $L=T-V$, where V depends only on q. The Eulerian form (30) may be used as a point of departure for the treatment of the non-holonomic case.

It appears that the Hamiltonian form also is given by the present method if p's as well as qt are taken as independent variables, and the operator is $\frac{\partial}{\partial t}+\sum_i u_i\frac{\partial}{\partial p_i}+v_i\frac{\partial}{\partial q_i}$. Then with

$$\left.\begin{aligned}\Omega&=\int\Sigma p_i dq_i-H\,dt\\ \mathscr{R}\Omega&=\int\Sigma dp_i dq_i+\frac{\partial H}{\partial p_i}dt\,dp_i+\frac{\partial H}{\partial q_i}dt\,dq_i\end{aligned}\right\} \qquad\qquad (31),$$

the reticular form showing only special combinations of differentials. If $\mathscr{R}\Omega$ is a complete form the associated terms, by the rule of p. 105, are $v_i dt dp_i$ and $-u_i dt dq_i$; and their identification with corresponding terms in (31) gives

$$v_i=\frac{\partial H}{\partial p_i},\qquad u_i=-\frac{\partial H}{\partial q_i} \qquad\qquad (32),$$

which are the Hamiltonian equations. These secure the vanishing of the temporal terms in $\frac{D}{Dt}\mathscr{R}\Omega$ when the non-temporal terms vanish. The differentials occurring in the latter in addition to the original $dp_i dq_i$ are $dp_i dp_j$, $dq_i dq_j$ and $dp_i dq_j$, each having one differential different from the term of origin. The types of non-temporal term are

$$\int\left(\frac{\partial u_i}{\partial p_i}+\frac{\partial v_i}{\partial q_i}\right)dp_i dq_i+\left(\frac{\partial v_i}{\partial q_j}+\frac{\partial u_j}{\partial p_i}\right)dp_i dq_j+\left(\frac{\partial v_i}{\partial p_j}-\frac{\partial v_j}{\partial p_i}\right)dp_i dp_j+\left(\frac{\partial u_j}{\partial q_i}-\frac{\partial u_i}{\partial q_j}\right)dq_i dq_j,$$

which vanish if

$$\left.\begin{array}{ll}\dfrac{\partial u_i}{\partial p_i}+\dfrac{\partial v_i}{\partial q_i}=0, & \dfrac{\partial v_i}{\partial q_j}+\dfrac{\partial u_j}{\partial p_i}=0\\ \dfrac{\partial v_i}{\partial p_j}-\dfrac{\partial v_j}{\partial p_i}=0, & \dfrac{\partial u_j}{\partial q_i}-\dfrac{\partial u_i}{\partial q_j}=0\end{array}\right\}\quad\ldots\ldots\ldots\ldots(33),$$

conditions manifestly fulfilled by (32). Quoting (16 c) we get at once $\dfrac{D\Omega}{Dt}=[\sum_i p_i v_i - H]$, the residue when the associated terms are shewn explicitly; an exact function appearing on the right hand because that is the form prior to the line-integral form. Direct work gives

$$\begin{aligned}\frac{D\Omega}{Dt}&=\int\sum_i\left(\frac{Dp_i}{Dt}+\sum_j p_j\frac{\partial v_j}{\partial q_i}\right)dq_i+\sum_{i,j}p_j\frac{\partial v_j}{\partial p_i}dp_i+\sum_i\left(p_i\frac{\partial v_i}{\partial t}-\frac{DH}{Dt}\right)dt\\ &=\int\sum_i\left(u_i+\sum_j p_j\frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_j}\right)dq_i+\sum_{i,j}p_j\frac{\partial}{\partial p_i}\frac{\partial H}{\partial p_j}dp_i+\frac{\partial}{\partial t}(\sum_i p_i v_i-H)\,dt\\ &=\int\sum_i\frac{\partial}{\partial q_i}\left(-H+\sum_j p_j\frac{\partial H}{\partial p_j}\right)dq_i+\sum_i\frac{\partial}{\partial p_i}\left(-H+\sum_j p_j\frac{\partial H}{\partial p_j}\right)dp_i+\frac{\partial}{\partial t}\left(-H+\sum_j p_j\frac{\partial H}{\partial p_j}\right)dt\\ &=\left[-H+\sum_j p_j\frac{\partial H}{\partial p_j}\right]=[-H+\sum_i p_i v_i]\quad\ldots\ldots\ldots\ldots(34).\end{aligned}$$

In the second step we used $\dfrac{DH}{Dt}=\dfrac{\partial H}{\partial t}$, a consequence of (32). The result (34) establishes contact with the further development of Hamilton's theory in connexion with the S function (see Routh's *Rigid Dynamics*, 3rd ed., p. 313).

§ 13. We proceed to enquire whether it is possible to treat forms of higher order in a manner comparable with the Hamiltonian form. As example we take the electrical surface-form, and write $\dfrac{D}{Dt}=\dfrac{\partial}{\partial t}+u\dfrac{\partial}{\partial x}+\ldots+U\dfrac{\partial}{\partial X}+\ldots$, treating XYZ as independent variables. Then

Ω being $\int X\,dy\,dz+\ldots+L_1\,dt\,dx+\ldots$,

$$\begin{aligned}\mathcal{R}\Omega=\int dX\,dy\,dz+\ldots+\left(\frac{\partial L_1}{\partial y}-\frac{\partial M_1}{\partial x}\right)dt\,dx\,dy+\ldots\\ +\int\left(\frac{\partial L_1}{\partial X}dX+\frac{\partial L_1}{\partial Y}dY+\frac{\partial L_1}{\partial Z}dZ\right)dt\,dx+\ldots\quad\ldots\ldots\ldots(35\,a).\end{aligned}$$

But if we make $\mathcal{R}\Omega$ a complete form, its value, by the method of p. 105, is

$$\mathcal{R}\Omega=\int dX\,dy\,dz+\ldots+(v\,dZ-w\,dY)\,dt\,dx+\ldots-U\,dt\,dy\,dz\ldots\quad\ldots\ldots\ldots(35\,b).$$

The two forms are identified by writing

$$\left.\begin{array}{l}U=\dfrac{\partial N_1}{\partial y}-\dfrac{\partial M_1}{\partial z},\ldots\\ \dfrac{\partial L_1}{\partial X}=0,\ \ \dfrac{\partial L_1}{\partial Y}=-w,\ \ \dfrac{\partial L_1}{\partial Z}=v,\ldots\end{array}\right\}\quad\ldots\ldots\ldots\ldots(36\,a)$$

The second group compels (uvw) to be independent of (XYZ). For we have $\frac{\partial w}{\partial X}=0$, $\frac{\partial v}{\partial X}=0$; and also $\frac{\partial v}{\partial Y}+\frac{\partial w}{\partial Z}=0$, with two others which involve $\frac{\partial u}{\partial X}=0$. From the second group therefore we infer that

$$L_1 = L + vZ - wY, \dots \qquad \text{(36 } b\text{)},$$

where LMN are functions of $(xyzt)$, not of (XYZ). To make $\frac{D}{Dt}\mathscr{R}\Omega$ vanish we have only to attend to its non-temporal terms, which are

$$\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)dx\,dy\,dz+\left(\frac{\partial U}{\partial X}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dX\,dy\,dz+\dots$$
$$+\int\left(\frac{\partial V}{\partial X}-\frac{\partial v}{\partial x}\right)dX\,dz\,dx+\left(\frac{\partial W}{\partial X}-\frac{\partial w}{\partial x}\right)dX\,dx\,dy+\dots.$$

But
$$U=\frac{\partial N_1}{\partial y}-\frac{\partial M_1}{\partial z}=\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}-X\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+Y\frac{\partial u}{\partial y}+Z\frac{\partial u}{\partial z},$$

or
$$U+X\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-Y\frac{\partial u}{\partial y}-Z\frac{\partial u}{\partial z}=\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},\dots \qquad \text{(37)},$$

a result in virtue of which the non-temporal integrands vanish. With (LMN) for $(V\alpha, V\beta, V\gamma)$, (37) is the equivalent of (27 b), the whole rate of increment of X, which was there represented by $\frac{\partial X}{\partial t}+u\frac{\partial X}{\partial x}+v\frac{\partial X}{\partial y}+w\frac{\partial X}{\partial z}$, being here represented by U. In relation to the fact that (LMN) or $(\alpha\beta\gamma)$ are introduced as functions of $xyzt$, we note that when $V\alpha = Va + wY - vZ$ is used,

$$\frac{\partial}{\partial X}\left(\tfrac{1}{2}\Sigma a^2-\tfrac{1}{2}\Sigma X^2\right)=-\xi, \text{ and } \frac{\partial}{\partial u}\left(\tfrac{1}{2}\Sigma a^2-\tfrac{1}{2}\Sigma X^2\right)=(Yc-Zb)/V;$$

i.e. we get electric force and mechanical momentum.

The fact that (uvw) in (35) are independent of (XYZ) turns on u being given both as $-\frac{\partial M_1}{\partial Z}$ and as $\frac{\partial N_1}{\partial Y}$, the proof involving a circle of comparisons of second differential coefficients such as $\frac{\partial^2 M_1}{\partial X \partial Z}$. This double value does not occur in the linear Hamiltonian form, which is therefore free from the limitation. For higher forms it is essential. The method of this paragraph applied to (16 a), for example, gives the condition (16 b) with U_{pqr} for $\frac{DX_{pqr}}{Dt}$, with $u_p \dots$ shown to be independent of X, and L_{pq} expressly introduced as a function of $x_p t$. The outcome of the enquiry is therefore that if non-temporal integrands are treated as independent variables, their appearance in temporal terms is restricted to the linear forms which belong to the associated terms. The quantities u_p then define a problem in relation to which $X \dots$ are to be found so as to make a given form invariant. As regards the functions L, in the hydrodynamical case we have the pressure, in the dynamical case the Lagrange function; these examples give the clue to their general significance.

While engaged on this paper I met with the short account of integral invariants given in Whittaker's *Analytical Dynamics*, and have since read the part of the *Mécanique Céleste*, Vol. III., in which M. Poincaré introduces the subject. His point of departure is a steady state of motion in which the quantities here called u are defined in terms of coordinates only. In the present paper a dependence of u on t also is supposed, and that dependence necessitates a consideration of the temporal terms used here. This fundamental difference does not however exclude points of contact with M. Poincaré's results. I may instance the conclusion of § 7, which is the equivalent of the statement (*op. cit.* p. 14), "Tout invariant intégral relatif est la somme d'une *intégrale de différentielle exacte* et d'un invariant intégral absolu." The words italicised are used in a special sense (defined on the same page), viz. they correspond to what is here called the reticular form and denoted by $\mathscr{R}\Omega$. The notation permits us to consider the operations $\mathscr{R}$ and D/Dt separately and in conjunction, and to draw the distinction between Ω and $\mathscr{R}\Omega$ which is necessary even for a closed circuit when Kelvin's constant circulation is admitted. It is hardly proper to regard the constants of the latter as constants of integration, for the integration is definite for the circuit. It appears more satisfactory to say that if Ω contains terms of the type $\mathscr{R}\Omega_{m-1}$, then while each quantity appearing in Ω is single-valued, those in the form Ω_{m-1} may be many-valued (analogue circulation-potential); and the value of Ω for a closed space may contain the cyclic constants of these latter as well as the integral of the characteristics of infinitesimal circuits as disclosed by $\mathscr{R}\Omega$.

IV. *On the application of Integral Equations to the Determination of Upper and Lower Limits to the value of a Double Integral.* By H. Bateman, M.A., Fellow of Trinity College, Cambridge.

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1. In a former paper* we considered the properties of a certain function $w(\lambda)$ connected with an integral equation of the type

$$f(s) = \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt \quad \text{.................................(1)},$$

in which $\kappa(s, t)$ is a real continuous symmetric function of s and t. $(a \leqslant s \leqslant b)$, $(a \leqslant t \leqslant b)$.

This function $w(\lambda)$ which may be called the *energy function* of the integral equation (1) is defined by the equation

$$w(\lambda) = \int_a^b f(s)\, \phi(s)\, ds \quad \text{.................................(2)}.$$

If $f(s)$ is continuous for $(a \leqslant s \leqslant b)$ the zeros and poles of the function $\lambda w(\lambda)$ occur alternately†. This result which is derived from the fact that $\lambda w(\lambda)$ increases continually with λ, will now be used to determine limits between which the value of the double integral

$$J = \int_a^b \int_a^b \kappa(s, t)\, x(s)\, x(t)\, ds\, dt \quad \text{.................................(3)},$$

must lie‡. The function $x(s)$ is supposed to be a real continuous function satisfying the condition

$$\int_a^b [x(s)]^2\, ds = 1 \quad \text{...(4)},$$

but is otherwise perfectly arbitrary. The limits that will be determined are independent of the functional form of $x(s)$.

* *Cambridge Phil. Trans.* vol. xx. no. xv. (1908) pp. 371—382.

† *Op. cit.* p. 374.

‡ This problem is solved by Hilbert, *Gött. Nachr.* (1904) for the case in which $\kappa(s, t)$ is a definite function, i.e. when the integral J is always positive. Another investigation is given by E. Holmgren, *Comptes Rendus*, t. CXLII. (1906) pp. 331—333.

We commence by defining a continuous function $f(s)$ by means of the equation

$$f(s) = x(s) - \lambda_0 \int_a^b \kappa(s, t)\, x(t)\, dt \quad \text{.......(5)},$$

where

$$\lambda_0 = \frac{1}{J}.$$

If $J \neq 0$ this equation implies that the function $\phi(s)$ which is the solution of (1) with the above value of $f(s)$, reduces to $x(s)$ when $\lambda = \lambda_0$. The value of $w(\lambda)$ for $\lambda = \lambda_0$ is thus given by

$$w(\lambda_0) = \int_a^b x(s) f(s)\, ds = \int_a^b [x(s)]^2\, ds - \lambda_0 \int_a^b \int_a^b \kappa(s, t)\, x(s)\, x(t)\, ds\, dt = 0,$$

in other words λ_0 is a zero of the function $\lambda w(\lambda)$.

Now since the zeros and poles of the function $\lambda w(\lambda)$ occur alternately there can only be *one* zero between each consecutive pair of poles. The value $\lambda = 0$ is evidently a zero of the function and is consequently the only zero lying between the numerically smallest negative pole and the smallest positive pole. It follows then that λ_0 cannot lie between these two poles, and so if λ_1 and λ_2 are used to denote these two poles respectively, we have the inequality

$$\frac{1}{\lambda_1} \leqslant \frac{1}{\lambda_0} \leqslant \frac{1}{\lambda_2},$$

which gives

$$\frac{1}{\lambda_1} \leqslant J \leqslant \frac{1}{\lambda_2} \quad \text{.......(6)}.$$

2. In order to determine the quantities λ_1 and λ_2 we must recall some properties of the integral equation (1).

It is known that the function $\phi(s)$ is given uniquely by a formula of the type*

$$\phi(s) = f(s) + \lambda \int_a^b \frac{D(\lambda; s, t)}{D(\lambda)} f(t)\, dt \quad \text{.......(7)},$$

where $D(\lambda)$ and $D(\lambda; s, t)$ are integral functions of λ. If, however, λ is a root of the equation

$$D(\lambda) = 0,$$

the homogeneous integral equation

$$\psi(s) = \lambda \int_a^b \kappa(s, t)\, \psi(t)\, dt \quad \text{.......(8)}$$

will possess at least one non-vanishing solution $\psi(s)$ and the function $\phi(s)$ as given by formula (7) will in general be infinite. Let λ be a p-fold root of $D(\lambda) = 0$, then it can be shown that there are m linearly independent solutions $\psi_1(s), \ldots \psi_m(s)$ of equation (8), $(m \leqslant p)$, and that this particular value of λ is a *simple* pole† of $\phi(s)$ unless all the conditions of the type

$$\int_a^b f(s)\, \psi_n(s)\, ds = 0, \qquad (n = 1, 2, \ldots m) \quad \text{.......(9)},$$

* Fredholm, *Acta Mathematica* (1903).

† A simple proof of this result is given by T. Boggio, *Comptes Rendus*, Oct. (1907).

are satisfied. When these conditions are satisfied the function $\phi(s)$, as determined by (7) for ordinary values of λ, remains finite for this singular value of λ and is a solution of (1). The solution of the integral equation is however really indeterminate for this value of λ since we may add any linear combination of the functions $\psi_n(s)$ to the particular solution found and still obtain a solution of the equation.

Since $\lambda w(\lambda) = \lambda \int_a^b f(s)\, \phi(s)\, ds$, it appears that $\lambda w(\lambda)$ can become infinite either when $\lambda = \pm \infty$ or when λ is a pole of $\phi(s)$, i.e. a singular value of λ. There are several possible cases that can arise.

(1) Let us suppose that the singular values of λ are both positive and negative *, then if λ_1' is the numerically smallest negative, λ_2' the smallest positive singular value, we have $\lambda_1 \leqslant \lambda_1'$, $\lambda_2 \geqslant \lambda_2'$ and so from the inequality (6) we may deduce that

$$\frac{1}{\lambda_1'} \leqslant J \leqslant \frac{1}{\lambda_2'} \quad \text{...(10).}$$

The quantities λ_1' and λ_2' are evidently independent of the form of the function $x(s)$ and so we have found a pair of limits of the type required.

(2) Suppose that the singular values of λ are all positive, then the only negative value of λ for which $\lambda w(\lambda)$ can possibly become infinite is $\lambda_1 = -\infty$, accordingly, if λ_2' is the smallest positive singular value we have the inequality

$$0 \leqslant J \leqslant \frac{1}{\lambda_2'} \quad \text{...(10)}'.$$

We may combine these two results in the following theorem.

If both positive and negative singular values of λ *exist, the double integral* J *lies between* $\frac{1}{\lambda_1'}$ *and* $\frac{1}{\lambda_2'}$ *where* λ_1' *and* λ_2' *are the smallest negative and positive singular values of* λ *respectively. The values* $\frac{1}{\lambda_1'}$ *and* $\frac{1}{\lambda_2'}$ *themselves being assumed when* $x(s)$ *is chosen to be a solution of the corresponding homogeneous integral equations*

$$\left.\begin{aligned} \psi_1(s) - \lambda_1' \int_a^b \kappa(s, t)\, \psi_1(t)\, dt = 0 \\ \psi_2(s) - \lambda_2' \int_a^b \kappa(s, t)\, \psi_2(t)\, dt = 0 \end{aligned}\right\} \quad \text{..........................(11)}$$

respectively.

If the singular values of λ *are all positive,* J *is either zero or positive and is less or equal to* $\frac{1}{\lambda_2'}$. *It follows from this that if* J *becomes negative there must be at least one negative singular value of* λ.

* It is known that they are all real.

(3) Next, suppose that for some of the numerically smallest singular values of λ the conditions of type

$$\int_a^b x(s)\,\psi_n(s)\,ds = 0 \quad \text{.......................................(12)}$$

are satisfied. In this case extra conditions are imposed upon the function $x(s)$ and so we can expect to obtain closer limits between which the value of J must lie.

If we multiply (5) by $\psi_n(s)$ and integrate remembering that $\psi_n(s)$ is a solution of an equation of type

$$\psi_n(s) - \mu \int_a^b \kappa(s, t)\,\psi_n(t)\,dt = 0,$$

we obtain

$$\int_a^b f(s)\,\psi_n(s)\,ds = 0 \quad \text{.......................................(13).}$$

This equation and the others which are obtained from the system (12) imply that the singular values of λ under consideration are not poles of the function $\phi(s)$ and so are not poles of $\lambda w(\lambda)$. We may therefore replace the inequality (10) by

$$\frac{1}{\lambda_1''} \leqslant J \leqslant \frac{1}{\lambda_2''} \quad \text{.......................................(14),}$$

where λ_1'' and λ_2'' are the numerically smallest negative and positive singular values of λ respectively for which the complete system of conditions of type (12) are not satisfied. Since $\lambda_1'' < \lambda_1'$, $\lambda_2'' > \lambda_2'$, the limits now obtained are much closer than the ones previously considered.

3. A double integral of the type

$$H = \int_a^b \int_a^b h(s, t)\,x(s)\,x(t)\,ds\,dt \quad \text{.................................(15)}$$

in which $h(s, t)$ is real and continuous but not symmetric may be reduced to the form (3) by means of the following artifice.

We evidently have

$$H = \int_a^b \int_a^b h(t, s)\,x(s)\,x(t)\,ds\,dt \quad \text{.............................(16),}$$

accordingly, if we put

$$2\kappa(s, t) = h(s, t) + h(t, s) \quad \text{....................................(17),}$$

we have

$$H = \int_a^b \int_a^b \kappa(s, t)\,x(s)\,x(t)\,ds\,dt,$$

and this is of the form (3) since $\kappa(s, t)$ is a symmetric function. The limits between which H must lie when $x(s)$ is a continuous function subject to the condition

$$\int_a^b [x(s)]^2\,ds = 1,$$

thus depend upon the singular values of λ belonging to an integral equation in which the characteristic function is determined by equation (17).

NOTE ADDED MAY 26, 1908.

The theory may be illustrated by means of the following example.

Let
$$h(s, t) = u(s)\, v(t),$$
then
$$\kappa(s, t) = \tfrac{1}{2}[u(s)\, v(t) + u(t)\, v(s)].$$

The singular values of λ may be found by assuming that a solution of the homogeneous equation
$$\psi(s) = \lambda \int_a^b \kappa(s, t)\, \psi(t)\, dt$$
is of the form
$$\psi(s) = \alpha u(s) + \beta v(s),$$
we must then have
$$\alpha u(s) + \beta v(s) \equiv \tfrac{1}{2}\lambda \int_a^b [u(s)\, v(t) + u(t)\, v(s)]\, [\alpha u(t) + \beta v(t)]\, dt.$$

Putting
$$c_{11} = \int_a^b [u(t)]^2\, dt,$$
$$c_{12} = \int_a^b u(t)\, v(t)\, dt,$$
$$c_{22} = \int_a^b [v(t)]^2\, dt,$$
and equating coefficients of $u(s)$ and $v(s)$, we obtain
$$(2 - \lambda c_{12})\, \alpha - \lambda c_{22} \beta = 0,$$
$$- \lambda c_{11} \alpha + (2 - \lambda c_{12})\, \beta = 0.$$

The singular values of λ are thus the roots of the equation
$$(2 - \lambda c_{12})^2 - \lambda^2 c_{11} c_{22} = 0.$$

The coefficient of λ^2 in this equation is $c_{12}^2 - c_{11} c_{22}$ and is negative on account of Schwarz's well known inequality *
$$\int_a^b [u(t)]^2\, dt \,.\int_a^b [v(t)]^2\, dt \geqq \left[\int_a^b u(t) \,.\, v(t)\, dt\right]^2,$$
consequently the roots are real and of opposite signs.

If λ_1 and λ_2 are the roots our theorem tells us that the double integral
$$\int_a^b \int_a^b u(s)\, v(t)\, x(s)\, x(t)\, ds dt, \text{ where } \int_a^b [x(s)]^2\, ds = 1,$$
lies between $\dfrac{1}{\lambda_1}$ and $\dfrac{1}{\lambda_2}$.

* This may be deduced from the theorem of § 2 by assuming $x(s) = u(s)$, $\kappa(s, t) = v(s)\, v(t)$.

Now the actual values of λ_1 and λ_2 are easily found to be given by

$$\frac{1}{\lambda_1} = \tfrac{1}{2}(c_{12} - \sqrt{c_{11}c_{22}}), \quad \frac{1}{\lambda_2} = \tfrac{1}{2}(c_{12} + \sqrt{c_{11}c_{22}}),$$

hence we have the inequality

$$c_{12} - \sqrt{c_{11}c_{22}} \leqq 2\int_a^b u(s)\, x(s)\, ds \int_a^b v(t)\, x(t)\, dt \leqq c_{12} + \sqrt{c_{11}c_{22}},$$

where

$$c_{11} = \int_a^b [u(t)]^2\, dt,$$

$$c_{12} = \int_a^b u(t)\, v(t)\, dt,$$

$$c_{22} = \int_a^b [v(t)]^2\, dt,$$

and $x(s)$ is any continuous function satisfying the relation

$$\int_a^b [x(s)]^2\, ds = 1.$$

When $u(s) = v(s)$ this reduces to the inequality of Schwarz.

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V. PLEMELJ'S CANONICAL FORM.

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AND

VI. THE OPERATOR RECIPROCANTS OF SYLVESTER'S THEORY OF RECIPROCANTS.

BY

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V. *Plemelj's Canonical Form.*

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INTRODUCTION.

In the following pages I propose to obtain Plemelj's canonical form for the principal part of the solving function of an integral equation of the second kind in the neighbourhood of a pole. It will be seen that the method differs essentially from that used by Plemelj *; accordingly, although the results obtained are identical, there may be some interest in this alternative procedure.

Let $\kappa(s, t)$ be a function of two real variables s and t, which is continuous† in the closed region defined by $a \leqslant s \leqslant b$, $a \leqslant t \leqslant b$; also let

$$\kappa\begin{pmatrix} s_1, & s_2, & \ldots, & s_n \\ t_1, & t_2, & \ldots, & t_n \end{pmatrix}$$

denote the determinant of which the qth element of the pth row is $\kappa(s_p, t_q)$. It has been shown by Fredholm‡ that

$$D(\lambda;\ s, t) = \kappa(s, t) - \lambda \int_a^b \kappa\begin{pmatrix} s, & s_1 \\ t, & s_1 \end{pmatrix} ds_1 + \ldots$$

$$+ \frac{(-\lambda)^n}{n!} \int_a^b \ldots \int_a^b \int_a^b \kappa\begin{pmatrix} s, & s_1, & \ldots, & s_n \\ t, & s_1, & \ldots, & s_n \end{pmatrix} ds_1 \ldots ds_n + \ldots \quad \ldots(1),$$

and $$D(\lambda) = 1 - \lambda \int_a^b \kappa(s_1, s_1)\, ds_1 + \frac{\lambda^2}{2!} \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, & s_2 \\ s_1, & s_2 \end{pmatrix} ds_1 ds_2 - \ldots$$

$$+ \frac{(-\lambda)^n}{n!} \int_a^b \ldots \int_a^b \int_a^b \kappa\begin{pmatrix} s_1, & s_2, & \ldots, & s_n \\ s_1, & s_2, & \ldots, & s_n \end{pmatrix} ds_1 \ldots ds_n + \ldots \ldots\ldots(2),$$

* Plemelj, 'Zur Theorie der Fredholmschen Funktionalgleichung, *Monatshefte für Mathematik und Physik* (1904), p. 93 (xv. *Jahrgang*).

† Everything that follows will be true, of course, for certain functions which are not continuous; but, in developing a part of the theory such as the present, it is desirable to avoid difficulties which properly belong to the integral calculus.

‡ The reader is referred to Fredholm's paper 'Sur une classe d'equations fonctionelles,' *Acta Mathematica*, vol. xxvii. (1903), p. 365.

are integral functions of λ; moreover, he has shown that, if

$$K_\lambda(s, t) = \frac{D(\lambda;\ s, t)}{D(\lambda)},$$

and λ is not a zero of (2),

$$\left.\begin{aligned} \kappa(s, t) &= K_\lambda(s, t) - \lambda \int_a^b \kappa(s, x) K_\lambda(x, t)\, dx, \\ \kappa(s, t) &= K_\lambda(s, t) - \lambda \int_a^b K_\lambda(s, x)\, \kappa(x, t)\, dx \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(3).$$

These characteristic relations, as they are called, are of fundamental importance in the theory of integral equations of the second kind. For, if we take the pair of adjoint equations

$$\left.\begin{aligned} f(s) &= \phi(s) - \lambda \int_a^b \kappa(s, t)\, \phi(t)\, dt, \\ g(t) &= \chi(t) - \lambda \int_a^b \chi(s)\, \kappa(s, t)\, ds \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(4),$$

multiply* along the first by $K_\lambda(\sigma, s)$, along the second by $K_\lambda(t\ \tau)$, and then integrate from a to b with regard to s and t respectively, we obtain after a little reduction

$$\phi(\sigma) = f(\sigma) + \lambda \int_a^b K_\lambda(\sigma, t) f(t)\, dt,$$

$$\chi(\tau) = g(\tau) + \lambda \int_a^b g(s) K_\lambda(s, \tau)\, ds.$$

It follows from this, that a knowledge of the function $K_\lambda(s, t)$ will enable us to obtain a solution† of the equations (4); on this account $K_\lambda(s, t)$ is known as the *solving function* corresponding to the *characteristic function* $\kappa(s, t)$. The integral function $D(\lambda)$ is called the *determinant* of the function $\kappa(s, t)$.

From (1) and (2) we obtain the identity

$$\frac{\partial}{\partial\lambda}(D(\lambda)) = -\int_a^b D(\lambda;\ s, s)\, ds,$$

which may be written*

$$\frac{\partial}{\partial\lambda}(\log D(\lambda)) = -\int_a^b K_\lambda(s, s)\, ds \quad\ldots\ldots\ldots\ldots(5).$$

In this last form we have an equation which enables us to calculate $D(\lambda)$ whenever $K_\lambda(s, t)$ is known (cf. § 3 below).

The equation (5) shows that if λ_0 is a zero of $D(\lambda)$, the function $K_\lambda(s, t)$ has necessarily a pole there. In the present paper we shall be concerned with the expansion of the solving function in positive and negative powers of $(\lambda_0 - \lambda)$; or, more precisely, we shall be concerned with those terms of this expansion which involve negative powers of $(\lambda_0 - \lambda)$.

* It is still assumed that λ is not a zero of $D(\lambda)$.

† The solution so obtained is easily shown to be unique.

When λ has the value λ_0 the equations (4) do not, in general, possess a solution; but the homogeneous equations

$$\phi(s) = \lambda_0 \int_a^b \kappa(s, t)\, \phi(t)\, dt \quad \text{..................................(5)},$$

$$\chi(t) = \lambda_0 \int_a^b \chi(s)\, \kappa(s, t)\, ds \quad \text{..................................(6)}$$

can now be solved. In fact, Fredholm has shown that, if m is the lowest integer for which constants $s_1, s_2, \ldots, s_m$; $t_1, t_2, \ldots, t_m$ can be chosen, so that

$$D\begin{pmatrix} \lambda; & s_1, s_2, \ldots, s_m \\ & t_1, t_2, \ldots, t_m \end{pmatrix} = D(\lambda)\, K_\lambda \begin{pmatrix} s_1, s_2, \ldots, s_m \\ t_1, t_2, \ldots, t_m \end{pmatrix}$$

does not vanish for $\lambda = \lambda_0$, then the m linearly independent functions

$$\phi_r(s) = \frac{D\begin{pmatrix} \lambda_0; & s_1, \ldots, s_{r-1}, s, s_{r+1}, \ldots, s_m \\ & t_1, \ldots, t_{r-1}, t_r, t_{r+1}, \ldots, t_m \end{pmatrix}}{D\begin{pmatrix} \lambda_0; & s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_m \\ & t_1, \ldots, t_{r-1}, t_r, t_{r+1}, \ldots, t_m \end{pmatrix}} \qquad (r = 1, 2, \ldots, m)$$

are solutions of (5); and the functions

$$\chi_r(t) = \frac{D\begin{pmatrix} \lambda_0; & s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_m \\ & t_1, \ldots, t_{r-1}, t, t_{r+1}, \ldots, t_m \end{pmatrix}}{D\begin{pmatrix} \lambda_0; & s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_m \\ & t_1, \ldots, t_{r-1}, t_r, t_{r+1}, \ldots, t_m \end{pmatrix}} \qquad (r = 1, 2, \ldots, m)$$

are solutions of (6). Further, if we take any solution whatever of (5), say $\phi(s)$, it can be shown* that

$$\phi(s) = \phi_1(s)\, \phi(s_1) + \phi_2(s)\, \phi(s_2) + \ldots + \phi_m(s)\, \phi(s_m);$$

and a corresponding result holds in regard to any solution $\chi(t)$ of (6).

In conclusion I quote the formula

$$\frac{\partial^n}{\partial \lambda^n}(D(\lambda)) = (-)^n \int_a^b \ldots \int_a^b \int_a^b D\begin{pmatrix} \lambda; & s_1, s_2, \ldots, s_n \\ & s_1, s_2, \ldots, s_n \end{pmatrix} ds_1 \ldots ds_n \quad \text{............(7)},$$

which is due to Fredholm†.

§ 1. Suppose now that we adopt the notation just explained; and let the expansion of $K_\lambda(s, t)$ in the neighbourhood of λ_0, be

$$K_\lambda(s, t) = \frac{f_\iota(s, t)}{(\lambda_0 - \lambda)^\iota} + \frac{f_{\iota-1}(s, t)}{(\lambda_0 - \lambda)^{\iota-1}} + \ldots + \frac{f_1(s, t)}{(\lambda_0 - \lambda)} + P(\lambda_0 - \lambda;\, s, t) \quad |\lambda_0 - \lambda| < L \text{ ...(8)},$$

where $P(\lambda_0 - \lambda;\, s, t)$ is a power series in $(\lambda_0 - \lambda)$ whose coefficients are functions‡ of s and t and $f_\iota(s, t)$ is not identically zero in the region under consideration. If we supply this expansion in the first of the characteristic relations (3), and equate the coefficients of the various powers of $(\lambda_0 - \lambda)$ on each side, we obtain, among other relations,

$$f_r(s, t) - \lambda_0 \int_a^b \kappa(s, x) f_r(x, t)\, dx = -\int_a^b \kappa(s, x) f_{r+1}(x, t)\, dx \qquad (r = 1, 2, \ldots, \iota) \text{ ...(9)},$$

* Cf. Plemelj *op. cit.* p. 120. The relation has been obtained directly by Bateman.

† *op. cit.* p. 371.

‡ They are, of course, continuous.

where it is supposed that $f_{\iota+1}(s, t)$ is identically zero. Multiply along each of these equations by $K_\lambda(\sigma, s)$, and integrate with respect to s between the limits a and b; then

$$\int_a^b K_\lambda(\sigma, s) f_r(s, t)\, ds - \lambda_0 \int_a^b \left[\int_a^b K_\lambda(\sigma, s)\, \kappa(s, x)\, ds\right] f_r(x, t)\, dx$$
$$= -\int_a^b \left[\int_a^b K_\lambda(\sigma, s)\, \kappa(s, x)\, ds\right] f_{r+1}(x, t)\, dx \qquad (r = 1, 2, \ldots, \iota) \ldots\ldots(9').$$

By the second of the characteristic relations (3), we may replace each of the quantities in the square brackets by

$$\frac{K_\lambda(\sigma, x) - \kappa(\sigma, x)}{\lambda};$$

and so, after a little reduction, (9′) becomes

$$\int_a^b K_\lambda(\sigma, x) f_{r+1}(x, t)\, dx - (\lambda_0 - \lambda) \int_a^b K_\lambda(\sigma, x) f_r(x, t)\, dx$$
$$= -\lambda_0 \int_a^b \kappa(\sigma, x) f_r(x, t)\, dx + \int_a^b \kappa(\sigma, x) f_{r+1}(x, t)\, dx,$$

which by (9)

$$= -f_r(\sigma, t).$$

Replacing σ by s, these equations may be written

$$\int_a^b \frac{K_\lambda(s, x) f_{r+1}(x, t)\, dx}{(\lambda_0 - \lambda)^{r+1}} - \int_a^b \frac{K_\lambda(s, x) f_r(x, t)\, dx}{(\lambda_0 - \lambda)^r} = -\frac{f_r(s, t)}{(\lambda_0 - \lambda)^{r+1}} \qquad (r = 1, 2, \ldots, \iota) \ldots\ldots(10).$$

Now add together the left and right sides of those of the equations (10) which follow the $(r-1)$th, and remember that $f_{\iota+1}(s, t)$ is zero; we thus obtain

$$\int_a^b K_\lambda(\sigma, x) f_r(x, t)\, dx = \sum_{n=r}^{\iota} \frac{f_n(s, t)}{(\lambda_0 - \lambda)^{n-r+1}} \qquad (r = 1, 2, \ldots, \iota) \quad \ldots\ldots(11).$$

§ 2. In the equations (11) supply the expansion (8), and equate the coefficients of the various powers of $(\lambda_0 - \lambda)$. Among other relations we obtain

$$\int_a^b f_q(s, x) f_r(x, t)\, dx = f_{q+r+1}(s, t) \qquad (q \leqslant \iota - r + 1),$$
$$= 0 \qquad (q > \iota - r + 1).$$

In particular, when $r = 1$ we have the set

$$\int_a^b f_q(s, x) f_1(x, t)\, dx = f_q(s, t) \qquad (q = 1, 2, \ldots, \iota); \quad \ldots(12);$$

and, when $q = 1$,

$$\int_a^b f_1(s, x) f_r(x, t)\, dx = f_r(s, t) \qquad (r = 1, 2, \ldots, \iota) \ldots\ldots(13).$$

Again, if we put $r = 2$,

$$\int_a^b f_q(s, x) f_2(x, t)\, dx = f_{q+1}(s, t) \qquad (q \leqslant \iota - 1)$$
$$= 0 \qquad (q > \iota - 1) \quad \ldots\ldots(14);$$

and, writing $q = 2$,

$$\int_a^b f_2(s, x) f_r(x, t)\, dx = f_{r+1}(s, t) \qquad (r \leqslant \iota - 1)$$
$$= 0 \qquad (r > \iota - 1) \quad \ldots\ldots(15).$$

It follows from this last set of equations that, when $\iota > 2$,

$$f_3(s, t) = \int_a^b f_2(s, x_1) f_2(x_1, t) dx,$$

$$f_4(s, t) = \int_a^b \int_a^b f_2(s, x_1) f_2(x_1, x_2) f_2(x_2, t) dx_1 dx_2,$$

....

$$f_\iota(s, t) = \int_a^b \dots \int_a^b \int_a^b f_2(s, x_1) f_2(x_1, x_2) \dots f_2(x_{\iota-2}, t) dx_1 \dots dx_{\iota-2},$$

$$0 = \int_a^b \dots \int_a^b \int_a^b f_2(s, x_1) f_2(x_1, x_2) \dots f_2(x_{\iota-1}, t) dx_1 \dots dx_{\iota-1}.$$

We proceed now to deduce the properties of the functions $f_r(s, t)$ which follow from these relations, and then to reduce each of them to a canonical form.

§ 3. In the first place, let us consider the function $f_1(s, t)$. Writing $q=1$ in (12) we see that

$$\int_a^b f_1(s, x) f_1(x, t) dx = f_1(s, t) \quad \text{.............................(12').}$$

Now Neumann has shown that, if $F_\lambda(s, t)$ is the solving function corresponding to the characteristic function $f_1(s, t)$, then for sufficiently small values of $|\lambda|$ ($< R$ say)

$$F_\lambda(s, t) = f_1(s, t) + \lambda \int_a^b f_1(s, x_1) f_1(x_1, t) dx_1 + \lambda^2 \int_a^b \int_a^b f_1(s, x_1) f_1(x_1, x_2) f_1(x_2, t) dx_1 dx_2 + \dots$$
$$+ \lambda^n \int_a^b \dots \int_a^b \int_a^b f_1(s, x_1) f_1(x_1, x_2) \dots f(x_n, t) dx_1 \dots dx_n + \dots.$$

Using (12'), we thus see that

$$F_\lambda(s, t) = f_1(s, t)[1 + \lambda + \lambda^2 + \dots + \lambda^n + \dots] \qquad (|\lambda| < R \leqslant 1)$$
$$= \frac{f_1(s, t)}{1-\lambda} \quad \text{...(16).}$$

But, since $F_\lambda(s, t)$ is known to be a meromorphic function and the function

$$\frac{f_1(s, t)}{1-\lambda}$$

coincides with it in a finite domain, it follows that (16) holds for all values of $\lambda \gtrless 1$.

Again, if we write $s = t$ in (8) and integrate between the limits a and b, we obtain

$$\int_a^b K_\lambda(s, s) ds = \int_a^b \frac{f_\iota(s, s) ds}{(\lambda_0 - \lambda)^\iota} + \dots + \int_a^b \frac{f_1(s, s) ds}{(\lambda_0 - \lambda)} + \int_a^b P(\lambda_0 - \lambda; s, s) ds \quad |\lambda_0 - \lambda| < L \dots(17).$$

The left-hand member of this equation is

$$-\frac{\partial}{\partial \lambda}(\log D(\lambda))$$

by (5); *i.e.* if p is the order of the zero λ_0 of $D(\lambda)$, is

$$\frac{p}{(\lambda_0 - \lambda)} + \text{power series in } (\lambda_0 - \lambda) \quad\ldots\ldots\ldots\ldots(18),$$

for sufficiently small values of $|\lambda_0 - \lambda|$. Comparing the coefficients of the various powers of $(\lambda_0 - \lambda)$ in each of the expansions (17) and (18), we deduce that

$$\int_a^b f_r(s, s)\, ds = 0, \qquad (\iota \geqslant r \geqslant 2)$$

$$\int_a^b f_1(s, s)\, ds = p \quad\ldots\ldots\ldots\ldots(19).$$

The equation just written enables us to determine $\Delta(\lambda)$, the determinant of $f_1(s, t)$. For, by (5) above, we have

$$-\frac{\partial}{\partial\lambda}(\log \Delta(\lambda)) = \int_a^b \frac{f_1(s, s)\, ds}{1 - \lambda}$$

$$= \frac{p}{1 - \lambda},$$

by (19). Integrating with respect to λ and remembering that, when $\lambda = 0$, $\Delta(\lambda) = 1$, we obtain

$$\Delta(\lambda) = (1 - \lambda)^p.$$

§ 4. In conformity with the notation explained above, let

$$\Delta\left(\lambda;\, \begin{matrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{matrix}\right) = \Delta(\lambda)\, F_\lambda \begin{pmatrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{pmatrix}.$$

By (7) we have

$$\int_a^b \Delta\left(\lambda;\, \begin{matrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{matrix}\right) ds_1 ds_2 \ldots ds_p = (-)^p \frac{\partial^p}{\partial\lambda^p}(\Delta(\lambda));$$

and so, supplying in the values of $\Delta(\lambda)$ and $F_\lambda(s, t)$,

$$\int_a^b f_1 \begin{pmatrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{pmatrix} ds_1 ds_2 \ldots ds_p = p\,!.$$

Thus the function

$$\Delta\left(\lambda;\, \begin{matrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{matrix}\right) = f_1 \begin{pmatrix} s_1, s_2, \ldots, s_p \\ s_1, s_2, \ldots, s_p \end{pmatrix} \quad\ldots\ldots\ldots\ldots(20)$$

is not identically zero. On the other hand

$$\Delta\left(\lambda;\, \begin{matrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{matrix}\right) = (1 - \lambda)^{p-n} f_1 \begin{pmatrix} s_1, s_2, \ldots, s_n \\ t_1, t_2, \ldots, t_n \end{pmatrix};$$

and consequently vanishes for $\lambda = 1$, if $p < n$. According to Fredholm's theory, it follows that if $s_1, s_2, \ldots, s_p$ are constants, selected so that (20) does not vanish, the functions

$$\phi_r(s) = \frac{f_1 \begin{pmatrix} s_1, \ldots, s_{r-1}, s, & s_{r+1}, \ldots, s_p \\ s_1, \ldots, s_{r-1}, s_r, & s_{r+1}, \ldots, s_p \end{pmatrix}}{f_1 \begin{pmatrix} s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \\ s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \end{pmatrix}} \qquad (r = 1, 2, \ldots, p)$$

are p linearly independent solutions of

$$\phi(s) = \int_a^b f_1(s, t)\,\phi(t)\,dt \quad \text{......(21)},$$

and the functions

$$\chi_r(t) = \frac{f_1\begin{pmatrix} s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \\ s_1, \ldots, s_{r-1}, t, s_{r+1}, \ldots, s_p \end{pmatrix}}{f_1\begin{pmatrix} s_1, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \\ s_r, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \end{pmatrix}} \qquad (r = 1, 2, \ldots, p)$$

are p linearly independent solutions of

$$\chi(t) = \int_a^b \chi(s) f_1(s, t)\,ds \quad \text{......(22)}.$$

Of course, these facts also follow from (12) and (13).

The equation (12) shows that

$$\phi(s) = f_1(s, t)$$

is a solution of (21). It follows therefore, by what was said above, that

$$f_1(s, t) = \phi(s) f_1(s_1, t) + \phi_2(s) f_1(s_2, t) + \ldots + \phi_p(s) f_1(s_p, t) \quad \text{......(23)}.$$

This equation, it is easily seen, is equivalent to

$$f_1(s, t) = f_1(s, s_1)\,\chi_1(t) + f_1(s, s_2)\,\chi_2(t) + \ldots + f_1(s, s_p)\,\chi_p(t) \quad \text{......(24)},$$

which may be obtained directly from (12). Returning to the equation (23), we have seen already that the functions $\phi_1(s), \ldots, \phi_p(s)$ are linearly independent. The same holds for the functions $f_1(s_1, t), \ldots, f_1(s_p, t)$; for, if any linear relation connects them, the determinant (20) would clearly vanish; and this is contrary to our hypothesis as to the choice of $s_1, s_2, \ldots, s_p$. In an exactly similar manner we may establish the linear independence of the functions $f_1(s, s_1), f_1(s, s_2), \ldots, f_1(s, s_p)$.

§ 5. In the case when the pole of $K_\lambda(s, t)$ is of the first order either (23) or (24) furnishes us with a canonical form for $f_1(s, t)$. For example, if we write

$$\phi_r(s) = \Phi_r^0(s), \quad f_1(s_r, t) = \Psi_r^0(t),$$

equation (23) may be written

$$f_1(s, t) = \Phi_1^0(s)\,\Psi_1^0(t) + \Phi_2^0(s)\,\Psi_2^0(t) + \ldots + \Phi_p^0(s)\,\Psi_p^0(t)\text{*}.$$

By (12′) we have

$$\int_a^b \Phi_r^0(s)\,\Psi_q^0(s)\,ds = \frac{f_1\begin{pmatrix} s_1, s_2, \ldots, s_{r-1}, s_q, s_{r+1}, \ldots, s_p \\ s_1, s_2, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \end{pmatrix}}{f_1\begin{pmatrix} s_1, s_2, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \\ s_1, s_2, \ldots, s_{r-1}, s_r, s_{r+1}, \ldots, s_p \end{pmatrix}},$$

which clearly

$$= 1 \qquad (q = r),$$
$$= 0. \qquad (q \neq r)$$

* This result is evidently true whether the pole be of the first order, or not: it agrees, in substance, with that announced by Bryon Heywood, *Comptes Rendus*, Nov. 25, 1907 (p. 909).

Moreover, as has been remarked already, the two sets of functions $\Phi_r^0(s)$, $\Psi_r^0(t)$ are each linearly independent.

§ 6. The case $\iota = 1$ having thus been disposed of we shall suppose in what follows that $\iota \geqslant 2$.

From (12) and (13) it appears that

$$\left.\begin{array}{l}\phi(s) = f_r(s, t)\\ \chi(t) = f_r(s, t)\end{array}\right\} \quad (\iota \geqslant r \geqslant 1)$$

are all solutions of (21) and (22) respectively. Thus*

$$f_r(s, t) = \phi_1(s) f_r(s_1, t) + \phi_2(s) f_r(s_2, t) + \ldots + \phi_p(s) f_r(s_p, t), \qquad (\iota \geqslant r \geqslant 1) \ldots (25)$$

and

$$f_r(s, t) = f_r(s, s_1)\chi_1(t) + f_r(s, s_2)\chi_2(t) + \ldots + f_r(s, s_p)\chi_p(t) \qquad (\iota \geqslant r \geqslant 1) \ldots (26).$$

Multiply the first of these equations by $f_2(s_q, s)$, and integrate with respect to s between the limits a and b. It follows from (15) that

$$f_{r+1}(s_q, t) = f_r(s_1, t)\int_a^b f_2(s_q, s)\phi_1(s)\,ds + \ldots + f_r(s_p, t)\int_a^b f_2(s_q, s)\phi_1(s)\,ds.$$
$$(q = 1, 2, \ldots, p)$$

Thus the functions

$$f_\iota(s_1, t),\ f_\iota(s_2, t),\ \ldots,\ f_\iota(s_p, t)$$

are linear functions of

$$f_{\iota-1}(s_1, t),\ f_{\iota-1}(s_2, t),\ \ldots,\ f_{\iota-1}(s_p, t);$$

these latter are, in turn, linear functions of

$$f_{\iota-2}(s_1, t),\ f_{\iota-2}(s_2, t),\ \ldots,\ f_{\iota-2}(s_p, t);$$

and so on, until finally we establish that each of these sets of functions consist of linear combinations of

$$f_1(s_1, t),\ f_1(s_2, t),\ \ldots,\ f_1(s_p, t),$$

or what amounts to the same thing, of

$$\chi_1(t),\ \chi_2(t),\ \ldots,\ \chi_p(t).$$

This last result might have been deduced directly from (26); for by writing

$$s = s_1,\ s_2,\ \ldots,\ s_p$$

in succession, we obtain the p relations

$$f_r(s_q, t) = f_r(s_q, s_1)\chi_1(t) + f_r(s_q, s_2)\chi_2(t) + \ldots + f_r(s_q, s_p)\chi_p(t) \qquad (q = 1, 2, \ldots, p) \ldots (27).$$

§ 7. The determinant of the coefficients of the functions $\chi(t)$ on the right-hand side of (27) is clearly

$$f_r\binom{s_1, s_2, \ldots, s_p}{s_1, s_2, \ldots, s_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

* Cf. B. Heywood in the place cited above.

When $r=\iota$, let $(p-c_1)$ be the order of the first minors* of (28) which do not all vanish. By the theory of simultaneous equations, it follows that $(p-c_1)$ of the functions

$$f_\iota(s_1, t),\ f_\iota(s_2, t),\ \ldots,\ f_\iota(s_p, t)$$

can be expressed as linear functions of the remaining c_1, which are themselves linearly independent. Let the latter be

$$f_\iota(s_{\kappa_1}, t),\ f_\iota(s_{\kappa_2}, t),\ \ldots,\ f_\iota(s_{\kappa_{c_1}}, t),$$

and let
$$\Psi_r^\rho(t)=f_{\iota-\rho}(s_{\kappa_r}, t)\qquad \begin{pmatrix} r=1, 2, \ldots, c_1 \\ \rho=0, 1, \ldots, \iota-1\end{pmatrix} \quad\ldots\ldots(29).$$

It appears at once from (12) and (14) that

$$\left.\begin{aligned} &\int_a^b \Psi_r^\rho(x) f_1(x, t)=\Psi_r^\rho(t), \\ &\int_a^b \Psi_r^\rho(x) f_2(x, t)=\Psi_r^{\rho-1}(t) \qquad (\rho \geqslant 1) \\ &\qquad\qquad\qquad\quad =0 \qquad\qquad\quad (\rho=0)\end{aligned}\right\} \quad\ldots\ldots(30).$$

The last pair of equations enables us to prove that the ιc_1 functions (29) are linearly independent. For, if there is any linear relation connecting them, there will be a certain number for which ρ is a maximum, say $\rho_0 (\leqslant \iota-1)$. The relation in question would then read

$$l_1\Psi_1^{\rho_0}(t)+l_2\Psi_2^{\rho_0}(t)+\ldots+l_{c_1}\Psi_{c_1}^{\rho_0}(t)+\ldots=0 \quad\ldots\ldots(31),$$

where the terms omitted after $l_{c_1}\Psi_{c_1}^{\rho_0}(t)$ are multiples of terms $\Psi_r^\rho(t)$ in which $\rho<\rho_0$, and it may possibly be that some (but not all) the constants $l_1, l_2, \ldots, l_{c_1}$ are evanescent. Multiply along (31) by $f_2(t, x_1)$, and integrate with respect to t between the limits a and b; using (30), we thus obtain

$$l_1\Psi_1^{\rho_0-1}(x_1)+l_2\Psi_2^{\rho_0-1}(x_1)+\ldots+l_{c_1}\Psi_{c_1}^{\rho_0-1}(x_1)+\ldots=0,$$

where the terms omitted are multiples of $\Psi_r^\rho(x_1)$ for which $\rho<\rho_0-1$. When we multiply along this last equation by $f_2(x_1, x_2)$, and integrate with respect to x_1 between the same limits, we have the relation

$$l_1\Psi_1^{\rho_0-2}(x_2)+l_2\Psi_2^{\rho_0-2}(x_2)+\ldots+l_{c_1}\Psi_{c_1}^{\rho_0-2}(x_2)+\ldots=0,$$

the terms which are now omitted being those for which $\rho<\rho_0-2$. Proceeding in this way we eventually obtain

$$l_1\psi_1^0(x_{\rho_0})+l_2\Psi_2^0(x_{\rho_0})+\ldots+l_{c_1}\Psi^0(x_{\rho_0})=0,$$

which cannot be true by our choice of the functions involved.

§ 8. Next write $r=\iota-1$ in (27) and (28); and let $p-k_2$ be the order of the first minors of the latter which do not all vanish. We can then express $p-k_2$ of the functions

$$f_{\iota-1}(s_1, t),\ f_{\iota-1}(s_2, t),\ \ldots,\ f_{\iota-1}(s_p, t) \quad\ldots\ldots(32)$$

as linear functions of the remaining k_2, among which no linear relation can exist. Evidently therefore,

$$\Psi_r^1(t) \qquad (r=1, 2, \ldots, c_1) \quad\ldots\ldots(33),$$

* The minors are supposed arranged in increasing order.

are linear combinations of these k_2 functions; moreover, from what has been said in § 6, it appears that

$$\Psi_r^0(t) \qquad (r=1, 2, \dots, c_1) \quad \dots\dots\dots\dots(34)$$

are also linear combinations of them. Since these $2c_1$ functions (33) and (34) are linearly independent, we must have $k_2 \geqslant 2c_1$, say $k_2 = 2c_1 + c_2$; also the functions (32) must all be expressible as linear combinations of (33), (34) and a certain c_2 selected from the set of k_2 above mentioned, say

$$f_{\iota-1}(s_{\kappa_{c_1+1}}, t), \quad f_{\iota-1}(s_{\kappa_{c_1+2}}, t), \quad \dots, \quad f_{\iota-1}(s_{\kappa_{c_1+c_2}}, t).$$

Referring to the results of § 7, we see that

$$f_\iota(s_{\kappa_{c_1+1}}, t), \quad f_\iota(s_{\kappa_{c_1+2}}, t), \quad \dots, \quad f_\iota(s_{\kappa_{c_1+c_2}}, t)$$

can be expressed as linear functions of (34), say

$$f_\iota(s_{\kappa_{c_1+q}}, t) = \sum_{r=1}^{c_1} l_{qr} \Psi_r^0(t).$$

Accordingly, if we suppose that

$$\Psi^0_{c_1+q}(t) = f_{\iota-1}(s_{\kappa_{c_1+q}}, t) - \sum_{r=1}^{c_1} l_{qr} \Psi_r^1(t),$$

we shall have

$$\int_a^b \Psi^0_{c_1+q}(x) f_2(x, t) = 0; \qquad (q=1, 2, \dots, c_2)$$

and the functions (32) can be expressed as linear combinations of the $2c_1 + c_1$ functions

$$\Psi_r^1(t) \quad (r=1, 2, \dots, c_1), \qquad \Psi_r^0(t) \quad (r=1, 2, \dots, c_1+c_2) \quad \dots\dots\dots(35).$$

These functions must be linearly independent; for, otherwise, the functions (32) would be linearly expressible in terms of a number of functions $< k_2$—which would necessitate the minors of (28) of order $p - k_2$ being all zero.

Again, if

$$\Psi^\rho_{c_1+q}(t) = f_{\iota-\rho-1}(s_{\kappa_{c_1+q}}, t) - \sum_{r=1}^{c_1} l_{qr} \Psi_r^{\rho+1}(t) \qquad \begin{matrix}(q=1, 2, \dots, c_2)\\ (\rho=0, 1, \dots, \iota-2)\end{matrix} \quad \dots(36),$$

the set of $\iota c_1 + (\iota - 1) c_2$ functions (29) and (36) are linearly independent. For, just as in the corresponding case in § 7, we have the relations

$$\int_a^b \Psi^\rho_{c_1+q}(x) f_1(x, t)\, dx = \Psi^\rho_{c_1+q}(t),$$

$$\begin{aligned}\int_a^b \Psi^\rho_{c_1+q}(x) f_2(x, t)\, dx &= \Psi^{\rho-1}_{c_1+q}(t) \qquad (\rho \geqslant 1)\\ &= 0 \qquad (\rho = 0).\end{aligned}$$

We may prove from this that any linear relation connecting the function (29) and (36) would evolve a non-evanescent relation connecting (35): this being impossible by what we have said, the result follows.

§ 9. The method of procedure will now be fairly obvious. We next write $r = \iota - 2$ in the determinant (28) and suppose that the minors of order $(p - k_3)$ are the first which do not all vanish. We can then express $(p - k_3)$ of the functions

$$f_{\iota-2}(s_1, t), \quad f_{\iota-2}(s_2, t), \quad \dots, \quad f_{\iota-2}(s_p, t) \quad \dots\dots\dots\dots(37)$$

as linear functions of the remaining k_3; and these latter are linearly independent. But the functions

$$\Psi_r^2(t) \quad (r=1, 2, \ldots, c_1), \qquad \Psi_r^1(t) \quad (r=1, 2, \ldots, c_1+c_2),$$
$$\Psi_r^0(t) \quad (r=1, 2, \ldots, c_1+c_2) \ldots (38)$$

are linearly independent, and (§§ 6, 7) are linear combinations of the functions (37). It follows that $k_3 \geqslant 3c_1+2c_2$, say $k_3 = 3c_1+2c_2+c_3$; and that the functions (37) can be expressed as linear functions of (38) and a certain c_3 of themselves, say

$$f_{\iota-2}(s_{\kappa_{c_1+c_2+1}}, t),\ f_{\iota-2}(s_{\kappa_{c_1+c_2+2}}, t),\ \ldots,\ f_{\iota-2}(s_{\kappa_{c_1+c_2+c_3}}, t).$$

Now we proved in the preceding paragraph that

$$f_{\iota-1}(s_{\kappa_{c_1+c_2+1}}, t),\ f_{\iota-1}(s_{\kappa_{c_1+c_2+2}}, t),\ \ldots,\ f_{\iota-1}(s_{\kappa_{c_1+c_2+c_3}}, t)$$

are linear functions of (38), say

$$f_{\iota-1}(s_{\kappa_{c_1+c_2+q}}, t) = \sum_{r=1}^{c_1} m_{qr}\Psi_r^1(t) + \sum_{r=1}^{c_1+c_2} n_{qr}\Psi_r^0(t) \ldots\ldots\ldots\ldots\ldots\ldots (39).$$

It follows therefore that, if

$$\Psi^\rho_{c_1+c_2+q}(t) = f_{\iota-\rho-2}(s_{\kappa_{c_1+c_2+q}}, t) - \sum_{r=1}^{c_1} m_{qr}\Psi_r^{\rho+2}(t) - \sum_{r=1}^{c_1+c_2} n_{qr}\Psi_r^{\rho+1}(t)$$
$$\begin{pmatrix} q = 1, 2, \ldots, c_3 \\ \rho = 0, 1, \ldots, \iota-3 \end{pmatrix} \ldots (40),$$

the functions (37) can all be expressed as linear combinations of the $3c_1+2c_2+c_3$ functions

$$\Psi_r^2(t) \quad (r=1, 2, \ldots, c_1), \qquad \Psi_r^1(t) \quad (r=1, 2, \ldots, c_1+c_2),$$
$$\Psi_r^0(t) \quad (r=1, 2, \ldots, c_1+c_2+c_3) \ldots (41);$$

also, by (12), (14) and (39), we have the relations

$$\int_a^b \Psi^\rho_{c_1+c_2+q}(x)\, f_1(x, t)\, dx = \Psi^\rho_{c_1+c_2+q}(t),$$
$$\int_a^b \Psi^\rho_{c_1+c_2+q}(x)\, f_2(x, t)\, dx = \Psi^{\rho-1}_{c_1+c_2+q}(t) \qquad (\rho \geqslant 1)$$
$$= 0 \qquad (\rho = 0).$$

Lastly the linear independence of the functions (41), and thence of (29), (36) and (40), follows by reasoning quite analogous to that employed in §§ 7, 8.

§ 10. Proceeding on these lines, it is evident that at the final stage we obtain the result, that the functions

$$f_1(s_1, t),\ f_1(s_2, t),\ \ldots,\ f_1(s_p, t) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

are each linear combinations of $k_\iota (\leqslant p)$ linearly independent functions

$$\Psi_r^\iota(t)\ (r=1, 2, \ldots, c_1),\ \ \Psi_r^{\iota-1}(t)\ (r=1, 2, \ldots, c_1+c_2),\ \ \Psi_r^{\iota-2}(t)\ (r=1, 2, \ldots, c_1+c_2+c_3), \ldots,$$
$$\Psi_r^0(t)\ (r=1, 2, \ldots, c_1+c_2 \ldots + c_\iota) \ldots (43),$$

where $$k_\iota = \iota c_1 + (\iota-1)c_2 + (\iota-2)c_3 + \ldots + c_\iota,$$

and we have

$$\left.\begin{aligned}\int_a^b \Psi_r^\rho(x) f_1(x, t)\,dx &= \Psi_r^\rho(t), \\ \int_a^b \Psi_r^\rho(x) f_2(x, t)\,dx &= \Psi_r^{\rho-1}(t) \qquad (\rho \geqslant 1) \\ &= 0 \qquad\qquad (\rho = 0)\end{aligned}\right\} \text{......................(44).}$$

We have seen that (42) are linearly independent; they cannot therefore be expressed as linear combinations of less than p functions. As we have already seen that $k_\iota \leqslant p$, it follows that

$$k_\iota = p.$$

§ 11. Returning now to (23), when we supply the values of the functions $f_1(s_r, t)$ in terms of the set (43), and collect the terms which involve each of the functions $\Psi_r^\rho(t)$, we obtain

$$f_1(s, t) = \sum_{\rho=0}^{\iota-1} \sum_{r=d_\rho+1}^{d_{\rho+1}} (\Phi_r^0(s)\Psi_r^{\iota-(\rho+1)}(t) + \Phi_r^1(s)\Psi_r^{\iota-(\rho+2)}(t) + \ldots + \Phi_r^{\iota-(\rho+1)}(s)\Psi_r^0(t)) \ldots (45),$$

where

$$\begin{aligned} d_\rho &= c_1 + c_2 + \ldots + c_\rho \qquad (\rho > 0) \\ &= 0 \qquad\qquad\qquad\quad (\rho = 0), \end{aligned}$$

and the functions $\Phi_r^\rho(s)$ are each linear combinations of

$$\phi_1(s), \quad \phi_2(s), \quad \ldots, \quad \phi_p(s).$$

We see from (45) that the p functions

$$f_1(s, s_1), \quad f_1(s, s_2), \quad \ldots, \quad f_1(s, s_p) \text{ .... (46)}$$

can be expressed as linear combinations of the p functions $\Phi_r^\rho(s)$. Consequently, if the latter were not linearly independent, we could express all the functions (46) in terms of a number $< p$ of them. But this would necessitate a linear relation connecting (46), which we have seen to be impossible (§ 4). It follows that the functions $\Phi_r^\rho(s)$, like the functions $\Psi_r^\rho(t)$, are linearly independent.

In the equation (45) replace t by x, multiply along by $f_2(x, t)$ and integrate with respect to x between the limits a and b. Using the relations (14) and (44), we obtain

$$f_2(s, t) = \sum_{\rho=0}^{\iota-2} \sum_{r=d_\rho+1}^{d_{\rho+1}} (\Phi_r^0(s)\Psi_r^{\iota-(\rho+2)}(t) + \Phi_r^1(s)\Psi_r^{\iota-(\rho+3)}(t) + \ldots + \Phi_r^{\iota-(\rho+2)}(s)\Psi_r^0(t)) \ldots (47).$$

Again, if we write x instead of t in this relation, multiply by $f_2(x, t)$ and integrate as before we obtain

$$f_3(s, t) = \sum_{\rho=0}^{\iota-3} \sum_{r=d_\rho+1}^{d_{\rho+1}} (\Phi_r^0(s)\Psi_r^{\iota-(\rho+3)}(t) + \Phi_r^1(s)\Psi_r^{\iota-(\rho+4)}(t) + \ldots + \Phi_r^{\iota-(\rho+3)}(s)\Psi_r^0(t));$$

and generally, by repeated use of this process, we have

$$f_q(s, t) = \sum_{\rho=0}^{\iota-q} \sum_{r=d_\rho+1}^{d_{\rho+1}} (\Phi_r^0(s)\Psi_r^{\iota-(\rho+q)}(t) + \Phi_r^1(s)\Psi_r^{\iota-(\rho+q+1)}(t) + \ldots + \Phi_r^{\iota-(\rho+q)}(s)\Psi_r^0(t)),$$

which is a canonical form for the coefficient of $(\lambda_0 - \lambda)^{-q}$ in the expansion (8).

§ 12. The principal part of the expansion of $K_\lambda(s, t)$ at its pole λ_0 is (§ 1)

$$\frac{f_\iota(s, t)}{(\lambda_0 - \lambda)^\iota} + \frac{f_{\iota-1}(s, t)}{(\lambda_0 - \lambda)^{\iota-1}} + \dots + \frac{f_1(s, t)}{(\lambda_0 - \lambda)};$$

accordingly if we supply the values of the various coefficients, obtained in the preceding paragraph, we shall obtain a canonical form for this principal part. The result is really equivalent to Plemelj's, and may be at once brought into line with it by noticing that this principal part, written in its canonical form, consists of that portion of

$$\sum_{\rho=0}^{\iota-1} \frac{\sum\limits_{r=d_\rho+1}^{d_{\rho+1}} \Phi_r(s)\Psi_r(t)}{(\lambda_0-\lambda)^{\iota-\rho}} \qquad (48),$$

which does *not* remain finite for $\lambda = \lambda_0$, where it is supposed that

$$\Phi_r(s) = \Phi_r^0(s) + (\lambda_0 - \lambda)\Phi_r^1(s) + \dots + (\lambda_0 - \lambda)^{\iota-\rho-1}\Phi_r^{\iota-\rho-1}(s),$$
$$\Psi_r(t) = \Psi_r^0(t) + (\lambda_0 - \lambda)\Psi_r^1(t) + \dots + (\lambda_0 - \lambda)^{\iota-\rho-1}\Psi_r^{\iota-\rho-1}(t).$$

There are certain relations between the functions $\Phi_r^\rho(s)$, $\Psi_n^\nu(t)$, which follow from (44). For, if we supply the canonical form of $f_1(x, t)$ in the first of these*, we obtain

$$\sum_{\rho=0}^{\iota-1}\sum_{r=d_\rho+1}^{d_{\rho+1}} \left(\Psi_r^{\iota-(\rho+1)}(t)\int_a^b \Phi_r^0(x)\Psi_n^\nu(x)\,dx + \Psi_r^{\iota-(\rho+2)}(t)\int_a^b \Phi_r^1(x)\Psi_n^\nu(x)\,dx\right.$$
$$\left. + \dots \Psi_r^0(t)\int_a^b \Phi_r^{\iota-(\rho+1)}(x)\Psi_n^\nu(x)\,dx\right) = \Psi_n^\nu(t).$$

Remembering the linear independence of the functions $\Psi_r^\rho(t)$, we see that $d_\rho < r \leqslant d_{\rho+1}$

$$\int_a^b \Phi_r^{\iota-(\rho+\mu+1)}(x)\Psi_n^\nu(x)\,dx = 1 \qquad (n = r),\ (\mu = \nu),$$

and
$$= 0$$
in all other cases.

§ 13. From the set of equations (9) take that one for which $r = 1$, and supply the values of $f_1(s, t)$ and $f_2(s, t)$ obtained in § 11. Recalling the fact that the functions $\Psi_r^\rho(t)$ are linearly independent, it is easily seen that, among other relations, we have

$$\Phi_r^0(s) = \lambda_0\int_a^b \kappa(s, x)\Phi_r^0(x)\,dx. \qquad (r = 1, 2, \dots, d_\iota).$$

Thus the d_ι linearly independent functions $\Phi_r^0(s)$ are solutions of (5).

Suppose now that we have any solution whatever, $\phi(s)$, of (5). Multiply along the equation

$$\phi(s) = \lambda_0\int_a^b \kappa(s, x)\phi(x)\,dx,$$

by $K_\lambda(\sigma, s)$, and integrate with respect to s between the limits a and b. By means of the second of the characteristic relations (3), we obtain

$$\phi(\sigma) = (\lambda_0 - \lambda)\int_a^b K_\lambda(\sigma, x)\phi(x)\,dx.$$

* It is supposed that ρ, r are replaced by ν, n respectively.

When we supply the expansion (8) on the right, we see that

$$\left.\begin{aligned}\int_a^b f_2(\sigma, x)\,\phi(x)\,dx &= 0,\\ \int_a^b f_1(\sigma, x)\,\phi(x)\,dx &= \phi(\sigma)\end{aligned}\right\}\quad\ldots\ldots(49).$$

From the first of these and (47) it appears that

$$\int_a^b \Psi_r^\rho(x)\,\phi(x)\,dx = 0,$$

unless $\Psi_r^\rho(t)$ is a multiplier of one of the functions $\Phi_r^0(s)$ in (45). Accordingly the second of the equations (49) shows that $\phi(\sigma)$ is a linear function of the d_ι functions $\Phi_r^0(\sigma)$. We have therefore proved that the number of linearly independent solutions of (5) is d_ι*, and we have seen that this is the number of terms in (48). It follows that the results obtained by this method are identical with Plemelj's.

* It is easily proved in a similar manner that this is the number of linearly independent solutions of (6).

VI. *The Operator Reciprocants of Sylvester's Theory of Reciprocants.*

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INTRODUCTION.

The subject of the change of the independent variable, when two variables are concerned, has been discussed in connection with a function of the differential coefficients

$$\frac{dy}{dx}, \ \frac{d^2y}{dx^2}, \ \frac{d^3y}{dx^3}, \ldots,$$

say

$$f\left(\frac{dy}{dx}, \ \frac{d^2y}{dx^2}, \ \frac{d^3y}{dx^3}, \ldots\right),$$

and the object of Sylvester, Hammond, Elliott, Rogers and Leudesdorf and others in Great Britain, including the present author, in researches some twenty years ago written in connection with the subject of reciprocants, was in the first place to evolve a theory of the functions which remain unaltered when the variables are simply interchanged, and in the second place to separate such forms into their categories, according as the invariability was maintained under linear and homographic transformations of different natures. The work thus intersected and elucidated that of Lie, Halphen and others on differential invariants and groups of transformations. In the course of the investigations various differential operators presented themselves as being effective, either as annihilators or generators or reversors, as had already been found to be the case in other invariant theories. These differential operators appear as effective instruments in the developments, but the true cause of this does not appear to have been reached. It is the object of this research to shew that the reason is that the instruments in question have in fact the same properties as the forms upon which they effectively operate, and that the forms which are invariant and the operators are properly to be regarded as *one* system of forms possessing the same properties. The present research is concerned with the development of this idea, and its particular application to the theory of reciprocants and differential invariants, and it will appear that the principle by which the "content" of this theory is enlarged is equally applicable to other invariant theories.

With regard to two variables, if the substitution be periodic, the invariant theory of the substitution is simple.

If for example

$$x=f(X, Y), \quad y=\phi(X, Y)$$

leads to

$$X=f(x, y), \quad Y=\phi(x, y),$$

and we write in a usual notation

$$\frac{dy}{dx}=t, \quad \frac{1}{2!}\frac{d^2y}{dx^2}=a_0, \quad \frac{1}{3!}\frac{d^3y}{dx^3}=a_1, \ldots$$

$$\frac{dY}{dX}=\tau, \quad \frac{1}{2!}\frac{d^2Y}{dX^2}=\alpha_0, \quad \frac{1}{3!}\frac{d^3Y}{dX^3}=\alpha_1, \ldots,$$

and

$$f_1(x, y, t, a_0, a_1, \ldots)=\phi_1(X, Y, \tau, \alpha_0, \alpha_1, \ldots),$$

it is clear that

$$f_1(x, y, t, a_0, a_1, \ldots) \pm \phi_1(x, y, t, a_0, a_1, \ldots)$$

is an absolute invariant of the given substitution of even or uneven order, according as the upper or lower sign be taken.

The most general substitution of this kind may be given one or other of the forms

$$\left.\begin{array}{l} x=f(X, Y) \\ X=f(x, y) \end{array}\right\}, \quad \left.\begin{array}{l} y=\phi(X, Y) \\ Y=\phi(x, y) \end{array}\right\},$$

and may therefore be defined by means of a single function f or ϕ. More generally we may take the canonical form of substitution to be

$$\left.\begin{array}{l} f(x, y)=\phi(X, Y) \\ f(X, Y)=\phi(x, y) \end{array}\right\},$$

for this leads by solution for x and y to a periodic substitution of the kind under consideration. The Sylvester substitution (the mere interchange of the variables) is then written

$$\left.\begin{array}{l} x=Y \\ X=y \end{array}\right\},$$

and the Halphenian substitution

$$x=\frac{1}{X}, \quad y=\frac{Y}{X}$$

in the form

$$\left.\begin{array}{l} y=\dfrac{Y}{X} \\ Y=\dfrac{y}{x} \end{array}\right\}.$$

It is my intention to further develop the invariant theories of these substitutions by including the theory of invariant differential operators.

I enlarge the subject-matter by adding to the elements to be dealt with the differential operations

$$\partial_x, \partial_y, \partial_t, \partial_{a_0}, \partial_{a_1}, \ldots,$$

and observe that if

$$f_1(x, y, t, a_0, a_1, \ldots \partial_x, \partial_y, \partial_t, \partial_{a_0}, \partial_{a_1}, \ldots)$$

$$=\phi_1(X, Y, \tau, \alpha_0, \alpha_1, \ldots \partial_X, \partial_Y, \partial_\tau, \partial_{\alpha_0}, \partial_{\alpha_1} \ldots),$$

$$f_1(x, y, t, a_0, a_1, \ldots \partial_x, \partial_y, \partial_t, \partial_{a_0}, \partial_{a_1}, \ldots)$$

$$\pm \phi_1(x, y, t, a_0, a_1, \ldots \partial_x, \partial_y, \partial_t, \partial_{a_0}, \partial_{a_1}, \ldots),$$

is an absolute invariant of the substitution whose order is even or uneven according as the upper or lower sign is taken.

It is my intention to particularly examine the invariant theories of the substitutions of Sylvester and Halphen, to shew how they may be treated in a simplified manner and to shew the remarkable parallelism that exists between them up to the point where they merge into the theory of the invariants of the homographic substitution

$$x = \frac{\lambda_1 X + \mu_1 Y + \nu_1}{\lambda X + \mu Y + \nu}, \quad y = \frac{\lambda_2 X + \mu_2 Y + \nu_2}{\lambda X + \mu Y + \nu}.$$

The Sylvester Substitution.

Art. 1. The substitution is

$$\left.\begin{aligned} x &= Y \\ X &= y \end{aligned}\right\},$$

and the reader is supposed to be familiar with the work of Sylvester and others on the theory of reciprocants.

To explain the notation that I have found it convenient to adopt, I suppose ξ, η to be corresponding increments of x, y, so that by Taylor's theorem

$$\eta = t\xi + a_0\xi^2 + a_1\xi^3 + \ldots = t_\xi = t\xi + a_\xi,$$
$$\xi = \tau\eta + \alpha_0\eta^2 + \alpha_1\eta^3 + \ldots = \tau_\eta = \tau\eta + \alpha_\eta,$$

and I write

$$\frac{1}{s}\eta^s = \frac{1}{s}(t\xi + a_0\xi^2 + a_1\xi^3 + \ldots)^s = t_{s0}\xi^s + t_{s1}\xi^{s+1} + t_{s2}\xi^{s+2} + \ldots,$$
$$\frac{1}{s}(\eta - t\xi)^s = \frac{1}{s}(a_0\xi^2 + a_1\xi^3 + a_2\xi^4 + \ldots)^s = a_{s0}\xi^{2s} + a_{s1}\xi^{2s+1} + a_{s2}\xi^{2s+2} + \ldots,$$
$$\frac{1}{s}\xi^s = \frac{1}{s}(\tau\eta + \alpha_0\eta^2 + \alpha_1\eta^3 + \ldots)^s = \tau_{s0}\eta^s + \tau_{s1}\eta^{s+1} + \tau_{s2}\eta^{s+2} + \ldots,$$
$$\frac{1}{s}(\xi - \tau\eta)^s = \frac{1}{s}(\alpha_0\eta^2 + \alpha_1\eta^3 + \alpha_2\eta^4 + \ldots)^s = \alpha_{s0}\eta^{2s} + \alpha_{s1}\eta^{2s+1} + \alpha_{s2}\eta^{2s+2} + \ldots.$$

I, in fact, introduce a double-suffix notation, thereby enabling the results to be extended and their expression to be simplified.

Observe that

$$t_{10} = t, \ t_{11} = a_{10} = a_0, \ t_{12} = a_{11} = a_1, \ldots \ t_{1,m} = a_{1,m-1} = a_{m-1}.$$
$$\tau_{10} = \tau, \ \tau_{11} = \alpha_{10} = \alpha_0, \ \tau_{12} = \alpha_{11} = \alpha_1, \ldots \ \tau_{1,m} = \alpha_{1,m-1} = \alpha_{m-1}.$$

I refer to Elliott's valuable paper of 1886* on the interchange of the variables, and remark that I have necessarily to traverse much of the same ground that he did; but differences will be found and the object in view is different.

Art. 2. The formulae for the change of the independent variable are

$$\tau = \frac{1}{t},$$

$$\alpha_0 = -\frac{a_0}{t^3},$$

$$\alpha_1 = -\frac{a_1}{t^4} + \frac{2a_0^2}{t^5},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

* *Proceedings of the Lond. Math. Soc.* vol XVIII. p. 142 *et seq.*

and generally, in the double-suffix notation,

$$\alpha_s = \alpha_{1s} = -t^{-2s-3}\left\{t^s a_{1s} - \binom{s+3}{1} t^{s-1} a_{2,s-1} + \binom{s+4}{2} t^{s-2} a_{3,s-2} - \ldots\right\}.$$

By easy algebra we reach the more general formula

$$\alpha_{rs} = (-)^r t^{-2s-3r}\left\{t^s a_{rs} - \binom{s+2r+1}{1} t^{s-1} a_{r+1,s-1} + \binom{s+2r+2}{2} t^{s-2} a_{r+2,s-2} - \ldots\right\},$$

wherein s must be a positive integer, but r is not so restricted.

Art. 3. When $r=0$, we must write

$$\log\frac{a_\xi}{a_0\xi^2} = \log\frac{\eta - t\xi}{a_0\xi^2} = \alpha_{01}\xi + \alpha_{02}\xi^2 + \alpha_{03}\xi^3 + \ldots,$$

and then
$$\alpha_{0s} = t^{-2s}\left\{t^s a_{0s} - \binom{s+1}{1} t^{s-1} a_{1,s-1} + \binom{s+2}{2} t^{s-2} a_{2,s-2} - \ldots\right\},$$

a formula of some importance.

Art. 4. If $F(\alpha_{10}, \alpha_{11}, \alpha_{12}, \ldots)$ be a homogeneous and isobaric function of $\alpha_{10}, \alpha_{11}, \alpha_{12}, \ldots$ of degree i and weight w, we know that

$$F(\alpha_{10}, \alpha_{11}, \alpha_{12}, \ldots) = (-)^i t^{-\mu} e^{-\frac{V}{t}} F(a_{10}, a_{11}, a_{12}, \ldots),$$

where $\mu = 3i + w$,

$$V = 4a_{20}\partial_{a_{10}} + 5a_{21}\partial_{a_{11}} + 6a_{22}\partial_{a_{12}} + \ldots,$$

the well-known operator which annihilates pure reciprocants.

It is easy to proceed to a generalization, and shew that

$$F(\alpha_{r0}, \alpha_{r1}, \alpha_{r2}, \ldots)$$

$$= (-)^i t^{-\mu} e^{-\frac{V_r}{t}} F(a_{r0}, a_{r1}, a_{r2}, \ldots),$$

where
$$V_r = (2r+2)\, a_{r+1,0}\,\partial_{a_{r1}} + (2r+3)\, a_{r+1,1}\,\partial_{a_{r2}} + (2r+4) a_{r+1,2}\,\partial_{a_{r3}} + \ldots$$

(becoming $V_1 = V$ when $r=1$)*.

This result shows that every function of

$$a_{r0}, a_{r1}, a_{r2}, \ldots,$$

which satisfies the partial differential equation $V_r = 0$ is a pure reciprocant.

In particular when $r=0$

$$a_{0p} = (-)^{p+1}\frac{1}{p}s_p,$$

where s_p is the sum of the pth powers of the root of the equation

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots = 0,$$

and the partial differential equations satisfied by pure reciprocants is

$$a_0\partial_{s_1} - 3a_1\partial_{s_2} + 6a_2\partial_{s_3} - 10a_3\partial_{s_4} + \ldots = 0.$$

* In fact if
$$U = a_0 + a_1u + a_2u^2 + \ldots,$$
$$V\frac{1}{r}U^r = U^r\partial_u(u^2U),$$

and thence
$$Va_{rs} = (2r+s+1)\, a_{r+1,s-1}.$$

Writing $s_{2m} = - s_{2m}'$ this is

$$a_0 \partial_{s_1} + 3a_1 \partial_{s_2'} + 6a_2 \partial_{s_3} + 10 a_3 \partial_{s_4'} + \ldots = 0,$$

and it will be found that the seminvariant operator is

$$\partial_{s_1} + 2s_1 \partial_{s_2'} + 3s_2' \partial_{s_3} + 4s_3 \partial_{s_4'} + \ldots.$$

Art. 5. In a paper of 1886* I discussed the operator which in the notation explained above may be written

$$\mu a_{m0} \partial_{a_n} + (\mu + \nu) a_{m1} \partial_{a_{n+1}} + (\mu + 2\nu) a_{m2} \partial_{a_{n+2}} + \ldots,$$

and when convenient denoted by

$$(\mu, \nu;\ m, n)_a.$$

Other operators that frequently present themselves are included in

$$\mu t_{m0} \partial_{a_n} + (\mu + \nu) t_{m1} \partial_{a_{n+1}} + (\mu + 2\nu) t_{m2} \partial_{a_{n+2}} + \ldots,$$

which may be denoted by

$$(\mu, \nu;\ m, n)_t.$$

It is to be observed that m may be any real number, positive or negative, and is not restricted to be an integer.

The numbers μ, ν are unrestricted numbers.

The number n may be -1, 0 or any positive integer.

The particular case corresponding to $m = 0$ will be dealt with in the proper place.

The only other operator worthy of attention for the present purpose is

$$\mu s_1 \partial_{a_n} - (\mu + \nu) s_2 \partial_{a_{n+1}} + (\mu + 2\nu) s_3 \partial_{a_{n+2}} - \ldots,$$

where

$$\log\left(1 + \frac{a_1}{a_0} u + \frac{a_2}{a_0} u^2 + \ldots\right) = s_1 u - \tfrac{1}{2} s_2 u^2 + \tfrac{1}{3} s_3 u^3 - \ldots.$$

The particular property and values of these operators consists in the fact that the eliminant of any two of them is an operator of the same nature. They thus constitute a group of infinite order.

Art. 6. Recalling the formulae

$$\eta = t\xi + a_0 \xi^2 + a_1 \xi^3 + \ldots = t_\xi = t\xi + a_\xi,$$

$$\eta = \tau\eta + \alpha_0 \eta^2 + \alpha_1 \eta^3 + \ldots = \tau_\eta = \tau\eta + \alpha_\eta,$$

it may be said that Elliott, in the paper quoted, shewed (with a non-essential change of his symbolism) that in any relation connecting ξ and η, which is such that we have terms involving

$$\xi^s \text{ and } \eta^r \frac{d\eta}{d\xi} \text{ only,}$$

we are at liberty to substitute

$$\partial_{a_{s-2}} \text{ for } \xi^s \text{ and } -\partial_{\alpha_{r-2}} \text{ for } \eta^r \frac{d\eta}{d\xi} \text{ respectively.}$$

* The Theory of a Multilinear Partial Differential Operator, with Applications to the Theories of Invariants and Reciprocants. *Proc. L.M.S.* Vol. XVIII. 1886.

Any relation is so expressible because

$$\eta^r = \eta^r \frac{d\eta}{d\xi} \cdot \frac{d\xi}{d\eta},$$

where

$$\frac{d\xi}{d\eta} = \tau + 2\alpha_0\eta + 3\alpha_1\eta^2 + \ldots.$$

It is convenient to denote $\frac{d\eta}{d\xi}$ by η'.

Transformation of ∂_{a_s}.

Art. 7. This is the theorem of Elliott, which I desire to exhibit in various forms.

Since

$$\xi^{s+2} = (s+2)(\tau_{s+2,0}\eta^{s+2} + \tau_{s+2,1}\eta^{s+3} + \ldots),$$

$$\frac{d\xi}{d\eta} = \tau + 2\alpha_0\eta + 3\alpha_1\eta^2 + \ldots,$$

we see that

$$\xi^{s+2} = (s+2)(\tau_{s+2,0}\eta^{s+2} + \tau_{s+2,1}\eta^{s+3} + \ldots)(\tau + 2\alpha_0\eta + 3\alpha_1\eta^2 + \ldots)\eta',$$

and by easy algebra

$$\xi^{s+2} = \{(s+3)\tau_{s+3,0}\eta^{s+2} + (s+4)\tau_{s+3,1}\eta^{s+3} + \ldots\}\eta',$$

whence writing

$$\partial_{a_{s-2}} \text{ for } \xi^s \text{ and } -\partial_{a_{s-2}} \text{ for } \eta^s\eta',$$

$$\partial_{a_s} = -(s+3)\tau_{s+3,0}\partial_{a_s} - (s+4)\tau_{s+3,1}\partial_{a_{s+1}} - \ldots,$$

or

$$\partial_{a_s} = -(s+3, 1;\ s+3, s)_\tau,$$

and putting $s = -1$,

$$\partial_t = -(2, 1;\ 2, -1)_\tau,$$

and arranging the dexter in powers of τ

$$\partial_t = -\tau^2\partial_\tau - \tau(3, 1;\ 1, 0)_\alpha - (4, 1;\ 2, 1)_\alpha,$$

where the last operator written is the pure reciprocant annihilator in respect of the elements

$$\alpha_0, \alpha_1, \alpha_2, \ldots,$$

and may be written V_α.

I also write

$$\begin{aligned} I_\tau &= \tau\partial_\tau + \alpha_0\partial_{\alpha_0} + \alpha_1\partial_{\alpha_1} + \alpha_2\partial_{\alpha_2} + \ldots, \\ W_\tau &= -\tau\partial_\tau + \alpha_1\partial_{\alpha_1} + 2\alpha_2\partial_{\alpha_2} + \ldots, \\ I_\alpha &= \alpha_0\partial_{\alpha_0} + \alpha_1\partial_{\alpha_1} + \alpha_2\partial_{\alpha_2} + \ldots, \\ W_\alpha &= \alpha_1\partial_{\alpha_1} + 2\alpha_2\partial_{\alpha_2} + \ldots, \end{aligned}$$

and then the above result may be written either as

$$\partial_t = -\tau^2\partial_\tau - \tau(3I_\alpha + W_\alpha) - V_\alpha,$$

or as

$$\partial_t = +\tau^2\partial_\tau - \tau(3I_\tau + W_\tau) - V_\alpha.$$

Art. 8. We may interchange the letters

$$t, a_0, a_1, \ldots \text{ with } \tau, \alpha_0, \alpha_1, \alpha_2, \ldots,$$

and thence we find the operator absolute reciprocants

$$(1 - t^2)\partial_t - t(3I_a + W_a) - V_a,$$

$$(1 + t^2)\partial_t - t(3I_t + W_t) - V_a,$$

of even order; and

$$(1 + t^2)\partial_t + t(3I_a + W_a) + V_a,$$

$$(1 - t^2)\partial_t + t(3I_t + W_t) + V_a,$$

of uneven order.

These results are typical of those that will be obtained in this paper. It is an obvious remark that any operator absolute reciprocant will produce an absolute reciprocant as the result of operating upon any absolute reciprocant whatever.

Other forms of operator absolute reciprocants are invariably obtainable from such transformations as

$$\partial_t = -\tau^2\partial_\tau - \tau(3I_a + W_a) - V_a,$$

$$\partial_t = +\tau^2\partial_\tau - \tau(3I_\tau + W_\tau) - V_a.$$

Ex. gr., multiply the first relation by t so that

$$t\partial_t = -\tau\partial_\tau - (3I_a + W_a) - \frac{V_a}{\tau},$$

and we arrive at the operator absolute reciprocants

$$(3I_a + W_a) + \frac{V_a}{t} \text{ of even order,}$$

$$2t\partial_t + (3I_a + W_a) + \frac{V_a}{t} \text{ of uneven order.}$$

Similarly we obtain the absolute reciprocants

$$2t\partial_t - (3I_t + W_t) - \frac{V_a}{t} \text{ of even order,}$$

$$(3I_t + W_t) + \frac{V_a}{t} \text{ of uneven order.}$$

When the operand is a homogeneous and isobaric absolute reciprocant

$$3I_t + W_t = 0,$$

and then

$$2t\partial_t - \frac{V_a}{t}, \qquad \frac{V_a}{t},$$

are generators which produce homogeneous and isobaric absolute reciprocants of the same and contrary orders respectively.

Art. 9. The dexter of the result

$$\partial_{a_s} = -(s+3,\ 1;\ s+3,\ s)_\tau$$

may be expanded in ascending powers of τ with the result

$$\begin{aligned}
\partial_{a_s} = & -\tau^{s+3}\partial_{a_s} \\
& -\binom{s+2}{0}\tau^{s+2}(s+4,\ 1;\ 1,\ s+1)_a \\
& -\binom{s+2}{1}\tau^{s+1}(s+5,\ 1;\ 2,\ s+2)_a \\
& -\binom{s+2}{2}\tau^{s}(s+6,\ 1;\ 3,\ s+3)_a \\
& - \dots\dots\dots\dots\dots\dots \\
& -\binom{s+2}{s+2}(2s+6,\ 1;\ s+3,\ 2s+3)_a.
\end{aligned}$$

Art. 10. An intermediate result is

$$\partial_{a_s}\alpha_n = -(n+3)\,\tau^{2s-n+3}\left\{\tau^{n-s-1}\alpha_{1,n-s-1} + \binom{s+2}{1}\tau^{n-s-2}\alpha_{2,n-s-2} + \ldots + \binom{s+2}{n-s-1}\alpha_{n-s,0}\right\}$$

leading to the formula

$$\begin{aligned}\partial_{a_s} = &-\tau^{s+3}\partial_{\alpha_s} - (s+4)\,\tau^{s+2}\alpha_{10}\partial_{\alpha_{s+1}} - (s+5)\,\tau^{s+1}\left\{\tau\alpha_{11} + \binom{s+2}{1}\alpha_{20}\right\}\partial_{\alpha_{s+2}}\\ &-(s+6)\,\tau^{s}\left\{\tau^2\alpha_{12} + \binom{s+2}{1}\tau\alpha_{21} + \binom{s+2}{2}\alpha_{30}\right\}\partial_{\alpha_{s+3}}\\ &-(s+7)\,\tau^{s-1}\left\{\tau^3\alpha_{13} + \binom{s+2}{1}\tau^2\alpha_{22} + \binom{s+2}{2}\tau\alpha_{31} + \binom{s+2}{3}\alpha_{40}\right\}\partial_{\alpha_{s+4}}\\ &-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

The accented Notation.

Art. 11. The notation becomes much simplified, in cases when we deal with homogeneous and isobaric algebraic quantities and operations by transforming to accented letters

$$t',\ a_0',\ a_1',\ldots\ \tau',\ \alpha_0',\ \alpha_1',\ldots$$

where

$$a_s' = \frac{a_s}{t^{\frac{1}{2}(s+3)}},\quad \alpha_s' = \frac{\alpha_s}{\tau^{\frac{1}{2}(s+3)}};$$

$$t' = \frac{t}{t^{\frac{1}{2}(-1+3)}} = 1,\quad \tau' = \frac{\tau}{\tau^{\frac{1}{2}(-1+3)}} = 1,$$

for t and τ then disappear from our identities

$$\begin{aligned}\alpha_0' &= -a_0',\\ \alpha_1' &= -a_1' + 2a_0'^2,\\ \alpha_2' &= -a_2' + 5a_0'a_1' - 5a_0'^3,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots;\end{aligned}$$

and writing further $\qquad \xi_1 = t^{\frac{1}{2}}\xi,\quad \eta_1 = \tau^{\frac{1}{2}}\eta$

we have

$$\eta_1 = \xi_1 + a_0'\,\xi_1^2 + a_1'\,\xi_1^3 + \ldots = \xi_1 + a_{\xi_1},$$

$$\xi_1 = \eta_1 + \alpha_0'\,\eta_1^2 + \alpha_1'\,\eta_1^3 + \ldots = \eta_1 + \alpha_{\eta_1},$$

$$\frac{1}{m}a_{\xi_1}{}^m = a_{m0}'\,\xi_1^{2m} + a_{m1}'\,\xi_1^{2m+1} + \ldots,$$

$$\frac{1}{m}\alpha_{\eta_1}{}^m = \alpha_{m0}'\,\eta_1^{2m} + \alpha_{m1}'\,\eta_1^{2m+1} + \ldots,$$

$$a_s' = -a'_{1,s} + \binom{s+3}{1}a'_{2,s-1} - \binom{s+4}{2}a'_{3,s-2} + \ldots.$$

It will be noted that to form a_s', a_s is divided by t raised to a power equal to half the characteristic

$$3i + w \ \text{ or } \ \mu$$

of a_s. Similarly in the case of any homogeneous and isobaric algebraic expression or operator we form the accented expression by dividing by t raised to a power equal to half the characteristic of such expression or operator.

In the identity $\eta = t\xi + \alpha_0\xi^2 + \alpha_1\xi^3 + \ldots$

the characteristic of η is $+1$,

and that of ξ is -1;

the degrees and weight of ξ are 0 and -1 respectively,

and those of η are $+1$ and -2 respectively,

the two identities in ξ and η are thus each homogeneous and isobaric.

The operator ∂_{a_s}

is of degree-weight $-1, -s$,

and its characteristic $-(s+3)$.

Hence $$\partial_{a_s'} = t^{\frac{1}{2}(s+3)}\,\partial_{a_s}.$$

The operator ∂_{α_s}

is of degree-weight $-(s+2),\ -2s-3$,

and its characteristic $+(s+3)$.

Hence $$\partial_{\alpha_s'} \mp \tau^{\frac{1}{2}(s+3)}\,\partial_{\alpha_s}.$$

In the relation $\partial_t = -\tau^2\partial_\tau - \tau(3I_\alpha + W_\alpha) - V_\alpha$;

we have an identity of degree-weight $-1, +1$,

and therefore of characteristic -2,

and we have $\partial_{t'} = -\partial_{\tau'} - (3I_{\alpha'} + W_{\alpha'}) - V_{\alpha'}$.

The absolute pure reciprocant $(4a_0a_2 - 5a_1^2) \div t^4$

becomes $4a_0'a_2' - 5a_1'^2$;

so that by obliteration of accents we pass to the non-absolute form.

The mixed homogeneous reciprocant

$$\frac{ta_1 - a_0^2}{t^3}$$

becomes $a_1' - a_0'^2$,

and is thus not homogeneous or isobaric in accented letters, but as usual the sum of the degree and weight of each term is a constant number. To reach the non-absolute form we have merely to obliterate accents and insert powers of t as required for homogeneity. The annexed table of degree, weight and characteristic will be found useful.

Observe the result

$$\partial_{a_s'} = -\partial_{a_s'} - \sum_{i=0}^{i=s+2}(s+i+4,\ 1;\ i+1,\ s+i+1)_{a'}$$

and note that $$\partial_{a_s'} \mp \left\{\partial_{a_s'} + \sum_{i=0}^{i=s+2}(s+i+4,\ 1;\ i+1,\ s+i+1)_{a'}\right\}$$

is an absolute reciprocant of even or uneven order.

	i	w	$3i+w$	$i+w$
x and ξ	0	-1	-1	-1
y and η	1	-2	$+1$	-1
t	1	-1	$+2$	0
τ	-1	$+1$	-2	0
a_s	1	s	$s+3$	$s+1$
α_s	$-s-2$	$2s+3$	$-s-3$	$s+1$
∂_{a_s}	-1	$-s$	$-s-3$	$-s-1$
∂_{α_s}	$s+2$	$-2s-3$	$+s+3$	$-s-1$
x_1 and ξ_1	$\frac{1}{2}$	$-\frac{3}{2}$	0	-1
y_1 and η_1	$\frac{1}{2}$	$-\frac{3}{2}$	0	-1
t'	0	0	0	0
τ'	0	0	0	0
a_s'	$-\frac{1}{2}(s+1)$	$\frac{3}{2}(s+1)$	0	$s+1$
α_s'	$-\frac{1}{2}(s+1)$	$\frac{3}{2}(s+1)$	0	$s+1$
$\partial_{a_s'}$	$\frac{1}{2}(s+1)$	$-\frac{3}{2}(s+1)$	0	$-s-1$
$\partial_{\alpha_s'}$	$\frac{1}{2}(s+1)$	$-\frac{3}{2}(s+1)$	0	$-s-1$

Art. 12. At this point it is convenient to study the transformation of the general operator.

The Transformation of $(\mu, \nu;\ m, n)_a$.

The symbolic form is readily found to be

$$\left(\frac{\mu}{m} - 2\nu\right) \xi^p a_\xi^m + \nu \xi^{p+1} a_\xi^{m-1} \partial_\xi a_\xi,$$

where
$$p = n - 2m + 2;$$

and that of a form
$$(\mu_1, \nu_1;\ m_1, n_1)_\alpha,$$

is
$$-\eta' \left\{\left(\frac{\mu_1}{m_1} - 2\nu_1\right) \eta^{p_1} \alpha_\eta^{m_1} + \nu_1 \eta^{p_1+1} \alpha_\eta^{m_1-1} \partial_\eta \alpha_\eta\right\},$$

where
$$p_1 = n_1 - 2m_1 + 2.$$

To effect the transformation of Sylvester we have to express the former form in terms of the latter form. Aided by the formulae

$$a_\xi = -\frac{1}{\tau}\alpha_\eta,$$

$$\xi = \tau\eta + \alpha_\eta,$$

$$\partial_\xi = \eta' \partial_\eta,$$

we find
$$(\mu, \nu;\ m, n)_a$$

$$= (-)^m \tau^{-m} \eta' \left\{\left(\frac{\mu}{m} - 2\nu\right)(\tau\eta + \alpha_\eta)^p \alpha_\eta^m (\tau + \partial_\eta \alpha_\eta) + \nu (\tau\eta + \alpha_\eta)^{p+1} \alpha_\eta^{m-1} \partial_\eta \alpha_\eta\right\},$$

and thence
$$(-)^m (\mu, \nu;\ m, n)_a$$

$$= \tau^{-m+p+1} \eta' \left(\frac{\mu}{m} - 2\nu\right)\left(1 + \frac{\alpha_\eta}{\tau\eta}\right)^p \eta^p \alpha_\eta^m$$

$$+ \tau^{-m+p} \eta' \left(\frac{\mu}{m} - 2\nu\right)\left(1 + \frac{\alpha_\eta}{\tau\eta}\right)^p \eta^p \alpha_\eta^m \partial_\eta \alpha_\eta$$

$$+ \tau^{-m+p+1} \eta' \nu \left(1 + \frac{\alpha_\eta}{\tau\eta}\right)^{p+1} \eta^{p+1} \alpha_\eta^{m-1} \partial_\eta \alpha_\eta.$$

If we write $$(1+u)^p = \Sigma C_{s,p} u^s,$$
the general term is found to be $\tau^{-m+p-s+1}\eta'$ multiplied by

$$\left(\frac{\mu}{m} - 2\nu\right) C_{s,p}\, \eta^{p-s} \alpha_\eta{}^{m+s} + \left\{\frac{\mu}{m} C_{s-1,p} - \nu\,(C_{s-1,p} - C_{s,p})\right\} \eta^{p-s+1} \alpha_\eta{}^{m+s-1} \partial_\eta \alpha_\eta;$$

and, comparing this with

$$-\eta'\left\{\left(\frac{\mu_1}{m_1} - 2\nu_1\right) \eta^{p_1} \alpha_\eta{}^{m_1} + \nu_1 \eta^{p_1+1} \alpha_\eta{}^{m_1-1} \partial_\eta \alpha_\eta\right\}$$

we find
$$m_1 = m + s,$$
$$\mu_1 = (m+s)\left\{(2C_{s-1,p} + C_{s,p})\frac{\mu}{m} - 2C_{s-1,p}\,\nu\right\},$$
$$\nu_1 = C_{s-1,p}\frac{\mu}{m} - (C_{s-1,p} - C_{s,p})\,\nu,$$
$$n_1 = n + s;$$

so that we may write
$$(-)^{m+1}(\mu,\ \nu;\ m,\ n)_a$$
$$= \tau^{n-3m+3}(\mu,\ \nu;\ m,\ n)_a$$
$$+ \tau^{n-3m+2}\left[(m+1)\left\{(2C_{0,p} + C_{1,p})\frac{\mu}{m} - 2C_{0,p}\,\nu\right\},\ C_{0,p}\frac{\mu}{m} - (C_{0,p} - C_{1,p})\,\nu;\ m+1,\ n+1\right]$$
$$+ \tau^{n-3m+1}\left[(m+1)\left\{(2C_{1,p} + C_{2,p})\frac{\mu}{m} - 2C_{1,p}\,\nu\right\},\ C_{1,p}\frac{\mu}{m} - (C_{1,p} - C_{2,p})\,\nu;\ m+2,\ n+2\right]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Art. 13. This is the general result, and since we may write the general term on the dexter either in the form

$$\tau^{n-3m-s+3}\left[(m+s)\left\{2C_{s-1,p}\left(\frac{\mu}{m} - \nu\right) + C_{s,p}\frac{\mu}{m}\right\},\ C_{s-1,p}\left(\frac{\mu}{m} - \nu\right) + C_{s,p}\,\nu;\ m+s,\ n+s\right],$$

or in the form

$$\tau^{n-3m-s+3}\left[(m+s)\left\{C_{s-1,p}\left(\frac{\mu}{m} - 2\nu\right) + C_{s,p+1}\frac{\mu}{m}\right\},\ C_{s-1,p}\left(\frac{\mu}{m} - 2\nu\right) + C_{s,p+1}\,\nu;\ m+s,\ n+s\right],$$

it is clear that we obtain special simplifications in the two cases

$$\frac{\mu}{m} - \nu = 0,$$

$$\frac{\mu}{m} - 2\nu = 0.$$

In the former
$$(-)^{m+1}(m,\ 1;\ m,\ n)_a$$
$$= \tau^{n-3m+3}(m,\ 1;\ m,\ n)_a$$
$$+ C_{1,p}\tau^{n-3m+2}(m+1,\ 1;\ m+1,\ n+1)_a$$
$$+ C_{2,p}\tau^{n-3m+1}(m+2,\ 1;\ m+2,\ n+2)_a$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

In the latter
$$\begin{aligned}&(-)^{m+1}(2m,\ 1;\ m,\ n)_a\\ &=\tau^{n-3m+3}(2m,\ 1;\ m,\ n)_a\\ &+C_{1,p+1}\tau^{n-3m+2}(2m+2,\ 1;\ m+1,\ n+1)_a\\ &+C_{2,p+1}\tau^{n-3m+1}(2m+4,\ 1;\ m+2,\ n+2)_a\\ &+\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Art. 14. It is necessary to determine all the cases in which the dexter reduces to a single operator.

There are two cases

$$\text{(i)}\quad p=n-2m+2=0,\quad \frac{\mu}{m}-\nu=0.$$

Then
$$(-)^{m+1}(m,\ 1;\ m,\ 2m-2)_a=\tau^{-m+1}(m,\ 1;\ m,\ 2m-2)_a$$

or to exhibit an absolute reciprocant

$$(-)^{m+1}t^{-\frac{1}{2}(m-1)}(m,\ 1;\ m,\ 2m-2)_a=\tau^{-\frac{1}{2}(m-1)}(m,\ 1;\ m,\ 2m-2)_a.$$

Here $2m-2$ may be -1, 0 or any positive integer and thus m must either be an integer or the half of an integer. We have the series of absolute reciprocants

$$\begin{aligned}(-)^{\frac{3}{2}}t^{\frac{1}{4}}(1,\ 2;\ \tfrac{1}{2},\ -1)_a&=\tau^{\frac{1}{4}}(1,\ 2;\ \tfrac{1}{2},\ -1)_a,\\ (1,\ 1;\ 1,\ 0)_a&=(1,\ 1;\ 1,\ 0)_a,\\ (-)^{\frac{1}{2}}t^{-\frac{1}{4}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a&=\tau^{-\frac{1}{4}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a,\\ -t^{-\frac{1}{2}}(2,\ 1;\ 2,\ 2)_a&=\tau^{\frac{1}{2}}(2,\ 1;\ 2,\ 2)_a,\\ \ldots\ldots\ldots&\ldots\ldots\ldots\end{aligned}$$

The square root of negative unity may always be removed by making use of the relation

$$t^{-\frac{3}{4}}a_0^{\frac{1}{2}}=(-)^{\frac{1}{2}}\tau^{-\frac{3}{4}}\alpha_0^{\frac{1}{2}}.$$

Thus when $m=\frac{1}{2}(2m'+1)$, where m' is an integer, we may write

$$(-)^{m'+1}t^{-\frac{1}{2}(m'+1)}a_0^{\frac{1}{2}}(m'+\tfrac{1}{2},\ 1;\ m'+\tfrac{1}{2},\ 2m'-1)_a=\tau^{-\frac{1}{2}(m'+1)}\alpha_0^{\frac{1}{2}}(m'+\tfrac{1}{2},\ 1;\ m'+\tfrac{1}{2},\ 2m'-1)_a,$$

and we may substitute for the first, third, fifth, &c. of the above series, the new series:—

$$\begin{aligned}-t^{-\frac{1}{2}}a_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ -1)_a&=\tau^{-\frac{1}{2}}\alpha_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ -1)_a,\\ t^{-1}a_0^{\frac{1}{2}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a&=\tau^{-1}\alpha_0^{\frac{1}{2}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a,\\ -t^{-\frac{3}{2}}a_0^{\frac{1}{2}}(5,\ 2;\ \tfrac{5}{2},\ 3)_a&=\tau^{-\frac{3}{2}}\alpha_0^{\frac{1}{2}}(5,\ 2;\ \tfrac{5}{2},\ 3)_a,\\ \ldots\ldots\ldots&\ldots\ldots\ldots\end{aligned}$$

A second series is obtained by putting

$$\text{(ii)}\quad p=n-2m+2=-1,\quad \frac{\mu}{m}-2\nu=0;$$

for then
$$(-)^{m+1}(2m,\ 1;\ m,\ 2m-3)_a=\tau^{-m}(2m,\ 1;\ m,\ 2m-3)_a,$$

which, in reciprocant form, is

$$(-)^{m+1}t^{-\frac{1}{2}m}(2m,\ 1;\ m,\ 2m-3)_a=\tau^{-\frac{1}{2}m}(2m,\ 1;\ m,\ 2m-3)_a;$$

and if $$m = m' + \tfrac{1}{2},$$

$$(-)^{m'+1} t^{-\frac{1}{2}(m'+2)} a_0^{\frac{1}{2}} (2m'+1,\ 1;\ m'+\tfrac{1}{2},\ 2m'-2)_a = \tau^{-\frac{1}{2}(m'+2)} \alpha_0^{\frac{1}{2}} (2m'+1,\ 1;\ m'+\tfrac{1}{2},\ 2m'-2)_\alpha;$$

and the series is

$$t^{-\frac{1}{2}}(2,\ 1;\ 1,\ -1)_a = \tau^{-\frac{1}{2}}(2,\ 1;\ 1,\ -1)_\alpha,$$

$$t^{-\frac{3}{2}} a_0^{\frac{1}{2}}(3,\ 1;\ \tfrac{3}{2},\ 0)_a = \tau^{-\frac{3}{2}} \alpha_0^{\frac{1}{2}}(3,\ 1;\ \tfrac{3}{2},\ 0)_\alpha,$$

$$-t^{-1}(4,\ 1;\ 2,\ 1)_a = \tau^{-1}(4,\ 1;\ 2,\ 1)_\alpha,$$

$$-t^{-2} a_0^{\frac{1}{2}}(5,\ 1;\ \tfrac{5}{2},\ 2)_a = \tau^{-2} \alpha_0^{\frac{1}{2}}(5,\ 1;\ \tfrac{5}{2},\ 2)_\alpha,$$

....

In no other case does the operator transform into a single operator.

Art. 15. I proceed to determine all the cases which lead to just two operators on the dexter. There are two cases

$$\text{(i)}\quad p = n - 2m + 2 = 1,\quad \frac{\mu}{m} - \nu = 0 \text{ gives}$$

$$(-)^{m+1}(m,\ 1;\ m,\ 2m-1)_a = \tau^{-m+2}(m,\ 1;\ m,\ 2m-1)_\alpha + \tau^{-m+1}(m+1,\ 1;\ m+1,\ 2m)_\alpha;$$

and if $$m = m' + \tfrac{1}{2};$$

$$(-)^{m'+1} a_0^{\frac{1}{2}}(m'+\tfrac{1}{2},\ 1;\ m'+\tfrac{1}{2},\ 2m')_a = \tau^{-m'} \alpha_0^{\frac{1}{2}}(m'+\tfrac{1}{2},\ 1;\ m'+\tfrac{1}{2},\ 2m')$$
$$+ \tau^{-m'-1} \alpha_0^{\frac{1}{2}}(m'+\tfrac{3}{2},\ 1;\ m'+\tfrac{3}{2},\ 2m'+1)_\alpha.$$

$$\text{(ii)}\quad p = n - 2m + 2 = 0 \text{ gives}$$

$$(-)^{m+1}(\mu,\ \nu;\ m,\ 2m-2)_a = \tau^{-m+1}(\mu,\ \nu;\ m,\ 2m-2)_\alpha$$
$$+ \left(\frac{\mu}{m} - \nu\right) \tau^{-m}(2m+2,\ 1;\ m+1,\ 2m-1)_\alpha;$$

and if $$m = m' + \tfrac{1}{2}$$

$$(-)^{m'+1} a_0^{\frac{1}{2}}(\mu,\ \nu;\ m'+\tfrac{1}{2},\ 2m'-1)_a = \tau^{-m'-1} \alpha_0^{\frac{1}{2}}(\mu,\ \nu;\ m'+\tfrac{1}{2},\ 2m'-1)_\alpha$$
$$+ \left(\frac{2\mu}{2m'+1} - \nu\right) \tau^{-m'-2} \alpha_0^{\frac{1}{2}}(2m'+3,\ 1;\ m'+\tfrac{3}{2},\ 2m')_\alpha.$$

Art. 16. For three operators on the dexter we have the two results

$$(-)^{m+1}(m,\ 1;\ m,\ 2m)_a$$
$$= \tau^{-m+3}(m,\ 1;\ m,\ 2m)_\alpha + 2\tau^{-m+2}(m+1,\ 1;\ m+1,\ 2m+1)_\alpha + \tau^{-m+2}(m+2,\ 1;\ m+2,\ 2m+2)_\alpha.$$

$$(-)^{m+1}(\mu,\ \nu;\ m,\ 2m-1)_a$$
$$= \tau^{-m+2}(\mu,\ \nu;\ m,\ 2m-1)_\alpha$$
$$+ \tau^{-m+1}\left\{(m+1)\left(3\frac{\mu}{m} - 2\nu\right),\ \frac{\mu}{m};\ m+1,\ 2m\right\}_\alpha$$
$$+ \left(\frac{\mu}{m} - \nu\right) \tau^{-m}(2m+4,\ 1;\ m+2,\ 2m+1)_\alpha.$$

Art. 17. Again for four operators on the dexter we reach the two results

$$(-)^{m+1}(m,\ 1;\ m,\ 2m+1)_a$$
$$=\tau^{-m+4}(m,\ 1;\ m,\ 2m+1)_a+3\tau^{-m+3}(m+1,\ 1;\ m+1,\ 2m+2)_a$$
$$+3\tau^{-m+2}(m+2,\ 1;\ m+2,\ 2m+3)_a+\tau^{-m+1}(m+3,\ 1;\ m+3,\ 2m+4)_a.$$

$$(-)^{m+1}(\mu,\ \nu;\ m,\ 2m)_a$$
$$=\tau^{-m+3}(\mu,\ \nu;\ m,\ 2m)_a$$
$$+\tau^{-m+2}\left\{(m+1)\left(4\frac{\mu}{m}-2\nu\right),\ \frac{\mu}{m}+\nu;\ m+1,\ 2m+1\right\}_a$$
$$+\tau^{-m+1}\left\{(m+2)\left(5\frac{\mu}{m}-4\nu\right),\ 2\frac{\mu}{m}-\nu;\ m+2,\ 2m+2\right\}_a$$
$$+\left(\frac{\mu}{m}-\nu\right)\tau^{-m}(2m+6,\ 1;\ m+3,\ 2m+3)_a.$$

Similarly two formulae may be written down, each of which involves a given number of operators on the dexter. Each of these necessarily leads to an absolute reciprocant operator.

The Transformation of the Operator $(\mu,\ \nu;\ m,\ n)_t$.

Art. 18. The symbolic expression is

$$\left(\frac{\mu}{m}-\nu\right)\xi^{n-m+2}\eta^m+\nu\xi^{n-m+3}\eta^{m-1}\eta'.$$

Also the symbolic expression of $(\mu',\ \nu';\ m',\ n')_a$

is
$$-\nu'\xi^{m'-1}\eta^{n'-m'+3}-\left(\frac{\mu'}{m'}-\nu'\right)\xi^{m'}\eta^{n'-m'+2}\eta'.$$

Hence it will be clear that $(\mu,\ \nu;\ m,\ n)_t$

$$=-\left\{\frac{\mu}{m}(n-m+3),\ \frac{\mu}{m}-\nu;\ n-m+3,\ n\right\}_\tau,$$

for as may be readily verified each of these operators leads to the same symbolic expression.

Most important consequences follow.

First observe that

$$(\mu,\ \nu;\ m,\ n)_t\mp\left\{\frac{\mu}{m}(n-m+3),\ \frac{\mu}{m}-\nu;\ n-m+3,\ n\right\}_t,$$

is an absolute reciprocant of even or of uneven order ascending as the upper or lower sign is taken.

We may put $m=\frac{1}{2}(n+3)$ and then the reciprocants are

$$\{\mu,\ \nu;\ \tfrac{1}{2}(n+3),\ n\}_t\mp\left\{\mu,\ \frac{2\mu}{n+3}-\nu;\ \tfrac{1}{2}(n+3),\ n\right\}_t.$$

Further putting $\mu=n+3,\ \nu=1,$

we find that $\{n+3,\ 1;\ \frac{1}{2}(n+3),\ n\}_t$

is a reciprocant of uneven order.

For
$$n = 2q - 3$$
this is
$$2qt_{q0}\partial_{a_{2q-3}} + (2q+1)\,t_{q1}\partial_{a_{2q-2}} + (2q+2)\,t_{q2}\partial_{a_{2q-1}} + \dots$$
for
$$q = 1$$
$$2t\partial_t + 3a_0\partial_{a_0} + 4a_1\partial_{a_1} + \dots = 3I_t + W_t,$$
which for an operand which is homogeneous and isobaric is equivalent to multiplication by $3i + w$.

These operators are useful for obtaining homogeneous and isobaric mixed reciprocants from pure reciprocants.

In particular we may notice
$$4\frac{t^2}{2}\partial_{a_1} + 5ta_0\partial_{a_2} + 6\left(\tfrac{1}{2}a_0{}^2 + ta_1\right)\partial_{a_3} + \dots$$
from its resemblance to V_a.

It will be remarked that
$$(1,\ 1;\ 1,\ 0)_t = t\partial_{a_0} + 2a_0\partial_{a_1} + 3a_1\partial_{a_2} + \dots$$
is converted into
$$-2\,(1,\ 0;\ 2,\ 0)_\tau = -2\left\{\tfrac{1}{2}\tau^2\partial_{\alpha_0} + \tau\alpha_0\partial_{\alpha_1} + \left(\tfrac{1}{2}\alpha_0 + \tau\alpha_1\right)\partial_{\alpha_2} + \dots\right\}.$$

Art. 19. I will now derive from these results those which affect well-known operators and discuss those which appear for the first time.

Writing $(1,\ 1;\ 1,\ 0)_a = a_0\partial_{a_0} + 2a_1\partial_{a_1} + 3a_2\partial_{a_2} + \dots = I_a + W_a,$

$$(2,\ 1;\ 2,\ 2)_a = 2\,.\,\tfrac{1}{2}a_0{}^2\partial_{a_2} + 3a_0a_1\partial_{a_3} + 4\left(\tfrac{1}{2}a_1{}^2 + a_0a_2\right)\partial_{a_4} + \dots = J_a$$
the general reciprocative relation
$$t^{-\frac{1}{2}(m-1)}\,(m,\ 1;\ m,\ 2m-2)_a = (-)^{m+1}\,\tau^{-\frac{1}{2}(m-1)}\,(m,\ 1;\ m,\ 2m-2)_\alpha$$
gives
$$I_a + W_a = I_\alpha + W_\alpha \quad \text{for} \quad m = 1,$$
$$t^{-\frac{1}{2}}J_a = -\tau^{-\frac{1}{2}}J_\alpha \quad \text{for} \quad m = 2;$$
the former relation shews that *quâ* the letters
$$a_0,\ a_1,\ a_2,\ \dots$$
the sum of the degree and weight of every term of every absolute reciprocant is a constant quantity; the latter shews that the operation of
$$t^{-\frac{1}{2}}J_a$$
invariably leaves the property of absolute reciprocance intact. J_a it will be remembered is the alternant of
$$(1,\ 1;\ 1,\ 0)_a \quad \text{and} \quad (4,\ 1;\ 2,\ 1)_a$$
and is also commutative with each. It thus occupies an important central position in the present theory. Its property, in relation to any reciprocant, which has just been established is new to the theory.

The other operators derived from the same formula are here presented for the first time.

In particular $$t^{-\frac{1}{2}}a_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ -1)_a$$

or $$t^{-\frac{1}{2}}a_0\left\{\partial_t+\frac{3}{2}\frac{a_1}{a_0}\partial_{a_0}+\frac{5}{8}\frac{4a_0a_2-a_1^2}{a_0^2}\partial_{a_1}+\ldots\right\}$$

is seen to be a generator for all reciprocants; moreover I have elsewhere established that

$$\frac{3}{2}\frac{a_1}{a_0}\partial_{a_0}+\frac{5}{8}\frac{4a_0a_2-a_1^2}{a_0^2}+\ldots$$

is a reversor to V_a the annihilator of pure reciprocants.

Art. 20. Again, in a result of Art. 14, put $m=1$ giving

$$t^{-\frac{1}{2}}(2a_0\partial_t+3a_1\partial_{a_0}+4a_2\partial_{a_1}+\ldots)=\tau^{-\frac{1}{2}}(2\alpha_0\partial_\tau+3\alpha_1\partial_{\alpha_0}+4\alpha_2\partial_{\alpha_1}+\ldots);$$

this operator, equivalent to ∂_x, is well known.

For $m=2$

$$t^{-1}(4,\ 1;\ 2,\ 1)_a=-\tau^{-1}(4,\ 1;\ 2,\ 1)_\alpha,$$

or $$t^{-1}V_a=-\tau^{-\frac{1}{2}}V_\alpha;$$

V_a being the well-known operator

$$4\,.\,\tfrac{1}{2}a_0^2\partial_{a_1}+5a_0a_1\partial_{a_2}+6\,(\tfrac{1}{2}a_1^2+a_0a_2)\,\partial_{a_3}+\ldots.$$

When m is the half of an integer we find the important reciprocant

$$t^{-\frac{3}{2}}a_0^{\frac{1}{2}}(3,\ 1;\ \tfrac{3}{2},\ 0)_a$$

and other new results in the theory.

In a result of Art. 15 put $m=1$ so that

$$(1,\ 1;\ 1,\ 1)_a=\tau(1,\ 1;\ 1,\ 1)_\alpha+(2,\ 1;\ 2,\ 2)_\alpha,$$

or in Sylvester's notation $$\Omega_a=\tau\Omega_\alpha+J_\alpha,$$

and if $$f(t)=\phi(\tau),$$

then $$\{f(t)+t\phi(t)\}\,\Omega_a+\phi(t)\,J_a,$$

and $$\{f(t)-t\phi(t)\}\,\Omega_a-\phi(t)\,J_a$$

are reciprocants of even and uneven order respectively.

As a particular case $$2t^{\frac{1}{2}}\Omega_a+t^{-\frac{1}{2}}J_a$$

is a reciprocant of even order, a result connected with the Theory of Principiants.

Art. 21. From the result of Art. 15 we find putting $m'=0$

$$a_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ 0)_a=\alpha_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ 0)_\alpha+\tau^{-1}\alpha_0^{\frac{1}{2}}(3,\ 2;\ \tfrac{3}{2},\ 1)_\alpha$$

shewing that $$2a_0^{\frac{1}{2}}(1,\ 2;\ \tfrac{1}{2},\ 0)_a+t^{-1}a_0^{\frac{1}{2}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a$$

and $$t^{-1}a_0^{\frac{1}{2}}(3,\ 2;\ \tfrac{3}{2},\ 1)_a$$

are reciprocants of even and uneven order respectively.

Passing on to another result of Art. 15 we find, for $m=1$,

$$(\mu,\ \nu;\ 1,\ 0)_a = (\mu,\ \nu;\ 1,\ 0)_a + (\mu-\nu)\,\tau^{-1} V_a$$

indicating a reciprocant $\qquad 2(\mu,\ \nu;\ 1,\ 0)_a + (\mu-\nu)\,t^{-1} V_a$

of even order.

This shews that I_a and W_a are each of them constant in every term of a pure reciprocant. Also that

$$2I_a + t^{-1} V_a$$
$$2W_a - t^{-1} V_a$$

are pure reciprocants of even order.

We are also led to reciprocant combinations of the two operators

$$(\mu,\ \nu;\ \tfrac{1}{2},\ -1)_a, \qquad (3,\ 1;\ \tfrac{3}{2},\ 0)_a$$

which may prove to be valuable as they are certainly interesting.

Art. 22. We may express these results in accented letters, remembering that $t'=1$, but $\partial_{t'} = t\partial_t$, by merely accenting the letters when t and τ do not appear as algebraic coefficients of the operators: thus

$$I_{t'} = -2I_{\tau'} - W_{\tau'}$$

and so forth, but

$$I_{a'} = I_{a'} + V_{a'},$$
$$V_{a'} = -V_{a'}$$

and so forth.

The accented notation is not applicable to non-homogeneous forms such as

$$\left(1 - \frac{1}{t^2}\right) V_a.$$

Art. 23. My object is to effect the transformation of all the operators that have appeared in the researches of previous investigators and to deduce the corresponding reciprocants; and thus to shew the fundamental reason of the effectiveness of such operators. Thus it was shewn by Leudesdorf* that the operators

$$(t^2 \pm t^k)\,\partial_t - V_a$$

performed upon any homogeneous reciprocant produce a reciprocant. I proceed by transformation to disclose the fundamental reason of this fact.

The operator $\qquad (t^2 \pm t^k)\,\partial_t - V_a$

becomes $\qquad \dfrac{\tau^{k-2} \pm 1}{\tau^k}\{\tau^2\partial_\tau - \tau(2\tau\partial_\tau + 3\alpha_0\partial_{\alpha_0} + 4\alpha_1\partial_{\alpha_1} + \ldots) - V_a\} + \dfrac{V_a}{\tau^2},$

and since $\qquad 2\tau\partial_\tau + 3\alpha_0\partial_{\alpha_0} + 4\alpha_1\partial_{\alpha_1} + \ldots = 3I_\tau + W_\tau,$

this is $\qquad \dfrac{1}{\tau^k}\{(\tau^k \pm \tau^2)\,\partial_\tau - (\tau^{k-1} \pm \tau)(3I_\tau + W_\tau) \mp V_a\};$

and since $\qquad 3I_t + W_t = -3I_\tau - W_\tau$

we find $\qquad (t^2 \pm t^k)\,\partial_t - V_a - t(3I_t + W_t) = \pm\dfrac{1}{\tau^k}\{(\tau^2 \pm \tau^k)\,\partial_\tau - V_a - \tau(3I_\tau + W_\tau)\}$

or $\qquad t^{-\frac{1}{2}k}\{(t^2 \pm t^k)\,\partial_t - V_a - t(3I_t + W_t)\}$

is a reciprocant of even or uneven order.

* Leudesdorf, "On some results connected with the Theory of Reciprocants," *Proc. L.M.S.*, p. 197.

Hence this operation is effective in producing reciprocants from reciprocants whether homogeneous or no.

Now in the case of homogeneity (implying isobarism)

$$3I_t + W_t = 0,$$

and then

$$(t^2 \pm t^k)\partial_t - V_a$$

is also effective, the theorem of Leudesdorf.

Art. 24. We find many similar instances in which a given operator is only a reciprocant for a given specification of operand. *Ex. gr.* It has been shewn that

$$2I_a + \frac{V_a}{t}$$

is in general a reciprocant of even order, but I_a itself is only a reciprocant of even order when the operand is a pure reciprocant for only then does V_a vanish as a portion of the operator. Such points as these arise naturally out of the discussion of particular operator reciprocants and are generally established at sight.

The operator $(1-t^2)\partial_t + t(3I_t + W_t) + V_a$.

Art. 25. Leudesdorf established that this operator caused every absolute orthogonal reciprocant to vanish. It is easy to shew from the preceding formulae that

$$(1-t^2)\partial_t + t(3I_t + W_t) + V_a = -(1-\tau^2)\partial_\tau - \tau(3I_\tau + W_\tau) - V_a,$$

so that

$$(1-t^2)\partial_t + t(3I_t + W_t) + V_a$$

is an absolute reciprocant of uneven order.

Art. 26. This leads at once to Sylvester's theorem which states that if R_a be an absolute orthogonal reciprocant $\partial_t R_a$ is a reciprocant. For

$$\{(1-t^2)\partial_t + t(3I_t + W_t) + V_a\} R_a = 0$$

or

$$\left(t + \frac{1}{t}\right)\partial_t R_a = -(3I_a + W_a) R_a - \frac{V_a}{t} R_a,$$

and if

$$R_a = qR_a \text{ where } q \text{ is } +1 \text{ or } -1$$

$$\left(\tau + \frac{1}{\tau}\right)\partial_\tau R_a = -(3I_a + W_a) R_a - \frac{V_a}{\tau} R_a.$$

Now we have shewn that $(3I_a + W_a) + \dfrac{V_a}{t}$

is a reciprocant of even order and $t + \dfrac{1}{t} = \tau + \dfrac{1}{\tau}$.

Hence

$$\partial_t R_a = q\partial_\tau R_a$$

or in other words

$$\partial_t R_a$$

is a reciprocant of the same order as R_a.

The operator $3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots$.

Art. 27. It is easy to shew that $3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots$

$$= \tau^{-1}(3\alpha_1\partial_{\alpha_0} + 4\alpha_2\partial_{\alpha_1} + 5\alpha_3\partial_{\alpha_2} + \ldots) - 2\tau^{-2}\alpha_0(3I_\alpha + W_\alpha) - 2\tau^{-3}\alpha_0 V_\alpha$$

and this may be also written $3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots$

$$= \tau^{-1}(3\alpha_1\partial_{\alpha_0} + 4\alpha_2\partial_{\alpha_1} + 5\alpha_3\partial_{\alpha_2} + \ldots) + 2\tau^{-1}\alpha_0\partial_\tau - 2\tau^{-3}\alpha_0(2,\ 1;\ 2,\ -1)_\alpha.$$

From the former we find that

$$t^{-\frac{1}{2}}(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots) - t^{-\frac{3}{2}}a_0(3I_a + W_a + t^{-1}V_a)$$

is a reciprocant of even order.

We note that if the operand be a pure reciprocant of characteristic zero, not involving t,

$$t^{-\frac{1}{2}}(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots)$$

is a generator, and that for *any* pure reciprocant

$$t^{-\frac{1}{2}}(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + \ldots) - t^{-\frac{3}{2}}a_0(3I_a + W_a)$$

is a generator.

From the latter we learn $\partial_t - t^{-2}(2,\ 1;\ 2,\ -1)_a$

is a reciprocant of even order.

Art. 28. *The operators*

$$h_a = t\partial_{a_0} + 2a_0\partial_{a_1} + 3a_1\partial_{a_2} + \ldots = (1,\ 1;\ 1,\ 0)_t,$$
$$H_a = t_{20}\partial_{a_0} + t_{21}\partial_{a_1} + t_{22}\partial_{a_2} + \ldots = (1,\ 0;\ 2,\ 0)_t.$$

Rogers* shewed that homographic invariants, viz. those unaltered by the substitution

$$(x,\ y) = \left(\frac{lx+m}{x+n},\quad \frac{l'y+m'}{y+n'}\right)$$

satisfy the two partial differential equations

$$h_a = 0, \qquad H_a = 0.$$

Now we find from Art. 18 $(1,\ 1;\ 1,\ 0)_t = -2(1,\ 0;\ 2,\ 0)_\tau$.

Thus all the solutions of $h_a = 0$ are simply transformations of the solutions of $H_a = 0$; moreover any reciprocant which satisfies $h_a = 0$, necessarily satisfies $H_a = 0$. We have the absolute reciprocants

$$h_a - 2A_a \text{ of even order,}$$
$$h_a + 2H_a \text{ of uneven order,}$$

and we can construct the homogeneous reciprocants

$$t^{-\frac{1}{2}}(th_a - 2H_a) \text{ of even order,}$$
$$t^{-\frac{1}{2}}(th_a + 2H_a) \text{ of uneven order.}$$

Art. 29. *The operator* $G_a = 4(a_0a_2 - a_1^2)\partial_{a_1} + 5(a_0a_3 - a_1a_2)\partial_{a_2} + \ldots$.

This is the well known generator employed by Sylvester.

It may be written $a_0(3a_1\partial_{a_0} + 4a_2\partial_{a_1} + 5a_3\partial_{a_2} + \ldots) - a_1(3I_a + W_a)$.

* Rogers, "Homographic and Circular Reciprocants," *Proc. L.M.S.*, vol. XVII. p. 220; "On Secondary Invariants," *ibid.* vol. XX. p. 161.

Thence by previous results

$$G_a = -\frac{\alpha_0}{\tau^3}\left\{\frac{1}{\tau}(3\alpha_1\partial_{a_0} + 4\alpha_2\partial_{a_1} + \ldots) - 2\frac{\alpha_0}{\tau^2}(3I_a + W_a) - 2\frac{\alpha_0}{\tau^3}V_a\right\}$$

$$+\left(\frac{\alpha_1}{\tau^4} - \frac{2\alpha_0^2}{\tau^5}\right)\left\{3\left(I_a + \frac{V_a}{\tau}\right) + \left(W_a - \frac{V_a}{\tau}\right)\right\}$$

or

$$G_a = -\frac{1}{\tau^4}G_\alpha + \frac{2}{\tau^6}(\tau\alpha_1 - \alpha_0^2)V_a$$

or

$$\frac{G_a}{t^2} = -\frac{G_\alpha}{\tau^2} + \frac{2}{\tau^4}(\tau\alpha_1 - \alpha_0^2)V_a$$

indicating invariants $\quad \dfrac{G_a}{t^2} - \dfrac{ta_1 - a_0^2}{t^4}V_a$ of uneven order,

$$\frac{ta_1 - a_0^2}{t^4}V_a \text{ of even order.}$$

$$\frac{G_a}{t^2} - \frac{ta_1 - a_0}{t^4}V_a$$

thus always produces an absolute invariant by acting upon any absolute invariants and in particular for forms which satisfy

$$V_a = 0$$

G_a is a generator.

The connexion of G_a with the Schwartzian derivative

$$ta_1 - a_0^2$$

is interesting.

Art. 30. The operator

$$S_a = \frac{a_1}{a_0}\partial_{a_0} + \frac{2a_0a_2 - a_1^2}{a_0^2}\partial_{a_1} + \frac{3a_0^2a_3 - 3a_0a_1a_2 + a_1^3}{a_0^3}\partial_{a_2} + \ldots.$$

This is a reversor in the theory discovered by Hammond. It is easy to prove the formula

$$S_a = -\tau^2 S_\alpha + 2\tau I_\alpha + V_\alpha$$

or

$$tS_a = -\tau S_\alpha + 2I_\alpha + \frac{V_\alpha}{\tau}.$$

The first of these shews the invariants

$$(1 - t^2)S_a + 2tI_a + V_a \text{ of even order,}$$

$$(1 + t^2)S_a - 2tI_a - V_a \text{ of uneven order,}$$

and the second $\quad 2I_a + \dfrac{V_a}{t}$ of even order,

$$2tS_a - 2I_a - \frac{V_a}{t} \text{ of uneven order,}$$

but since $\dfrac{V_a}{t}$ is an invariant of uneven order we deduce

$$tS_a - I_a$$

an invariant of uneven order.

This result is of much interest. It shews that S_a is a generator for all forms such that

$$I_a = 0.$$

Ex. gr. S_a generates an invariant from

$$\frac{4a_0a_2 - 5a_1^2}{a_0^2};$$

the operation, in fact, produces $\quad \dfrac{12}{a_0^4}(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3).$

The operation of $tS_a - I_a$ upon $4a_0a_2 - 5a_1^2$ produces

$$12\frac{t}{a_0^2}(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3) + 2(ta_1 - a_0^2)\frac{4a_0a_2 - a_1^2}{a_0^2}.$$

We may write

$$S_a = s_1\partial_{a_0} - s_2\partial_{a_1} + s_3\partial_{a_2} - \ldots,$$

where $\quad \log\dfrac{1}{a_0}(a_0 + a_1u + a_2u^2 + \ldots) = s_1u - \tfrac{1}{2}s_2u^2 + \tfrac{1}{3}s_3u^3 - \ldots.$

Art. 31. Consider now the operator

$$S_a' = \partial_{a_0} + s_1\partial_{a_1} - s_2\partial_{a_2} + s_3\partial_{a_3} - \ldots.$$

It is easy to shew that

$$\begin{aligned} s_1\partial_{a_1} - s_2\partial_{a_2} + s_3\partial_{a_3} - \ldots = &- \tau^3(\sigma_1\partial_{a_1} - \sigma_2\partial_{a_2} + \sigma_3\partial_{a_3} - \ldots) \\ &+ \tau^2(2\alpha_0\partial_{a_1} + \alpha_1\partial_{a_2} - \alpha_3\partial_{a_4} - 2\alpha_4\partial_{a_5} - \ldots) \\ &+ \tau(8, 1;\ 2, 2)_a \\ &+ (6, 1;\ 3, 3)_a, \end{aligned}$$

and we know that

$$\partial_{a_0} = -\tau^3\partial_{a_0} - \tau^2(4, 1;\ 1, 1)_a - 2\tau(5, 1;\ 2, 2)_a - (6, 1;\ 3, 3)_a.$$

Hence $\quad S_a' = -\tau^3S_a' - 2\tau^2\Omega_a - \tau J_a,$

or $\quad t^{\frac{3}{2}}S_a' = -\tau^{\frac{3}{2}}S_a' - 2\tau^{\frac{1}{2}}\Omega_a - \dfrac{J_a}{\tau^{\frac{1}{2}}},$

shewing that $\quad 2t^{\frac{3}{2}}S_a' + 2t^{\frac{1}{2}}\Omega_a + \dfrac{J_a}{t^{\frac{1}{2}}}$

and therefore also $\quad t^{\frac{3}{2}}S_a' + t^{\frac{1}{2}}\Omega_a$

is a reciprocant of uneven order.

In accented notation this is $\quad S_{a'} + \Omega_{a'}.$

Art. 32. The operator

$$\begin{aligned} T_a = \frac{3}{2}\frac{a_1}{a_0}\partial_{a_0} + \frac{5}{8}\frac{4a_0a_2 - a_1^2}{a_0^2}\partial_{a_1} + \frac{7}{16}\frac{8a_0^2a_3 - 4a_0a_1a_2 + a_1^3}{a_0^3}\partial_{a_2} \\ + \frac{9}{128}\frac{64a_0^3a_4 - 16(a_0^2a_2^2 + 2a_0^2a_1a_3) + 24a_0a_1^2a_2 - 5a_1^4}{a_0^4}\partial_{a_3} \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

is important because it is a *reversor* in the theory of pure reciprocants; this is additional to the two reversors discovered by Sylvester and Hammond which are given in Sylvester's *Lectures on Reciprocants.*

Its form is $$3g_1\partial_{a_0}+5g_2\partial_{a_1}+7g_3\partial_{a_2}+\ldots,$$

where $$\left(1+\frac{a_1}{a_0}u+\frac{a_2}{a_0}u^2+\ldots\right)^{\frac{1}{2}}=1+g_1u+g_2u^2+g_3u^3+\ldots.$$

In Art. 19 I shewed that $t^{-\frac{1}{2}}a_0(\partial_t+T_a)$ is a reciprocant.

In fact it may be shewn that

$$T_a=-\tau^2T_\alpha+\tau(3I_\alpha+W_\alpha)+V_\alpha,$$

and thence that $2tT_a-(3I_a+W_a)$ is a reciprocant of uneven order. This shews that T_a is a generator for all forms, such that $3I_a+W_a=0$.

Ex. gr. Let the operand be

$$\frac{4a_0a_2-5a_1^2}{a_0^{\frac{8}{3}}},$$

the operation of T_a yields

$$14a_0^{-\frac{14}{3}}(a_0^2a_3-3a_0a_1a_2+2a_1^3).$$

Art. 33. I next remark that the relation

$$(\mu,\nu;\ m,n)_t$$
$$=-\frac{1}{m}\{\mu(n-m+3),\ \mu-m\nu;\ n-m+3,\ n\}_\tau$$

holds for the substitution

$$t=\frac{1}{\tau},$$

$$a_0=-\frac{\alpha_0}{\tau^3},$$

$$a_1=-\frac{\alpha_1}{\tau^4}+\frac{2\alpha_0^2}{\tau^5},$$

....

$$a_s=-\tau^{-2s-3}\left\{\tau^s\alpha_{1s}-\binom{s+3}{1}\tau^{s-1}\alpha_{2,s-1}+\binom{s+4}{2}\tau^{s-2}\alpha_{3,s-2}+\ldots\right\}.$$

Therefore making a unit increase of suffix, that is, changing

$$t,\ a_0,\ a_1,\ a_2,\ \ldots\ a_s,\ \ldots \qquad \tau,\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots\ \alpha_s\ \ldots$$

into $$a_0,\ a_1,\ a_2,\ a_3,\ \ldots\ a_{s+1}\ \ldots \qquad \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ \alpha_{s+1}\ \ldots,$$

we find that the relation $$(\mu,\nu;\ m,n+1)_a$$
$$=-\frac{1}{m}\{\mu(n-m+3),\ \mu-m\nu;\ n-m+3,\ n+1\}_\alpha,$$

holds for the advanced substitution

$$a_0 = \frac{1}{\alpha_0},$$

$$a_1 = -\frac{\alpha_1}{\alpha_0^3},$$

$$a_2 = -\frac{\alpha_2}{\alpha_0^4} + \frac{2\alpha_1^2}{\alpha_0^5},$$

$$\cdots\cdots\cdots\cdots$$

which is derived from the formulae for the interchange of the variable by merely making a unit increase of suffix.

As this substitution must not be confused with that which interchanges the variables x, y, I write the theorem in the form

$$(\mu, \nu;\ m, n+1)_a$$
$$= -\frac{1}{m}\{\mu(n-\mu+3),\ \mu - m\nu;\ n-m+3,\ n+1\}_b,$$

where

$$a_0 = \frac{1}{b_0},$$

$$a_1 = -\frac{b_1}{b_0^3},$$

$$a_2 = -\frac{b_2}{b_0^4} + \frac{2b_1^2}{b_0^5}.$$

$$\cdots\cdots\cdots\cdots$$

This most important transformation arranges the whole of the members of the group of the multilinear operators dealt with into pairs, the components of each of which are transformable into one another.

A pair contains identical operators in the particular case

$$\{n+3,\ 1;\ \tfrac{1}{2}(n+3),\ n+1\}_a = -\{n+3,\ 1;\ \tfrac{1}{2}(n+3),\ n+1\}_b,$$

a formula including a singly-infinite system of operators.

The most interesting of this nature are

$$(2,\ 1;\ 1,\ 0),$$

$$(3,\ 1;\ \tfrac{3}{2},\ 1),$$

$$(4,\ 1;\ 2,\ 2).$$

Art. 34. Putting $m=1$, $n=0$ in the general formula, we find

$$(\mu, \nu;\ 1, 1)_a = -(2\mu,\ \mu-\nu;\ 2, 1)_b,$$

and now putting $\mu = 2$, $\nu = 1$

$$(2, 1;\ 1, 1)_a = -(4, 1;\ 2, 1)_b,$$

a wonderful result for now writing

$$a_0 = c_0,\quad a_1 = 2c_1,\quad \ldots\quad a_s = (s+1)\,c_s,\quad \ldots.$$

We find that the transformation

$$c_0 = \frac{1}{b_0},$$

$$2c_1 = -\frac{b_1}{b_0^3},$$

$$3c_2 = -\frac{b_2}{b_0^4} + \frac{2b_1^2}{b_0^5},$$

$$4c_3 = -\frac{b_3}{b_0^5} + \frac{5b_1b_2}{b_0^6} - \frac{5b_1^3}{b_0^7},$$

$$\dots\dots\dots\dots\dots\dots$$

gives
$$(1, 1;\ 1, 1)_c = -(4, 1;\ 2, 1)_b,$$
or
$$\Omega_c = -V_b,$$
or in words the seminvariant operator Ω_c, *quâ* the elements $c_0, c_1, c_2, \dots$, is converted by the above transformation into the pure reciprocant operator V_b, *quâ* the elements $b_0, b_1, b_2, \dots$.

Hence the above transformation converts all seminvariants into pure reciprocants, and the inverse transformation

$$b_0 = \frac{1}{c_0},$$

$$b_1 = -\frac{2c_1}{c_0^3},$$

$$b_2 = -\frac{3c_2}{c_0^4} + \frac{8c_1^2}{c_0^5},$$

$$b_3 = -\frac{4c_3}{c_0^5} + \frac{30c_1c_2}{c_0^6} - \frac{40c_1^3}{c_0^7},$$

$$\dots\dots\dots\dots\dots\dots$$

all pure reciprocants into seminvariants.

Art. 35. This result will now be utilized in order to throw new light upon the structure of those differential forms which are unaltered by the homographic transformation. These have been studied principally by Halphen and Sylvester, and have been called by the latter Principiants.

Principiants were shewn by Sylvester to be the simultaneous solutions of the equations

$$(1, 1;\ 1, 1)_a = 0,$$

$$(4, 1;\ 2, 1)_a = 0.$$

It is first necessary to shew that in certain cases the solutions of $(\mu, \nu;\ m, n+1)_a = 0$ are easily obtained.

Digression on the solutions of $(\mu, \nu;\ m, n+1)_a = 0$.

A pocket solution of this partial differential equation is obtainable in two cases:

(i) when $m = 1$, (ii) when $\nu = 0$.

Case 1. $m=1$.

By substituting for the elements a_0, a_1, a_2, ... certain numerical multiples of them, the operator is at once transformable to $(1, 0;\ 1,\ n+1)_a = P$ suppose.

Let
$$U = a_0 + a_1 u + a_2 u^2 + \ldots,$$
so that
$$PU = a_0 u^{n+1} + a_1 u^{n+2} + a_2 u^{n+3} + \ldots = u^{n+1} U.$$
Then
$$Pf(U) = f'(U)\, u^{n+1} U,$$
and we can determine $f(U)$ so that
$$f'(U)\, U = 1,$$
for then
$$f(U) = \log U.$$
Hence
$$P \log (a_0 + a_1 u + a_2 u^2 + \ldots) = u^{n+1},$$
and if
$$\log (a_0 + a_1 u + a_2 u^2 + \ldots) = B_0 + B_1 u + B_2 u^2 + \ldots,$$
$$(1, 0;\ 1, n+1)_a B_s = 0, \text{ if } s \neq n+1,$$
$$(1, 0;\ 1, n+1)_a B_{n+1} = 1,$$
and I say that $\log (a_0 + a_1 u + a_2 u^2 + \ldots)$ is the pocket solution of $(1, 0;\ 1, n+1)_a = 0$, under the condition above stated.

Case 2. $\mu = 1$, $\nu = 0$.

Put
$$(1, 0;\ m, n+1)_a = Q,$$
and
$$\frac{1}{m} U^m = M_0 + M_1 u + M_2 u^2 + \ldots.$$
Since
$$QU = M_0 u^{n+1} + M_1 u^{n+2} + M_2 u^{n+3} + \ldots = u^{n+1} \frac{1}{m} U^m,$$
$$Qf(U) = f'(U) \frac{1}{m} U^m u^{n+1},$$
and the dexter is equal to u^{n+1} if
$$f(U) = -\frac{m}{m-1} U^{-(m-1)}.$$
Hence
$$(1, 0;\ m, n+1)_a U^{-(m-1)} = -\frac{m-1}{m} u^{n+1},$$
or if
$$U^{-(m-1)} = C_0 + C_1 u + C_2 u^2 + \ldots,$$
$$(1, 0;\ m, n+1)_a C_s = 0, \text{ when } s \neq n+1,$$
$$(1, 0;\ m, n+1)_a C_{n+1} = -\frac{m-1}{m},$$
and I call $(a_0 + a_1 u + a_2 u^2 + \ldots)^{-(m-1)}$ the pocket solution of $(1, 0;\ m, n+1)_a = 0$, under the condition above stated.

The relation
$$(\mu, \nu;\ m, n+1)_a$$
$$= -\frac{1}{m} \{\mu (n-m+3), \mu - m\nu;\ n-m+3, n+1\}_b$$
now enables us to determine all forms of operator $(\mu, \nu;\ m, n+1)_a$ for which pocket solutions exist.

These are

$$\text{I.}\quad (\mu, \nu;\ 1, n+1).$$
$$\text{II.}\quad (1, 0;\ m, n+1).$$
$$\text{III.}\quad (\mu, \nu;\ n+2, n+1).$$
$$\text{IV.}\quad (m, 1;\ m, n+1).$$

Art. 36. To resume the argument of Art. 35 I remark that the simultaneous solutions of the two equations

$$(1, 1;\ 1, 1)_a = 0,$$
$$(4, 1;\ 2, 1)_a = 0,$$

also satisfy the equation

$$(2, 1;\ 2, 2)_a = 0,$$

since the alternant of $(1, 1;\ 1, 1)_a$ and $(4, 1;\ 2, 1)_a$ is $(2, 1;\ 2, 2)_a$.

Transforming these three equations we obtain

$$(1, 0;\ 2, 1)_b = 0,$$
$$(2, 1;\ 1, 1)_b = 0,$$
$$(1, 0;\ 2, 2)_b = 0,$$

and it will be noted that pocket solutions can be given of each of these three equations.

The first equation has the solution

$$\frac{1}{b_0 + b_1 u + b_2 u^2 + b_3 u^3 + \ldots},$$

and since
$$\frac{b_0}{b_0 + b_1 u + b_2 u^2 + b_3 u^3 + \ldots} = 1 - k_1 u + k_2 u^2 - k_3 u^3 + \ldots,$$

where k_s is the homogeneous product sum of order s of the roots of the equation

$$b_0 \nu^r - b_1 \nu^{r-1} + b_2 \nu^{r-2} - b_3 \nu^{r-3} + \ldots = 0,$$

we see that every solution is a function of $k_2, k_3, k_4, \ldots$.

The third equation has the same pocket solution, but here k_2 is not a solution; hence every solution is a function of $k_1, k_3, k_4, k_5, \ldots$.

Hence the simultaneous solutions of the two equations

$$(1, 0;\ 2, 1)_b = 0,$$
$$(1, 0;\ 2, 2)_b = 0,$$

are functions of $k_3, k_4, k_5, \ldots$.

Art. 37. What functions of these elements then satisfy the second equation

$$(2, 1;\ 1, 1)_b = 0\,?$$

The equation is

$$N_b = 2b_0 \partial_{b_1} + 3b_1 \partial_{b_2} + 4b_2 \partial_{b_3} + \ldots = 0,$$

and writing
$$b_0 + b_1 u + b_2 u^2 + \ldots = B,$$

$$N_b \frac{b_0}{B} = -\frac{b_0}{B^2}(2b_0 u + 3b_1 u^2 + 4b_2 u^3 + \ldots)$$

$$= -\frac{b_0}{B^2}(2uB + u^2 \partial_u B)$$

$$= -2u\frac{b_0}{B} + u^2 \partial_u \frac{b_0}{B},$$

$$\therefore\ N_b(1 - k_1 u + k_2 u^2 - k_3 u^3 + \ldots) = (-2u + u^2 \partial_u)(1 - k_1 u + k_2 u^3 - k_3 u^3 + \ldots),$$

$$\therefore\ N_b k_1 = -2,\ \ N_b k_2 = k_1,\ \ N_b k_3 = 0,\ \ N_b k_4 = -k_3,$$

and
$$N_b k_s = -(s-3)\,k_{s-1}.$$

Hence N_b operating upon a function of $k_3, k_4, k_5, \ldots$

is equivalent to
$$k_3 \partial_{k_4} + 2k_4 \partial_{k_5} + 3k_5 \partial_{k_6} + \ldots,$$

and this establishes that principiants are seminvariants, *quâ* the elements

$$k_3,\ k_4,\ k_5,\ \ldots.$$

It will be remembered that Sylvester and Hammond shewed that principiants were seminvariant functions

(1) of seminvariantive elements,

(2) of reciprocantive elements.

In each case the law defining the elements was one which connected three successive elements by a formula involving a partial differential operation—nothing more was known of their internal structure. The present research shews that the seminvariantive elements above referred to are by the 'advanced' substitution converted into the well-known forms

$$k_3,\ k_4,\ k_5,\ \ldots,$$

which can be written down *currenti calamo* in terms of the functions $b_0, b_1, b_2, \ldots$.

In fact Sylvester's elements $A_0, A_1, A_2, \ldots$ are by that substitution converted into

$$-k_3,\ k_4,\ -k_5,\ \ldots.$$

Art. 38. This most interesting development has a parallel in the homographic reciprocants of Rogers; he shewed that such satisfy the two partial differential equations

$$(1, 1;\ 1, 0)_t = 0,$$

$$(1, 0;\ 2, 0)_t = 0.$$

Now observe that by the transformation theorem

$$(1, 1;\ 1, 0)_t = -2(1, 0;\ 2, 0)_\tau,$$

shewing that every reciprocant that satisfies

$$(1, 1;\ 1, 0)_t = 0$$

also satisfies
$$(1, 0;\ 2, 0)_t = 0,$$

and that in any case every solution of the former is by an interchange of the variables converted into a solution of the latter and *vice versâ*.

The solution of the simultaneous equations

$$(1, 1;\ 1, 0)_t = 0,$$

$$(1, 0;\ 2, 0)_t = 0,$$

is easily obtained. The latter possesses the pocket solution

$$(t + a_0 u + a_1 u^2 + \dots)^{-1},$$

in such wise that writing

$$t\,(t + a_0 u + a_1 u^2 + \dots)^{-1} = 1 - j_1 u + j_2 u^2 - j_3 u^3 + \dots,$$

every solution is a function of $\quad t,\ j_2,\ j_3,\ j_4,\ \dots,$

j_s being the homogeneous product sums of the roots of an equation

$$tx^n - a_0 x^{n-1} + a_1 x^{n-2} - \dots = 0.$$

Now it is easy to shew that

$$(1, 1;\ 1, 0)_t \frac{t}{t + a_0 u + a_1 u^2 + \dots} = (u^2 \partial_u - u) \frac{t}{t + a_0 u + a_1 u^2 + \dots},$$

and thence that

$$(1, 1;\ 1, 0)_t\ j_{s+1} = -(s-1)\, j_s.$$

Hence $(1, 1;\ 1, 0)_t$ operating upon a function of $t,\ j_2,\ j_3,\ j_4, \dots$ is equivalent to

$$j_2 \partial_{j_3} + 2 j_3 \partial_{j_4} + 3 j_4 \partial_{j_5} + \dots.$$

We have thus confirmed the result of Rogers that the simultaneous solutions of the equations

$$(1, 1;\ 1, 0)_t = 0,$$

$$(1, 0;\ 2, 0)_t = 0,$$

are seminvariant functions of the elements $j_2,\ j_3,\ j_4, \dots$.

These solutions are necessarily reciprocants because the interchange of x and y converts

$$(1, 1;\ 1, 0)_t \text{ into } (1, 0;\ 2, 0)_\tau,$$

and

$$(1, 0;\ 2, 0)_t \text{ into } (1, 1;\ 1, 0)_\tau.$$

Apart altogether from invariant theory it is not, I think, possible to over-estimate the importance of the transformation involved in the formulae for the interchange of the variables. The group of multilinear operators now possess the two cardinal properties:

(i) they form a group *quâ* alternation;

(ii) they form a group *quâ* the transformation which interchanges the variables, the formulae being regarded merely as giving a substitution proceeding from one suffixed set of letters a to another suffixed set b, and having no essential connexion with variables x and y.

I hope to discuss subsequently the operator invariants appertaining to the transformation of Halphen.

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THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS.

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VII. *The solution of Linear Differential Equations by means of Definite Integrals**.

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1. The solution of a linear differential equation by means of a definite integral of a specified type, e.g.

$$f(x) = \int \kappa(x, t)\, \phi(t)\, dt,$$

is a problem which is closely allied to that of the inversion of a definite integral. The integral equation connecting $f(x)$ and $\phi(t)$ may be regarded as one expression of a linear relation between the two functions. In order to solve the second problem we must obtain an equation in which $\phi(t)$ is derived directly from $f(x)$ by means of a distributive operation. The first problem will be considered as solved when a functional equation to be satisfied by $\phi(t)$ is obtained by means of a distributive operation† from the differential equation satisfied by $f(x)$.

The function $\kappa(x, t)$ will be regarded as the *nucleus* of a transformation which makes the function $f(x)$ correspond to $\phi(t)$, the differential equation satisfied by $f(x)$ to a functional equation (usually a differential equation) satisfied by $\phi(t)$. When definite limits are assigned in the integral there is a further correspondence between the linear conditions satisfied by $f(x)$ and the linear conditions satisfied by $\phi(t)$.

To illustrate this let us put

$$f(x) = \int_a^b \kappa(x, t)\, \phi(t)\, dt,$$

and suppose that

$$\int_c^d f(x)\, \chi(x)\, dx = \lambda,$$

then if

$$g(t) = \int_c^d \chi(x)\, \kappa(x, t)\, dx,$$

* This paper has originated from one portion of a Smith's prize essay (1905). The work has now been brought to a more complete stage but the main ideas have been unaltered.

† The general theory of distributive operations has been developed by Pincherle. S. Pincherle e U. Amaldi, "Le operazione funzionali distributive," Bologna (1901), *Encyklopädie der Mathematischen Wissenschaften*, Band II. 1, Heft 6.

and all the functions under consideration are continuous, we must have

$$\int_a^b g(t)\,\phi(t)\,dt = \lambda.$$

The linear condition to be satisfied by $\phi(t)$ is thus derived immediately from the one satisfied by $f(x)$.

2. The transformation of a differential equation or functional equation by means of the nucleus $\kappa(x, t)$ depends upon the formation of a relation of the type

$$L_x[\kappa(x, t)] = M_t[\kappa(x, t)],$$

where
$$L_x(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + \ldots\ldots p_n u$$

is the given differential equation, and $M_t(u)$ is either a differential equation or linear functional equation.

For if the integral
$$f(x) = \int \kappa(x, t)\,\phi(t)\,dt$$

can be differentiated n times by the rule of Leibnitz, we have

$$L_x(f) = \int L_x[\kappa(x, t)]\,.\,\phi(t)\,dt$$

$$= \int M_t[\kappa(x, t)]\,.\,\phi(t)\,dt.$$

In order then that f may be a solution of $L_x(f) = 0$, the function $\phi(t)$ must be chosen so that the quantity on the right hand side is zero. If M_t is a differential equation we may take $\phi(t)$ to be an integrating factor, so that the quantity under the integral sign is a perfect differential. If M_t is a *difference equation* we choose $\phi(t)$ so that the quantity under the integral sign is a *perfect difference*. In either case $\phi(t)$ is a solution of the equation adjoint to $M_t(u) = 0$.

Finally, to ensure that $L_x(f) = 0$, the limits or path of integration must be chosen so that the integral on the right hand side is zero. The introduction of double circuit integrals has made the method much more effective, for we are now able to obtain, in general, representations of all the solutions of an equation. The choice of a contour appropriate to a particular solution of the equation has become quite an art*.

In the present paper we shall endeavour to trace out the connection between relations of the type†

$$L_x(u) = M_t(v)$$

and the theory of definite integrals. The principal objects that will be kept in view are

* The following list of papers dealing with this part of the subject may prove useful:
Jordan, *Cours d'Analyse*, t. III. pp. 240—276.
Pochhammer, *Math. Ann.*, t. XXXV. pp. 470—494, 495—526; t. XXXVII. pp. 500—511.
Hobson, *Phil. Trans. Roy. Soc.* (A), vol. CLXXXVII. p. 493.
Klein, "Vorlesungen über die hypergeometrische Funktion" (lithographed), Göttingen (1894); *Math. Ann.* (1891), t. XXXVIII. pp. 144—152.
Barnes, *Proc. Lond. Math. Soc.*, Ser. 2, vol. VI. Part 2 (1908), p. 141.

† The importance of relations of this type in the present subject was pointed out by Schlesinger in his *Theorie der linearen Differentialgleichungen*.

the formation of inversion formulae, i.e. the discovery of a transformation which is the inverse of a given one and the construction of periodic transformations*.

3. The following properties of two adjoint differential expressions will be required in the course of the work; for proofs we may refer to Forsyth's *Theory of Differential Equations*, Part III. vol. IV. pp. 252—254.

(1) If $M_t(u)$ and $\bar{M}_t(v)$ are two adjoint differential equations we have the identity

$$vM_t(u) - u\bar{M}_t(v) = \frac{d}{dt} R(u, v),$$

where $R(u, v)$ is the bilinear concomitant.

(2) If $M_t(u) = 0$ be a composite equation, i.e. if

$$M(u) = m_1 . m_2 \ldots m_r(u),$$

then $\bar{M}_t(u)$ will also be a composite equation and will be composed of the adjoint factors in the reverse order.

The whole theory of definite integrals is closely related to the theory of functional relations of the type

$$L_x(u) = M_t(v).$$

In the case when $L_x(u)$ and $M_t(v)$ are differential equations this analogy may be developed as follows:

First, consider the property that transformations of the type

$$f(x) = \int \kappa(x, t)\, \phi(t)\, dt = K(\phi)$$

form a group†.

Let $\kappa(x, t)$ be the nucleus of a transformation from a differential equation $L_x(u) = 0$ satisfied by $f(x)$ to a differential equation $\bar{M}_t(v) = 0$ satisfied by $\phi(t)$ and $h(t, y)$ the nucleus of a transformation from $\bar{M}_t(v) = 0$ to $\bar{N}_y(w) = 0$. Then we have

$$L_x(\kappa) = M_t(v),$$

$$\bar{M}_t(h) = N_y(w),$$

where κ and v are functions of x and t,

h „ w „ „ „ t „ y.

Let $$g(x, y) = \int \kappa(x, t)\, h(t, y)\, dt,$$

where the path of integration is at first unspecified.

* This problem has been discussed by Levi-Civita, "Sur les groupes d'opérations fonctionnelles et sur l'inversion des intégrales définies," *Rendiconti Instituto Lombardo*, t. XXVII. (1895), pp. 529—544, 565—577.

† Pincherle, *Acta Mathematica*, vol. X. p. 153.

We have in general
$$L_x(g) = \int L_x(\kappa)\, h(t, y)\, dt$$
$$= \int M_t(v)\, h(t, y)\, dt$$
$$= \int v \,.\, \bar{M}_t(h)\, dt + \int \frac{d}{dt} R(v, h)\, dt$$
by the first property of the adjoint equation.

Now let the path of integration be chosen so that
$$\int \frac{dR}{dt}\, dt \equiv 0,$$
then
$$L_x(g) = \int v \,.\, \bar{M}_t(h)\, dt = \int v \,.\, N_y(w)\, dt$$
$$= N_y \int v(x, t)\, w(t, y)\, dt$$
$$= N_y(e).$$

Hence there is a relation of the type
$$L_x(g) = N_y(e),$$
or a transformation connecting L_x *and* N_y *of which* $g(x, y)$ *is the nucleus.*

It should be noticed that the existence of the relation
$$L_x(\kappa) = M_t(v)$$
implies that $v(x, t)$ is the nucleus of a transformation from $M_t(v) = 0$ to $\bar{L}_x(z) = 0$.

Next, let
$$L_x(\kappa) = M_t(v),$$
$$P_x(\kappa) = Q_t(v),$$
then
$$[L_x + P_x](\kappa) = [M_t + Q_t](v).$$
Now the equation adjoint to $[M_t + Q_t](v) = 0$ is $[\bar{M}_t + \bar{Q}_t](v) = 0$, hence $L_x + P_x$ is transformed into $\bar{M}_t + \bar{Q}_t$ by means of the transformation with the nucleus $\kappa(x, t)$. This shows that the operation by which we pass from L_x to M_t is in general *distributive.*

This remark is important because it enables us to construct very general relations of the type
$$L_x(\kappa) = M_t(v)$$
bit by bit. For every integral of the given type will in general furnish us with a relation of this form, and if suitable changes are made to keep v and κ the same, we may multiply all the different relations that are known by arbitrary constants and add them together. This is really another step towards the solution of the problem of representing a function $f(s)$ by means of a definite integral of the type
$$f(s) = \int \kappa(s, t)\, \phi(t)\, dt,$$
because the solution of
$$\lambda L_x(u) + \mu P_x(u) = 0$$
is in general not a linear combination of the solutions of $L_x(u) = 0$ and $P_x(u) = 0$, thus the

representation of a function which is linearly independent of those whose representations are already known, can be obtained.

Next consider the effect of various operations upon the equation

$$L_x(\kappa) = M_t(v).$$

Let P_x be a differential operator such that

$$P_x(v) = Q_t(w),$$

then
$$P_x L_x(\kappa) = M_t P_x(v) = M_t Q_t(w).$$

Now the differential equation adjoint to $M_t Q_t(w)$ is $\bar{Q}_t \bar{M}_t(w)$, hence $P_x . L_x$ corresponds to $\bar{Q}_t \bar{M}_t$ in the transformation of nucleus κ. The most interesting case occurs when $w = v$, because then the transformation from $P_x . L_x$ to $\bar{Q}_t \bar{M}_t$ is of exactly the same type as before.

In the case of the Laplace transformation $\kappa = v = e^{xt}$, and we have

$$\psi(x) . e^{xt} = \psi\left(\frac{d}{dt}\right) . e^{xt}.$$

The equation adjoint to $\psi\left(\frac{d}{dt}\right)$ is $\psi\left(-\frac{d}{dt}\right)$, accordingly

$$P_x . [\psi(x) . u] \text{ corresponds to } \bar{Q}_t . \psi\left(-\frac{d}{dt}\right) v,$$

and
$$\psi(x) . P_x(u) \quad\text{,,}\qquad\text{,,}\quad \psi\left(-\frac{d}{dt}\right) \bar{Q}_t(v).$$

4. The problem of the inversion of a definite integral may be considered as partially solved when a transformation depending upon a function $\bar{\kappa}(t, x)$ can be found such that*

$$\bar{M}_t \{\bar{\kappa}(t, x)\} \equiv \bar{L}_s \{\bar{v}(t, x)\}.$$

The first inversion formula of this type was discovered by Cauchy† for the case of the Laplace transformation.

If the quantities a_{nm} are quite arbitrary and

$$L_x(u) \equiv \sum_m \sum_n a_{nm} x^n \frac{d^m u}{dx^m},$$

$$M_t(u) \equiv \sum_m \sum_n a_{nm} t^m \frac{d^n u}{dt^n},$$

$$u = \kappa(x, t) = e^{xt},$$

we have
$$L_x(u) = M_t(u),$$
$$\bar{L}_x(v) = \bar{M}_t(v),$$
where
$$v = e^{-xt}.$$

It is difficult to construct reciprocal relations of the above type by a direct method and so we are obliged to use definite integrals to guide us to the result.

Consider, for instance, the case in which the nucleus is of the form $F(x + y)$, we may then proceed as follows.

* See a paper by the author, *Proceedings of the London Mathematical Society*, ser. 2, vol. IV. Parts 6 and 7, p. 486.

† *Exercices de Math.*, vol. II. Paris (1827), p. 157.

Let
$$f(x) = \int e^{xt}\phi(t)\,dt,$$
then we shall in general have a relation of the type
$$L_x(f) = \bar{M}_t(\phi),$$
where $\bar{M}_t$ is derived from L_x by means of the Laplace transformation, i.e.
$$\bar{M} = \Lambda\,.\,(L).$$

Now alter the variable in the equation $\bar{M}_t(\phi) = 0$ putting
$$\phi(t) = h(t)\,.\,g(t),$$
and let
$$g(t) = \int e^{ty}\psi(y)\,dy,$$
then
$$f(x) = \iint e^{t(x+y)}\,h(t)\,\psi(y)\,dy\,dt,$$
accordingly, if
$$F(x+y) = \int e^{t(x+y)}\,h(t)\,dt,$$
we have
$$f(x) = \int F(x+y)\,\psi(y)\,dy.$$

This suggests that there is a transformation depending on the nucleus $F(x+y)$ by means of which we can pass from the differential equation $L_x(f)$ satisfied by $f(x)$ to the differential equation $\bar{N}_y(\psi) = 0$ satisfied by ψ, in accordance with the group property investigated above.

When we are given the function $F(x+y)$ a differential equation satisfied by $h(t)$ can be deduced by means of the Laplace transformation or in some cases the function $h(t)$ itself may be derived from $F(x)$ by means of Pincherle's inversion formula
$$F(x) = \int_c e^{tx}h(t)\,dt,$$
$$h(t) = \frac{1}{2\pi i}\int_0^\infty e^{-xt}F(x)\,dx.$$

The transformation from $L_x(f)$ to $\bar{N}_y(\psi)$ is then built up as follows:

(1) A Laplace transformation from $L_x(f)$ to $\bar{M}_t(\phi)$.

(2) A change in the dependent variable from $\phi(t)$ to $h(t)\,g(t)$.

(3) A Laplace transformation from the new equation to $\bar{N}_y(\psi)$.

Inverting the order of proceedings we can build up the transformation from $\bar{N}_y(\psi)$ to $L_x(f)$ as follows:

(1) An inverse Laplace transformation.

(2) A change in the dependent variable from $g(t)$ to $\dfrac{\phi(t)}{h(t)}$.

(3) An inverse Laplace transformation.

It is clear that the nucleus of the inverse transformation is also of the form $\bar{F}(x+y)$*.

If H denote the operation of multiplying the dependent variable by $h(t)$, the transformation and its inverse may be written in the symbolical forms

$$T = \Lambda H \Lambda,$$
$$T^{-1} = \Lambda^{-1} H^{-1} \Lambda^{-1}.$$

We shall now follow out the work in detail in the case when $F(x+y) = (x+y)^{-n-1}$. The definite integral to be used in solving differential equations is then of the form

$$f(x) = \int \frac{\psi(y)}{(x+y)^{n+1}}.$$

Integrals of this type were used by Euler to solve linear differential equations, the transformation arising from the nucleus $(x+y)^{-(n+1)}$ has consequently been named after him † although the analysis of the transformation is chiefly due to Heine ‡.

It will be convenient to replace y by $-t$, the nucleus then takes the form $(x-t)^{n+1}$ usually used in Euler's transformation.

Let the differential equation satisfied by $f(x)$ be

$$P_x(w) = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} a_{m,\kappa} x^{\kappa} \frac{d^m w}{dx^m}.$$

The equation derived from this by means of the transformation of Laplace is

$$\bar{Q}_z(u) = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} (-1)^{\kappa} a_{m,\kappa} \frac{d^{\kappa}}{dz^{\kappa}} (z^m u).$$

Changing the sign of the independent variable and writing $u = x^n v$, we obtain the equation

$$R_z(v) = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} (-1)^m a_{m,\kappa} \frac{d^{\kappa}}{dz^{\kappa}} (z^{m+n} v).$$

The lowest power of z in this equation is in general z^{n-q}, but there are two cases in which it is of a higher order:

(1) when $a_{0,q} = 0$, (2) when n is a positive integer less than $q-1$.

* The general inversion formula given by Sonine, *Acta Mathematica*, vol. IV. p. 171, is closely connected with this result.

Let
$$k(t) = 1 + c_1 t + c_2 t^2 + \dots,$$
$$\frac{1}{k(t)} = 1 + d_1 t + d_2 t^2 + \dots,$$
$$\sigma(x) = x^{-p} \sum_{m=0}^{\infty} \frac{c_m x^m}{\Gamma(m-p+1)},$$
$$p+q=1,$$
$$\psi(x) = x^{-q} \sum_{n=0}^{\infty} \frac{d_n x^n}{\Gamma(n-q+1)},$$
then, if
$$f(x) - f(a) = \int_a^x \psi(x-\lambda) \phi(\lambda) \, d\lambda,$$
$$\phi(\lambda) = \int_a^{\lambda} \sigma(\lambda - x) f'(x) \, dx.$$

The function $\sigma(x)$ is obtained from $t^{p-1} k\left(\frac{1}{t}\right)$ and $\psi(x)$ from $\frac{t^{q-1}}{k\left(\frac{1}{t}\right)}$ by inverse Laplace transformations. If we put

$$h(t) = \frac{t^{q-1}}{k\left(\frac{1}{t}\right)},$$
$$\frac{1}{h(t)} = t^p \cdot k\left(\frac{1}{t}\right),$$

and remark that if $\overline{M}_t(v)$ is the Laplace transformed equation of $L_x(u)$, then $\overline{M}_t(tu)$ is the Laplace transformed equation of $L_x\left(\frac{du}{dx}\right)$, the analogy is complete.

† Cf. Schlesinger, *Theorie der linearen Differentialgleichungen*, vol. II. p. 415.

‡ *Crelle*, Band 60 (1862), p. 252; Band 61 (1863), p. 356; Band 62 (1863), p. 110. *Handbuch der Kugelfunktionen*, vol. I. 2nd ed. Berlin (1881), Part III. p. 466.

Dealing first with the case when n is not a positive integer $<(s-1)$ let $z^{n-\rho}$ be the lowest power of z in $R(v)$. Dividing throughout by $z^{n-\rho}$ the equation takes the form

$$R_1(v) = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} \sum_{r=0}^{\kappa} (-1)^m a_{m,\kappa} \frac{\kappa!}{r!(\kappa-r)!} \frac{(m+n)\dots(m+n-r+1)}{1} z^{m+\rho-r} \frac{d^{\kappa-r}v}{dz^{\kappa-r}}.$$

The equation obtained from this by Laplace's transformation is the adjoint of

$$S_t(w) = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} \sum_{r=0}^{\kappa} (-1)^m a_{m,\kappa} \frac{(m+n)\dots(m+n-r+1)}{r!(\kappa-r)!} (\kappa)!\, t^{\kappa-r} \frac{d^{m+\rho-r}w}{dt^{m+\rho-r}},$$

and will be called Euler's transformed equation.

Since $$\frac{d^{m+\rho-r}}{dt^{m+\rho-r}}(x-t)^{\rho-n-1} = \frac{(n-\rho+1)(n-\rho+2)\dots(n+m-r)}{(x-t)^{m+n+1-r}}$$
$$= \frac{(n-\rho+1)(n-\rho+2)\dots n\,.(x-t)^r}{(m+n)(m+n-1)\dots(m+n-r+1)} \frac{d^m}{dt^m}\left[\frac{1}{(x-t)^{n+1}}\right],$$

we find that

$$S_t[(x-t)^{\rho-n-1}] = \sum_{m=0}^{p} \sum_{\kappa=0}^{q} \sum_{r=0}^{\kappa} (n-\rho+1)\dots n\,.(-1)^m \frac{(\kappa)!}{(r)!(\kappa-r)!} a_{m,\kappa} t^{\kappa-r}(x-t)^r . \frac{d^m}{dt^m}(x-t)^{-n-1}$$
$$= \sum_{m=0}^{p} \sum_{\kappa=0}^{q} (n-\rho+1)\dots n\,.\,a_{m,\kappa} x^\kappa \frac{d^m}{dx^m}(x-t)^{-n-1}$$
$$= (n-\rho+1)\dots n\,.\,P_x[(x-t)^{-n-1}].$$

This relation $$(n-\rho+1)\dots n\, P_x[(x-t)^{-n-1}] \equiv S_t[(x-t)^{\rho-n-1}]$$

indicates that there is a transformation depending upon the nucleus $(x-t)^{-n-1}$ from $P_x(w)$ to $\bar{S}_t(w)$, it also indicates the existence of a transformation depending on the nucleus $(x-t)^{\rho-n-1}$ from $S_t(u)$ to $\bar{P}_x(u)$ *.

In the case when n is a *positive integer* less than ρ the equation just obtained reduces to

$$S_t[(x-t)^{\rho-n-1}] = 0,$$

and the method breaks down. It appears then that the lowest differential coefficient in S_t is at least of order $\rho-n$.

To obtain the right relation in this case we write $n+\epsilon$ in place of n and equate coefficients of ϵ when ϵ is very small. The relation is

$$\left[S_t + \epsilon\frac{\partial}{\partial n}S_t\right][(x-t)^{\rho-n-1}\{1-\epsilon\log(x-t)\}] \equiv (n-\rho+\epsilon+1)\dots(n+\epsilon)\,P_x[(x-t)^{-n-\epsilon+1}].$$

Now $$\left(\frac{\partial}{\partial n}S_t\right)(x-t)^{\rho-n-1} \equiv 0,$$

because the lowest differential coefficient in $\frac{\partial}{\partial n}S_t$ is also at least of order $\rho-n$, thus on equating coefficients of ϵ we obtain

$$S_t[(x-t)^{\rho-n-1}\operatorname{Log}(x-t)] = (-1)^{\rho-n}\,n!\,(\rho-n-1)!\,P_x[(x-t)^{-n-1}],$$

one of the numbers $(n-\rho+\epsilon+1)\dots(n+\epsilon)$ on the right hand side of the previous equation being equal to ϵ.

As before there is a transformation depending on the nucleus $(x-t)^{-n-1}$ from $P_x(w)$ to $\bar{S}_t(w)$, so that in all cases $\bar{S}_t(w)$ may be called Euler's transformed equation. The transformation from $S_t(u)$ to $\bar{P}_x(u)$, however, now depends on the nucleus

$$(x-t)^{\rho-n-1}\log(x-t).$$

* Cf. Schlesinger, *Theorie der linearen Differentialgleichungen*, vol. II. p. 416.

The equation $S_t(u)$ does not contain a differential coefficient of order less than $\rho - n$, it may be reduced to the usual form by writing $\frac{d^{\rho-n}u}{dt^{\rho-n}} = v$. The quantity u may be expressed as a definite integral by means of the transformation and v derived from it by differentiation.

If X_n denotes the operation of multiplying the dependent variable of a differential equation by x^n and Λ^{-1} denotes the inverse of Laplace's transformation, i.e. a Laplace's transformation combined with a change in sign of the independent variable, Euler's transformation E_n can be expressed in the symbolical form

$$E_n = \Lambda^{-1} X_n \Lambda.$$

Hence

$$\begin{aligned} E_n E_m &= \Lambda^{-1} X_n \Lambda \,.\, \Lambda^{-1} X_m \Lambda \\ &= \Lambda^{-1} X_n X_m \Lambda = \Lambda^{-1} X_{m+n} \Lambda \\ &= E_{m+n}, \end{aligned}$$

so that *the operation E_n possesses the addition theorem*

$$E_n E_m = E_{n+m}.$$

The transformation inverse to E_n is E_{-n} according to what has gone before, so we may expect to find relations of the type

$$f(x) = \int (x-t)^{n-1} \phi(t)\, dt,$$

$$\phi(t) = \lambda \int (x-t)^{-n-1} f(x)\, dx.$$

Cauchy's integral

$$2\pi i f(x) = \int_c \frac{f(t)}{t-x}\, dt$$

is the most famous formula of this kind. Webb * has shown that when $n = 0$ the general hypergeometric equation of the pth order is transformed into itself by means of Euler's transformation, so that relations of the type

$$f(x) = \int \frac{\phi(t)}{t-x}\, dt$$

connecting two different solutions of the equation may be sought.

To illustrate this we may consider the hypergeometric equation of the fourth order satisfied by the functions

$$P_n(x) P_m(x), \quad P_n(x) Q_m(x), \quad P_m(x) Q_n(x), \quad Q_m(x) Q_n(x),$$

it is easy to prove that

$$P_n(x) Q_m(x) = \tfrac{1}{2} \int_{-1}^{+1} \frac{P_n(\mu) P_m(\mu)}{x-\mu}\, d\mu,$$

where m and n are integers and $m \geqslant n$ †.

* *Phil. Trans. Roy. Soc.*, Ser. A, vol. CCIV. pp. 481—497.

† This may be deduced very easily from the integral

$$Q_m(x) = \tfrac{1}{2} \int_{-1}^{+1} \frac{P_m(\mu)}{x-\mu}\, d\mu$$

by induction, using the addition formula for $P_n(x)$.

The addition formula for the operation E_n is closely connected with the fact that E_n and the operation of differentiation obey the commutative law*, and is in harmony with Riemann's representation † of a fractional differentiation

$$D^s \alpha(x) = \frac{1}{\Gamma(m-s)} D^m \int^x (x-z)^{m-s-1} \alpha(z)\, dz.$$

In this formula the variable enters into the limits of the integral. In general the formulae for the transformation of the differential equations are the same whether the variable appears in the limits or not, provided the integral can be obtained as the limit of a contour integral. Thus we have an illustration of the inversion formula $\frac{1}{E_n} = E_{-n}$ in Abel's formula

$$f(x) = \frac{\sin n\pi}{\pi} \int_0^x \frac{\phi'(a)}{(x-a)^{1-n}}\, da,$$

$$\phi(a) = \int_0^a \frac{f(x)\, dx}{(a-x)^n}, \qquad 1 > n > 0,$$

when the last equation is differentiated with regard to a. Another formula of inversion has been obtained by the author ‡.

If
$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{(1-2ts+s^2)^\nu}, \qquad (\nu > 0),$$

then
$$\phi(t) = \frac{1}{i\pi} \int_{t-i\sqrt{1-t^2}}^{t+i\sqrt{1-t^2}} (1-2ts+s^2)^{\nu-1} [\nu f(s) + s f'(s)]\, ds.$$

To reduce this to the standard form we must put

$$s + \frac{1}{s} = 2x,$$

$$2^\nu s^\nu f(s) = F(x),$$

we then get
$$F(x) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{(x-t)^\nu},$$

$$\phi(t) = \frac{1}{2\pi i} \int_C (x-t)^{\nu-1} F'(x)\, dx,$$

where C is a certain contour which starts at the point $x = t$ and finishes up there. If we put

$$s = t + i\sqrt{1-t^2} \cos\alpha,$$

the contour is traced out by the point

$$x = \tfrac{1}{2}\left[t + i\sqrt{1-t^2}\cos\alpha + \frac{1}{t + i\sqrt{1-t^2}\cos\alpha}\right],$$

as α varies between 0 and π. It is evidently a loop which cuts the axis at the point

$$x = \tfrac{1}{2}\left(t + \frac{1}{t}\right),$$

i.e. at a point beyond the singularity $x = 1$.

* Pincherle, *Encyklopädie der Mathematischen Wissenschaften*, Band II. 1, Heft 6.

† *Werke*, ed. Dedekind und Weber, p. 331.

‡ *Proceedings of the London Mathematical Society*, Ser. 2, vol. IV. Parts 6, 7, p. 469.

Another special case of the inversion formula is

$$f(s)=\int_0^\infty \frac{\phi(t)}{(s-t)^{n+1}},$$

$$\phi(t)=-\frac{n}{4\pi\sin n\pi}\int_C f(s)(s-t)^{n-1}\,ds,$$

where C is a double loop contour round the points 0 and t. The restrictions to be laid on $f(s)$ and $\phi(t)$ in order that this formula may hold have not been found. The theorem is certainly true when

$$f(s)=(-s)^{m-1}\frac{\Gamma(n+m)\,\Gamma(1-m)}{\Gamma(n+1)},\quad \phi(t)=t^{n+m-1},$$

for we have Pochhammer's formula *.

There is evidently much room for research in this direction, if a general inversion formula could be found the result would be of considerable importance in the theory of integral equations.

5. Transformations of the type

$$f(x)=\int \kappa(x-t)\,\phi(t)\,dt$$

have been applied to linear differential equations by Mellin † and Cailler ‡.

Mellin's result is that if $\kappa(u)$ satisfies the equation

$$\left[F\left(\frac{d}{du}\right)-uG\left(\frac{d}{du}\right)\right]\kappa=0,$$

which is of Laplace's type, then there is a transformation depending on the nucleus $\kappa(x-t)$ from

$$\left[P\left(\frac{d}{dx}\right)G\left(\frac{d}{dx}\right)+Q\left(\frac{d}{dx}\right)F\left(\frac{d}{dx}\right)-xQ\left(\frac{d}{dx}\right)G\left(\frac{d}{dx}\right)\right]f=0,$$

an equation of Laplace's type, to

$$\left[P\left(\frac{d}{dt}\right)-tQ\left(\frac{d}{dt}\right)\right]\phi=0,$$

which is also an equation of Laplace's type.

From our point of view this transformation depends upon the fact that the partial differential equation

$$\left[P\left(\frac{d}{dx}\right)G\left(\frac{d}{dx}\right)+Q\left(\frac{d}{dx}\right)F\left(\frac{d}{dx}\right)-xQ\left(\frac{d}{dx}\right)G\left(\frac{d}{dx}\right)\right]w$$
$$=\left[P\left(-\frac{d}{dt}\right)G\left(-\frac{d}{dt}\right)-Q\left(-\frac{d}{dt}\right)tG\left(-\frac{d}{dt}\right)\right]w,$$

* *Math. Ann.* xxxv. (1890), pp. 495—526.

† *Acta Societatis scientiarum Fennicae*, t. XXI. No. 6 (1896).

‡ Darboux's *Bulletin*, XXIII.

is satisfied by $w=\kappa(x-t)$. If we put $G\left(-\frac{d}{dt}\right)w=v$ the expression on the right hand side may be written

$$\left[P\left(-\frac{d}{dt}\right)-Q\left(-\frac{d}{dt}\right)t\right]v,$$

which is adjoint to the equation satisfied by ϕ.

M. Cailler's result is somewhat simpler, it depends upon the fact that the partial differential equation

$$\left[xG\left(\frac{d}{dx}\right)+F\left(\frac{d}{dx}\right)\right]w=\left[tG\left(-\frac{d}{dt}\right)+P\left(-\frac{d}{dt}\right)\right]w \text{ ....(1)},$$

is satisfied by $w=\kappa(x-t)=\kappa(z)$, if

$$\left[zG\left(\frac{d}{dz}\right)+F\left(\frac{d}{dz}\right)-P\left(\frac{d}{dz}\right)\right]w=0 \text{ ....}(\kappa),$$

it thus gives a transformation from

$$\left[xG\left(\frac{d}{dx}\right)+F\left(\frac{d}{dx}\right)\right]f=0$$

to *

$$\left[G\left(\frac{d}{dt}\right)t+P\left(\frac{d}{dt}\right)\right]\phi=0$$

by means of a nucleus $\kappa(x-t)=\kappa(z)$ which satisfies equation (κ).

If the equations satisfied by f, ϕ and κ are

$$\Sigma(a_r+b_rx)\frac{d^{n-r}f}{dx^{n-r}}=0,$$

$$\Sigma(a_r'+b_r't)\frac{d^{n-r}\phi}{dt^{n-r}}=0,$$

$$\Sigma(a_r''+b_r''z)\frac{d^{n-r}\kappa}{dz^{n-r}}=0,$$

respectively, the relation between the parameters is

$$b_r=b_r'=b_r'',$$

$$a_r''=a_r+a_r'-(n-r+1)\,b_{r-1}.$$

Transformations which are applicable to equations not of Laplace's type may be constructed by means of the artifice explained at the beginning. Mellin has given the formula

$$\kappa(x-t)=\kappa(z),$$

$$\left[F\left(\frac{d}{dz}\right)-zG\left(\frac{d}{dz}\right)\right]\kappa=0,$$

$$\left[P_0\left(\frac{d}{dt}\right)+P_1\left(\frac{d}{dt}\right)t+\ldots P_n\left(\frac{d}{dt}\right)t^n\right]\phi=0,$$

$$\Big\{P_0\left(\frac{d}{dx}\right)G^n\left(\frac{d}{dx}\right)+P_1\left(\frac{d}{dx}\right)G^{n-1}\left(\frac{d}{dx}\right)(xG-F)+P_2\left(\frac{d}{dx}\right)G^{n-2}\left(\frac{d}{dx}\right)(xG+G'-F)(xG-F)$$
$$+\ldots P_n\left(\frac{d}{dx}\right)[xG+(n-1)G'-F]\ldots[xG-F]\Big\}f=0.$$

* This equation is adjoint to the one on the right hand side of (1).

If we denote the last equation by $L_x(f)$, it is clear that the equation

$$L_x(\kappa) \equiv \left[P_0\left(-\frac{d}{dt}\right) + tP_1\left(-\frac{d}{dt}\right) + \dots t^n P_n\left(-\frac{d}{dt}\right)\right] G^n\left(-\frac{d}{dt}\right)\kappa$$

is satisfied by $\kappa(x-t)$.

This can be regarded as the fundamental relation of type

$$L_x(\kappa) = M_t(v)$$

on which the transformation depends.

As we have remarked before this relation is the same when the variable x appears in the upper limit of the integral. Integrals of the type

$$f(x) = \int_0^x \kappa(x-t)\,\phi(t)\,dt$$

may be conveniently studied in connection with a theorem of Borel's* which states that if

$$u(z) = \int_0^\infty e^{-zt}\,\kappa(t)\,dt, \qquad v(z) = \int_0^\infty e^{-zt}\,\phi(t)\,dt,$$

then

$$u(z)\,v(z) = \int_0^\infty e^{-zt} f(t)\,dt,$$

where $f(t)$ is connected with $\kappa(t)$ and $\phi(t)$ by the above equation.

If we know the differential equations satisfied by $\kappa(t)$ and $\phi(t)$ we can find the differential equations satisfied by $u(z)$ and $v(z)$ and deduce a differential equation satisfied by $u(z)\ v(z)$. The function $f(t)$ will then satisfy the equation obtained from this last one by the inverse of Laplace's transformation.

As an illustration of the above formula we may take

$$\kappa(t) = J_m(t), \quad \phi(t) = \frac{J_n(t)}{t},$$

we then have

$$u(z) = \frac{[\sqrt{1+z^2} - z]^m}{\sqrt{1+z^2}},$$

$$v(z) = \frac{1}{n}[\sqrt{1+z^2} - z]^n,$$

$$\therefore\ u(z)\,v(z) = \frac{1}{n}\frac{[\sqrt{1+z^2} - z]^{m+n}}{\sqrt{1+z^2}},$$

which corresponds to $f(t) = \frac{1}{n} J_{m+n}(t)$, hence we have the formula†

$$\int_0^x J_m(x-t)\,J_n(t)\,\frac{dt}{t} = \frac{1}{n} J_{m+n}(x).$$

6. The transformations depending on the nucleus $\kappa(xt)$ may evidently be derived from those depending on a nucleus $F(u+v)$ by writing $x = e^u$, $t = e^v$. They have been

* *Leçons sur les Séries Divergentes*, p. 104. A more general proof is given by E. Cunningham, *Proc. Lond. Math. Soc.* Ser. 2, vol. III. (1904), p. 161.

† For another derivation of this result see a paper by the author. *Proc. Lond. Math. Soc.* Ser. 2, vol. III. Part 2.

studied in detail by Mellin (*op. cit.* pp. 39—46), who gives a formula for a very general type of transformation. This formula, however, does not cover all possible cases and we shall find it more convenient to construct relations of the type $L_x(u) = M_t(u)$ directly by special artifices.

In the first place it is clear that if

$$u = \kappa(xt) = \kappa(z),$$

$$F\left(x\frac{d}{dx}\right)u = F\left(t\frac{d}{dt}\right)u = F\left(z\frac{d}{dz}\right)u.$$

Now the equation

$$\left[x^n F\left(x\frac{d}{dx}\right) + G\left(x\frac{d}{dx}\right)\right]u = \left[G\left(t\frac{d}{dt}\right) + t^{-n}H\left(t\frac{d}{dt}\right)\right]u$$

is satisfied by $u = \kappa(xt) = \kappa(z)$, if

$$\left[z^n F\left(z\frac{d}{dz}\right) - H\left(z\frac{d}{dz}\right)\right]u = 0,$$

accordingly there is a transformation depending on the nucleus $\kappa(xt)$ from

$$\left[x^n F\left(x\frac{d}{dx}\right) + G\left(x\frac{d}{dx}\right)\right]u = 0$$

to the differential equation which is the adjoint equation of

$$\left[t^n G\left(t\frac{d}{dt}\right) + H\left(t\frac{d}{dt}\right)\right]u = 0.$$

These equations are all of Pfaff's type*.

Application to the hypergeometric equation. The equation

$$x^2(x-1)\frac{\partial^2 u}{\partial x^2} + \{(a+b+1)x^2 - cx\}\frac{\partial u}{\partial x} + abxu = t(1-t)\frac{\partial^2 u}{\partial t^2} + (d-ct)\frac{\partial u}{\partial t}$$

is satisfied by $u = \kappa(xt) = \kappa(z)$, if

$$z(z-1)\frac{d^2u}{dz^2} + \{(a+b+1)z - d\}\frac{du}{dz} + abu = 0.$$

Now $t^{d-1}(1-t)^{c-d-1}$ is an integrating factor of

$$t(1-t)\frac{\partial^2 u}{\partial t^2} + (d-ct)\frac{\partial u}{\partial t},$$

and the boundary condition

$$\left[t^d(1-t)^{c-d}\frac{du}{dt}\right]_{t_0}^{t_1} = 0$$

is satisfied by $t_0 = 0$, $t_1 = 0$ provided $d > 0$, $c > 0$, thus we obtain the equation

$$\int_0^1 t^{d-1}(1-t)^{c-d-1}F(a, b, d, xt)\,dt = \frac{\Gamma(d)\,\Gamma(c-d)}{\Gamma(c)}F(a, b, c, x).$$

When $d = b$, $F(a, b, d, xt) = F(a, b, b, xt) = (1-xt)^{-a}$, and the above reduces to Euler's formula

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}F(a, b, c, x).$$

* Cf. Boole's *Differential Equations*, p. 420.

If in the previous equation we put $x=1$ and replace $F(a, b, c, 1)$ by its value

$$\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-b)\,\Gamma(c-a)},$$

we find that

$$\int_0^1 t^{d-1}(1-t)^{c-d-1} F(a, b, d, t)\,dt = \frac{\Gamma(d)\,\Gamma(c-d)\,\Gamma(c-a-b)}{\Gamma(c-b)\,\Gamma(c-a)}.$$

Again, the partial differential equation

$$\left[x^n F\left(x\frac{d}{dx}\right) + x^{-n} G\left(x\frac{d}{dx}\right)\right] u = -\left[t^n F\left(t\frac{d}{dt}\right) + t^{-n} G\left(t\frac{d}{dt}\right)\right] u$$

is satisfied by $u=\kappa(xt)=\kappa(z)$, if

$$\left[z^n F\left(z\frac{d}{dz}\right) + G\left(z\frac{d}{dz}\right)\right] u = 0.$$

Hence in this case there is a transformation depending on the nucleus $\kappa(xt)$ from a differential equation

$$\left[x^{2n} F\left(x\frac{d}{dx}\right) + G\left(x\frac{d}{dx}\right)\right] u = 0$$

of Pfaff's type to the adjoint equation.

Example. The equation

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} + \left(x - \frac{n^2}{x}\right) u = -t\frac{d^2u}{dt^2} - \frac{du}{dt} - \left(t - \frac{n^2}{t}\right) u$$

is satisfied by $u=\kappa(xt)$, if

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z-n^2)\,u = 0.$$

A solution of this equation is given by $u=J_{2n}(2\sqrt{z})$, while

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} + \left(x - \frac{n^2}{x}\right) u = 0$$

is a self adjoint differential equation satisfied by $J_n(x)$. We are thus led to the formula

$$J_n(x) = \int_0^\infty J_{2n}(2\sqrt{xt})\,J_n(t)\,dt,$$

which appears to hold for real + values of x and for values of n such that

$$R(n) > -\tfrac{1}{2}\text{*}.$$

* This formula is such a simple one that an independent method of obtaining it will not be out of place here.

Starting with the formula (Nielsen, *Handbuch der Cylinderfunktionen*, p. 184)

$$\int_0^\infty e^{-z^2t} J_\nu(xz)\,J_\nu(yz)\,z\,dz = \frac{1}{2t} e^{-\frac{x^2+y^2}{4t}-\frac{\nu\pi i}{2}} J_\nu\left(\frac{ixy}{2t}\right),$$

$$R(t)>0, \qquad R(\nu)>-1,$$

we put $y=\dfrac{a^2}{x}$, and integrate with regard to x between 0 and ∞, then if

$$f(z) \equiv \int_0^\infty J_\nu(xz)\,J_\nu\left(\frac{a^2z}{x}\right) dx,$$

we have

$$\int_0^\infty e^{-z^2t} f(z)\,z\,dz = \frac{1}{2t} e^{-\frac{\nu\pi i}{2}} J_\nu\left(\frac{ia^2}{2t}\right) \int_0^\infty e^{-\frac{1}{4t}\left(x^2+\frac{a^4}{x^2}\right)} dx$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{\nu\pi i}{2}} J_\nu\left(\frac{ia^2}{2t}\right) e^{-\frac{a^2}{2t}}.$$

Now

$$\int_0^\infty e^{-z^2t} J_{2\nu}(2az)\,dz = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, J_\nu\left(\frac{ia^2}{2t}\right) e^{-\frac{a^2}{2t}-\frac{\nu\pi i}{2}},$$

and Lerch has shown (*Acta Math.*, 1903) that there is only one continuous function $f(z)$, such that

The formula

$$J_{2n}(2u) = \int_0^\infty J_n\left(\frac{u^2}{v}\right) J_n(v)\, dv$$

admits of some interesting applications.

If in the expansion

$$\sum_{s=0}^{\infty} (\nu+s) P_s^\nu(\cos\theta) J_{\nu+s}(ax) J_{\nu+s}(\alpha x) = \frac{(\frac{1}{2}a\alpha x)^\nu}{\Gamma(\nu)} \frac{J_\nu[x\sqrt{a^2 - 2a\alpha\cos\theta + \alpha^2}]}{[a^2 - 2a\alpha\cos\theta + \alpha^2]^{\frac{\nu}{2}}}$$

we put $\alpha = \frac{z^2}{a}$ and integrate with regard to a between 0 and ∞, we obtain

$$\sum_{s=0}^{\infty} \frac{\nu+s}{x} P_s^\nu(\cos\theta) J_{2\nu+2s}(2xz) = \left(\frac{x}{2}\right)^\nu \frac{z^{2\nu}}{\Gamma(\nu)} \int_0^\infty \frac{J_\nu\left[x\sqrt{a^2 + \frac{z^4}{a^2} - 2z^2\cos\theta}\right]}{\left[a^2 + \frac{z^4}{a^2} - 2z^2\cos\theta\right]^{\frac{\nu}{2}}} da.$$

Now if $F(a)$ is an even function

$$\int_0^\infty F\left(a - \frac{z^2}{a}\right) da = \int_0^\infty F(a)\, da,$$

$$\therefore \int_0^\infty \frac{J_\nu\left[x\sqrt{a^2 + \frac{z^4}{a^2} - 2z^2\cos\theta}\right]}{\left[a^2 + \frac{z^4}{a^2} - 2z^2\cos\theta\right]^{\frac{\nu}{2}}} da = \int_0^\infty \frac{J_\nu\left(x\sqrt{a^2 + 4z^2\sin^2\frac{\theta}{2}}\right)}{\left(a^2 + 4z^2\sin^2\frac{\theta}{2}\right)^{\frac{\nu}{2}}} da.$$

Again,

$$\int_0^\infty \frac{J_\nu(x\sqrt{a^2+b^2})}{(a^2+b^2)^{\frac{\nu}{2}}} da = \sqrt{\frac{\pi}{2x}} \frac{J_{\nu-\frac{1}{2}}(bx)}{b^{\nu-\frac{1}{2}}},$$

for if we write the integral on the left hand side equal to $f(x)$, we have

$$\int_0^\infty e^{-x^2 t} x^{\nu+1} f(x)\, dx = \int_0^\infty \int_0^\infty e^{-x^2 t} \frac{J_{\nu+1}(x\sqrt{a^2+b^2})}{(a^2+b^2)^{\frac{\nu}{2}}} x^{\nu+1} dx\, da,$$

but

$$\int_0^\infty e^{-x^2 t} J_\nu(xz)\, x^{\nu+1} dx = \frac{z^\nu}{(2t)^{\nu+1}} e^{-\frac{z^2}{4t}},$$

$$\int_0^\infty e^{-z^2 t} f(z)\, z\, dz = \text{a given function of } t,$$

accordingly, we must have

$$zf(z) = J_{2\nu}(2az).$$

This gives the equation

$$J_{2\nu}(2az) = z\int_0^\infty J_\nu(xz) J_\nu\left(\frac{a^2 z}{x}\right) dx.$$

Putting $az = u$, $xz = v$, this takes the simpler form

$$J_{2\nu}(2u) = \int_0^\infty J_\nu\left(\frac{u^2}{v}\right) J_\nu(v)\, dv \quad \text{.........(1).}$$

If on the other hand we write $a^2 z = b$ the formula may be written

$$J_{2\nu}(2\sqrt{bz}) = z\int_0^\infty J_\nu(xz) J_\nu\left(\frac{b}{x}\right) dx.$$

Now Hankel has shown that if

$$f(z) = \int_0^\infty J_\nu(xz)\, \phi(x)\, x\, dx,$$

then

$$\phi(x) = \int_0^\infty J_\nu(xz) f(z)\, z\, dz.$$

Applying this to the present case we obtain

$$\frac{1}{x} J_\nu\left(\frac{b}{x}\right) = \int_0^\infty J_\nu(xz) J_{2\nu}(2\sqrt{bz})\, dz.$$

Putting $xz = t$, $b = xy$, we get

$$J_\nu(y) = \int_0^\infty J_{2\nu}(2\sqrt{yt}) J_\nu(t)\, dt \quad \text{............(2),}$$

the formula in question.

consequently the double integral becomes

$$\frac{1}{(2t)^{\nu+1}}\int_0^\infty e^{-\frac{a^2+b^2}{4t}}\,da = \frac{\sqrt{\pi}}{2^{\nu+1}t^{\nu+\frac{1}{2}}}e^{-\frac{b^2}{4t}}.$$

Now
$$\int_0^\infty e^{-x^2t}x^{\nu+\frac{1}{2}}J_{\nu-\frac{1}{2}}(bx) = \frac{b^{\nu-\frac{1}{2}}}{(2t)^{\nu+\frac{1}{2}}}e^{-\frac{b^2}{4t}},$$

hence since there is only one function $f(x)$ which leads to a given function of t, we must have

$$f(x) = \sqrt{\frac{\pi}{2}}\,\frac{1}{b^{\nu-\frac{1}{2}}}x^{-\frac{1}{2}}J_{\nu-\frac{1}{2}}(bx),$$

i.e.
$$\int_0^\infty \frac{J_\nu\left(x\sqrt{a^2+b^2}\right)}{(a^2+b^2)^{\frac{\nu}{2}}}\,da = \sqrt{\frac{\pi}{2x}}\,\frac{J_{\nu-\frac{1}{2}}(bx)}{b^{\nu-\frac{1}{2}}}.$$

This gives
$$\int_0^\infty \frac{J_\nu\left[x\sqrt{a^2+\frac{z^4}{a^2}-2z^2\cos\theta}\right]}{\left[a^2+\frac{z^4}{a^2}-2z^2\cos\theta\right]^{\frac{\nu}{2}}}\,da = \sqrt{\frac{\pi}{2x}}\,\frac{J_{\nu-\frac{1}{2}}(2xz\sin\theta)}{(2z\sin\theta)^{\nu-\frac{1}{2}}}.$$

We thus arrive at the expansion

$$\sum_{s=0}^\infty (\nu+s)\,P_s^\nu(\cos\theta)\,J_{2\nu+2s}(2xz) = \frac{\sqrt{\pi}}{2^{\nu+\frac{1}{2}}}\,\frac{(xz)^{\nu+\frac{1}{2}}}{\Gamma(\nu)}\,\frac{J_{\nu-\frac{1}{2}}(2xz\sin\theta)}{(2\sin\theta)^{\nu-\frac{1}{2}}}.$$

Putting $z=1$ it takes the simpler form

$$\frac{J_{\nu-\frac{1}{2}}(2x\sin\theta)}{(x\sin\theta)^{\nu-\frac{1}{2}}} = \frac{\Gamma(\nu)}{\sqrt{\pi}}\left(\frac{2}{x}\right)^{2\nu}\sum_{s=0}^\infty(\nu+s)\,P_s^\nu(\cos\theta)\,J_{2\nu+2s}(2x).$$

This formula is a particular case of a more general expansion given in Nielsen's *Handbuch der Cylinderfunktionen*. The simplification that occurs in this particular case appears to have been overlooked.

7. We shall now consider the problem of constructing transformations which are simply periodic.

Let $L_s(u)$ and $\overline{L}_s(u)$ be two adjoint linear differential expressions and let $w(s, t)$ be a solution of the partial differential equation*

$$L_s(w) = \overline{L}_t(w).$$

Consider the integral
$$f(s) = \int w(s, t)\,\phi(t)\,dt,$$

the path of integration and the function $\phi(t)$ being at present arbitrary except that they must be such as to allow the integral to be differentiated a suitable number of times by the rule of Leibnitz. We then have

$$\begin{aligned} L_s(f) &= \int L_s(w)\,\phi(t)\,dt \\ &= \int \overline{L}_t(w)\,\phi(t)\,dt \\ &= \int w(s, t)\,L_t(\phi)\,dt - \int \frac{d}{dt}R(\phi, w)\,dt, \end{aligned}$$

where R is the bilinear concomitant.

* Levi Civita, *op. cit.*

Now let the path of integration and the function ϕ be chosen so that the second integral is zero for all values of s in a given interval or domain, then we have the relation

$$L_s(f) = \int w(s, t) L_t(\phi) dt,$$

which indicates that the distributive operations $L_s(\phi)$ and

$$W_s(\phi) = \int w(s, t) \phi(t) dt$$

obey the commutative law.

In what follows we shall suppose the integral is taken along a real path from a to b and that the linear conditions imposed on ϕ are such that the above equation is satisfied for all values of s in the interval (a, b).

Now let $\phi(t)$ be a solution of the differential equation

$$L_t(u) + \lambda u = 0,$$

and let us suppose that the linear conditions to be satisfied by $\phi(t)$ are such that they can only be satisfied by a solution of the above equation for certain particular isolated values of λ. We shall call these the conditions "C" and shall assume that they are sufficient to determine a solution of the differential equation uniquely except for an arbitrary constant multiplier.

Further, let $w(s, t)$ be chosen so that it satisfies the conditions C when considered as a function of s; the function $f(s)$ will then also satisfy these conditions, provided $\phi(t)$ and $w(s, t)$ are continuous. Moreover, since

$$L_s(f) = \int_a^b w(s, t) L_t(\phi) dt$$

$$= -\lambda \int_a^b w(s, t) \phi(t) dt = -\lambda f,$$

$f(s)$ is a solution of $L_s(f) + \lambda f = 0$, hence it must be a constant multiple of $\phi(s)$ and so we have the relation

$$\phi(s) = \mu \int_a^b w(s, t) \phi(t) dt,$$

where the value of μ depends in some way on the corresponding value of λ.

This is a homogeneous integral equation of the first kind which is satisfied by all the solutions of $L_s(f) + \lambda f = 0$ which satisfy the given conditions. The values of μ corresponding to the different singular values of λ are the characteristic values of μ for the integral equation.

If the function $w(s, t)$ be properly chosen the solutions of the adjoint integral equation

$$\psi(t) = \mu \int_a^b \psi(s) w(s, t) ds$$

will be solutions of the adjoint differential equation

$$\bar{L}_t(u) + \lambda u = 0,$$

and will satisfy a set of linear conditions which are satisfied by $w(s, t)$ when considered as a function of t. We shall come across some illustrations of this presently. The conditions to be satisfied in this case may be regarded as *adjoint* to the previous set.

For the sake of simplicity we shall now suppose that $L_s(u)$ is a differential equation of the second order. Such an equation can be made self adjoint by multiplying throughout by a suitable factor $p(s)$. Let

$$H_s(u) = p(s) L_s(u),$$

then, since

$$v(s) p(s) L_s(u) - u(s) \overline{L}_s [v(s) p(s)] = \frac{d}{ds} R[u, vp],$$

and

$$v(s) H_s(u) - u(s) H_s(v) = \frac{dR}{ds},$$

we must have

$$H_s(v) = \overline{L}_s [v(s) p(s)].$$

Thus the partial differential equation satisfied by w may be written

$$\frac{1}{p(s)} H_s [w(s, t)] = H_t \left[\frac{w(s, t)}{p(t)}\right].$$

Now write

$$\kappa(s, t) = \sqrt{\frac{p(s)}{p(t)}} w(s, t),$$

then $\kappa(s, t)$ is a solution of

$$M_s[\kappa(s, t)] = M_t[\kappa(s, t)],$$

where $M_s(u)$ denotes the self adjoint differential expression

$$\frac{1}{\sqrt{p(s)}} H_s \left[\frac{u}{\sqrt{p(s)}}\right].$$

This partial differential equation is symmetrical in s and t and it will be convenient to assume that $\kappa(s, t)$ is also symmetrical in s and t. The relations between the linear conditions satisfied by $w(s, t)$, when considered as a function of s and when considered as a function of t, are obtained at once from the expression of $w(s, t)$ in terms of $\kappa(s, t)$. In the case of the function $\kappa(s, t)$ the two sets of conditions are the same.

If the conditions satisfied by $\kappa(s, t)$ are the following,

$$\kappa(a, t) \equiv 0, \quad \kappa(b, t) \equiv 0, \quad \kappa(s, a) \equiv 0, \quad \kappa(s, b) \equiv 0,$$

the conditions satisfied by $w(s, t)$ are evidently of the same type.

If $\kappa(s, t)$ remains finite at $s = a$ and $s = b$, but $p(s)$ becomes zero when $s = a$ like $(s-a)^\rho$, then it is clear that

$$(s-a)^{\frac{\rho}{2}} w(s, t) \text{ is finite when } s = a,$$

$$(t-a)^{-\frac{\rho}{2}} w(s, t) \text{ is finite when } t = a.$$

If $\kappa(s, t)$ becomes infinite like $(s-a)^{-r}$ when $s = a$, and like $(t-a)^{-r}$ when $t = a$, then $w(s, t)$ will become infinite like $(s-a)^{-r-\frac{\rho}{2}}$ when $s = a$, and like $(t-a)^{-r+\frac{\rho}{2}}$ when $t = a$.

The integral equation

$$\chi(s) = \mu \int_a^b \kappa(s, t)\, \chi(t)\, dt$$

is transformed by means of the substitution

$$\chi(t) = \sqrt{p(t)}\, \phi(t)$$

into
$$\phi(s) = \mu \int_a^b w(s, t)\, \phi(t)\, dt.$$

Now $\chi(t)$ is a solution of an equation of the type

$$M_t(\chi) + \lambda\chi = 0,$$

i.e. of
$$\frac{1}{\sqrt{p(t)}} H_t \left[\frac{\chi}{\sqrt{p(t)}}\right] + \lambda\chi = 0,$$

i.e. of
$$\sqrt{p(t)}\, L_t \left[\frac{\chi}{\sqrt{p(t)}}\right] + \lambda\chi = 0,$$

or
$$L_t[\phi] + \lambda\phi = 0,$$

and satisfies the conditions C.

On the other hand, if we put

$$\psi(t) = \sqrt{p(t)}\, \chi(t),$$

we get
$$\psi(s) = \mu \int_a^b \psi(t)\, w(t, s)\, dt,$$

and
$$\frac{1}{\sqrt{p(t)}} \overline{L}_t [\sqrt{p(t)}\, \chi(t)] + \lambda\chi = 0,$$

i.e.
$$\overline{L}_t(\psi) + \lambda\psi = 0.$$

Hence the two adjoint linear differential equations

$$L_t(\phi) + \lambda\phi = 0,$$
$$\overline{L}_t(\psi) + \lambda\psi = 0,$$

possess solutions satisfying the two adjoint sets of linear conditions for the same particular values of λ and are solutions of the two adjoint integral equations

$$\phi(s) = \mu \int_a^b w(s, t)\, \phi(t)\, dt,$$

$$\psi(t) = \mu \int_a^b \psi(s)\, w(s, t)\, dt.$$

Moreover, since
$$\chi(s) = \mu \int_a^b \kappa(s, t)\, \chi(t)\, dt,$$

where $\kappa(s, t)$ is a *symmetric* function, the singular values of μ are all real.

8. It is easy to construct a partial differential equation of the type

$$P_s(w) = Q_t(w)$$

that is satisfied by the function $w = e^{pst}$ by taking

$$P_s(w) = \sum_m \sum_n a_{mn} p^n s^n \frac{d^m w}{ds^m},$$

$$Q_t(w) = \sum_m \sum_n a_{mn} p^m t^m \frac{d^n u}{dt^n},$$

where the quantities a_{mn} are constants.

Let $P_s(w) \equiv Q_s(w) \equiv (1-s^2)\frac{d^2w}{ds^2} + \{n - m - (m+n)s\}\frac{dw}{ds} + \{p^2s^2 - p(m-n)s + \lambda\} w,$

then the above requirements are evidently satisfied. The equation $P_s(w)$ is not self adjoint, but if we write

$$\kappa(s, t) = \sqrt{\alpha(s)\,\alpha(t)} \,.\, w(s, t),$$

$$\alpha(s) = (1+s)^{n-1}(1-s)^{m-1},$$

we have $$M_s(\kappa) = M_t(\kappa),$$

where $M_s(u)$ denotes the self adjoint expression

$$M_s(u) = \sqrt{\alpha(s)}\, P_s\left[\frac{u}{\sqrt{\alpha(s)}}\right].$$

Since $M_s(u)$ is a self adjoint differential equation whose first term is $(1-s^2)\frac{du^2}{ds^2}$, we have

$$M_s(u) = \frac{d}{ds}\left[(1-s^2)\frac{du}{ds}\right] + qu,$$

where q is some function of s. Accordingly, if v is any function of t,

$$vM_t(u) - uM_t(v) = \frac{d}{dt}\left[(1-t^2)\left(v\frac{du}{dt} - u\frac{dv}{dt}\right)\right] = -\frac{d}{dt}\left[(1-t^2)\,u^2\frac{d}{dt}\left(\frac{v}{u}\right)\right] = \frac{dR}{dt}.$$

In this equation put

$$u = \kappa(s, t) = (1+s)^{\frac{n-1}{2}}(1+t)^{\frac{n-1}{2}}(1-s)^{\frac{m-1}{2}}(1-t)^{\frac{m-1}{2}} e^{pst},$$

and choose $v(t)$ to be a function which has the forms

$$(1+t)^{\frac{n-1}{2}}[a_0 + a_1(1+t) + \ldots\ldots],$$

$$(1-t)^{\frac{m-1}{2}}[b_0 + b_1(1-t) + \ldots\ldots],$$

in the neighbourhood of the points $t = -1$, $t = +1$ respectively. The quantity R will then vanish at both limits provided $n > 0$, $m > 0$. Accordingly, if

$$\chi(s) = \int_{-1}^{+1} \kappa(s, t)\, v(t)\, dt,$$

$$M_s(\chi) = \int_{-1}^{+1} M_s(\kappa)\, v(t)\, dt$$

$$= \int_{-1}^{+1} M_t(\kappa)\, v(t)\, dt$$

$$= \int_{-1}^{+1} \kappa(s, t)\, M_t(v)\, dt + \int_{-1}^{+1} \frac{dR}{dt}\, dt,$$

i.e. $$M_s(\chi) = \int_{-1}^{+1} \kappa(s, t)\, M_t(v)\, dt.$$

If λ is chosen so that a function of type v can be found for which

$$M_t(v) = 0,$$

we shall have $M_s(\chi) = 0$, and since χ satisfies the same linear conditions as v it must be a multiple of v, consequently

$$\chi(s) = \mu \int_{-1}^{+1} \kappa(s, t) \chi(t) \, dt.$$

Putting $\chi(s) = \sqrt{\alpha(s)} \phi(s)$, we find that $\phi(s)$ is a solution of $P_s(\phi) = 0$, which remains finite at the limits $s = \pm 1$ and is a solution of an integral equation of the type

$$\phi(s) = \mu \int_{-1}^{+1} e^{pst} (1+t)^{n-1} (1-t)^{m-1} \phi(t) \, dt.$$

Again, if we put $$\psi(s) = \sqrt{\alpha(s)} \chi(s),$$

we find that $\psi(s)$ vanishes to the orders $(n-1)$, $(m-1)$ at the points $s = -1$, $s = +1$ respectively, and is a solution of the equation adjoint to $P_s(u) = 0$. This equation may be written

$$\overline{P}_s(u) = (1 - s^2) \frac{d^2u}{ds^2} - [n - m - (m + n - 4) s] \frac{du}{ds} + [p^2 s^2 - p(m-n) s + m + n - 2 + \lambda] u = 0,$$

and the corresponding integral equation is

$$\psi(t) = \mu \int_{-1}^{+1} \chi(s) e^{pst} (1+t)^{n-1} (1-t)^{m-1} \, ds.$$

If we put $2 - n$ for n, $2 - m$ for m and change the sign of p, this differential equation takes the same form as $P_s(u)$ except that the value of λ is different.

Combining the two results we have the following theorem.

If λ be chosen so that the differential equation

$$(1 - s^2) \frac{d^2u}{ds^2} + \{n - m - (m+n) s\} \frac{du}{ds} + \{p^2 s^2 - p(m-n) s + \lambda\} u = 0$$

possesses a solution $u = \phi(s)$ which is finite for $s = \pm 1$, $(n \geqslant 0, m \geqslant 0)$, $\phi(s)$ will satisfy an integral equation of the type

$$\phi(s) = \mu \int_{-1}^{+1} e^{pst} (1+t)^{n-1} (1-t)^{m-1} \phi(t) \, dt;$$

if on the other hand, λ be chosen so that $u = \chi(s)$ is zero to the orders $1 - n$, $1 - m$ at the points $s = -1$, $s = +1$ respectively with $(n \leqslant 2, m \leqslant 2)$, $\chi(s)$ will satisfy an integral equation of the type

$$\psi(s) = \nu \int_{-1}^{+1} e^{-pst} (1+s)^{1-n} (1-s)^{1-m} \psi(t) dt.$$

When $n = 0$ the first part of the theorem still holds provided $\phi(t)$ is chosen to be a solution of the differential equation which vanishes to the first order at the point $t = -1$, similarly, if $m = 0$, $\phi(t)$ must vanish to the first order at $t = +1$.

The second part of the theorem holds for $n = 2$ if $\psi(t)$ remains finite for $t = -1$, similarly, it holds for $m = 2$, if $\psi(t)$ remains finite for $t = +1$.

The particular case in which $m=n$ has been discussed by Max Abraham (*Math. Ann.* Bd. 52, pp. 81—112), and the asymptotic expansions of the functions

$$\theta(s)=\int_{-\infty}^{+1} e^{pst}(1+t)^{n-1}(1-t)^{n-1}\phi(t)\,dt,$$

$$\omega(s)=\int_{1}^{\infty} e^{-pst}(1+s)^{1-n}(1-s)^{1-m}\psi(t)\,dt,$$

have been investigated. The functions $\phi(t)$ and $\psi(t)$ are now the analytic functions defined by the equations

$$\phi(s)=\mu\int_{-1}^{+1} e^{pst}(1+t)^{n-1}(1-t)^{n-1}\phi(t)\,dt,$$

$$\psi(s)=\mu\int_{-1}^{+1} e^{-pst}(1+s)^{1-n}(1-s)^{1-n}\psi(t)\,dt,$$

for all finite values of s. The derivation of the principal terms in the asymptotic expansions of $\theta(s)$ and $\omega(s)$ as performed by Abraham must be considered as an important departure in applications of integral equations to analysis.

It should be noticed that when $m=n=\frac{1}{2}$ the differential equation (1) reduces to the equation of the elliptic cylinder. The above theorem is then a particular case of the following, which was communicated to the author by E. T. Whittaker four years ago.

It is known that the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\lambda^2 u=0$$

is satisfied by *

$$u=\int_0^{2\pi} e^{\lambda(x\cos\alpha+y\sin\alpha)} f(\alpha)\,d\alpha.$$

Let us put $x=\cosh\omega\cos\phi$, $y=\sinh\omega\sin\phi$, then the differential equation becomes

$$\frac{\partial^2 u}{\partial \omega^2}+\frac{\partial^2 u}{\partial \phi^2}+\lambda^2(\cos^2\phi-\cosh^2\omega)\,u=0,$$

and is satisfied by $u=E(\omega)\,F(\phi)$ if

$$\frac{d^2E}{d\omega^2}-(\lambda^2\cosh^2\omega-p)\,E=0,$$

$$\frac{d^2F}{d\phi^2}+(\lambda^2\cos^2\phi-p)\,F=0.$$

The definite integral is now

$$u=\int_0^{2\pi} e^{\lambda\{\cosh\omega\cos\phi\cos\alpha+\sinh\omega\sin\phi\sin\alpha\}} f(\alpha)\,d\alpha,$$

and is in general a periodic function of ϕ of period 2π. We shall now endeavour to choose $f(\alpha)$ so that the integral may satisfy the second differential equation, but it is clear that p must be chosen so that the equation possesses a periodic solution.

* E. T. Whittaker, *Math. Ann.* (1903), pp. 333—355.

We have $$\frac{\partial^2 u}{\partial \phi^2} = \int_0^{2\pi} [\lambda^2 \{\sinh \omega \cos \phi \sin \alpha - \cosh \omega \sin \phi \cos \alpha\}^2$$
$$- \lambda \{\cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha\}] e^{\lambda \{\cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha\}} f(\alpha)\, d\alpha.$$

Now $(\sinh \omega \cos \phi \sin \alpha - \cosh \omega \sin \phi \cos \alpha)^2$
$$= (\sinh \omega \sin \phi \cos \alpha - \cosh \omega \cos \phi \sin \alpha)^2 + \cos^2 \alpha - \cos^2 \phi.$$

Therefore $$\frac{\partial^2 u}{\partial \phi^2} = \int_0^{2\pi} \left[\lambda^2 (\cos^2 \alpha - \cos^2 \phi) + \frac{\partial^2}{\partial \alpha^2}\right] e^{\lambda \{\cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha\}} f(\alpha)\, d\alpha,$$

and so $$\frac{\partial^2 u}{\partial \phi^2} + (\lambda^2 \cos^2 \phi - p)\, u$$
$$= \int_0^{2\pi} [(\lambda^2 \cos^2 \alpha - p) f(\alpha) + f''(\alpha)]\, e^{\lambda z} + \left[-f'(\alpha)\, e^{\lambda z} + f(\alpha) \frac{\partial}{\partial \alpha} e^{\lambda z}\right]_0^{2\pi},$$

where $$z = \cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha.$$

If then $f(\alpha)$ is chosen to be a solution of

$$\frac{d^2 f}{d\alpha^2} + (\lambda^2 \cos^2 \alpha - p) f = 0,$$

which has the period 2π, the quantity between the limits will disappear and u will be a solution of the differential equation

$$\frac{d^2 u}{d\phi^2} + (\lambda^2 \cos^2 \phi - p)\, u = 0.$$

Let this solution be denoted by $F(\phi)$, then we must have

$$E(\omega)\, F(\phi) = \int_0^{2\pi} e^{\lambda \{\cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha\}} F(\alpha)\, d\alpha.$$

This is a homogeneous integral equation of the second kind for the determination of $F(\phi)$.

If we put $\omega = i\psi$, the integral on the right hand side is a symmetrical function of ϕ and ψ, accordingly, $E(\omega)$ should be a constant multiple of $F(\psi)$. We thus have the formula

$$\mu F(\phi)\, F(\psi) = \int_0^{2\pi} e^{\lambda \{\cos \psi \cos \phi \cos \alpha + i \sin \psi \sin \phi \sin \alpha\}} F(\alpha)\, d\alpha.$$

The solutions of this equation for the different possible values of μ are the periodic solutions of

$$\frac{d^2 F}{d\phi^2} + (\lambda^2 \cos^2 \phi - p)\, u = 0$$

for the different possible values of p.

If $F_m(\phi)$ and $F_n(\phi)$ are the solutions for two different values of μ we have

$$\int_0^{2\pi} F_m(\phi)\, F_n(\phi)\, d\phi = 0,$$

this suggests the existence of an expansion of the type

$$e^{\lambda \{\cosh \omega \cos \phi \cos \alpha + \sinh \omega \sin \phi \sin \alpha\}} = \Sigma c_n(\lambda) E_n(\omega) F_n(\phi) F_n(\alpha)\ *.$$

The equation of the elliptic cylinder may also be dealt with as follows.

The equation

$$x(x-1)\frac{\partial^2 V}{\partial x^2} + (x - \tfrac{1}{2})\frac{\partial V}{\partial x} + (a + bx)V = y(y-1)\frac{\partial^2 V}{\partial y^2} + (y - \tfrac{1}{2})\frac{\partial V}{\partial y} + (a + by)V$$

is satisfied by $V = f(xy) = f(z)$, if

$$z\frac{\partial^2 f}{\partial z^2} + \tfrac{1}{2}\frac{\partial f}{\partial z} + bf = 0.$$

This gives $$f = A\cos 2\sqrt{bz} + B\sin 2\sqrt{bz}.$$

Now an integrating factor of the expression on the right hand side is

$$\frac{1}{\sqrt{y(1-y)}}\phi(y),$$

where $\phi(y)$ is a solution of

$$y(y-1)\frac{\partial^2 \phi}{\partial y^2} + (y - \tfrac{1}{2})\frac{\partial \phi}{\partial y} + (a + by)\phi = 0.$$

The condition to be satisfied at the limits is

$$\left[\sqrt{y(1-y)}\left(V\frac{\partial \phi}{\partial y} - \phi\frac{\partial V}{\partial y}\right)\right] = 0.$$

Put $y = \sin^2 t$, $x = \sin^2 s$, then two independent solutions of the equation are

$$\phi(t) = \sum_0^\infty A_p \sin^{2p} t, \qquad A_0 = 1,$$

$$\psi(t) = \sum_0^\infty B_p \sin^{2p+1} t.$$

The limits may be taken to be 0 and 1 and we have the equations

$$\phi(s) = \int_0^{\frac{\pi}{2}} \cos(2\sqrt{b}\sin s \,.\, \sin t)\,\phi(t)\,dt,$$

$$\psi(s) = \int_0^{\frac{\pi}{2}} \sin(2\sqrt{b}\sin s \,.\, \sin t)\,\phi(t)\,dt,$$

provided a is such that $\dfrac{\partial \phi}{\partial t}$ is zero when $t = \dfrac{\pi}{2}$.

* We may obtain an expansion for $F(\phi)$ in powers of λ by writing

$$E(\omega) = \mu_0 + \lambda\mu_1 + \lambda^2\mu_2 + \dots,$$

$$F(\phi) = f_0(\phi) + \lambda f_1(\phi) + \lambda^2 f_2(\phi) + \dots,$$

and comparing coefficients of λ, on the two sides of the integral equation, this gives

$$E(\omega) = 2\pi\left[1 + \frac{\lambda^2}{2^2}\cosh 2\omega - \frac{\lambda^4}{2^3 \,.\, 4^3}(y\cosh 4\omega - 2) + \dots\right],$$

but we are unable to establish the convergence of the series. Similarly

$$F(\phi) = 2\pi\left[1 + \frac{\lambda^2}{2^2}\cos 2\phi - \frac{\lambda^4}{2^3 \,.\, 4^3}(y\cos 4\phi - 2) + \dots\right].$$

If we take the more general equation

$$x(x-1)\frac{\partial^2 V}{\partial x^2}+n(x-\tfrac{1}{2})\frac{\partial V}{\partial x}+(a+bx)V=y(y-1)\frac{\partial^2 V}{\partial y^2}+(y-\tfrac{1}{2})\frac{\partial V}{\partial y}+(a+by)V=0,$$

we shall have a solution $V=f(xy)=f(z)$ if

$$z\frac{\partial^2 f}{\partial z^2}+\frac{n}{2}\frac{\partial f}{\partial z}+bf=0.$$

This is satisfied by
$$\frac{J_{\frac{n}{2}-1}(2\sqrt{bz})}{(2\sqrt{bz})^{\frac{n}{2}-1}},$$

accordingly the solutions of the integral equation

$$\phi(x)=\int_0^1\frac{J_m 2\sqrt{bxy}}{(bxy)^{\frac{m}{2}}}y^m(1-y)^m\phi(y)\,dy,$$

are solutions of the differential equations of type

$$x(x-1)\frac{d^2\phi}{dx^2}+(2x-1)(m+1)\frac{d\phi}{\partial x}+(a+bx)\phi=0$$

for appropriate values of a. When $m=0$ the integral equation takes the simple form

$$\phi(x)=\int_0^1 J_0(2\sqrt{bxy})\,\phi(y)\,dy.$$

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THE IRREDUCIBLE CONCOMITANTS OF TWO QUADRATICS IN n VARIABLES.

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VIII. *The Irreducible Concomitants of two Quadratics in n Variables.*

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[Communicated by E. W. Barnes, Sc.D., Trinity College.]

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CONTENTS.

I. Introduction.

§ 1. In the following pages it is proposed to discuss one phase of the general problem of Gordan's theorem on irreducible concomitants* applied to two quadratics in n variables. We shall solve the problem in one sense, but shall not arrive at what should be the final result. In fact we shall obtain a finite set of forms in terms of which every concomitant may be rationally and integrally expressed, but the forms will not be the most useful from a geometrical point of view, though for values of n less than 5 they are either identical with or simpler than those usually given. The symbolic method will be exclusively used, so that we must at the outset establish certain identities of which the binary and ternary identities are particular cases. We shall then state and prove a series of propositions resulting in the determination of a finite set of forms necessarily including the irreducible system. Finally we shall touch on the problem of reducing the number of these forms to a minimum, investigating in particular the invariants, covariants, and contravariants of the irreducible system, and making a few applications of the theory to special cases.

* Grace and Young, *Algebra of Invariants*, ch. vi.

§ 2. *Notation.* The general quantic of order p in n variables is conveniently written

$$f = \Sigma \frac{p!}{r_1!\, r_2! \dots r_n!} a_{r_1 r_2 \dots r_n} x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}; \ \sum_1^n r_k = p,$$

the summation extending to all possible different values of the integers r_k which satisfy the condition.

In agreement with the symbols introduced for binary and ternary forms* we shall represent f by the umbral expression

$$\left(\sum_{k=1}^n \alpha_k x_k\right)^p = \alpha_x^p,$$

so that

$$\alpha_1^{r_1} \alpha_2^{r_2} \dots \alpha_r^{r_n} = a_{r_1 r_2 \dots r_n},$$

and any product of αs of degree unequal to p is meaningless.

As in the case of binary forms we introduce symbols $\beta, \gamma, \dots$ equivalent to α and let

$$f = \alpha_x^p = \beta_x^p = \gamma_x^p = \dots.$$

If in writing any function of the coefficients a in terms of $\alpha, \beta, \gamma, \dots$ we never allow the degree of the symbols α in any term to exceed p, we may use the symbolic instead of the algebraic form of any function of the coefficients a, without fear of introducing any inconsistencies or contradictions *.

§ 3. *The Coordinates.* Let

$$a_x, b_x, c_x, \dots k_x, u_x^{(1)}, u_x^{(2)}, \dots u_x^{(r-1)} \dots\dots (1),$$

be n linear n-ary forms written symbolically. Then their jacobian with regard to x may be conveniently written

$$(abc \dots k u_1 \dots u^{(r-1)}) \dots\dots (2).$$

Suppose that x represents point coordinates in S_{n-1} dimensions and suppose that the quantities $a, b, \dots k$ are fixed in value, but $u, \dots u^{(r-1)}$ are variable.

We may look on the linear forms (1) as representing certain linear S_{n-2}s in S_{n-1} space.

The quantities

$$u^{(1)} \equiv u_1, u_2, \dots u_n$$
$$u^{(2)} \equiv u_1^{(2)} \dots\dots u_n^{(2)}$$
$$\dots\dots\dots\dots$$

may be considered as S_{n-2}-coordinates—the reciprocals of point-coordinates, and the relation

$$u_x = 0,$$

as the incidence of a certain S_{n-2} whose coordinates are (u) with the point (x).

Now a linear space S_{n-r} is the intersection of $r-1$ linear spaces S_{n-2} and therefore should be completely defined by $r-1$ sets of variables (u). It is easily proved † to have coordinates given by quantities

$$(u^{(1)} u^{(2)} \dots u^{(r-1)})$$

$\binom{n}{r-1}$ in number, being the minors formed from the last $r-1$ columns of the determinant (2).

* Grace and Young, *Algebra of Invariants*, ch. I. † *ibid.*, last chapter.

The S_{n-r} is the reciprocal of the S_{r-1} space in $n-1$ dimensions and the number of coordinates necessary and sufficient to define either of these is the same. The condition that the one should be incident with the other is of type

$$\Sigma\, p_i q_i = 0,$$

where p_i and q_i are corresponding coordinates.

§ 4. If

$$\begin{array}{rllllll} u_1 \equiv v_1 & \text{represent} & n & \text{coordinates for spaces of type} & & & S_{n-2} \\ (u_1 u_2) \equiv v_2 & \text{,,} & nC_2 & \text{,,} & \text{,,} & \text{,,} & S_{n-3} \\ (u_1 u_2 u_3) \equiv v_3 & \text{,,} & nC_3 & \text{,,} & \text{,,} & \text{,,} & S_{n-4} \\ \vdots & \cdots\cdots & & & & & \\ (u_1 u_2 \ldots u_{n-1}) \equiv v_{n-1} \equiv x & \text{,,} & n & \text{,,} & \text{,,} & \text{,,} & S_0 \text{ or point,} \end{array}$$

we have a complete set of linear coordinates for S_{n-1} space.

§ 5. *Fundamental Theorem.* Any concomitant of any system of forms whose coefficients may be typified by a and whose variables are included in the set $v_1, v_2, \ldots x$ is a function of type $F(a, v_1, v_2, \ldots x)$ or is obtained from this by polarization. Further, any concomitant of a set of forms $a_x{}^n, b_x{}^m, \ldots$ is expressible as a sum of terms composed entirely of factors such as

$$(a_1 a_2 \ldots a_{n_1} v_{n-n_1}),$$

(where $a_1, a_2 \ldots$ are symbols of type $a, b, \ldots$), or else can be derived from such terms by polarization.

This theorem, which may be proved like that for ternary forms, will be assumed, and we shall only deal with concomitants which are not polarized from factors such as $(a_1 a_2 \ldots a_{n_1} v_{n-n_1})$.

Thus we may assume that the number of different u symbols is limited to $n-1$ and that, as far as these symbols are concerned, a single term concomitant consists of brackets containing

$$u_1 u_2 \ldots u_r$$

where r may be $0, 1, \ldots n-1$; so that the number of symbols u_r occurring in a concomitant is not less than the number of symbols u_s provided $r < s$. Moreover when the symbols u occur in this way they are arranged most suitably for geometrical applications, since each set of symbols u in any bracket represents a definite type of coordinate S_p and since there is one, and one only, set of S_p-coordinates for each value of $p < n-1$.

§ 6. In the search for an irreducible system of concomitants for given ground forms it is necessary to apply identities which change the constituent symbols of a bracket factor. It would therefore appear best that as far as possible identities should only be used which leave the symbols of type u unaltered in a bracket, or at the most only disarrange them temporarily.

For example, in ternary forms there are two u symbols u_1 and u_2, but they occur in the form

$$u_1 = u$$
$$(u_1 u_2) = x.$$

We need not use the identity

$$(abc)\, d_x \equiv (abc)\,(du_1 u_2) = (abd)\,(cu_1 u_2) - (abu_1)\,(cdu_2) + (abu_2)\,(cdu_1),$$

for the theory may be developed without decomposing x in this way.

But in quaternary forms * it is useful sometimes to apply the identity

$$(abcu_1)\,(u_2\alpha) - (abcu_2)\,(u_1\alpha) + (bcp)\,(a\alpha) + (cap)\,(b\alpha) + (abp)\,(c\alpha) = 0,$$

where $(u_1 u_2) = p$, the line coordinate.

This identity is used to decompose p but sooner or later a similar identity may be employed to unite the symbols u_1, u_2 and form them into p again.

§ 7. The question arises, is it possible to maintain this grouping of u symbols in dealing with forms of higher dimensions? This paper does not answer this question for the grouping of the u symbols will be disregarded except in the single case of x. We shall obtain a set of forms P_1 of type

$$F(u_1, u_2, \ldots u_{n-2}, x) \quad\ldots\ldots(1)$$

in terms of which all concomitants of two quadratics may be expressed rationally and integrally; whereas the canonical set P_2 of irreducible forms would be of type

$$F(u_1, \overline{u_1 u_2}, \ldots v_{n-2}, x) \quad\ldots\ldots(2).$$

The forms (1) for the special case of quaternary forms are simpler than the irreducible system given by Gordan in *M. A.* Bd. LVI., and it might be comparatively easy to obtain the system (2) from (1), but for higher dimensions great complications arise.

Two reasons may be urged in favour of the system (1). Type (1) is identical with (2) as regards the irreducible invariants, covariants and contravariants since u_1 and x are the only variables occurring, and secondly the terms of type (1) are simpler than those of type (2).

On the other hand the irreducible concomitants of type $F(v_r)$ are not given explicitly by forms (1). We might call these, pure concomitants as opposed to mixed concomitants. From a geometrical standpoint the pure concomitants are the most interesting as the interpretation of mixed concomitants is usually very obscure. Analytically the set of forms (1) is preferable to (2) because they are simpler.

§ 8. If it were possible, and it may be, to build up all irreducible forms of type $F(v_r)$ of (2) from forms of (1) for all values of r, it would appear that forms (1) are best to take for the irreducible system. If not, then seeing that forms (1) do not give the most satisfactory solution of Gordan's theorem we are led to the further question, whether it is possible to extend the method used by Gordan for the case when $n = 4$ to the general case and thus obtain forms of type (2), which are the proper solution of the problem. This problem has not been attempted in these pages.

§ 9. To illustrate that the system P_1 of type (1) is simpler than P_2 of type (2), consider this example. For quaternary forms there is a certain quadratic complex $F_1{}^2$ which is equivalent, neglecting reducible terms to

$$(a_1 b_1 b_2 b_3)\,(a_2 bp)\,(a_2 b_1 b_2 b_3)\,(a_1 bp)\dagger.$$

* Gordan, *Math. Ann.* Bd. LVI. § 15.

† *ibid.* Bd. LVI. p. 3.

This is a member of the irreducible system of type (2). But the methods of the following paper, viz.:—those which lead to the irreducible system (1), would reduce this concomitant to terms of type

$$(a_1 a_2 b u_i)(a_1 a_2 b u_j) \times (b_1 b_2 b_3 u_k)(b_1 b_2 b_3 u_l),$$

and reducible terms in which p has not been decomposed.

In this expression i, j, k, l stand for 1, 1, 2, 2 in some order, and $p = (u_1 u_2)$.

Thus in the system (1) F_1^2 would be considered reducible, but we should have terms in (1) of type

$$(a_1 a_2 b u_1)(a_1 a_2 b u_2)$$

which do not come in (2).

These are simpler than F_1^2. On the other hand it is not at all easy to reverse the steps and find what combinations of these terms, involving u_1 and u_2 separated from each other, are required to give the terms involving no variable but p, which are irreducible if p is not to be decomposed.

II. Notation.

§ 10. Assuming the truth of the fundamental theorem there are two kinds of symbolic factors in a concomitant, the bracket factor and the linear factor a_x.

In a bracket factor n symbols must occur and they are either of type u or of type a; u standing for the variable and a for the coefficient.

Thus each bracket is of form

$$(a_1 a_2 \ldots a_{n_1} u_1 u_2 \ldots u_{n_2});\ n_1 + n_2 = n.$$

If $n_2 = n - 1$ we call this a_x so that the 2nd type of factor is really a special form of the 1st type. If k symbols $a_1 a_2 \ldots a_k$ stand in a bracket, the expression $a_1 a_2 \ldots a_k$ will be called the 'product' of the symbols a.

We shall invariably use the following convention. Arabic letters stand for ordinary symbols, n of which compose a bracket; Greek letters, capitals and the symbol v stand for products of these symbols.

Thus we abbreviate $(a_1 a_2 \ldots a_{n_1} u_1 u_2 \ldots u_{n_2})$ to $(\alpha_{n_1} v_{n_2})$ where α_{n_1} stands for n_1 a symbols and v_{n_2} for n_2 u symbols. We define α_0 as unity.

The Greek letters require a suffix to denote the number or extent of Arabic letters which they include.

If we wish to divide up the a symbols more particularly we use more than one Greek letter to represent them. Thus we might write the bracket as $(\alpha_{m_1} \beta_{m_2} \gamma_{m_3} v_{n_2})$ where $m_1 + m_2 + m_3 = n_1$, but in all cases the full expression for the bracket is written out in the same order as the condensed expression.

The bracket $(\alpha_r \beta_k \gamma_s)$ means that there are r a symbols followed by k like symbols followed by s like symbols. These symbols belong to definite ground forms $f_1, f_2, \ldots$ which are of course known.

§ 11. The bracket $(\alpha_r\beta_k\gamma_s)$ is really a determinant of the $(r+k+s)$th or nth order; each symbol stands for a column and the order of the symbols is the order of the columns.

It follows that
$$(\alpha_r\beta_k\gamma_s) = (-)^{kr}(\beta_k\alpha_r\gamma_s)$$
$$= (-)^{kr+s(k+r)}(\gamma_s\beta_k\alpha_r),$$

since these operations are identical with those of interchanging columns of a determinant.

Let $\alpha_r \equiv a_1a_2 \dots a_r$ when written in full.

Then $\alpha_r \equiv \epsilon_k a_{k_1}a_{k_2}\dots a_{k_r}$ where $k_1, k_2 \dots$ are the suffixes $1, 2, \dots r$ in some order and $\epsilon_k^2 = 1$.

In fact the sign ϵ_k is $(-)^q$ where q is the number of inversions of pairs of letters necessary to get from the order $1, 2, \dots r$ to $k_1, k_2, \dots k_r$.

This is obvious from the definition of the symbol α_r remembering that $(\alpha_r A)$ is a determinant.

Thus the full expression for α_r may be written in $r!$ orders.

§ 12. *Determinantal permutations.* Consider the determinant

$$\begin{vmatrix} a_1^{(1)} & a_2^{(1)} \dots a_r^{(1)} & b_1^{(1)} & b_2^{(1)} \dots b_s^{(1)} \\ a_1^{(2)} & a_2^{(2)} \dots a_r^{(2)} & b_1^{(2)} & b_2^{(2)} \dots b_s^{(2)} \\ \vdots & & & \\ a_1^{(r+s)} & \dots\dots & \dots & b_s^{(r+s)} \end{vmatrix}.$$

Let it be typified by $(a_1a_2\dots a_rb_1\dots b_s)$ or $(\alpha_r\beta_s)$. We may expand it in a form in which each term consists of a pair of minors, one containing nothing but symbols from the first r rows and the other, symbols from the last s rows.

The number of terms is $\binom{r+s}{r}$.

If the first term is called $(\alpha_r)(\beta_s)$ another term is $-(a_1a_2\dots a_{r-1}b_1)(a_rb_2\dots b_s)$, another is $+(a_1a_2\dots a_{r-2}b_2b_1)(a_ra_{r-1}b_3\dots b_s)$, and so on.

Thus we have a series of $\binom{r+s}{r}$ pairs of brackets of symbols, each pair having a definite sign and a definite order (though each bracket has either $r!$ or $s!$ equivalent orders) and this series is formed completely and uniquely on being given a single term of it, $(\alpha_r)(\beta_s)$.

Such an operation is called a determinantal permutation of the two groups of letters α_r and β_s.

For example, given $a_1a_2a_3$, b_1b_2, there are $\binom{5}{2}$ terms in the determinantal permutations and the full series is

$$\begin{array}{ll} a_1a_2a_3,\ b_1b_2 & a_2a_3b_1,\ b_2a_1 \\ a_1a_2b_1,\ b_2a_3 & a_2a_3b_2,\ a_1b_1 \\ a_1a_2b_2,\ a_3b_1 & a_1b_1b_2,\ a_2a_3 \\ a_1a_3b_1,\ a_2b_2 & a_2b_1b_2,\ a_3a_1 \\ a_1a_3b_2,\ b_1a_2 & a_3b_1b_2,\ a_1a_2. \end{array}$$

The number of equivalent arrangements of each term is $3!\,2! = 12$.

Observation. If r or $s=1$ it may be necessary to use negative signs for certain terms.

For example, the series due to a, b is

$$a,\ b, \quad -b,\ a.$$

§ 13. The method is applicable to groups of three or more letters. Just as we may expand the determinant

$$(a_1 a_2 a_3 b_1 b_2 c_1)$$

in the form of a series

$$\Sigma\,(a_1 a_2 a_3)\,(b_1 b_2)\,(c_1),$$

so we may permute the three sets of symbols

$$a_1 a_2 a_3,\ b_1 b_2,\ c_1$$

determinantally. In this case there are $\dfrac{6!}{3!\,2!\,1!}$ terms in the series. The general case is obvious.

§ 14. *Application to the theory of invariants.* Suppose that we have a product of symbolic letters $(A\alpha_r)(\beta_s B)\,M$ where A stands for $n-r$ symbols of type u or a, where B stands for $n-s$ similar symbols, and where M stands for bracket or x factors.

Let
$$\alpha_r = a_1 a_2 \ldots a_r,$$
$$\beta_s = b_1 b_2 \ldots b_3.$$

We may permute α_r, β_s determinantally into the $\dbinom{r+s}{r}$ different terms, one of which is θ_r, ϕ_s say. Corresponding to this we may write down the series

$$\Sigma\,(A\theta_r)(\phi_s B)\,M$$

consisting of $\dbinom{r+s}{r}$ terms.

This series is unique on being given a single term of it $(A\alpha_r)(\beta_s B)$ because the permutations of the letters θ_r or ϕ_s amongst themselves do not alter the value of the term.

This process of forming a series from a given term will be called the $R\,(\alpha_r, \beta_s)$ process and will be written as

$$\Sigma\,(A\dot{\alpha}_r)(\dot{\beta}_s B)\,M \text{ or } \Sigma\,(A\dot{\alpha}_r)\left(\frac{\alpha_r\beta_s}{\alpha_r}B\right)M,$$

where in the first form a dot is placed over each symbol which undergoes a change. If α_r is written in the form $\gamma_{r_1}\delta_{r_2}\ldots (r_1 + r_2 \ldots = r)$ all the symbols γ, δ, ... will receive a dot.

We shall sometimes write $R\,(r, s)$ for $R\,(\alpha_r, \beta_s)$ where there is no ambiguity.

§ 15. Similarly if we start with several factors

$$(A\alpha_r)(\beta_s B)(\gamma_t C)\ldots$$

we form a series by the $R\,(r, s, t, \ldots)$ process and call it

$$\Sigma\,(A\dot{\alpha}_r)(\dot{\beta}_s B)(\dot{\gamma}_t C)\ldots,$$

where the $\dfrac{(r+s+t+\ldots)!}{r!\,s!\,t!\ldots}$ terms in the series correspond to the equal number of determinantal permutations of α_r, β_s, γ_t, ...

For example, the operation of $R(1, 1)$ on $(Aa)(bB)$ gives

$$(Aa)(bB)-(Ab)(aB).$$

Again, if
$$\alpha_{n+1}=a_1a_2a_3\ldots a_{n+1},$$
$$\Sigma(\dot{\alpha}_n)\,\dot{a}_{n+1,x}$$
is
$$(\alpha_n)\,a_{n+1,x}-(\alpha_{n-1}a_{n+1})\,a_{n,x}+\ldots.$$

Once more
$$R(2, 2) \text{ on } (Aa_1a_2)(a_3a_4B)$$
is
$$\Sigma(A\dot{a}_1\dot{a}_2)(\dot{a}_3\dot{a}_4B),$$
and is
$$(Aa_1a_2)(a_3a_4B)-(Aa_1a_3)(a_2a_4B)+(Aa_1a_4)(a_2a_3B)\ldots$$
when written in full.

Note. If θ_r, ϕ_s is a determinantal permutation of α_r, β_s the series is expressible either as

$$\Sigma(A\dot{\alpha}_r)(\dot{\beta}_sB) \text{ or as } \Sigma(A\dot{\theta}_r)(\dot{\phi}_sB).$$

§ 16. *Properties of the process $R(r, s)$.* (1) If any number of these operations are applied to some or all of the symbols

$$+\,a_1a_2\ldots a_p$$

which originally occur in this order, and if any term in the result is

$$(-)^q a_{i_1}a_{i_2}\ldots a_{i_p}$$

where $i_1i_2\ldots i_p$ are the numbers $1, 2, \ldots p$ in some order, the sign of this term is $(-)^q$ where q is the number of interchanges two and two required to pass from the original term to this or *vice versâ*.

This follows at once from the theory of determinants.

(2) Again, the result of applying the operation $R(r, s)$ to the symbols of the bracket $(\alpha_r\beta_s\theta)$ is a series of $\binom{r+s}{r}$ terms each equal to $(\alpha_r\beta_s\theta)$.

For a typical term in the series is

$$(-)^q(\alpha_r'\beta_s'\theta)$$

where q is the number of inversions necessary to pass from the order of symbols in $\alpha_r\beta_s$ to that in $\alpha_r'\beta_s'$.

But by the theory of determinants

$$(\alpha_r'\beta_s'\theta)=(-)^q(\alpha_r\beta_s\theta),$$
$$\therefore\ (-)^q(\alpha_r'\beta_s'\theta)=(\alpha_r\beta_s\theta).$$

Thus each term is the same.

Lemma I. If to the first $r+s-1$ symbols of α_r, $\beta_{s-1}b$ we apply the operation $R(r, s-1)$ and if we operate on each term of the resulting series with $R(r+s-1, 1)$, we obtain the terms of the series due to $R(r, s)$ each s times over.

Let $\theta_r\phi_{s-1}$ be a determinantal permutation of (α_r, β_{s-1}). Suppose that the terms of $(\dot{\alpha}_r, \dot{\beta}_{s-1}b)$ are written down in a column, and that rows are made opposite each term by operating with $R(r+s-1, 1)$ upon the term, in such a way that $r+s$ columns are formed. There will then be a rectangular diagram of $\binom{r+s-1}{r}$ rows and $r+s$ columns. The first column contains terms of type $\theta_r\phi_{s-1}b$; the next $s-1$ columns may be taken as due to $(\theta_r\dot{\phi}_{s-1}, \dot{b})$ and the last r columns as due to $(\dot{\theta}_r\phi_{s-1}, \dot{b})$.

Terms of	$s-1$ columns due to	r columns due to
$(r, s-1)$	$(\theta_r\dot{\phi}_{s-1}, \dot{b})$	$(\dot{\theta}_r\phi_{s-1}, \dot{b})$.

The first s columns are identical because the last s symbols of each term form the second member of the pair θ_r, ϕ_s. Cp. (2) of this paragraph.

In the first column a typical member is $\theta_{r-1}c$, say, and there are just s members having θ_{r-1} common to each other, c taking up the remaining possible s values. Let b be the $(r+s)$th symbol. Then in any row which begins $\theta_{r-1}c$, one and only one first member $\theta_{r-1}b$ of a term is found in the last r columns, viz.:—the one where c and b have been interchanged. Thus in the last r columns the term $(\theta_{r-1}b, \ldots)$ occurs s times, once on each row beginning $(\theta_{r-1}c, \ldots)$.

We can therefore pick out of the last r columns the terms $s\Sigma(\theta_{r-1}b, \ldots)$, where θ takes all possible different values.

Now θ_{r-1} is formed in $\binom{r+s-1}{r-1}$ ways. Therefore we have picked out $s\binom{r+s-1}{r-1}$ terms from the last r columns. But the number of terms in these r columns is

$$r\binom{r+s-1}{s-1} = s\binom{r+s-1}{r-1}.$$

Hence we have accounted for all the terms of these r columns in

$$s\Sigma(\dot{\theta}_{r-1}b, \dot{\phi}_s).$$

Thus the full operation leads to the terms

$$s\Sigma(\dot{\alpha}_r, \dot{\beta}_{s-1}b) + s\Sigma(\dot{\alpha}_{r-1}b, \dot{\beta}_s\dot{a}),$$

the first series being due to the first s columns, and the second, to the next r columns.

These series are equivalent to $s\Sigma(\dot{\alpha}_r, \dot{\beta}_{s-1}\dot{b})$. Q. E. D.

For example, let $\alpha_r = 1, 2$ and $\beta_s = 3, 4$. The first operation is $(\dot{1}\dot{2}, \dot{3}4)$, the second is

$$R(ijk, l); \qquad i, j, k, l = 1, 2, 3, 4 \text{ in some order.}$$

1st Operation	1 Column	2 Columns	
12, 34	-12, 43	14, 23	42, 13
23, 14	-23, 41	24, 31	43, 21
31, 24	-31, 42	34, 12	41, 32

The second column reproduces the first, and the last two $(r=2)$ give terms of $(\dot{1}4, \dot{2}\dot{3})$ twice. The result is $2(\dot{1}\dot{2}, \dot{3}4)$.

Lemma II. If we have $r+1+s$ symbols $\alpha_r a \beta_s$ and form the series $(\dot{\alpha}_r\dot{a}, \dot{\beta}_s)$, and if upon the first $r+1$ symbols of each term we operate with $(r, 1)$ and then prefix the last of these $r+1$ symbols in each term to the s remaining symbols, we form a series of terms equivalent to

$$(s+1)\Sigma(\dot{\alpha}_r, \dot{a}\dot{\beta}_s).$$

As before let the terms of $(\dot{\alpha}_r\dot{a}, \dot{\beta}_s)$ be arranged in a column, and the series $(r, 1)$ due to each term be set out in a row opposite the particular term. There will then be $r+1$ columns and $\binom{r+s+1}{s}$ rows.

Consider any term of the first column, $(\theta_r c, \mu)$ say. There are $s+1$ terms in this column with the same θ_r but different c and in the corresponding rows there is one and only one term in which θ_r is the leading portion. Hence we can find $s+1$ terms beginning θ_r.

The number of ways of forming θ_r is $\binom{r+s+1}{r}$.

We have therefore accounted for $(s+1)\binom{r+s+1}{r}$ terms. But this is the total number of terms since it is equal to $(r+1)\binom{r+s+1}{s}$. Thus the total number of terms may be arranged as

$$(s+1) \text{ times } \Sigma(\dot{\theta}_r, \dot{\phi}_{s+1})$$

or

$$(s+1)\,\Sigma(\dot{\alpha}_r, \dot{a}\dot{\beta}_s). \qquad \text{Q. E. D.}$$

For example, take a_1a_2, a_3, a_4. The diagram is

$a_1a_2a_3, a_4$	$a_2a_3a_1a_4$	$a_3a_1a_2a_4$
$a_1a_4a_2, a_3$	$a_4a_2a_1a_3$	$a_2a_1a_4a_3$
$a_4a_3a_2, a_1$	$a_3a_2a_4a_1$	$a_2a_4a_3a_1$
$a_1a_3a_4, a_2$	$a_3a_4a_1a_2$	$a_1a_3a_4a_2$.

§ 17. *Fundamental Identities.* There are two ways of reducing a symbolical expression if it is a concomitant of given ground forms,

(1) by the interchange of equivalent symbols;

(2) by the use of the fundamental identities.

(1) If α_p and α'_p are two sets of p equivalent symbols belonging to a form

$$f = a_x{}^m = a'_x{}^m = \dots,$$

we may interchange α_p and α'_p without affecting the value of the concomitant.

If

$$\alpha_p \equiv \alpha_{p_1}\alpha_{p_2}\dots\alpha_{p_k},$$
$$\alpha'_p \equiv \alpha'_{p_1}\alpha'_{p_2}\dots\alpha'_{p_k},$$

the interchange of α_p and α'_p is a condensed way of saying that α_{p_r} is interchanged with α'_{p_r} for $r = 1, 2, \dots k$.

(2) We shall now proceed to establish the linear identities.

§ 18. As in the case of binary forms we have the identity

$$(a_2a_3\dots a_{n+1})\,a_{1x} - (a_1a_3\dots a_{n+1})\,a_{2x} + (a_1a_2a_4\dots a_{n+1})\,a_{3x} - \dots = 0,$$

the series containing $n+1$ terms.

This is deduced from the vanishing determinant

$$\begin{vmatrix} a_1^{(1)} & a_2^{(1)} & \dots & a_{n+1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_1^{(r)} & a_2^{(r)} & \dots & a_{n+1}^{(r)} \\ \vdots & \vdots & & \vdots \\ a_{1x} & a_{2x} & \dots & a_{n+1,x} \end{vmatrix},$$

where $a_{1x} \equiv a_1^{(1)} x_1 + a_1^{(2)} x_2 + \dots + a_1^{(n)} x_n$.

We observe that we may write this identity as $(-)^{n+1} \Sigma (\dot{\boldsymbol{\alpha}}_n) \dot{a}_{n+1,x} = 0$ on writing α_{n+1} for $a_1 a_2 \dots a_{n+1}$ and letting $\alpha_{n+1} = \alpha_r \alpha_{r+1} \dots \alpha_{n+1}$ for all values of r.

If we write β for x where β is of extent $n-1$ in letters of type u or a we may express this identity as

$$\Sigma (\dot{\boldsymbol{\alpha}}_n) (\dot{a}_{n+1} \beta) = 0 \text{ ....Identity I.}$$

If we write u_1 for a_{n+1}, we have

$$\Sigma (\dot{\boldsymbol{\alpha}}_n) (\dot{u}_1 \beta) = 0$$

or

$$\Sigma (\dot{\boldsymbol{\alpha}}_{n-1} \dot{u}_1) (\dot{a}_n \beta) = 0.$$

Hence

$$\Sigma (\dot{\boldsymbol{\alpha}}_{n-1} u_1) (\dot{a}_n \beta) = (\alpha_n) (u_1 \beta) \text{ ....Identity II.}$$

A special case of this is

$$(bcu)\, a_x + (cau)\, b_x + (abu)\, c_x = (abc)\, u_x.$$

Writing u_2 for a_n, we have by Identity II

$$\Sigma (\dot{\alpha}_{n-1} u_1) (\dot{u}_2 \beta) - (\alpha_{n-1} u_2) (u_1 \beta) = 0$$

or

$$-\Sigma (\dot{\boldsymbol{\alpha}}_{n-2} \dot{u}_2 u_1) (\dot{a}_{n-1} \beta) - (\alpha_{n-1} u_2) (u_1 \beta) = 0$$

or

$$-\Sigma (\dot{\boldsymbol{\alpha}}_{n-2} u_2 u_1) (\dot{a}_{n-1} \beta) + (\alpha_{n-1} u_1) (u_2 \beta) - (\alpha_{n-1} u_2) (u_1 \beta) = 0,$$

hence

$$\Sigma (\dot{\boldsymbol{\alpha}}_{n-2} u_1 u_2) (\dot{a}_{n-1} \beta) + \Sigma (\alpha_{n-1} \dot{u}_1) (\dot{u}_2 \beta) = 0 \text{ ....Identity III.}$$

§ 19. In general if

$$\alpha_p = a_1 a_2 \dots a_p$$

and

$$U_q = u_1 u_2 \dots u_q,$$

where

$$p + q - 1 = n,$$

$$\Sigma (\dot{\boldsymbol{\alpha}}_{p-1} U_q) (\dot{a}_p \beta) + (-)^q \Sigma (\alpha_p \dot{U}_{q-1}) (\dot{u}_q \beta) = 0 \text{ ....Identity A,}$$

q being the number of displacements of a_p in order to get from the order of letters in the first term of the first series to that in the first term of the second.

This may be proved by induction, observing that it is already established for $q = 1$ or 2.

Assume $\Sigma (\dot{\boldsymbol{\alpha}}_p U_{q-1}) (\dot{a}_{p+1} \beta) + (-)^{q-1} \Sigma (\alpha_{p+1} \dot{U}_{q-2}) (\dot{u}_{q-1} \beta) = 0$; write u_q for a_{p+1}.

Then

$$\Sigma (\dot{\boldsymbol{\alpha}}_p U_{q-1}) (\dot{u}_q \beta) + (-)^{q-1} \Sigma (\alpha_p u_q \dot{U}_{q-2}) (\dot{u}_{q-1} \beta) = 0,$$

or

$$(-)^q \Sigma (\dot{\boldsymbol{\alpha}}_{p-1} U_{q-1} \dot{u}_q) (\dot{a}_p \beta) + (-)^{q-1} \Sigma (\alpha_p u_q \dot{U}_{q-2}) (\dot{u}_{q-1} \beta) = 0,$$

or

$$(-)^q \Sigma (\dot{\boldsymbol{\alpha}}_{p-1} U_q) (\dot{a}_p \beta) + (-)^{2q} (\alpha_{p-1} a_p U_{q-1}) (u_q \beta) + (-)^{q-1} \Sigma (\alpha_p u_q \dot{U}_{q-1}) (\dot{u}_{q-1} \beta) = 0,$$

or

$$\Sigma (\dot{\boldsymbol{\alpha}}_{p-1} U_q) (\dot{a}_p \beta) + (-)^q \Sigma (\alpha_p \dot{U}_{q-1}) (\dot{u}_q \beta) = 0,$$

which is Identity A. It is therefore true by induction.

§ 20. We are now in a position to establish the fundamental linear identity of which all the previous identities are particular cases.

Theorem. The effect of the operation $R(r, s, t, \ldots)$ upon a single term of symbolic brackets

$$(\alpha_r A)(\beta_s B)(\gamma_t C) \ldots$$

is to form a series which may be equated to a series, each term of which has a bracket of type

$$(\alpha_r \beta_s \gamma_t \ldots A'),$$

unless $\Sigma r > n$. In this case the second series is replaced by zero.

We shall first prove it for two brackets and shew that if $r + s < n$

$$\Sigma(\dot{\alpha}_r A)(\dot{\beta}_s B) = \epsilon \Sigma(\alpha_r \beta_s \dot{\theta})(\dot{\phi}),$$

where θ and ϕ contain the symbols of A and B and where $\epsilon^2 = 1$.

This has just been proved for the case when $s = 1$; it is then Identity A. Assume it true when β is of extent $s - 1$, and let A be U_q.

Then we assume

$$\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_{s-1} B) + \epsilon \Sigma(\alpha_p \beta_{s-1} \dot{U}_{q-s+1})(\dot{V}_{s-1} B) = 0,$$

where $U_{q-s+1} V_{s-1} \equiv U_q$, and $\epsilon = (-)^{q(s-1)}$.

Let $B = b_s B'$ and $\beta_{s-1} b_s = \beta_s$.

Then $$\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_{s-1} b_s B') + \epsilon \Sigma(\alpha_r \beta_{s-1} \dot{U}_{q-s+1})(\dot{V}_{s-1} b_s B') = 0.$$

Now permute $(\alpha_r \beta_{s-1}, b_s)$ determinantally for each term of these two series.

Consider the first series. We have a term $(\alpha_r U_q)(\beta_{s-1} b_s B')$ to which $R(\alpha_r, \beta_{s-1})$ is applied followed by $R(\alpha_r \beta_{s-1}, b_s)$. The result is s times the result of operating with

$$R(\alpha_r, \beta_s). \qquad \text{[Lemma I.}$$

Thus the first series is replaced by

$$s\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_s B').$$

Consider the second series. The term $(\alpha_r \beta_{s-1} U_{q-s+1})(V_{s-1} b_s B')$ is now replaced by

$$\Sigma(\dot{\alpha}_r \dot{\beta}_{s-1} U_{q-s+1})(V_{s-1} \dot{b}_s B'),$$

which is equal to $(-)^{q-1} \Sigma(\alpha_r \beta_s \dot{U}_{q-s})(\dot{u}_{q-s+1} V_{s-1} B')$ by Identity A.

Thus the second series is formed from the single term $\epsilon(\alpha_r \beta_s U_{q-s})(V_s B')$ by applying

$$R(U_{q-s+1}, V_{s-1}),$$

and then by making the first set of letters in the pair one less in number by operating with $R(q-s, 1)$ upon this first set and prefixing the one end letter to the second set.

Therefore the whole operation is equivalent to obtaining the series

$$(-)^{q-1} s\epsilon \Sigma(\alpha_r \beta_s \dot{U}_{q-s})(\dot{V}_s B'). \qquad \text{[Lemma II.}$$

But $\epsilon = (-)^{q(s-1)}$, hence $(-)^q \epsilon = (-)^{qs}$.

Combining these results we have

$$\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_s B') + (-)^{qs} \Sigma(\alpha_r \beta_s \dot{U}_{q-s})(\dot{V}_s B') = 0 \quad \ldots\ldots\ldots \text{Identity B.}$$

Thus since this is true for $s=1$ and since the assumption of its truth for the value $s-1$ implies its truth for s, it is by induction generally true, provided $r+s \ngtr n$.

If $r+s=n$ the second series is the term

$$(\alpha_r \beta_s)(UB').$$

The application of $R(\alpha_r \beta_s, b)$ to this gives zero by the first identity. Similarly for values of $r+s>n$ the second series is always zero.

Thus $$\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_s B')=0 \text{ if } r+s>n,$$

and $$\Sigma(\dot{\alpha}_r U_q)(\dot{\beta}_s B')+(-)^{qs}\Sigma(\alpha_r \beta_s \dot{U}_{q-s})(\dot{V}_s B')=0 \text{ ....Identity B},$$

if $r+s \ngtr n$.

§ 21. The general identity may be stated thus:

$$\Sigma(\dot{\alpha}_{r_1} A_1)(\dot{\alpha}_{r_2} A_2)\dots(\dot{\alpha}_{r_p} A_p)+\epsilon\Sigma(\alpha_r \dot{A}_{11})(\dot{A}_{r_2} A_2)\dots(\dot{A}_{r_p} A_p)=0,$$

where $\alpha_r \equiv \alpha_{r_1}\alpha_{r_2}\dots\alpha_{r_p}$ and $A_1 \equiv A_{11} A_{r_2} A_{r_3} \dots A_{r_p}$.

The proof follows directly from the case for $p=2$. In fact the first series is obtained by applying the process $R(r_1+r_2, r_3, \dots r_p)$ to each term of the series obtained from $R(r_1, r_2)$. The effect of the latter operation may be written as

$$(\alpha_{r_1}\alpha_{r_2}\dot{A}_{12})(\dot{A}_{13} A_2)(\alpha_{r_3} A_3)\dots$$

on using the previous identity.

The process $R(r_1+r_2, r_3 \dots r_p)$ is equivalent to

$$R(r_1+r_2+r_3, r_4, \dots r_p) R(r_1+r_2, r_3).$$

Proceeding in this way and applying Identity B at each step we obtain what we want.

We call this general identity, Identity C. As before, if $r_1+r_2+r_3\dots>n$ the second series is replaced by zero.

§ 22. *Observation.* The effect of this operation on $\overline{p-1}$ brackets is to replace the original permuted symbols by an equal number of symbols out of the remaining bracket without altering the other symbols of these $\overline{p-1}$ brackets. We may also choose any one of the p brackets as that into which all α_r is to come, provided that the one chosen is not of type a_x. The identity applies if any of the suffixes $r_2, r_3, \dots r_p$ are unity and the corresponding brackets replaced by factors a_x.

But the identity fails if all the suffixes $r_1 \dots r_p$ are unity and the corresponding brackets are of type a_x. It applies however if one or more of the corresponding brackets is not of type a_x.

For example the identity does not apply to

$$\Sigma \dot{a}_x \dot{b}_x \dot{c}_x,$$

but does, to $\Sigma(\dot{a}pqr)\dot{b}_x\dot{c}_x$ in quaternary forms.

§ 23. There is a useful special case of this identity. If we have the series

$$\Sigma(\dot{a}_1\theta_1)(\dot{a}_2\theta_2)\ldots(\dot{a}_{n-1}\theta_{n-1}),$$

we may write it as equal to

$$(\alpha_{n-1}\theta_1\theta_2\ldots\theta_{n-1}), \quad \alpha_{n-1}\equiv a_1a_2\ldots a_{n-1},$$

where the determinant is of an order reciprocal to that of $(a_1\theta_1)$. The proof is exactly the same as that for ternary forms, in which case the identity is

$$a_pb_q-a_qb_p=(\overline{ab}\,pq).$$

If $\theta_1=p_1p_2\ldots p_{n-1}$, we may write

$$(\alpha_{n-1}\theta_1\theta_2\ldots\theta_{n-1})=-\Sigma(p_1\alpha_{n-1})(p_2\theta_2)\ldots(p_{n-1}\theta_{n-1}).$$

Also $$(\alpha_{n-1}\theta_1\theta_2\ldots\theta_{n-1})=-(\theta_1\alpha_{n-1}\theta_2\ldots\theta_{n-1}).$$

III. Complete system for linear forms and for one quadratic.

§ 24. *Linear Forms.* Let $f=a_x=b_x=c_x=\ldots$ be a linear form.

Since the bracket $(ab\,\theta_{n-2})=-(ba\,\theta_{n-2})$ and since the symbols a and b are interchangeable it follows that the only concomitant of a linear form is itself.

Similarly the complete system for k linear forms

$$f_1=a_{1x},\ f_2=a_{2x},\ \ldots f_k=a_{kx}$$

is given by the various values of

$$(a_1a_2\ldots a_pu_1u_2\ldots u_{n-p}),$$

where p takes the values 1, 2, ... k or n whichever is least.

§ 25. *Quadratic Forms.* The theory of quadratic forms of more than two dimensions, treated symbolically, depends upon the following theorem, to which the preceding identities have been leading. In fact it gives as an immediate consequence the complete system for a single quadratic and is the underlying principle to work upon in finding all possible irreducible forms for two quadratics.

Theorem. If $f=a^2_{1x}=a^2_{2x}=\ldots$ is a quadratic in n variables and if $P=(\alpha_r\gamma_s)\,g_2g_3\ldots g_w$ is a concomitant of this quadratic and any other forms (where $\alpha_r\equiv a_1a_2\ldots a_r$, γ_s refers to symbols u or b not belonging to f, and each g stands for a symbolic factor), then P is expressible as a sum of terms Q of type $(\alpha_r\gamma_s)(\alpha_r\delta_s)\,g_3'\ldots g_w'$, where the letters $a_1a_2\ldots a_r$, complementary to those of α_r in the first bracket, figure together in another bracket.

Let r_1 of these as occur in g_2,

r_2 ,, ,, ,, ,, ,, g_3,

and so on.

Consider the series $\Sigma(\alpha_r\gamma_s)(\dot{\alpha}_{r_1}\ldots)(\dot{\alpha}_{r_2}\ldots)\ldots g_w$.

If the order of permuted symbols in any term of this series is $(-)^q\,\theta_{r_1}\theta_{r_2}\ldots$ we may interchange these symbols back to the order in the first term, $\alpha_{r_1}\alpha_{r_2}\ldots$, by q interchanges. The complementary symbols all lie in the first bracket and when they are interchanged the bracket is unaltered except that it is multiplied by $(-)^q$. The whole effect of the interchange is to reproduce the first term, since the sign is $(-)^{2q}$ and therefore positive.

The series is therefore equal to $\binom{r}{r_1,\ r_2,\ \dots} P$.

By Identity C this operation also leads to the product of $(\alpha_r \gamma_s)$ and a series in which the symbols α_r all figure in one bracket for each term, since $r \ngtr n$, unless all the symbols complementary to α_r occur with x (§ 22).

Therefore $\binom{r}{r_1,\ r_2,\ \dots} P = \Sigma (\alpha_r \gamma_s)(\alpha_r \delta_s) g_3' \dots g_w'$ where δ_s may contain symbols belonging to f.

In the exceptional case $P = 0$ if $r > 1$ for

$$(aa'\theta)\, a_x a_x'\, M = 0,$$

as is seen by interchanging a and a'. If $r = 1$, P is of form $(a\gamma_s)\, a_x M$ which satisfies the theorem.

In general if $\alpha_r \delta_s \equiv \alpha_{r+k} \delta_{s'}$ where $\delta_{s'}$ is independent of f, we may transform P again to contain another bracket containing α_{r+k}. Ultimately we express P as $\Sigma (\alpha_{r'} \gamma_{s''})(\alpha_{r'} \delta_{s''}) M$ where $r' \geqslant r$ and γ and δ contain no symbol of f.

Suppose that P is a single term concomitant of f and other forms. Then this theorem shews that P may be expressed as $\Sigma P_1 Q_1$ where P_1 consists of two bracket or x factors, each containing the same set of symbols belonging to f. Similarly Q_1 is expressible as $\Sigma P_2 Q_2$ where P_2 is of type P_1. Proceeding in this way we finally express P as $\Sigma P_1 \dots P_k Q$ where P_r consists of two bracket or x factors containing the same α_r, and nothing more, belonging to f, and where Q is independent of symbols of f.

We say that P is prepared mod. f, when written in this way, and express it as

$$P = \Sigma P_a;$$

$$P_a = (\alpha_{r_1} \gamma_{s_1})(\alpha_{r_1} \delta_{s_1})(\alpha'_{r_2} \gamma'_{s_2})(\alpha'_{r_2} \delta'_{s_2}) \dots Q,$$

where γ, δ, … Q are independent of symbols of f.

When we are collecting together symbols α_r out of brackets $g_p g_q g_r \dots g_u$ say into g_p, as far as all the other brackets are concerned the initial and final expressions are alike. The only alteration in $g_p \dots g_u$ is that certain symbols from g_p have taken the place of whatever symbols of α_r were in them originally. Cp. § 22.

§ 26. *Complete system for one quadratic.* Let $f = a^2_{1x} = a^2_{2x} = \dots$ be the quadratic.

The only possible constituents of a concomitant are $(\alpha_r U_{n-r})$ and a_x, where r may be 2, 3, … n; in fact if $r = 1$ the two are the same.

If $P = (\alpha_r U_{n-r}) M$ is a concomitant, where r is the greatest suffix of α in P and $r > 2$, P is expressible as $(\alpha_r U_{n-r})(\alpha_{r'} U'_{n-r'}) M'$ where $r' \geqslant r$, by the previous theorem.

If $r' = r$, P has now a factor $(\alpha_r U_{n-r})(\alpha_r U'_{n-r})$ and is reducible. If $r' > r$ we proceed in the same way and ultimately obtain a factor of form

$$(\alpha_s U_{n-s})(\alpha_s U'_{n-s}),$$

since s cannot be greater that n.

M' may be treated similarly to P.

Thus every form is reducible to products of forms such as $(\alpha_r U_{n-r})(\alpha_r U'_{n-r})$ and a^2_x.

Or, neglecting polar forms, the complete irreducible system of a quadratic is

$$\alpha_n^2,\ (\alpha_{n-1} u_1)^2,\ (\alpha_{n-2} u_1 u_2)^2,\ \dots\ a_x^2.$$

There is one invariant, α_n^2, the discriminant and the other concomitants considered geometrically are the equations of the quadric in point, line, plane, ... S_{n-2} coordinates.

This is the irreducible system of type (2) § 7. Cp. Gordan, *M. A.*, Bd. LVI.

IV. THE SYSTEM FOR TWO QUADRATICS.

§ 27. *Preliminary Theorems.* We shall now find a finite system of forms in terms of which all concomitants of two quadratics may be expressed. It will be assumed, in accordance with § 7, that any bracket factor in a form containing less than $n-1$ u symbols may be transformed by means of the identities, but that factors of type a_x must always remain unaltered as far as x is concerned.

As regards the actual number of irreducible forms, the cases of $n = 2$, 3, 4 have been completely worked out.

For	$n = 2$	there are	6	concomitants	*,
	$n = 3$	,, ,,	20	,,	*,
	$n = 4$	,, ,,	580	,,	†.

We know *a priori* that the number in the general case is finite and we shall prove that the number $(n+1)$ of pure invariants is equal to the number of covariants and also contravariants.

§ 28. *Notation.* Let
$$f_1 = a^2_{1x} = a^2_{2x} = \dots a_x^2 = a_x'^2 = \dots,$$
$$f_2 = b^2_{1x} = b^2_{2x} = \dots b_x^2 = b_x'^2 = \dots,$$
be the two quadratics.

We shall invariably use the symbols a, α, A to belong to f_1; the symbols b, B, β, γ, δ, ζ will invariably refer to f_2; the symbols u, v will refer to the variables $u_1 u_2 \dots u_{n-1}$, x; the symbols θ, ϕ, ψ will usually refer to a, b or u.

The convention of § 10 concerning Greek and Arabic letters holds.

A bracket will usually be written in the form

$$(\alpha_p \beta_q v)$$

* Grace and Young, *Algebra of Invariants.* † Gordan, *M. A.*, Bd. LVI.

and in this case it is understood that the suffix of v is $n-p-q$, and when v occurs several times in a concomitant it need not refer to the same set of u symbols each time.

Suppose we have a symbol α_{r_k} in a concomitant. If we wish to consider this symbolic product as made up of k' sets of fewer as we write it

$$\alpha_{r_k} \equiv \alpha_{r_{k1}} \alpha_{r_{k2}} \dots \alpha_{r_{kk'}}.$$

If the symbols θ_p and θ_q occur simultaneously they usually refer to $p+q$ different symbols of type a, b or u, unless a definite statement is made to the contrary.

Finally, we use C, C', ... to stand for the total number of symbols in a bracket which are not of type b, so that any bracket must be of form $(\beta_r C)$, β_n, or (C).

The symbols g, g_1, $g_2 \dots g'$, denote bracket or x factors and whenever we write a single term concomitant as

$$P \equiv g_1 g_2 \dots g_r M,$$

g_k is understood to mean the kth bracket from the left of the last concomitant in which that particular bracket is fully written out.

The symbols M, M', ... are used to indicate that part of a single term concomitant which is not explicitly written out.

Sometimes the symbol X is used to denote the whole product of factors of type a_x or b_x in a concomitant.

§ 29. *Reducibility.* We shall consider a concomitant P of two quadratics to be reducible if it may be expressed in the form $\Sigma P_1 Q_1$, where P_1 and Q_1 contain no symbol of type a or b in common.

For example, P_1 might be $(\alpha_r \beta_k v_s)(\alpha_r \beta_k v_s')$, where v_s and v_s' are not necessarily the same, but each consist of s symbols of type u.

We shall denote the number of bracket factors of P by w, the weight of P.

§ 30. *Theorem I.* Let P be any concomitant of two quadratics. If P contain a factor α_n or β_n it is reducible.

For in the first case P is of form

$$\alpha_n \,.\, M,$$

and the n symbols a complementary to those of α_n either lie in M in the form a_x or one at least is situated in an ordinary bracket. In the former case P is zero as is seen by interchanging two as, and in the latter case P may be prepared in the form

$$\Sigma \alpha_n^2 \,.\, M' \text{ or } \alpha_n^2 \,.\, \Sigma M',$$

which is reducible.

Hence P is reducible if it contain the factor α_n.

Similarly it is if β_n figure in it.

§ 31. *Theorem II.* Again, if P contain two bracket factors of type

$$(\beta_p v)(\beta_q v), \qquad (1)$$

or

$$(\alpha_p v)(\alpha_q v), \qquad (2)$$

P is reducible provided that one symbol or more is common to β_p, β_q or α_p, α_q in the respective cases.

In the first case take P to be

$$(\beta_p \beta_r v)(\beta_q \beta_r v) M,$$

where β_p, β_q, β_r are entirely different.

If $p=0$, $q=0$, P is at once reducible.

If $p>0$ and $r>0$ we may express P as a sum of terms

$$Q=(\beta_p \beta_r v)(\beta_p \beta_r \beta_{q_1} v) M,$$

where β_{q_1} is part or none of β_q, (§ 25).

If $q_1=0$, Q is reducible. But if $q_1>0$ we may bracket the complementary symbols to β_{q_1} with $\beta_q \beta_r$ in g_1, since the same symbols occur all together in g_2. Thus Q is expressible as a sum of terms

$$R=(\beta_p \beta_r \beta_{q_1} v)(\beta_p \beta_r \beta_{q_1} v') M \text{ or } (Bv)(Bv') M,$$

all of which are reducible.

Hence P is reducible.

Exactly the same argument applies to a concomitant with factors of type (2).

This argument is useless if the symbols u are not allowed to be separated (Cp. type (2), § 7).

§ 32. Suppose that we consider concomitants in their prepared forms mod. f_1. Then as a rule a single term concomitant of this type would be

$$P=(\alpha_r \beta_k \gamma_s v_t)(\alpha_r \beta_k \delta_{s_1} v_{t_1}) M \quad \text{.................................}(1),$$

where β_k represents whatever b symbols are common to the two brackets written out in full; where α refers to f_1; β, γ, δ to f_2, and v to the variables.

We may suppose that $s \geqslant s_1$.

The exceptional cases are (1) when $r=1$, $k=s_1=0$,

and $$P=(a\gamma_s v_t)\, a_x M \quad \text{.......................................}(1a),$$

and (2) when $P=a_x^2 M=$ reducible.

If we call any concomitant of type (1)

$$P(r;\ k,\ s,\ s_1)$$

we shall prove the following theorem:

Theorem III. Every term of type $P(r;\ k,\ s,\ s_1)$ where $s \geqslant s_1$, $r>1$ is either zero or expressible as a sum of terms of the following types,

I. $P(r';\ l,\ m,\ n)$, $r'>r$,

II. $P(r;\ k',\ m',\ n')$, $k'>k$,

III. $P(r;\ k,\ m'',\ s_1')$, $s_1'<s_1$,

the suffixes r, k, s, s_1 being supposed to admit these inequalities, and s_1 being at least equal to 2.

If P is reducible we shall say $P \equiv R$.

If P is expressible in terms of types I—III, except for terms of type Q, we shall say

$$P \equiv \Sigma Q \text{ mod. I—III.}$$

Thus the theorem to be proved is that

$$P(r;\ k,\ s,\ s_1) \equiv 0 \text{ mod. I—III,}$$

provided $r > 0$, $s \geqslant s_1 > 1$.

§ 33. We take P in the form

$$P = (\alpha_r \beta_k \gamma_s v)(\alpha_r \beta_k \delta_{s_1} v) M \quad \text{.......................(1)}$$
$$= g_1 g_2 M,$$

and suppose $\alpha_r \beta_k \equiv \theta$.

Consider the s symbols complementary to γ_s. Each of them must occur in M, either in a bracket or in the form g_x. If two occur in the latter form P is zero since

$$(bb'\phi) b_x b_x' M = 0.$$

Thus if $s > 1$ one at least of the s symbols of γ_s occurs in a bracket. P may therefore be prepared (§ 25) so that all the symbols complementary to γ_s occur in this bracket; so that $P = \Sigma P_1$ where

$$P_1 = (\theta \gamma_s v)(\theta \delta_{s_1} v)(\gamma_s BC) M \quad \text{.......................(2),}$$

where B denotes symbols of f_2 and C, symbols of f_1 or v.

If P contain two brackets only and $s_1 > 1$, P is therefore zero. We assume P to have more than two brackets and investigate terms such as P_1.

The symbol B may contain some in common with δ_{s_1}. We therefore write P_1 as

$$(\theta \gamma_s v)(\theta \delta_{s_1} v)(\gamma_s \delta_{s_{11}} \beta_{k_1} C) M \quad \text{.......................(2'),}$$

where $\beta_{k_1} \delta_{s_{11}} \equiv B$ and $\delta_{s_{11}} \delta_{s_{12}} \equiv \delta_{s_1}$.

§ 34. *Lemma I.* If in the form (2') $s_{11} > 0$, $P_1 \equiv 0$ mod. III and the theorem is proved.

For if $s_{11} > 0$, since the symbols $\gamma_s \delta_{s_{11}}$ are all equivalent b symbols bracketed in g_3, we may bracket the complementary symbols γ_s and $\delta_{s_{11}}$ out of g_1 and g_2 into g_1. The result expresses P_1 as

$$\sum_z \left(\frac{\theta v}{z} \gamma_s \delta_{s_{11}}\right)(\theta \delta_{s_{12}} \dot{z} v)(\gamma_s \delta_{s_{11}} \beta_{k_1} C) M, \qquad (\S\ 25),$$

where z denotes s_{11} symbols chosen from θ and v.

If z contain any symbol of θ whether of type a or b, the bracket g_2 contains two like symbols and the term is zero; for $(bb\phi) M = 0$.

Hence the only possible non-zero terms of this summation are those in which z consists entirely of vs and these only come when the number of vs in g_1 is originally $\geqslant s_{11}$.

Thus P_1 is replaced by $\sum_v (\theta \gamma_s \delta_{s_{11}} v)(\theta \delta_{s_{12}} v) g_3 M$.

Each of these terms is of type $P(r;\ k, \ldots, s_{12})$ with $s_{12} < s_1$, that is to say, type III. Hence $P_1 \equiv 0$ mod. III. Q. E. D.

The theorem is now proved unless $s_{11}=0$. We therefore take P to be of the form

$$P=(\alpha_r\beta_k\gamma_s v)(\alpha_r\beta_k\delta_{s_1}v)(\gamma_s\beta_{k_1}C)M \text{ ....(3)},$$

where β_{k_1} represents all the b symbols included with γ_s in g_3 and where β_k, γ_s, δ_{s_1}, β_{k_1} are entirely different.

§ 35. *Lemma II.* If in the form (3) $k_1 < s_1$, $P\equiv 0$ mod. I—III.

For since s_1 and s are >0 we may apply the operation $R(\delta_{s_1}, \gamma_s)$ to g_2 and g_3 and obtain by Identity B, § 20,

$$\Sigma g_1(\alpha_r\beta_k\dot{\delta}_{s_1}v)(\dot{\gamma}_s\beta_{k_1}C)M+\Sigma g_1(\alpha_r\beta_k\dot{z}v)\left(\delta_{s_1}\gamma_s\frac{\beta k_1 C}{z}\right)M=0.$$

The first series consists of the term P and terms in which some of the symbols of γ_s and δ_{s_1} have been interchanged. Such terms contain r like symbols a and more than k like symbols b in g_1 and g_2. They are therefore of type II. In the second series z represents s_1 symbols from $\beta_{k_1}C$ and if $k_1 < s_1$, z must include symbols of C, i.e. symbols of type a or u. If z include p a symbols, α_p say, g_2 contains $\alpha_r\alpha_p$ and g_1, α_r. Since $r>0$ we may prepare g_1 and g_2 so that each contains $\alpha_r\alpha_p$. Such terms are of type I. In the remaining terms z includes some of β_{k_1}, $\beta_{k_{11}}$ say, and some u symbols. Thus

$$P+\Sigma g_1(\alpha_r\beta_k\beta_{k_{11}}v)(\delta_{s_1}\gamma_s\beta_{k_{12}}C)M\equiv 0 \text{ mod. I, II.}$$

where $$\beta_{k_{11}}\beta_{k_{12}}\equiv\beta_{k_1}.$$

Since $k_{11}\ngtr k_1 < s_1$, all these terms except P are of type III.

Therefore $P\equiv 0$ mod. I—III. Q. E. D.

Thus Lemmas I and II cover all cases except when

$$P=(\theta\gamma_s v)(\theta\delta_{s_1}v)(\gamma_s\beta_{k_1}C)M, \quad k_1\geqslant s_1>1.$$

The symbols complementary to β_{k_1} are more than one in number and therefore may be bracketed together in a fourth bracket or else lie in factors b_x; so that $P=0$. Hence the theorem is proved for concomitants of not more than three brackets.

We therefore consider with perfect generality forms such as

$$P=(\theta\gamma_s v)(\theta\delta_{s_1}v)(\gamma_s\beta_{k_1}C)(\beta_{k_1}BC')M \text{ ....(4)},$$

where $s\geqslant s_1$, $k_1\geqslant s_1$, $\theta=\alpha_r\beta_k$.

Here B may include some symbols of δ_{s_1}. Let $\delta_{s_1}\equiv\delta_{s_{11}}\delta_{s_{12}}$ and $B=\delta_{s_{11}}B'$. Then we write P as

$$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\gamma_s\beta_{k_1}C)(\beta_{k_1}\delta_{s_{11}}B'C')M \text{ ....(4')}.$$

§ 36. *Lemma III.* If in the form (4′) $s_{11}>0$, $P\equiv 0$ mod. I—III.

For since in (4′) the symbols β_{k_1} and $\delta_{s_{11}}$ are bracketed in g_4, the complementary symbols $\delta_{s_{11}}$ and β_{k_1} may be taken from g_2 and g_3 and bracketed in g_3—supposing that $k_1>0$ and $s_{11}>0$ (§ 25).

In this way we express P as

$$\Sigma(\theta\gamma_s v)(\theta\delta_{s_{12}}v\dot{z})\left(\frac{\gamma_s C}{z}\beta_{k_1}\delta_{s_{11}}\right)(\beta_{k_1}\delta_{s_{11}}B'C')M,$$

where z consists of s_{11} symbols from $\gamma_s C$.

If z include any symbols of γ_s the corresponding term has r like as, more like bs in $g_1 g_2$ and is therefore of type II.

If z include any symbol a from C, g_2 will contain more than $r\,a$ symbols; we may then bracket the set complementary to these in g_1 since $r > 0$ (§ 25) and obtain nothing but terms of type I. The only other possibility is for z to be of type v entirely. The corresponding terms will be of type $P(r;\ k,\ s,\ s_{12})$ or type III, since $s_{12} < s_1$.

Thus $P \equiv 0$ mod. I—III. Q. E. D.

§ 37. Hence every form P satisfies the theorem except those of form (4′) with $s_{11} = 0$. If P contains four brackets only and $s_{11} = 0$, the symbols complementary to δ_{s_1} must be entirely in factors like d_x, and since $s_1 > 1$, P is zero. Assume P to have five brackets at least and let the symbols complementary to δ_{s_1} in g_2 be bracketed in a fifth bracket. We consider forms of type

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{k_1}\gamma_s C)(\beta_{k_1} B C')(\delta_{s_1} B' C'')\,M \qquad (5).$$

Let the symbols common to B and B' be β_p and let $B \equiv \beta_p\beta_l$, $B' \equiv \beta_p\beta_m$.

Then
$$P = (\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{k_1}\gamma_s C)(\beta_{k_1}\beta_p\beta_l C')(\beta_p\beta_m\delta_{s_1} C'')\,M \qquad (A)$$
$$= g_1 g_2 g_3 g_4 g_5 M,$$

where $r > 0$, $s \geqslant s_1$, $k_1 \geqslant s_1$, $s_1 > 1$.

Either $p = 0$ or is > 0.

(1) Let $p > 0$. We may then bracket β_{k_1} with β_p from $g_3 g_5$ in g_5 since the complementary symbols are bracketed in g_4 and both k_1 and p are > 0. P is then replaced by terms of type

$$g_1 g_2 (\gamma_s C z_{k_1})(\beta_{k_1}\beta_p\beta_l C')\left(\beta_p\beta_{k_1}\frac{\beta_m\delta_{s_1} C''}{z_{k_1}}\right) M \qquad (6).$$

If z contain any symbol from δ_{s_1} Lemma I applies.

We therefore consider terms of type (6) with $z = \beta_{m_1} C_1''$, where $\beta_{m_1}\beta_{m_2} = \beta_m$, $C'' = C_1'' C_2''$. On writing C for CC_1'' and C'' for C_2'' and $\beta_{p'}$ for $\beta_p\beta_{k_1}$ (6) takes the form

$$g_1 g_2 (\beta_{m_1}\gamma_s C)(\beta_{p'}\beta_l C')(\beta_{p'}\beta_{m_2}\delta_{s_1} C'')\,M \qquad (6').$$

In this form we may consider $m_1 \geqslant s_1$, otherwise Lemma II applies. Thus if $m < s_1$, it follows that $m_1 < s_1$ and $P \equiv 0$ mod. I—III.

(2) But suppose that $p = 0$. If we apply the process $R(\beta_{k_1},\ \delta_s)$ to g_3, g_5 we obtain the following identity:

$$\Sigma g_1 g_2 (\dot\beta_{k_1}\gamma_s C)\, g_4 (\beta_m \dot\delta_s C)\, M + \Sigma g_1 g_2 (\dot z \gamma_s C)\, g_4 \left(\beta_{k_1}\delta_s \frac{\beta_m C}{z}\right) M = 0.$$

In every term of the first series except the first ($= P$) some symbols of δ_s occur in g_3 with γ_s and Lemma I applies. Every term of the second series is either $\equiv 0$ mod. I—III or of form (6′) as in the previous case.

Therefore the last identity expresses

$$P \equiv \Sigma \text{ forms } (6') \text{ mod. I—III}.$$

§ 38. Thus both cases come to the same thing and we must consider terms of type

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{m_1}\gamma_s C)(\beta_{p'}\beta_l C')(\beta_{p'}\beta_{m_2}\delta_{s_1} C'')\, M \quad \text{.....................(6')},$$

where m_1 and s are both $\geqslant s_1$, and p' is at least $= k_1$ and therefore > 1.

We may therefore bracket δ_{s_1} from g_2 with $\beta_{p'}$ in g_4 since they are also bracketed in g_5. If $l < s_1$ the resulting terms must each have some symbols from C' to take the place of δ_{s_1} in g_2 and, as before, are all of type I and III. Thus if $l < s_1$ the theorem is proved.

Take $l \geqslant s_1$ and let $\beta_l \equiv \beta_{s_1}\beta_{l_2}$. The operation here indicated replaces (6′) by a series of forms

$$\Sigma\,(\theta\gamma_s v)(\theta\dot{\beta}_{s_1} v)(\beta_{m_1}\gamma_s C)(\beta_{p'}\delta_{s_1}\dot{\beta}_{l_2}\dot{C}')(\beta_{p'}\delta_{s_1}\beta_{m_2} C'')\, M.$$

Terms of this series in which some symbols of C' appear in g_2 are of types I, III, as before. All other terms are of type

$$g_1(\theta\beta_{s_1} v)\, g_3\,(\beta_{p'}\delta_{s_1}\beta_{l_2} C')(\beta_{p'}\delta_{s_1}\beta_{m_2} C'')\, M.$$

Interchange the equivalent symbols β_{s_1} and δ_{s_1} in this and we have the result that the concomitant P is $\equiv 0$ mod. I—III + terms of type

$$g_1(\theta\delta_{s_1} v)\, g_3\,(\beta_{p'}\beta_{s_1}\beta_{l_2} C')(\beta_{p'}\beta_{s_1}\beta_{m_2} C'')\, M$$

or

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{m_1}\gamma_s C)(\beta_{p''}\beta_{l_2} C')(\beta_{p''}\beta_{m_2} C'')\, M \quad \text{........................(7)},$$

where $\beta_{p''} \equiv \beta_{p'}\delta_{s_1}$, and where the symbols complementary to $\delta_{s_1}\beta_{m_1}\beta_{m_2}$ and β_{l_2} lie in M.

Since $p'' > 0$ we may bracket $\beta_{p''}\beta_{l_2}$ in g_5 as well as g_4 and express (7) as terms such as

$$g_1 g_2 g_3\,(\beta_{p''}\beta_{l_2} C')(\beta_{p''}\beta_{l_2}\beta_{m_{21}} C''')\, M,$$

where $\beta_{m_{21}}$ is part of β_{m_2}.

Finally if $m_{21} > 0$ we may bracket the set complementary to $\beta_{m_{21}}$ of g_5 in g_4 and writing B for $\beta_{p''}\beta_{l_2}\beta_{m_{21}}$ obtain in all cases terms of type

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{m_1}\gamma_s C)(BC_1)(BC_2)\, M \quad \text{...........................(8)},$$

where $\theta = \alpha_r\beta_k$, $r > 0$; s, $m_1 \geqslant s_1$; and the number of bs in B is $\geqslant p''$, i.e. > 0.

§ 39. This form (8) reproduces form (3) with one difference, namely that in (8) there are two brackets more prepared mod. f_2 than there were in (3). To see this, suppose that $2k$ brackets were prepared mod. f_2 in (3). Such brackets must lie in M. All the processes through which (3) went to arrive at the form (8) never altered a bracket in M except such as included any symbols complementary to the symbols δ_{s_1}, β_{k_1} and later β_l and β_m; such brackets therefore could not be prepared mod. f_2. Thus the final form of M in (8) is the same as the form of M in (3) as regards brackets prepared mod. f_2. Thus (8) includes $2k$ prepared brackets and the two new ones g_4 and g_5.

If in the form (3) the number of brackets is $2k + 3 + k'$, the number of brackets in (8) in which symbols complementary to δ_{s_1} and β_{m_1} may lie is $k' - 2$, whereas it was k' in the case of (3).

Similarly if we operate on (8) as we did on (3) we obtain terms of types I—III or else terms such as

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{\mu_1}\gamma_s C)\, M \quad \text{...................................(8')},$$

$$\mu_1 \geqslant s_1,$$

where the number of brackets prepared mod. f_2 in M is two more than in (8), i.e. is $2k + 4$.

§ 40. Proceeding in this way we come after a finite number of stages to one at which no terms except I—III appear, or if any others do they are of form (8′) where all but one or else two brackets of M are prepared mod. f_2, according as k' is odd or even.

(i) If there is one bracket unprepared it must contain all symbols complementary to δ_{s_1} and β_{μ_1} which do not lie in x factors. Hence either the term is zero, since $\mu_1 \geqslant s_1 > 1$ or we may bracket all β_{μ_1} in this bracket and obtain the term

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{\mu_1}\gamma_s C)(\beta_{\mu_1}\delta_{s_{11}} C') M_b d_x d_x' \dots a_x \dots,$$

where M_b denotes brackets prepared mod. f_2, and $dd' \dots \delta_{s_{11}} = \delta_{s_1}$.

Since $s_1 > 1$, either $s_{11} > 0$ or more than one d occurs with x.

Thus either Lemma III applies or the term is zero.

Hence if k' is odd $P \equiv 0$ mod. I—III.

(ii) If there are two brackets left unprepared in M, the argument is exactly the same as that which led up to the form (5) and we either have zero terms or those to which Lemma III applies or else forms

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\beta_{\mu_1}\gamma_s C')(\beta_{\mu_1} BC'')(\delta_{s_1} B'C''') M_b CX \quad \dots\dots\dots\dots\dots\dots (9),$$

where X denotes only factors of type a_x or b_x, C denotes brackets with no b, and M_b, brackets prepared mod. f_2.

If any symbols B, B' appear in g_4, g_5 they are either complementary to one another or else their complements are in X.

We may therefore write (9) as

$$g_1 g_2 g_3 (\beta_{\mu_1}\beta_p\beta_l C'')(\delta_{s_1}\beta_p\beta_m C''') M_b CX \quad \dots\dots\dots\dots\dots\dots\dots\dots (10),$$

where the complements of β_l and β_m occur in the form b_x.

If $l > 1$ or $m > 1$ this form is therefore zero.

Hence $l \not> 1$, $m \not> 1$. $\therefore$ $l < s_1$ and $m < s_1$. But if $m < s_1$ this term $\equiv 0$ mod. I—III (§ 35).

Therefore in all cases whether the number of brackets in P is odd or even, $P \equiv 0$ mod. I—III, which proves the theorem.

This theorem is true of $P(r;\, k, s, s_1)$ provided that $r > 0$, $s \geqslant s_1$, $s_1 > 1$.

§ 41. Suppose that we apply this theorem to each term I—III which arises from this reduction of $P(r;\, k, s, s_1)$. Since the terms which arise are of the same type $P(-;\, -, -, -)$, with either an increase in r, or $r + k$, or no change in $r + k$ and a diminution in s_1, it follows that after a finite number of steps, in which we apply the theorem to whatever term arises, we ultimately express $P(r;\, k, s, s_1)$ in terms of

(1) $P(n;\, 0, 0, 0)$,
(2) $P(r';\, n - r', 0, 0)$,
(3) $P(r';\, k', s', 1)$.

The first two of these sets are terms with invariants α_n^2 or $(a_{r'}\beta_{n-r'})^2$ for factors. Thus they are reducible. The third set which arises from s_1 diminishing to its lower limit may be taken one step further.

§ 42. Take any form P of type (3) and apply the theorem to it.

Let $$P = (\alpha_r \beta_k \gamma_s v)(\alpha_r \beta_k d v) M.$$

I. Suppose $s > 1$. As in the general case we may bracket γ_s again in P and take P as

$$(\alpha_r \beta_k \gamma_s v)(\alpha_r \beta_k d v)(\beta_{k_1} \gamma_s C) M.$$

If the complement of d lie in g_3, Lemma I reduces P to the form $P(r;\ k,\ s+1,\ 0)$, or $(\gamma_{s+1} y_{s+1}) M$, say.

If $k_1 = 0$, Lemma II applies.

If $k_1 > 1$, either $P = 0$ or β_{k_1} may be bracketed in another bracket.

If $k_1 = 1$, either β_{k_1} is in a bracket factor or with x.

In the latter case $P = (\alpha_r \beta_k \gamma_s v)(\alpha_r \beta_k d v)(b \gamma_s C) b_x M$ say.

On applying $R(d,\ b)$ to g_2, g_3 we obtain the identity

$$P - g_1 (\alpha_r \beta_k b v)(d \gamma_s C) b_x M + \Sigma g_1 (\alpha_r \beta_k \dot{z} v)\left(d b \frac{\gamma_s C}{z}\right) b_x M = 0,$$

or $$P - (\gamma_s \eta_s) M + \text{I, II, III} = 0,$$

where $(\gamma_s \eta_s) \equiv (\theta \gamma_s v)(\theta v b) b_x$, which we shall call type IV, and where III has the factor

$$(\gamma_s y_s) \equiv (\theta \gamma_s v)(\theta v).$$

In the other cases β_{k_1} (whether k_1 is one or more) is bracketed twice and we have forms such as

$$P = (\theta \gamma_s v)(\theta d v)(\beta_{k_1} \gamma_s C)(\beta_{k_1} B C') M.$$

If d occurs in g_4 Lemma III applies and reduces P to types I—III. If not, either d lies in another bracket g_5 or with x. The latter form is of type IV.

Thus $P \equiv Q$ mod. I—IV, where Q is

$$(\theta \gamma_s v)(\theta d v)(\beta_{k_1} \gamma_s C)(\beta_{k_1} \beta_p \beta_l C')(\beta_p \beta_m d C'') M,$$

as in the general case.

§ 43. If we operate on this form as in the general case we obtain the same results unless at any time the symbol d or the symbols β_{k_1}, β_{m_1}, β_{μ_1} occur with x. Such cases only lead to forms I—IV as they are the same as those discussed above (§ 42). The main case is the same as when $s_1 > 1$ until we come to form (10) in the general theorem. In this case since m may be 1 and $s_1 = 1$ we have to consider the form

$$(\theta \gamma_s v)(\theta d v)(\beta_{\mu_1} \gamma_s C)(\beta_{\mu_1} \beta_p \beta_l)(\beta_p d b C'') b_x M,$$

in which $b \equiv \beta_m$ $(m = 1)$, and l is either zero or unity, μ_1 is at least unity. Cp. § 40.

If $p > 0$, we bracket $\beta_{\mu_1} \beta_p$ in g_5 from g_3 and g_5 and as before obtain terms of types I—III, and

$$g_1 g_2 (b \gamma_s C)(\beta_{\mu_1} \beta_p \beta_l C')(\beta_{\mu_1} \beta_p d C''') b_x M,$$

in which g_3 is of a form already discussed $(k_1 = 1)$, so that the term is reducible to I—IV.

Hence the theorem applies to $P(r;\ k,\ s,\ 1)$ if $s > 1$ if we include IV with I—III.

§ 44. II. Take $s=1$. P may be either

$$(\alpha_r\beta_k gv)(\alpha_r\beta_k dv)\, g_x M,$$

or

$$(\alpha_r\beta_k gv)(\alpha_r\beta_k dv)(g\beta_{k_1}C)\, M.$$

The second case may be treated exactly as in the case when $s>1$. The first case is of the form $(d_1 z_1)\, M$, say.

Lastly, if $s_1=0$, P necessarily has the factor $(\gamma_s y_s)$ (see beginning of § 42), or else $s=0$ and P has the factor

$$(\alpha_r\beta_k v)(\alpha_r\beta_k v').$$

§ 45. Combining all these results, we see that the first two factors of $P(r; k, s, s_1)$ may be taken to be either a concomitant of weight two of type

$$(\alpha_r\beta_k v)(\alpha_r\beta_k v'),\quad r>0 \text{ and } <n+1,$$

or else

$$(\theta\gamma_s v)(\theta v),$$

or

$$(\theta\gamma_s v)(\theta dv),$$

and in this case the complement of d occurs with x.

We may state these results in the following form:

Theorem IV. If a concomitant of two quadratics is expressible as

$$P=(\alpha_r Bv)(\alpha_r B'v')\, M,$$

it is expressible in terms of

$$\text{(A)}\quad (\alpha_r\beta_k\gamma_s v)(\alpha_r\beta_k dv)\, d_x M,$$

$$\text{(B)}\quad (\alpha_r\beta_k\gamma_s v)(\alpha_r\beta_k v)\, M,$$

where both k and s may be zero, but r must be >0 (§ 32).

Form (A) is written $(\gamma_s\eta_s)\, M$, and (B) $(\gamma_s y_s)\, M$, but we usually denote both y and η by ξ, so that $(\gamma_s\xi_s)\, M$ stands for (A) and (B).

§ 46. *Observation.* By no possibility can a term $(\gamma\xi)$ turn up of the form

$$(a\gamma v)(av) \text{ or } (a\gamma v)\, a_x,$$

for the number of different symbols u is $n-2$ when x is looked upon as an indecomposible coordinate, so that $(av)=0$ since two at least of the symbols included in v must be identical.

The notation $(\gamma\xi)\equiv(\gamma\theta v)(\theta v\hat{b})\,\hat{b}_x$ is used for $(\gamma\xi)$ and the symbol $\hat{b}$ means that b may either be present or absent from the term, so that both forms (A) and (B) are included.

§ 47. *Theorem V.* If by any process we have arrived at concomitants of the type

$$P\equiv\prod_{r=1}^{q}(\gamma_{s_r}\xi_{s_r})\, M \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and if M contain two brackets prepared mod. f_1, P is expressible as a sum of terms

$$\prod_{1}^{q+1}(\gamma_{s_r}\xi_{s_r})\, M' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and reducible terms.

Let us write $(\alpha_r\beta_k\gamma_s v)(\alpha_r\beta_k\delta_{s_1}v)M$ for M, where $r>0$ by hypothesis, and let P be written as

$$(\alpha_r\beta_k\gamma_s v)(\alpha_r\beta_k\delta_{s_1}v)M\Pi_b{}^qX \qquad (3).$$

We shall say that a concomitant having two initial factors of this type and q factors each involving ξ, is of type

$$P(r;k,s,s_1)\Pi_b{}^q.$$

If, however, instead of factors $(\beta_s\xi_s)$ Π has factors $(\alpha\beta\xi)$ we shall write it as Π_{ab}.

We have $r>0$ by hypothesis and we may suppose $s \geqslant s_1$.

Our object will be to prove that P is expressible in the following types:

I. $P(r'; \ldots\ldots)\Pi^q,\ r'>r,$

II. $P(r;k',\ldots)\Pi^q,\ k'>k,$

III. $P(r;k,\ldots,s_1')\Pi^q,\ s_1'<s_1,$

IV. $P(r;k,s',\ldots)\Pi^q,\ s'<s.$

§ 48. Suppose that $s_1=0$, then the two first factors are of form $(\gamma_s y_s)$ and the proposition is proved. Assume $s_1>0$. Then $s>0$.

The symbols complementary to γ_s of P lie either wholly in X and Π or not, X denoting all factors a_x or b_x. If not, we may bracket them in an ordinary bracket outside Π without altering the form of Π except possibly introducing symbols a or u for whatever symbols of γ were originally in Π (§ 22).

Therefore Π_b may become Π_{abu}. But u does not affect Π,

for
$$\begin{aligned}(\alpha\beta u\xi) &= (\alpha\beta v\theta v')(\theta v''\hat{b})\hat{b}_x \\ &= (\alpha\beta\xi') \text{ say.}\end{aligned}$$

Hence Π_{abu} is of type Π_{ab}.

Thus P may be taken in the form either

$$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\beta_{k_1}\gamma_s C)M\Pi_{ab}{}^q,$$

or
$$g_1g_2M \text{ [all } \gamma_s \text{ in } X\Pi].$$

As in the previous work none of δ_{s_1} lie in g_3 else Lemma I applies. After using Lemma II, either β_{k_1} may be bracketed again without altering the type of $\Pi_{ab}{}^q$ or all β_{k_1} lies in X and Π.

Next Lemma III applies and either δ_{s_1} may be bracketed again without altering the type of $\Pi_{ab}{}^q$ or all δ_{s_1} lies in X and Π.

Thus P is reduced to types I—III (by application of the Lemmas) and the four following types:

(1) $$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\beta_{k_1}\gamma_s C)(\beta_{k_1}\beta_p\beta_l C')(\beta_p\beta_m\delta_{s_1}C'')M\Pi_{ab}{}^qX,$$

as in § 37.

(2) When P admits of previous treatment up to the 4th bracket but the symbols complementary to δ_{s_1} lie in Π or X.

(3) When the symbols complementary to β_{k_1} lie in ΠX.

(4) When the complements of γ_s lie in ΠX.

In the 3rd case all terms are of form

$$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\beta_{k_1}\gamma_s C)M\{\beta_{k_1} \text{ in } \Pi X\}.$$

Apply the process $R(\delta_{s_1}, \gamma_s)$ to g_2 and g_3.

This gives

$$\Sigma g_1(\theta\dot{\delta}_{s_1}v)(\beta_{k_1}\dot{\gamma}_s C)M\{\ldots\}+\Sigma g_1(\theta\dot{z}_{s_1}v)\left(\delta_{s_1}\gamma_s\frac{\beta_{k_1}C}{z_{s_1}}\right)M\{\ldots\}=0,$$

and as before all the terms of these series are of types I—III except P and those in which z_{s_1} is entirely chosen from β_{k_1}. Thus if $\beta_{s_1}\equiv z_{s_1}=$ part of β_{k_1}

$$P\equiv \text{I—III mod. } \Sigma g_1(\theta\dot{\beta}_{s_1}v)(\delta_{s_1}\gamma_s\dot{\beta}_{k_1-s_1}C)M\{\beta_{k_1} \text{ in } \Pi X\}.$$

Interchange β_{s_1} and δ_{s_1}.

Therefore $$P\equiv\Sigma g_1(\theta\delta_{s_1}v)(\gamma_s\beta_{k_1}C)M\{\delta_{s_1} \text{ in } \Pi X\} \text{ mod. I—III}.$$

Thus terms of classes (2) and (3) above are either I—III or $(\theta\gamma_s v)(\theta\delta_{s_1}v)M\Pi X$, where all the symbols complementary to δ_{s_1} lie in ΠX. Unless some lie in Π either $P=0$ or $s_1<2$ and we obtain from the first two brackets an extra factor for Π and the theorem is proved. Thus some of δ_{s_1} lie in Π.

Let δ_{σ_1} be a part of δ_{s_1} which lies in a definite factor ξ.

P must then be of form

$$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\delta_{\sigma_1}\epsilon_{\sigma_2}\theta' v)(\theta' v\hat{b})\hat{b}_x MX\Pi^{q-1} \quad\ldots\ldots\ldots\ldots(4),$$

where the particular ξ factor is $(\delta_{\sigma_1}\epsilon_{\sigma_2}\xi)$ and ϵ stands for a or b symbols, where $\theta'=\alpha_{r_1}\beta_{k_1}$ and $\hat{b}$ implies that b is present supposing ξ to be of form η, or absent, supposing ξ to be of form y (see § 45). The two cases may be worked together.

§ 49. In θ' r_1 is >0 but k_1 may be zero (§ 45).

I. Suppose $k_1>0$. Let $\delta_{s_1}\equiv\delta_{\sigma_1}\delta_{\sigma_2}$. We may bracket β_{k_1} and δ_{σ_1} in g_4 since they also occur in g_3 of (4) and replace P by terms

$$\Sigma(\theta\gamma_s v)(\theta\delta_{\sigma_2}\dot{\alpha}_{r_{11}}\dot{\hat{b}}v)(\delta_{\sigma_1}\epsilon_{\sigma_2}\alpha_{r_1}\beta_{k_1}v)(\dot{\alpha}_{r_{12}}\beta_{k_1}\delta_{\sigma_1}\dot{\hat{b}}\dot{v})\dot{b}_x M,$$

where $\alpha_{r_{11}}\alpha_{r_{12}}=\alpha_{r_1}$ and $r_{11}\not>\sigma_1$.

If $r_{11}=0$ each of these terms is of type III unless $\sigma_1=1$ and the symbol b has been changed from the bracket g_4 to g_2.

This exception is

$$g_1(\theta\delta_{\sigma_2}\hat{b}v)(\beta_{k'}\epsilon_{\sigma_2}\alpha_{r_1}v)(\alpha_{r_1}\beta_{k'}v)\hat{b}_x M \quad\ldots\ldots\ldots\ldots(5),$$

where $\beta_{k'}\equiv\beta_{k_1}\delta_{\sigma_1}$.

If $r_{11}>0$ we may prepare g_1g_2 mod. f_1 and obtain

$$\Sigma(\alpha_r\alpha_{r_{11}}\dot{\beta}_{k_2}\dot{\gamma}_{s_2}\dot{v})(\alpha_r\alpha_{r_{11}}\beta_k\delta_{\sigma_2}\hat{b}v)(\alpha_{r_{12}}\beta_{k'}\epsilon_{\sigma_2}\dot{\beta}_{k_3}\dot{\gamma}_{s_3}v)(\alpha_{r_{12}}\beta_{k'}\hat{b}v)\hat{b}_x M\ldots\ldots\ldots(6),$$

where $\beta_{k_2}\beta_{k_3}=\beta_k$, $\gamma_{s_2}\gamma_{s_3}=\gamma_s$ and $k_3+s_3\not>r_{11}$.

Each term of this series is of type I since the 3rd and 4th brackets of (6) are of type $(\alpha\beta\xi)$.

Hence if any of δ_{s_1} occur in Π, $P\equiv 0$ mod. I—III or is of type (5), and (5) is of type III unless b turns up in g_2.

But in (5) the symbols complementary to δ_{σ_2} by hypothesis lie in ΠX, or $\sigma_2=0$. If they lie in X there must be at least two symbols of type b in g_2 found also with X, for there is one already, namely b, so that the term is zero. If $\sigma_2=0$ the first two factors of (5) with b_x form a new ξ factor and the theorem is proved. The only alternative is for δ_{σ_2} to lie in Π entirely. The preceding reduction then applies again since $k'>k>0$, and expresses P in terms of I—III and also terms corresponding to (5), namely,

$$g_1(\theta\delta_{\sigma_3}\hat{b}\hat{b}'v)\,\hat{b}_x\hat{b}_x'M\Pi^qX,$$

where $\sigma_3<\sigma_2<s_1$.

Such terms are either zero or

$$g_1(\theta\delta_{\sigma_3}\hat{b}v)\,\hat{b}_x\,M\Pi^qX.$$

Unless $\sigma_3=0$ we continue in this way applying the same arguments till finally we must have exhausted the symbols δ from g_2 and we have the same case as when $\sigma_2=0$ above.

Thus P is expressible as I—III or $\Pi^{q+1}M$.

II. Suppose that $k_1=0$, P is then of form

$$(\theta\gamma_s v)(\theta\delta_{s_1}v)(\delta_{\sigma_1}\epsilon_{\sigma_2}\alpha_{r_1}v)(\alpha_{r_1}v\hat{b})\,\hat{b}_x M\Pi_b{}^{q-1}X \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4').$$

Since g_4 cannot be of form a_x (§ 46) it is always possible to use $R(\delta_{\sigma_1},\,\alpha_{r_1}v\hat{b})$ on g_2 and g_4. The result of this gives terms such as

$$\Sigma\,(\theta\gamma_s v)(\theta\delta_{\sigma_2}\dot{\alpha}_{r_{11}}\dot{\delta}_{\sigma_{11}}v)(\delta_{\sigma_1}\epsilon_{\sigma_2}\alpha_{r_1}v)(\dot{\alpha}_{r_{12}}\dot{\delta}_{\sigma_{12}}\dot{v}\dot{\hat{b}})\,M=0,$$

all of which except P have been discussed in the previous case of $k_1>0$, since in these $\delta_{\sigma_{12}}$ is common to g_3 and g_4.

Thus the theorem is proved for concomitants coming under cases (2) and (3).

§ 50. Now in this proof for case (2) we have not used the fact that $s_1 \leqslant s$ except in saying that terms occur of type III or $P(r;\,k,\,\ldots,\,s_1')\,\Pi$, $s_1'<s_1$. The proof therefore applies to case (4) (in which the symbols γ_s bear the same relation to Π that δ_{s_1} bore in case (2)) provided that we introduce a type IV or $P(r;\,k,\,s',\,\ldots)\,s'<s$. If in IV $s'<s_1$ we interchange the first two brackets and call the term type III.

Hence concomitants coming under cases (2), (3), (4) are expressible in types I—IV and Π^{q+1}.

§ 51. The only other type to be considered is that of class (i).

Let $$P=(\theta\gamma_s v)(\theta\delta_{s_1}v)(\gamma_s\beta_{k_1}C)(\beta_{k_1}\beta_p\beta_l C')(\delta_{s_1}\beta_p\beta_m C'')\,XM\Pi_{ab}{}^q\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
be such a concomitant.

As in the previous theorem if $p>0$ we bracket $\beta_p\beta_{k_1}$ from g_5, g_3 in g_5 since their complements are found in g_4.

But if $p=0$ we apply the process $R(\beta_{k_1},\,\delta_{s_1})$ on g_3, g_5 and in both cases obtain terms of types I—III, and

$$\Sigma g_1g_2(\gamma_s\beta_{m_1}C)(\beta_{p'}\beta_{m_2}C')(\delta_{s_1}\beta_{p'}\beta_l C'')\,MX\Pi_{ab}{}^q\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

where $\beta_{m_1}\beta_{m_2}\equiv\beta_m$ and $\beta_{p'}\equiv\beta_p\beta_{k_1}$, so that $p'>0$ in both cases.

Since $s_1 > 0$ we may bracket δ_{s_1} and $\beta_{p'}$ in g_4 as before, and by repeating precisely the reduction of the previous theorem we obtain a sum of terms

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\gamma_s\beta_{m_1} C)(BC_1')(BC_1'')\,MX\Pi^q \quad\ldots\ldots\ldots\ldots\text{(iii)},$$

to which terms (ii) are equivalent, mod. I—III (§ 38).

Now $B = \beta_p\beta_{k_1}\beta_{s_1}\beta_{l_2}\beta_{m_{21}}$. Hence a certain number of symbols a, u from C' and C'' in (ii) have been displaced in (iii) by symbols complementary to β_l and β_m, and have taken up the places vacated by the latter symbols in $MX\Pi_{ab}{}^q$. Some of these a and u symbols may therefore lie in Π^q of (iii).

As regards u symbols this makes no difference to the form of Π_{ab} for any factor, since $(\alpha\beta v\xi)$ is of type $(\alpha\beta\xi)$ (§ 48).

Now suppose that P had originally k pairs of factors prepared mod. f_2. They must occur in the portion of P which we call M and in (iii) we have $k+1$ such pairs, for none of the operations through which P has gone have altered the original k pairs. If we write the original k pairs as

$$\prod_1^k (B_m C_m)(B_m C_m')$$

we should now write (iii) as

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\gamma_s\beta_{m_1} C)\prod_1^{k+1}(B_m C_m)(B_m C_m')\,MX\Pi_{ab}{}^q \quad\ldots\ldots\ldots\text{(iii}\,a\text{)}.$$

§ 52. This form (iii a) is of exactly the same type as the original concomitant except that it has two more brackets prepared mod. f_2, and $\Pi_{ab}{}^q$ for $\Pi_b{}^q$.

If we call it Q, we may sum up the previous work by saying that

P, the original concomitant $\equiv \Sigma Q$ mod. I—IV, $M\Pi_{ab}{}^{q+1}$, or $\equiv \Sigma Q$ for short.

Similarly since Q is of the same type as P it is expressible as

$$Q \equiv \Sigma Q',$$

where Q' has $2k+4$ brackets prepared mod. f_2.

Similarly $\qquad Q' \equiv \Sigma Q'',\ Q'' \equiv \Sigma Q''',\ \ldots\ldots$

We proceed in this way obtaining more brackets prepared mod. f_2 at each stage till either all the terms $Q^{(m)}$ are $\equiv 0$ by Lemmas I—III or till terms are either of this form or such that it is impossible to form more brackets prepared mod. f_2.

§ 53. The latter case means that we replace P by terms of type (iii a) in which M contains one or no bracket involving b symbols.

If M contains one such bracket, we write (iii a) as

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\gamma_s\beta_{\mu_1} C)(\beta_{\mu_{11}}\beta_\nu C')\,P_b YX\Pi_{ab}{}^q \quad\ldots\ldots\ldots\ldots\text{(iv)},$$

where P_b denotes brackets prepared mod. f_2 and where Y contains a and u symbols only, and $\beta_{\mu_{11}}$ is part of β_{μ_1} and $\delta_{s_{11}}$ of δ_{s_1}.

If $\mu_{11} > 0$ we bracket β_{μ_1} in g_4 as in g_3, and obtain terms

$$g_1 g_2 g_3 (\beta_{\mu_1}\delta_{s_{11}}\beta_{\nu'} C)\,P_b XY\Pi_{ab}{}^q.$$

If $s_{11} > 0$ Lemma III applies.

If $s_{11} = 0$ then the symbols complementary to δ_s lie in ΠX and the case has been discussed.

Thus if $\mu_{11} > 0$, (iii a) $\equiv 0$ mod. I—IV, Π^{q+1}.

Take $\mu_{11} = 0$. Then all the symbols complementary to β_{μ_1} lie in ΠX and we have case (2) of § 48.

The same applies if there is no bracket g_4, i.e. if (iii a) is replaced by

$$(\theta\gamma_s v)(\theta\delta_{s_1} v)(\gamma_s\beta_{\mu_1} C) P_b X Y \Pi_{ab}{}^q.$$

Thus in all cases

$$P \equiv 0 \text{ mod. I—IV, } \Pi^{q+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

§ 54. If we perform the same reductions over and over again as many times as is possible upon terms of types I—IV, as in the previous theorem (III), the relation (v) implies that

$P(r;\, k, s, s_1)\,\Pi_b{}^q$ is expressible in terms of

(1) $P(r;\, k, 0, 0)\,\Pi_{ab}{}^q$, i.e. reducible terms,

(2) $P(r;\, k, s, 0)\,\Pi_{ab}{}^q$,

for we have proved relation (v) supposing $r > 0$, $s \geqslant s_1$, $s_1 > 0$.

In fact we obtain ultimately nothing but type $\Pi_{ab}{}^{q+1}$. For (1) and (2) are of this type and are reducible if in the factor $(\gamma_s \xi_s)$, $s = 0$.

§ 55. Finally we may express a term $\Pi_{ab}{}^{q+1} M$ as a sum of terms $\Pi_b{}^{q+1} M$ and thus establish the theorem completely.

There is no reason to suppose that the a symbols arising explicitly in Π_{ab} are all different. Suppose that two ξ factors of Π_{ab} have symbols α_p in common.

We may take these two factors to be

$$(\alpha_{r_{11}}\alpha_p\beta_t\theta v)(\theta v\hat{b})(\alpha_{r_{12}}\alpha_p\beta_{t_1}\theta' v)(\theta' v\hat{b}')\,\hat{b}_x\hat{b}'_x,$$

where $\theta = \alpha_{r_1}\beta_{k_1}$ and $r_1 > 0$ by hypothesis.

Bracketing α_p with α_{r_1} in the second bracket since they are bracketed in the first, we replace these four brackets by

$$\Sigma\,(\alpha_{r_{11}}\beta_t\alpha_{r_1+p}\beta_{k_1}v)\left(\alpha_{r_1+p}\frac{\beta_{k_1}v\hat{b}}{z_p}\right)(\alpha_{r_{12}}\dot{z}_p\beta_{t_1}\theta' v)(\theta' v\hat{b}')\,\hat{b}_x\hat{b}'_x,$$

or by

$$\Sigma\,(\alpha_{r_{11}}\beta_t\xi)(\alpha_{r_{12}}\beta\xi),$$

where no a is common to the two ξ factors.

For if z_p contain $\hat{b}$ and b exist, the last two factors are still of the form $(\alpha\beta\xi)$. They certainly are of this form for other possible values of z_p.

Doing the same for every pair of like sets of a symbols explicitly stated in Π_{ab}, we express the concomitant as

$$\Pi^{q+1}(\alpha_{s_{r_1}}\gamma_{s_{r_2}}y_{s_r})\,M,$$

where all the αs found explicitly in Π are different.

§ 56. Consider a typical factor of Π_{ab} with no a explicitly found in common to two brackets,

$$(\alpha_p \beta_q \alpha_{r_1} \beta_{k_1} v)(\alpha_{r_1} \beta_{k_1} v \hat{b}) \hat{b}_x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The complementary symbols of α_p by hypothesis do not occur in Π^{q+1} but must occur in MX, supposing P to be $\Pi^{q+1}MX$.

If $p > 0$, since $r_1 > 0$ we may bracket α_p and α_{r_1} together in the second of the factors (1).

The result is to obtain a factor

$$(\alpha_p \alpha_{r_1} \beta_q \beta_{k_1} v)(\alpha_p \alpha_{r_1} \beta_{k_{11}} v \hat{b}) \hat{b}_x \text{ or } (\beta_q \beta_{k_{12}} \xi),$$

where $\beta_{k_1} \equiv \beta_{k_{11}} \beta_{k_{12}}$ and some symbols, $\beta_{k_{12}}$, of β_{k_1} have taken the place of α_p in MX.

If we do the same for each factor of Π_{ab} containing as explicitly we thus revert to the form Π_b.

§ 57. Thus we have expressed the concomitant

$$\prod_1^q (\beta_{s_r} \xi_{s_r}) M$$

in terms of concomitants $\prod_1^{q+1} (\beta_{s_r} \xi_{s_r}) M'$ and reducible terms, on the assumption that M contains a pair of brackets prepared mod. f_1. This proves the theorem V.

Corollary. The only type of reducible term which has arisen in the course of the proof of the theorem is $\prod_1^{q+1} (\beta_{s_r} \xi_{s_r}) M$, where one (or more) suffix s_r is zero. The reducible factor may be conveniently written ξ_0 or $(\theta v)(\theta v' \hat{b}) \hat{b}_x$.

V. The irreducible system.

§ 58. We are now in a position to find a system which includes all irreducible forms for two quadratics.

If P is any concomitant it may be prepared mod. f_1. Factors of P may be paired off either as

$$(\alpha_r \beta v)(\alpha_r \beta' v) \qquad r = 1, \ldots n \qquad (1),$$

$$\text{or } (a \beta v) a_x \qquad (2),$$

$$\text{or } a_x^2 \qquad (3).$$

Taking the first alternative we may apply theorem IV (§ 45) and obtain a factor $(\beta_s \xi)$ instead of these two brackets, with reducible terms which are included in $(\beta_s \xi)$ if s may be zero.

P may now be replaced by

$$(\beta \xi) M \text{ or } (\beta \xi) M_a,$$

where M_a is prepared mod. f_1, since M contains no a symbol in common with $(\beta \xi)$.

If M_a contains a bracket pair of form (1) the last theorem proves that we may take P to be of form

$$\Sigma (\beta_{s_1} \xi_{s_1})(\beta_{s_2} \xi_{s_2}) M, \text{ where } s_1, s_2 \geqslant 0, \nless n - 1.$$

Similarly if M be prepared mod. f_1 and any pair of brackets of type (1) arise we may apply the theorem and obtain a third ξ factor or reducible terms with factor ξ_0.

Proceeding in this way we must necessarily use up all pairs of brackets of type (1).

It follows that any single term concomitant of two quadratics may be expressed in the form ΣQ, where

$$Q = \prod_{r=1}^{q} (\beta_{s_r} \xi_{s_r}) \prod_{1}^{q'} \{(a_k \beta v)\, a_{kx}\}\, M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

where M contains no symbol of f_1.

§ 59. *Theorem VI.* In the forms (A) we may assume all the symbols included in the sets $\beta_{s_1} \ldots \beta_{s_q}$ to differ.

For if β_{s_1} and β_{s_2} had some in common, the corresponding ξ factors would be of the form

$$(\beta_p \beta_{s_{11}} \theta v)(\theta v \hat{b})(\beta_p \beta_{s_{21}} \theta' v)(\theta' v \hat{b}')\, \hat{b}_x \hat{b}_x' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$\theta = \alpha_r \beta_k, \qquad \theta' = \alpha_{r'} \beta_{k'}.$$

If θ' contain no b symbol, i.e. $k' = 0$, we apply $R(\beta_p, g_4)$ to g_1 and g_4 and obtain terms of type

$$(\alpha_{p_1} \beta_{p_2} \beta_{s_{11}} \theta v \hat{b}')(\theta v \hat{b})(\beta_p \beta_{s_{21}} \theta' v)(\alpha \beta_{p_1} v \hat{b}')\, \hat{b}_x \hat{b}_x'.$$

Here g_1 contains $\alpha_{p_1} \alpha_r$,

g_2 " α_r,

g_3 " α_{p_1}.

We may therefore bracket $\alpha_r \alpha_{p_1}$ from g_2 and g_3 in g_2 and obtain

$$\Sigma (\alpha' \beta_{p_2} \beta_{s_{11}} \beta_k v \hat{b}')(\alpha' \beta_{k_1} v \hat{b})(\beta_p \beta_{s_{21}} \beta_{k_2} \hat{b} \alpha v)(\alpha \beta_{p_1} v \hat{b}')\, \hat{b}_x \hat{b}_x' \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where $\alpha' = \alpha_{p_1} \alpha_r$ and $\beta_k = \beta_{k_1} \beta_{k_2}$.

These terms are the same in type as (i) where θ' contains some bs. Thus both cases unite and we consider the case $k' > 0$.

Since g_3 of (i) contains $\beta_p \beta_{k'}$ we may bracket the complementary set in g_4 and obtain terms such as

$$(\beta_{s_{11}} \hat{b}' \alpha_{r_1'} \theta v)(\theta v \hat{b})(\beta_p \beta_{s_{21}} \alpha_{r'} \beta_{k'} v)(\alpha_{r_2'} \beta_p \beta_{k'} v \hat{b}')\, \hat{b}_x \hat{b}_x',$$

where $\alpha_{r_1'} \alpha_{r_2'} = \alpha_{r'}$.

If $r_1' = 0$ this is of type $(\beta\xi)(\beta'\xi)$ where β and β' have no common symbol. For $\hat{b}'$ in the first bracket does not alter the type of ξ. In fact $(\gamma_s \hat{b} \xi)$ is either $(\gamma_s b \xi)$ or $(\gamma_s u \xi)$, i.e. $(\gamma_s \xi)$.

If $r_1' > 0$ we prepare g_1 and g_2 mod. f_1 and obtain

$$(\alpha_{r_1'} \alpha_r \beta_k \beta_{s_{11}} \hat{b}' v)(\alpha_{r_1'} \alpha_r \dot{\beta}_{k_1} \dot{v} \dot{\hat{b}})(\beta_p \beta_{s_{21}} \alpha_{r_2'} \dot{\beta}_{k_2} \beta_{k_1'} \dot{\hat{b}} \dot{v})(\alpha_{r_2'} \beta_p \beta_{k'} v \hat{b}')\, \hat{b}_x \hat{b}_x' \quad \ldots\ldots\ldots\text{(iii)},$$

which is of form $(\beta\xi)(\beta'\xi)$ where β and β' have k_2 symbols in common, and $k_2 \leqslant r_1' \leqslant p$.

Repeating this reduction we obtain more like as in the first two brackets and more like bs in the second two. This may go on till either β_k or $\alpha_{r'}$ is exhausted.

If $\alpha_{r'}$ is exhausted $g_3 g_4$ are of type $(\beta_p \beta_q v)(\beta_p \beta_r v)$ and the concomitant is reducible (§ 31).

We need not perform the reduction but put $g_3 g_4$ in with M of form (A) § 57 and this theorem is still true.

If β_k is exhausted, another application of this process eliminates all like bs from the two ξ factors since β_k supplies them in (iii).

A similar treatment applies to all other pairs of ξ factors with like symbols. Thus finally Q is expressible as a sum of terms of type A where no two b symbols in $\beta_{s_1} \ldots \beta_{s_q}$ are the same.

Q. E. D.

Hence any single term concomitant of two quadratics f_1 and f_2 is expressible as a sum of terms Q of type

$$Q = \prod_1^q (\gamma_{s_r} \xi_{s_r}) \prod_{k=1}^{q'} \{(a_k \beta v)\, a_{kx}\}\, M \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{A}'),$$

where all the Σs_r b symbols included explicitly in the first product are different and where M is independent of a.

It follows that the complements of the b symbols γ_{s_r} are either in the second product or in M.

§ 60. If a term Q contain a pair of brackets not included in $\prod\limits_1^q (\gamma\xi)$ with one or more identical b symbol in them, Q is reducible.

For these brackets must be of type

either $$(\beta_r \beta_p a v)(\beta_r \beta_q a' v)\, a_x a_x',$$

or $$(\beta_r \beta_p a v)(\beta_r \beta_q v)\, a_x,$$

or $$(\beta_r \beta_p v)(\beta_r \beta_q v),$$

and by preparing them mod. f_2—which is possible since $r > 0$—we express Q as a sum of terms of types

$$(\beta_{r'} a v)(\beta_{r'} a' v)\, a_x a_x' M',$$

$$(\beta_{r'} a v)(\beta_{r'} v)\, a_x M',$$

$$(\beta_{r'} v)(\beta_{r'} v')\, M',$$

all of which are reducible, being of type $R \times M'$. Cp. § 29.

§ 61. Consider any one bracket g_m of (A′) not of type $(\gamma\xi)$. It must contain two or more symbols of type a or b. It cannot contain more than one a since the partner occurs in the form a_x; it must therefore have at least one b.

It may therefore be written

$$(\beta_p \beta_{p'} a v), \text{ or else } (\beta_p \beta_{p'} v),$$

where the complementary a, if a exist, occurs in the form a_x, (cp. form (A′)); and where the symbols β_p are complementary to symbols in $\Pi(\gamma\xi)$ and $\beta_{p'}$ to symbols in the other brackets or X. But in the last paragraph we have shewn that no b may be common to two of these brackets not of type $(\gamma\xi)$. Hence the partners of the p' symbols $\beta_{p'}$ must occur in X. Hence $p' < 2$, else the form vanishes.

Thus a typical final bracket g_m of Q in form (A′) is

$$(\beta_p \hat{b} \hat{a} v)\, \hat{b}_x \hat{a}_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where b and a may be present or absent and where β_p is complementary to p symbols amongst $\gamma_{s_1} \dots \gamma_{s_q}$.

If $p=0$ this form (1) is a complete concomitant, and Q is therefore reducible.

§ 62. Hence some symbols from $\gamma_{s_1} \dots \gamma_{s_q}$ must occur in each bracket of the final set in Q. As at least one of the symbols γ_s contributes to β_p, suppose that γ_{s_1} is one, the order of $s_1, s_2 \dots s_q$ being immaterial. We may then bracket all the symbols complementary to γ_{s_1} in this bracket g_m and express Q as a sum of terms

$$\left[\prod_1^q (\gamma_{s_r}\xi_{s_r})\right](\gamma_{s_k}\beta\hat{a}\hat{b}v)\,\hat{a}_x\hat{b}_x M' \quad \dots\dots\dots(1).$$

If M' contain any brackets, similar remarks apply to them, and to any definite bracket of it $g_{m'}$ say.

We therefore find that $g_{m'}$ necessarily contains some symbols from $\gamma_{s_2} \dots \gamma_{s_q}$, unless $q=1$, and selecting one particular set represented, γ_{s_2} say, we bracket them in $g_{m'}$ and obtain a sum of terms

$$Q' = \prod_1^q (\gamma_{s_r}\xi_{s_r})(\gamma_{s_1}\beta\hat{a}\hat{b})\,\hat{a}_x\hat{b}_x(\gamma_{s_2}\beta'\hat{a}'\hat{b}')\,\hat{a}_x'\hat{b}_x' M'' \quad \dots\dots\dots(2)$$

$$= \prod_1^q . \, g_m g_{m'} \hat{a}_x\hat{b}_x\hat{a}_x'\hat{b}_x' \text{ say.}$$

These terms are reducible. For if β and β' contain a common symbol the reduction is already established (§ 60).

If not let the process $R(\gamma_{s_2}, g_m)$ be applied to the brackets $(\gamma_{s_2}\xi_2)$ and g_m. This leads to the identity

$$Q' + \Sigma Q'' = 0;$$

where Q'' is the same as Q' except that certain symbols γ_{s_2} from $(\gamma_{s_2}\xi_2)$ have been interchanged with an equal number of symbols of g_m. Thus every term Q'' has some symbols alike in g_m and $g_{m'}$, viz., some of γ_{s_2}, and is therefore reducible (§ 60). Hence Q' is reducible by means of factors of lower weight.

If $q=1$ the remark at the end of § 61 applies to $g_{m'}$.

The only alternative to these is that in (1) M' has no bracket factors.

Thus no irreducible term Q can have more than one bracket factor other than those included in

$$\prod_1^q (\gamma_{s_r}\xi_{s_r})$$

when Q is written in the form (A′). We shall call this bracket, if it exist, the *Final Bracket*, and it may always be taken to contain one complete set of symbols γ_{s_k} at least, as was shewn above for g_m.

§ 63. Hence we have shewn that every possible irreducible concomitant including one or more pairs of brackets prepared mod. f_1 is expressible either as

$$(\xi_0) \equiv (\alpha_r\beta_k v)(\alpha_r\beta_k v\hat{b})\,\hat{b}_x,$$

or

$$\prod_{r=1}^q (\gamma_{s_r}\xi_{s_r}) . \left(\frac{\gamma_{s_1}\gamma_{s_2}\dots\gamma_{s_q}}{b_1 b_2 \dots b_i}\hat{a}\hat{b}v\right)\hat{a}_x\hat{b}_x b_{1x}\dots b_{ix} \quad \dots\dots\dots(I),$$

for the latter is the most general form with only one final bracket. Moreover any one of $\gamma_{s_1}, \ldots, \gamma_{s_q}$ may be taken to stand completely in the final bracket, § 62.

Here $b_1 \ldots b_i$ denote some symbols from $\gamma_{s_1} \ldots \gamma_{s_q}$.

§ 64. Case of no final bracket.

When a concomitant contains no bracket except those of type $(\gamma\xi)$ the symbols complementary to (γ) must occur with x. If the suffix of γ is greater than unity the concomitant is zero, for

$$(bb'\gamma'\xi)\, b_x b_x' = 0.$$

Further if there are two or more ξ factors the concomitant is only a product of simpler ones

$$(g\xi)\, g_x \,.\, (g'\xi')\, g_x' \,\ldots .$$

If there are no ξ factors the only irreducible concomitants are f_1, f_2.

Thus the only possible irreducible concomitants with no final bracket are

(1) $(\xi_0) \equiv (\theta v)(\theta v\hat{b})\, \hat{b}_x,$ where $\theta = \alpha_r \beta_k,$

(2) $(g\xi)\, g_x \equiv (\theta g v)(\theta v \hat{b})\, g_x \hat{b}_x,$

(3) $f_1 = a_x^2, \quad f_2 = b_x^2$ ....(3).

§ 65. We may now write down a set of forms which is bound to contain all possible irreducible forms. We shall consider them in ascending order of total number of bracket factors. This number is called the weight (§ 29).

By § 63 it will follow that any concomitant of weight w is a rational integral function of the concomitants of weight $w, w-1, \ldots 3, 2, 1, 0$ retained in the system.

Moreover every form of the system except forms (3) of § 64 have a final bracket.

1. Zero weight. $f_1 = a_x^2$, $f_2 = b_x^2$.

2. Weight one. There can be no ξ factor, since any ξ factor is of weight two, and therefore the final bracket alone exists and we have, by § 61,

$$(ab\, v_{n-2})\, a_x b_x.$$

3. Weight two. There must be one ξ factor and no final bracket. The only possibilities are (§ 64)

$$(\theta v)(\theta v'), \quad (\theta v)(\theta v' b)\, b_x, \quad (\theta v b)(\theta v' b')\, b_x b_x'.$$

4. Weight three. There must be one ξ factor and one final bracket,

$$(\gamma_s \xi_s)(\gamma_s \hat{a} \hat{b} v)\, \hat{a}_x \hat{b}_x$$

is the only possibility (§ 63).

5. Any even weight greater than two is impossible for there is necessarily a final bracket for concomitants of weight > 2 and therefore an odd number of brackets [see form I].

6. $w = 5$.

$$(\gamma_{s_1} \xi_{s_1})(\gamma_{s_2} \xi_{s_2})(\gamma_{s_1} \beta \hat{a} \hat{b} v)\, \hat{a}_x \hat{b}_x \hat{b}'_x,$$

where $\beta b' \equiv \gamma_{s_2}$. There are two types since β may be γ_{s_2} or $\gamma_{\overline{s_2 - 1}}$. And so on.

§ 66. The number of ξ factors is $\not> n-1$. For if there were q ξ factors, the concomitant would be

$$P = \prod_1^q (\gamma_{s_r}\xi_{sr})(\gamma_{s_k}\beta\hat{a}\hat{b}v)\,\hat{a}_x\hat{b}_x b_{1x} b_{2x} \ldots b_{ix} \quad \text{.......................... by (I)},$$

where $\beta b_1 b_2 \ldots b_i$ are complementary to $\gamma_{s_1} \ldots \gamma_{s_{k-1}}\gamma_{s_{k+1}} \ldots \gamma_{s_q}$ and no two symbols $b_1 \ldots b_i$ are complementary to two in one ξ factor, else $P = 0$.

Hence $\Sigma s_r - i \not> n-1$, otherwise the final bracket would be of form β_n and the concomitant reducible.

Also the concomitant is reducible if any symbol $b_1 \ldots b_i$ is complementary to one of $\gamma_{s_1} \ldots \gamma_{s_q}$, i.e. if s_1 say $=1$ and $b_1 \equiv \gamma_{s_1}$, for P would have the factor $(b_1\xi_1)\,b_{1x}$. Hence $i < n$, otherwise there would be too many symbols to be included in the final bracket.

Thus $\Sigma s_r \not> 2n-1$.

Further at least one symbol complementary to γ_{s_r} lies in the final bracket and therefore $q \not> n-1$, else the final bracket would be of form β_n.

Thus $w = 2q+1$ provided $q > 0$ and hence

$$w \not> 2n-1.$$

This proves that the system of § 65 is finite as well as complete, for the number of different forms which (I) can take up for any definite value of w is manifestly finite.

Note. We may choose s_k to be the greatest suffix in the ξ factors. If $s_k = n-1$ there can only be one ξ factor for the extent of β is at most $n-1-s_k$.

§ 67. Summing up, we have shewn that every possibly irreducible concomitant of two quadratics is expressible as a sum of terms

$$\Xi \,.\, F \,.\, X,$$

Ξ denoting a product of q factors $(\theta\gamma_s v)(\theta v\hat{b})\,\hat{b}_x$,

F denoting $\left(\dfrac{\gamma_{s_1} \ldots \gamma_{s_q}}{b_1 \ldots b_i}\,\hat{a}\hat{b}v\right)$,

X denoting $\hat{a}_x\hat{b}_x b_{1x} \ldots b_{ix}$;

with the exception of those of weight two, $(\theta v\hat{b})(\theta v\hat{b}')\,\hat{b}_x\hat{b}_x'$.

Further that the symbols $b_1 b_2 \ldots b_i$ refer to i different sets $\gamma_{s_1}\gamma_{s_2} \ldots \gamma_{s_i}$ and $s_1 > 1, \ldots s_i > 1$, $q > i$ and $\not> n-1$.

Hence we have incidentally proved Gordan's Theorem for this case and have also established a method of writing down a set of forms which is bound to contain all irreducible ones.

On being given n it is comparatively simple to write down the various types which occur. For example $n = 5$.

§ 68. Application to Quinary Forms.

The ξ factors may contain symbols γ of suffix 1, 2, 3 or 4. Let (s) denote a ξ factor of suffix s and [ ] denote the final bracket.

The irreducibles are included in

(1) $w=0$, f_1, f_2,

(2) $w=1$, $(ab\upsilon)\, a_x b_x$,

(3) $w=2$, $(\theta\upsilon\hat{b})(\theta\upsilon'\hat{b}')\,\hat{b}_x\hat{b}_x'$,

(4) $w=3$, $(\gamma_s\xi)(\gamma_s\hat{a}\hat{b}\upsilon)\,\hat{a}_x\hat{b}_x$,

(5) $w=5$, $(\gamma_{s_1}\xi)(\gamma_{s_2}\xi)(\gamma_{s_1}\gamma_{s_2}\hat{a}\hat{b}\upsilon)\,\hat{a}_x\hat{b}_x$,

$(\gamma_{s_1}\xi)(\gamma_{s_2}\xi)(\gamma_{s_1}\gamma_{s_{21}}\hat{a}\hat{b}_x\upsilon)\,\hat{a}_x\hat{b}_x g_x$, where $\gamma_{s_{21}} g \equiv \gamma_{s_2}$,

(6) $w=7$, $(\gamma_{s_1}\xi)(\gamma_{s_2}\xi)(\gamma_{s_3}\xi)(\gamma_{s_1}\gamma_{s_2}\gamma_{s_3}\hat{a}\hat{b}\upsilon)\,\hat{a}_x\hat{b}_x$,

„ „ „ $(\gamma_{s_1}\gamma_{s_{21}}\gamma_{s_{31}}\hat{a}\upsilon)\,\hat{a}_x\hat{\gamma}_{s_{22}\cdot x}\hat{\gamma}_{s_{32}\cdot x}$,

(7) $w=9$, $(g_1\xi)(g_2\xi)(g_3\xi)(g_4\xi)(g_1g_2g_3g_4\hat{a}\hat{u})\,\hat{a}_x$.

(5), (6), (7) give the following nine types:

(3) (2) [3, 1] 1,

where [3, 1] 1 means that γ_3 and one of γ_2 occur in the final bracket, the other of γ_2 occurring with x,

(3) (1) [3, 1],

for if (3) occurs as a ξ factor its complement may be written in the final bracket (*note*, § 66) and there is only room for one more symbol of ξ in [ ], since $n-1=4$ in this case,

(2) (2) [2, 2],

(2) (2) [2, 1] 1,

(2) (1) (1) [2, 1, 1],

(2) (1) [2, 1],

(1) (1) (1) (1) [1, 1, 1, 1] *,

(1) (1) (1) [1, 1, 1],

(1) (1) [1, 1].

§ 69. As regards the terms left in the irreducible system many are reducible. It is however difficult to establish any general principles of reducing these terms further, but we shall give one proposition regarding the nature of the symbols $\alpha_r\beta_k$ paired in a ξ factor.

Theorem. Any ξ factor $(\gamma_{s_p}\xi)$ must have at least one symbol a. Let it then be written

$$(\alpha_{r_p}\beta_{k_p}\gamma_{s_p}\upsilon)(\alpha_{r_p}\beta_{k_p}\upsilon\hat{b})\,\hat{b}_x.$$

Then the concomitant (I) is reducible unless one and only one symbol k_p is zero.

This at once diminishes the number of possible irreducible forms of low weight considerably.

Proof. Suppose that $\prod_1^q(\gamma_s\xi_s)\,F.X$ is written

$$(\alpha_r\beta_k\gamma_s\upsilon)(\alpha_r\beta_k\hat{b}\upsilon)(\alpha_{r_1}\beta_{k_1}\gamma_{s_1}\upsilon)(\alpha_{r_1}\beta_{k_1}\hat{b}_1\upsilon)\dots(\gamma_s\gamma_{s_{11}}\gamma_{s_{21}}\dots\hat{b}'\hat{a}\upsilon)\,X,$$

where $\gamma_{s_{r1}}$ is complementary to all or all but one of the symbols γ_{s_r} and where the complements of $b, b_1\dots$ occur with x if the symbols exist at all.

* In general the concomitants of highest weight are always of this type.

Then either one of the suffixes k is zero or we may reduce this to forms in which the first k is zero.

If $k \neq 0$ we may bracket β_k with γ_s in the final bracket since they are bracketed in g_1, and obtain a set of forms

$$g_1(\alpha_r\gamma_{s_{111}}\gamma_{s_{211}}\ldots\hat{a}\hat{b}v)\,g_3g_4\ldots(\beta_k\gamma_s\gamma_{s_{112}}\gamma_{s_{212}}\ldots C')\,X,$$

where $$\gamma_{s_{111}}\gamma_{s_{112}}\equiv\gamma_{s_{11}},\quad \gamma_{s_{r11}}\gamma_{s_{r12}}\equiv\gamma_{s_{r1}}.$$

Since $k_1 > 0$ we may bracket $\beta_{k_1}\gamma_{s_1}$ in g_4 as in g_3 and do the same sort of thing with $k_2, k_3 \ldots$.

Thus we express the original form as a sum of terms

$$(\alpha_r\beta_{k'}v)(\alpha_{rr_{11}r_{21}}\ldots\hat{a}\hat{b}v)(\alpha_{r_1}\beta_{k_1}\gamma_{s_1}v)(\alpha_{r_{12}}\beta_{k_1}\gamma_{s_1}\hat{b}_1v)\ldots(\beta_k\gamma_s\alpha_{r_{13}}\alpha_{r_{23}}\ldots C')\,X,$$

where $$\alpha_{r_{k1}}\alpha_{r_{k2}}\alpha_{r_{k3}}\alpha_{r_{k4}}\equiv\alpha_{r_k},\quad \beta_{k'm}=\beta_{km}\gamma_{sm},\quad m=0,\,1,\,\ldots,$$

and $\alpha_{r_{k4}}$ occurs in X.

Terms in which $\hat{b}_1$ has been transferred to g_2 have been omitted since they are either of the form just written or zero.

For $$(\theta C\hat{b}\hat{b}')\,\hat{b}_x\hat{b}_x' \text{ is either } 0 \text{ or } (\theta C\hat{b})\,\hat{b}_x.$$

We now bracket $\alpha_{r_{11}}\alpha_{r_{21}}\ldots$ with α_r in g_1 and obtain terms

$$(\alpha_{r'}\beta_{k'_{01}}v)(\alpha_{r'}\hat{b}v)(\alpha_{r_{12}}\alpha_{r_{13}}\alpha_{r_{14}}\beta_{k'_{02}}\beta_{k'_1}v)(\alpha_{r_{12}}\beta_{k'_1}\hat{b}_1v)\ldots(\beta_{k'}\alpha_{r_{13}}\ldots C')\,X,$$

where $$\beta_{k'_{01}}\beta_{k'_{02}}\ldots\equiv\beta_{k'}\equiv\beta_k\gamma_s:\ \text{call } \beta_{k'_{01}},\ \beta_{k''}.$$

Now prepare g_3g_4, g_5g_6, … mod. f_1.

Thus we have terms

$$(\alpha_{r'}\beta_{k''}v)(\alpha_{r'}\hat{b}v)(\alpha_{r'_1}\beta_{k'_{02}}\beta_{k'_1}v)(\alpha_{r'_1}\beta_{k'_{11}}\hat{b}_1v)\ldots(\beta_{k''}\beta_{k'_{02}}\ldots\beta_{k'_{12}}\ldots C')\,X.$$

These are of original type with $k=0$ in g_1g_2.

Hence we may always take one k at least to be zero.

But two suffixes k must not be zero.

For consider the form which is now proved to include all possible irreducibles,

$$(\alpha_r\gamma_sv)(\alpha_r\hat{b}v)(\alpha_{r_1}\beta_{k_1}\gamma_{s_1}v)(\alpha_{r_1}\beta_{k_1}\hat{b}_1v)\ldots(\gamma_{s_{01}}\gamma_{s_{11}}\ldots C)\,X$$

where $\gamma_s\gamma_{s_1}\ldots$ are entirely different.

Since the symbols complementary to γ_s occur in the final bracket partly we may collect them all into it and consider terms such as

$$(\alpha_r\gamma_sv)(\alpha_r\hat{b}v)\,g_3g_4\ldots(\gamma_s\gamma_{s_{11}}\ldots C)\,X,$$

where $s_{11}\ldots$ are all >0.

If any one of $k_1, k_2, \ldots$ is zero, k_1 say, this is reducible. For since $s_{11}>0$ we may collect γ_s and $\gamma_{s_{11}}$ from g_1 and g_3 into g_3 and obtain

$$(\alpha_rzv)(\alpha_r\hat{b}v)\left(\frac{\alpha_{r_1}v\gamma_{s_{12}}}{z}\gamma_{s_{11}}\gamma_s\right)(\alpha_{r_1}\hat{b}_1v)\ldots X,$$

where $\gamma_{s_{11}}\gamma_{s_{12}}\equiv\gamma_{s_1}$. But either $s_{12}=0$ or 1 since the complementary symbols to it occur in X. Therefore $\gamma_{s_{12}}=\hat{b}'$ say.

These terms are now of type

$$(\alpha_r \alpha_{r_{11}} \hat{b}' v)(\alpha_r \hat{b} v) \dots X.$$

Preparing the first two brackets mod. f_1 we express these as

$$\Sigma (\alpha_r \alpha_{r_{11}} \hat{b}' v)(\alpha_r \alpha_{r_{11}} \hat{b} v) \hat{b}_x \hat{b}_x' M.$$

These are all reducible. Q. E. D.

VI. Special types of Irreducible Concomitants.

§ 70. *Invariants.* These have bracket factors alone of type $(\alpha_r \beta_s)$ and therefore have only ξ factors provided $s = 0$, for $(\gamma_s \xi_s)$ contains s u symbols or $s-1$ u symbols and one x. Hence the only irreducible invariants are the $\overline{n+1}$ terms

$$(\alpha_r \beta_{n-r})^2, \qquad r = 0, 1, \dots n.$$

In fact they form the coefficients of κ/λ in the discriminant of

$$\kappa f_1 + \lambda f_2.$$

The proof of theorem III above applies to invariants and reduces them to be rational integral functions of these $\overline{n+1}$ very directly.

§ 71. *Covariants.* The only factors in a covariant are of type $(\alpha\beta)$ or a_x, b_x.

Hence the ξ factors must either be invariants or $(\theta b)(\theta b') b_x' \equiv (b\xi)$.

The typical possible irreducible covariant has q ξ factors and a final bracket. Since $s = 1$, for each ξ factor the covariant is either reducible to type

$$(\theta b)(\theta b') b_x b_x',$$

or each symbol bracketed with ξ occurs in the final bracket. Thus the covariant is of type

$$\prod_{p=1}^{q} (b_p \xi_p)(\beta_q \alpha \beta) X,$$

where $b_1 \dots b_q \equiv \beta_q$, and α, β have complementary symbols in X.

Thus α consists of one a symbol at most,

and β " " b " "

But if α did not appear the final bracket would be of form β_n and the covariant would be reducible.

Thus there is one a in the final bracket.

Since the sum of the suffixes of the bracket is n, therefore $q + 1 = n$ or $n - 1$ and the only possibilities are

$$\left[\prod_1^{n-1} (b_p \xi_p)\right] (\beta_{n-1} a) a_x; \quad \beta_{n-1} \equiv b_1 b_2 \dots b_{n-1},$$

and

$$\left[\prod_1^{n-2} (b_p \xi_p)\right] (\beta_{n-2} ab) b_x a_x,$$

where $(b_p \xi_p) = (\alpha_{r_p} \beta_{k_p} b_p)(\alpha_{r_p} \beta_{k_p} b_p') b'_{p,x}$ and $r_p > 0$.

Calling $\alpha_{r_p}\beta_{k_p}$, θ_p, the covariant is either

$$\prod_1^{n-1}(b_p\theta_p)\,.\,(\beta_{n-1}a)\prod_1^{n-1}(b_p'\theta_p)\,\Pi b'_{px}a_x,$$

or

$$\prod_1^{n-2}(b_p\theta_p)\,(\beta_{n-2}ab)\prod_1^{n-2}(b_p'\theta_p)\,\Pi b'_{px}a_xb_x.$$

These are equivalent to a numerical multiple of

$$(\beta_{n-1}\,.\,\theta_1\,.\,\theta_2\ldots\theta_{n-1})(\theta_1b_1')(\theta_2b_2')\ldots(\theta_{n-1}b'_{n-1})(\beta_{n-1}a)\,a_x\ldots,$$

and

$$(\beta_{n-1}\theta_1\ldots\theta_{n-2}x)(\theta_1b_1')\ldots(\beta_{n-2}ab)\,X\ \ (\S\ 23).$$

If $\theta_1=\alpha_r\beta_k$ and $\theta_2=\alpha_r'\beta_k'$, i.e. if two symbols θ are constituted in like manner from a and b, the covariant is zero.

For if $$C=(\beta_{n-1}\theta_1\theta_2\ldots)(\beta_{n-1}a)(\theta_1b_1')(\theta_2b_2')\,b_{1x}'b_{2x}'M,$$

it is unaltered by interchanging α_r and α_r', β_k and β_k', b_1' and b_2'. These interchanges are equivalent to interchanging θ_1 and θ_2, b_1' and b_2'. C becomes

$$(\beta_{n-1}\theta_2\theta_1\ldots)(\beta_{n-1}a)(\theta_2b_2')(\theta_1b_1')\,b_{2x}'b_{1x}'M,$$

which is the same as before except for the first bracket.

Since $(\beta_{n-1}\theta_2\theta_1\Theta)=-(\beta_{n-1}\theta_1\theta_2\Theta)$, this interchange implies that $C=-C$.

Thus $C=0$.

Hence for non-zero covariants all the symbols θ must differ in kind. But any θ may only be $\alpha_r\beta_k$, where $r=1, 2, \ldots n-1$. Thus there are $n-1$ different kinds of θ.

Further if $r=1$ the covariant is reducible.

In fact C is now $(a\beta_{n-2}b)(a\beta_{n-2}b')\,M$.

Preparing this mod. f_2 we introduce b into g_2 for a or b'. In the first case g_2 is of form β_n and in the second case $g_1g_2\equiv(a\beta_{n-2}b)^2$. Thus C is reducible, for both these cases are reducible.

Thus r may only be $2, 3, \ldots n-1$ and therefore each symbol θ may have $n-2$ values.

Hence since the θs in an irreducible covariant are all different, their number is $\not> n-2$.

There is therefore just one possibility namely

$$(\beta_{n-1}a)(\beta_{n-1}\theta_1\theta_2\ldots\theta_{n-2}x)(\theta_1b_1')\ldots(\theta_{n-2}b_{n-2}')\,b_{1x}'\ldots,$$

where each θ is different and $\theta_r=\alpha_{r+1}\beta_{n-r-2}$, $r=1, 2, \ldots n-2$.

This may be reducible too but it certainly is not for $n=3$.

Thus the number of irreducible covariants is $n+1$.

For there are the $n-2$ quadratic covariants

$$(\theta b)(\theta b')\,b_xb_x',\quad \theta=\alpha_r\beta_k,\quad r=2\ldots n-1,$$

besides f_1 and f_2, and this covariant of the nth order just found.

These $n-2$ quadratic covariants together with $\alpha_n^2f_2$ and $\beta_n^2f_1$ form the n coefficients of powers of κ/λ in the expression for the point equation of the system of quadrics

$$\kappa u_\alpha^2+\lambda u_\beta^2.$$

The covariant of the nth order represents the point equation of the polyhedron self conjugate to both of the quadrics f_1 and f_2.

§ 72. *Contravariants.* We shall now find the irreducible contravariants of f_1 and f_2. They will turn out to be the $n+1$ reciprocals of the irreducible covariants.

These concomitants are made up of factors $(\alpha_r\beta_{n-r})$ and $(\alpha_r\beta_{n-r-1}u)$.

Thus a ξ factor must be of the form

$$(\theta b)(\theta u), \text{ i.e. } (b\xi), \qquad (s=1),$$

or else

$$(\theta u)^2.$$

The general irreducible form is

$$\prod_1^q (b_p\xi_p)\,F,$$

where F is the final bracket, since there are no x symbols.

Moreover F is of form $(\alpha\beta u)$ or $(\alpha\beta)$. But since no x symbols arise, α must be non-existent and therefore F is (βu) or (β). The latter is reducible. Thus F is (βu).

Hence a contravariant is either $(\theta u)^2$ or

$$\prod_1^{n-1}(b_p\xi_p)(\beta_{n-1}u), \quad \beta_{n-1}\equiv b_1b_2\dots b_{n-1},$$

i.e.

$$\prod_1^{n-1}\{(b_p\theta_p)(\theta_p u)\}(\beta_{n-1}u), \qquad (\S\ 63),$$

where $\theta_p=\alpha_{r_p}\beta_{k_p}$ and one and only one of the suffixes k_p is zero.

The latter type of contravariant is

$$(\beta_{n-1}\theta_1\theta_2\dots\theta_{n}) \text{ [illegible] } (\theta_1 u)\dots, \qquad (\S\ 23),$$

or $(\theta_1\theta_2\dots\theta_n)(\theta_1 u)\dots(\theta_n u)$, where [illegible] q may be $0, 1, \dots \overline{n-1}$.

Suppose that both θ_1 and [illegible] Let $\theta_1\equiv\alpha_p\beta_q$, $\theta_2\equiv\alpha_p'\beta_q'$, and

[illegible] $)\dots(\theta_n u)$.

Interchange α_p [illegible] ariant is unaltered in value.

But this pr[illegible] and θ_2.

Thus [illegible] $)\dots(\theta_n u)$.

But [illegible] $\theta_n)$.

Henc[illegible]

The[illegible] -zero term. But θ_r can only have the n values [illegible]

It f[illegible] of this type is

where θ_r [illegible] values $1, 2, \dots n-1$.

Thus, since the only other irreducible contravariants are $(\theta u)^2$, i.e. the n terms obtained by taking the tangential equation of the quadrics

$$\kappa f_1 + \lambda f_2,$$

there are only $n+1$ possible irreducible contravariants—all but one being of class 2, the last being of class n.

§ 73. The last one represents the tangential equation of the n-cornered polyhedron self conjugate to both quadrics.

This is most easily seen by taking the jacobian of the n quadrics $(\theta_r u)^2$.

All these quadrics have a common self conjugate polyhedron provided that there is a proper n-cornered polyhedron self conjugate to the first two as we see by taking it for the polyhedron of reference and writing

$$f_1 \equiv \sum_{r=1}^{n} k_r x_r^2, \quad u_\alpha^2 = \lambda \Sigma u_r^2/k_r,$$

$$f_2 \equiv \sum_{r=1}^{n} x_r^2; \qquad u_\beta^2 = \lambda' \Sigma u_r^2.$$

The jacobian of n linearly independent quadrics of type f_1 is proportional to $\prod_1^n x_r$ and that of the reciprocal quadrics is proportional to Πu_r.

But the jacobian of the n quadrics $(\theta_r u)^2$ where $\theta_r = \alpha_r \beta_{n-1-r}$, $r = 0, 1, \ldots, n-1$, is $(\theta_1 u)(\theta_2 u) \ldots (\theta_1 \theta_2 \ldots \theta_n)$, which is the irreducible contravariant in question.

Thus it represents the polyhedron of reference.

Similarly the irreducible covariant of the nth order is the jacobian of the n irreducible covariants of the second order, all of whi[illegible] a common self conjugate polyhedron. It therefore represents the point equation of th[illegible]nd the contravariants and covariants are entirely reciprocal.

§ 74. Once more, consider concomitan[illegible]factors alone of types $(\alpha\beta)$ and $(\alpha\beta u_1 u_2)$.

Let $(u_1 u_2) = p$; these concomitants are [illegible]plexes of lines.

Our reductions may have separated [illegible]ust always be as many u_1s as u_2s in a concomitant of th[illegible]

The only possibilities are $(\theta p)^2$, $(\theta$ [illegible] $_s . v)$, since there are no x factors.

Now $(\gamma_s \xi_s)$ may be $(\theta\gamma_2)(\theta p)$, [illegible]

The final bracket may contain [illegible]

1. If it contain p, there are [illegible] u_1 or u_2 in the ξ factors. Hence $n-2$ mu[illegible]

2. If it have u_1 or u_2 singl[illegible]s u_1 or u_2 in the ξ factors. Hence $n-1$ [illegible]

It follows that unless n is even there are no concomitants of this type except such as can be expressed in terms of invariants, $(\theta p)^2$, and of $(\theta u_1)(\theta u_2)$.

If n is even we may have

$$\Pi(\gamma_s \xi_s)(\Pi\gamma_s . p) \text{ where } s=1 \text{ or } 2 \text{ and } \Sigma s = n-2,$$

or

$$\Pi(\gamma_s \xi_s)(\Pi\gamma_s u_2) \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad = n-1,$$

and where the number of symbols u_1 occurring in a bracket with no symbol u_2 is equal to that of u_2 with no symbol u_1.

§ 75. Further than this it is difficult to find irreducible forms for special sets of u symbols.

§ 76. In conclusion it is interesting to observe that supposing we were to consider x as made up of $\overline{n-1}$ u symbols and were to allow it to be decomposed (§ 7), the complete system for two quadratics might be established on the same lines as above more readily and simply. In fact, every constituent of a concomitant might now be considered as a bracket factor

$$(\alpha_r \beta_k v_s),$$

where $r+k+s=n$ and s might take the values $0, 1, \ldots n-1$.

The general irreducible term would be simply

$$\prod_1^q (\gamma_{s_r} \xi_{s_r})(\gamma_{s_1} \ldots \gamma_{s_q} v) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where

$$(\gamma_s \xi_s) = (\alpha_r \beta_k \gamma_s v_t)(\alpha_r \beta_k v'_{t+s});$$

q could not be $>(n-1)$ and the number of terms to be considered is obviously much less than in the former case since each ξ factor is now of one type only instead of two.

Further, the number of symbols u may be unlimited, so that (1) practically gives the irreducible system for two quadratics and any number of linear forms. For in all the work, except for contravariants, we never use the fact that the number of different u symbols is $n-1$ or less, except when $n-1$ of them occur together and are called x, and in § 46 which would be unnecessary if x were decomposible.

§ 77. For example, take ternary forms.

Let $u_1, u_2 \ldots$ be the variables and $(u_1 u_2) = x$.

The only possible irreducibles except those of type (1) are

$$f_1 = a_x^2,\ f_2 = b_x^2,\ (aa'u_i)(aa'u_j),\ (abu_i)(abu_j),\ (bb'u_i)(bb'u_j),\ a_\alpha^2,\ a_\beta^2,\ b_\alpha^2,\ b_\beta^2.$$

The possible ξ factors are

$$\left.\begin{array}{ll} (1) & (\alpha b)(\alpha u) \\ (2) & (ab_1 b)(ab_1 u) \\ (3) & (au_1 b)\, a_x \end{array}\right\} \equiv (b\xi),$$

$$(4) \quad (\alpha\beta)\, a_x \qquad \equiv (\beta\xi).$$

The final bracket may contain two b symbols at most, by (1); for $\Sigma s \not> n-1$.

Therefore the only possible irreducible forms of type (1) are

(i) $(b\xi)\, b_x$, (ii) $(\beta\xi)(\beta u)$, (iii) $(b\xi)(b'\xi)(bb'u)$.

By § 69 the latter form must contain $(\alpha b)(\alpha u)$ once and once only. The only possibilities for it are therefore

$$(\alpha b)(\alpha u)(ab_1b')(ab_1u)(bb'u) \quad\ldots\ldots(5),$$

and

$$(\alpha b)(\alpha u)(aub')(ax)(bb'u) \quad\ldots\ldots(6).$$

By bracketing bb' in the third bracket of (6) we at once reduce it. Thus (5) alone remains and we have the set

(i) $(b\xi)(bx)$, i.e. $(\alpha b)(\alpha u)b_x$, $(ab_1b)(ab_1u)b_x$, $(aub)a_xb_x$,

(ii) $(\beta\xi)(\beta u)$, i.e. $(a\beta)a_x(\beta u)$,

(iii) $(\alpha b)(\alpha u)(ab_1b')(ab_1u)(bb'u)$.

But $(ab_1b)(ab_1u)b_x = (ab_1b)(bb_1u)a_x$ mod. reducible terms.

Hence every irreducible is included in the four invariants and the set of forms

$$(\alpha u_i)(\alpha u_j),\quad (abu_i)(abu_j),\quad (\beta u_i)(\beta u_j),\quad (au_iu_j)(au_ku_l),\quad (bu_iu_j)(bu_ku_l),\quad (\alpha b)(\alpha u_i)(bu_ju_k),$$
$$(abu_i)(au_ju_k)(bu_lu_m),\quad (a\beta)(\beta u_i)(au_ju_k),\quad (\alpha b)(\alpha u_i)(ab_1b')(ab_1u_j)(bb'u_k),$$

where i, j, k, l, m take up all possible values $1, 2, \ldots p$, p being the number of variables u. Thus there are only these 13 types of concomitants of two quadratics and any number of linear forms.

If this number p is 2 and $x=(uu')$ we see that the usual 20 irreducibles may be expressed in terms of these 13 forms

$$a_\alpha^2,\ a_\beta^2,\ b_\alpha^2,\ b_\beta^2,$$
$$(\alpha u)^2,\quad (abu)^2,\quad (\beta u)^2,\quad a_x^2,\quad b_x^2,$$
$$(\alpha b)(\alpha u)b_x,\quad (abu)a_xb_x,\quad (a\beta)(\beta u)a_x,$$

and

$$(\alpha b)(\alpha u)(ab_1b')(ab_1u)(bb'u),$$

and of terms derived from these by polarization with regard to

$$u'\frac{\partial}{\partial u}.$$

[§ 78.] Several papers have been written at various times dealing with the geometry of two quadrics in n dimensions, among which the most important is by Segre (*Mem. R. Accademia delle Scienze di Torino*, serie II, vol. XXXVI (1884), pp. 3—86).

Quite recently a paper has been published by Brusotti* in which the symbolic notation is employed and the geometry connected with the irreducible forms of type

$$(\alpha_r\beta_s v_t)^2;\quad t=0,\ 1,\ \ldots n-1,$$

is developed. A full list of references to other papers on two quadrics is given.

[ ] Added April 21, 1909.

* Brusotti, *Palermo, Rendiconti*, XXIII, p. 265.

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ON UNIFORM OSCILLATION

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IX. *On Uniform Oscillation.* By W. H. Young, Sc.D., F.R.S.

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§ 1. In a paper* published in the *Proceedings of the London Mathematical Society* last year I initiated what in order to distinguish its nature from that of the valuable work of Borel and others may be called a direct attack on the theory of non-converging series. The applications of results there obtained which I have myself already made† to the Theory of Fourier Series and of Non-differentiable Functions seem of themselves sufficient to justify the use of this new mode of procedure. In the paper in question I introduced the concepts of uniform and non-uniform oscillation above and below, shewing that at a point where the upper (lower) oscillation is uniform, the upper (lower) function of the series is upper (lower) semi-continuous, and that such points necessarily fill up the continuum excepting at most a set of the first category.

The possibility of the presence of the exceptional set of the first category often prevents our drawing certain general conclusions as to the presence or not of exceptional points of divergence or oscillation in discussing a series which is not known to converge. It becomes therefore of importance to devise tests for the uniformity of the oscillation of a series. On the other hand from the practical point of view, we naturally wish to have some guide as to how to construct series which oscillate uniformly, either above or below or both, throughout an interval.

One way in which uniformly oscillating series naturally arise is when we integrate term-by-term an oscillating series of functions whose partial summations have in their ensemble a finite upper (lower) bound. The integrated series then oscillates uniformly above (below). I have already had occasion to discuss the properties of such series elsewhere. The main object of the present note is to give a general test for uniformity above (below) of a series, which corresponds precisely to the test for uniform convergence of series known as the Dirichlet Test.

The theorem proved is the following:

If the series of constant terms

$$a_0 + a_1 + \dots$$

have a finite upper (*lower*) *limit, the series*

$$a_0 f_0(x) + a_1 f_1(x) + \dots$$

oscillates uniformly above (*below*), *provided the functions* $f_0, f_1, \dots$ *form a monotone non-increasing sequence of non-negative continuous functions; and a similar statement is true if the constants be replaced by continuous functions of any number of variables, provided the series so obtained itself oscillates uniformly above* (*below*).

* "On Oscillating Successions of Continuous Functions," *Proc. L. M. S.*, Ser. 2, vol. vi. (1908), pp. 298—320.

† "Note on Trigonometrical Series," and "On the Derivates of Non-differentiable Functions," *Mess. of Math.* (1908), pp. 44—48 and 65—69.

Even when the upper (lower) function is upper (lower) semi-continuous, there may be points at which the oscillation is not uniform above (below), though I have proved* that these form at most a set of the first category. There is one case however in which such "invisible" points of non-uniform oscillation cannot occur, and the final theorems of the present paper are concerned with this fact. I point out that if the successive partial summations $s_0, s_1, \ldots$ are all of them monotone increasing (decreasing) functions of x, then such invisible points of non-uniform oscillation are absent. Thus at a point at which the upper (lower) function is upper (lower) semi-continuous, the oscillation above (below) is uniform. This result is implicitly contained in the paper above quoted, but I venture to call attention to it here and to the particular case of it when there is convergence, because I have been unable to find this latter in any of the most recent text-books which deal with uniform convergence. It is possible therefore that it is here stated for the first time.

It will be noticed that I have confined my attention to oscillating series. It is obvious that the results which have been obtained must have their parallel in the Theory of the Oscillation of Infinite or Improper Integrals.

In conclusion I may remark that the very fact that the results here obtained are generalisations of, and lead immediately to, corresponding results for series which do not oscillate but converge, forms of itself evidence that the introduction of the peak and chasm functions and of the concepts of uniform and non-uniform oscillation was not an artificial one, but one enjoined by the very nature of mathematical reasoning, that it constitutes in fact an inevitable step in the development of the Theory of Limits.

§ 2. I begin by filling a small lacuna in the theory as so far presented, by proving a theorem from which follows as a corollary that the relative values of the peak, chasm, upper and lower functions of a series are unaffected by the omission of any finite number of terms of a non-converging series. The necessity for this theorem at once became evident as soon as I passed from the consideration of successions of functions to that of series.

Theorem 1. *If*

$$u_0 + u_1 + \ldots$$

and

$$U_0 + U_1 + \ldots$$

are two series of continuous functions of any number of variables x, of which the latter converges uniformly at any particular point, then the upper, lower, peak and chasm functions of the series

$$(U_0 + u_0) + (U_1 + u_1) + \ldots$$

are got by adding the sum of the second series to the corresponding functions of the first series.

In particular therefore the differences of the upper, lower, peak and chasm functions of a series of continuous functions are unaltered by the omission of any finite number of terms of the series.

This Theorem is an immediate consequence of the following Lemma.

* *Loc. cit.*

Lemma. *If we omit the restriction that the second series converges uniformly at the point in question, and denote the upper, lower, peak and chasm functions of the three series by*

$$f_1,\ f_2,\ \pi \text{ and } \chi,\ F_1,\ F_2,\ \Pi \text{ and } \mathrm{X},\ A,\ B,\ C \text{ and } D,$$

then we have

$$f_1 + F_2 \leqslant A \leqslant f_1 + F_1,$$
$$f_2 + F_2 \leqslant B \leqslant f_2 + F_1,$$
$$\pi + \mathrm{X} \leqslant C \leqslant \pi + \Pi,$$
$$\chi + \mathrm{X} \leqslant D \leqslant \chi + \Pi.$$

The proof of these inequalities is immediate. It will be sufficient to prove the third. We have

$$U_0 + u_0 + U_1 + u_1 + \ldots + U_n + u_n = (U_0 + U_1 + \ldots + U_n) + (u_0 + u_1 + \ldots + u_n);$$

thus the upper bound of the left-hand side in any region Q

$$\leqslant \text{upper bound of } (U_0 + U_1 + \ldots + U_n) + \text{upper bound of } (u_0 + u_1 + \ldots + u_n).$$

Hence, letting n increase indefinitely, and denoting by M_Q the highest limit,

$$M_Q \text{ of } (U_0 + u_0 + \ldots) \leqslant M_Q \text{ of } (U_0 + \ldots) + M_Q \text{ of } (u_0 + \ldots).$$

Letting the region Q shrink up to a point, we get, by the definition of the peak function,

$$C \leqslant \Pi + \pi.$$

Similarly, since

upper bound of the left-hand side of the above equality

$$\geqslant \text{lower bound of } (U_0 + U_1 + \ldots + U_n) + \text{upper bound of } (u_0 + u_1 + \ldots + u_n),$$
$$C \geqslant \mathrm{X} + \pi,$$

which proves the third inequality in the enunciation.

§ 3. We now proceed to obtain certain extensions of theorems respecting series of constant terms. As the case in which such a series does not converge or properly diverge is not always clearly discussed in the text-books, and as there is more than one method of defining the upper and lower limits of such a series, we begin with a short preliminary discussion of what we mean by oscillation of a series of constant terms.

When we say that a series of constant terms

$$a_0 + a_1 + a_2 + \ldots$$

oscillates, we mean that, bracketing the terms suitably together, so as to give a law by which the positive integer n describes a suitable sequence, the sum s_n of the first n terms may be made to have a unique limit, which varies with the mode of bracketing. The highest and lowest limits* so obtainable are called *the upper limit and the lower limit respectively of the series,* and their difference is called *the magnitude of the oscillation of the series.*

An alternative definition more in consonance with the modern theory of limits is the following:

* A set of limits, in the language of the Theory of Sets of Points, must be *a closed set* and may be *any* closed set; in particular the limits of an oscillating series may consist of a finite number of values, or may fill up a closed continuum, but, being a closed set, always have a highest and lowest.

Let U_i and L_i be the upper and lower bounds of all the partial summations from and after the ith, and let U be the lower bound of the quantities U_i, and L the upper bound of the quantities L_i, then U is called the upper limit and L the lower limit of the series.

To shew the equivalence of these two definitions, we notice that, from their definition,

$$U_1 \geqslant U_2 \geqslant U_3 \geqslant \ldots$$

form a monotone decreasing sequence and have therefore their lower bound U as unique limit.

Since for all values of $n \geqslant i$

$$s_n \leqslant U_i,$$

it follows that, denoting by V the upper limit of the series defined in the first manner,

$$V \leqslant U_i,$$

and therefore $$V \leqslant U.$$

But, since a limit of limits of quantities is itself a limit of those quantities, U is a limit of the quantities s_n, so that it cannot be greater than V, whence

$$V = U.$$

Similarly the lower limits defined in the two manners are equal, which proves the equivalence of the two definitions.

Hence also, *if U_0 is finite, U is finite or $-\infty$ (in which case the series diverges properly to $-\infty$), and conversely, if U is finite or $-\infty$, U_0 is finite.*

The first of these statements is obvious, since U is the limit of the monotone decreasing sequence U_0, U_1, ... The second follows from the definition of U_0 as the upper bound of all the quantities s_n, so that it is either itself one of the quantities s_n, and therefore finite, or is one of their limits, and therefore, their highest possible limit U.

§ 4. Theorem 2. *If the series*

$$a_0 + a_1 + \ldots \quad \ldots\ldots\ldots\ldots\ldots(1)$$

has M for the magnitude of its oscillation, and

$$k_0 \geqslant k_1 \geqslant k_2 \geqslant \ldots$$

is a monotone decreasing sequence of positive quantities, whose unique limit is k, then

I. *The series*

$$a_0 k_0 + a_1 k_1 + \ldots \quad \ldots\ldots\ldots\ldots\ldots(2)$$

has an oscillation whose magnitude is kM (provided kM is definite), and therefore converges if, and only if, k is zero, or the a-series converges;

II. *When M is infinite, but the lower (upper) limit of the a-series is finite, so is the lower (upper) limit of the (a, k)-series, or else the latter diverges properly to $+\infty$ $(-\infty)$; the other extreme limits of both series are infinite with the same sign;*

III. *When the lower and upper limits of the a-series are respectively $-\infty$ and $+\infty$, one at least, but not necessarily both, of the extreme limits of the (a, k)-series is infinite, if k be not zero; but if k be zero the (a, k)-series may be convergent;*

IV. *If k be not zero, and the first series diverges to $+\infty$ $(-\infty)$ so does the second series; but if k be zero, we cannot make any such statement.*

For, denoting by $s_0, s_1, \ldots$ the partial summations of the a-series, and by S_0, S_1 those of the (a, k)-series,

$$S_n = a_0k_0 + \ldots + a_nk_n = k_0s_0 + k_1(s_1 - s_0) + \ldots + k_n(s_n - s_{n-1})$$
$$= s_0(k_0 - k_1) + s_1(k_1 - k_2) + \ldots + s_{n-1}(k_{n-1} - k_n) + s_nk_n.$$

Since the quantities in brackets are all positive or zero, we have, using u_0 and l_0 for the upper and lower bounds of the partial summations of the a-series, for all values of n,

$$l_0k_0 \leqslant S_n \leqslant u_0k_0,$$

so that all the limits of the (a, k)-series lie between l_0k_0 and u_0k_0.

But, as pointed out at the end of the preceding article, if the upper limit of the a-series is finite, so is u_0, and therefore the partial summations S_n of the (a, k)-series are bounded above, so that the (a, k)-series has a finite upper limit unless, as in Ex. 1 below, it diverges properly to $-\infty$. This proves one half of the first statement of II, and similarly the other half follows. To prove (I), we have from the above, denoting by M' the oscillation of the (a, k)-series,

$$M' \leqslant (u_0 - l_0)\,k_0.$$

Now omit a_0 from the first series, and therefore a_0k_0 from the second series; in this way we do not alter the oscillation of either series. We change, however, s_i into $s_{i+1} - a_0$, and therefore alter u_0 and l_0 into $u_1 - a_0$ and $l_1 - a_0$, by which $u_0 - l_0$ becomes $u_1 - l_1$. Hence, by the above reasoning,

$$M' \leqslant (u_1 - l_1)\,k_1.$$

Similarly for all integers n,

$$M' \leqslant (u_n - l_n)\,k_n,$$

whence,

$$M' \leqslant (u - l)\,k \leqslant Mk.$$

Since the oscillation is a non-negative quantity, this shews that if

$$k = 0,$$

the (a, k)-series converges.

If k is different from zero, write

$$k_n = k + e_n,$$

then the quantities e_n have zero as limit, and therefore the series

$$a_0e_0 + a_1e_1 + \ldots$$

converges; let its sum be E. We have

$$a_0k_0 + a_1k_1 + \ldots + a_nk_n = a_0e_0 + a_1e_1 + \ldots + a_ne_n + k(a_0 + a_1 + \ldots + a_n),$$

letting n proceed towards infinity along various sequences, the right-hand side of this equality has for upper and lower limit respectively $E + ku$ and $E + kl$, which are therefore respectively the upper and lower limits of the left-hand side, that is of the (a, k)-series, so that the oscillation of this latter series is precisely kM, which proves I.

To prove the final statement in II. Suppose for definiteness that the upper limit of the a-series is $+\infty$. Then we proceed to shew that the upper limit of (2) is also $+\infty$.

We have, denoting the partial summations of the series (2) by $S_0, S_1, \ldots$

$$\begin{aligned} a_0 + a_1 + \ldots + a_n &= \frac{a_0 k_0}{k_0} + \frac{a_1 k_1}{k_1} + \ldots + \frac{a_n k_n}{k_n} \\ &= \frac{S_0}{k_0} + \frac{S_1 - S_0}{k_1} + \ldots + \frac{S_n - S_{n-1}}{k_n} \\ &= S_0\left(\frac{1}{k_0} - \frac{1}{k_1}\right) + \ldots + S_{n-1}\left(\frac{1}{k_{n-1}} - \frac{1}{k_n}\right) + S_n \frac{1}{k_n}, \\ &\leqslant L_0\left(\frac{1}{k_0} - \frac{1}{k_n}\right) + U_0 \frac{1}{k_n}, \end{aligned}$$

where L_0 is the lower bound and U_0 the upper bound of all the partial summations of the (a, k)-series, so that L_0 is finite by what has been proved above. Hence, if k is not zero, since one of the limits of the left-hand side is $+\infty$, it follows that U_0 is $+\infty$, whence also the upper limit of the (a, k)-series is $+\infty$.

That when $k=0$, the (a, k)-series may converge even when the upper and lower limits of the a-series are $+\infty$ and $-\infty$ respectively, is shewn by considering the series whose general term is

$$a_n = (-2)^n,$$

and the sequence of multipliers

$$k_n = (\tfrac{1}{2})^{2n}.$$

To prove the final statement III we have as in the preceding proof the double inequality

$$U_0\left(\frac{1}{k_0} - \frac{1}{k_n}\right) + L_0 \frac{1}{k_n} \leqslant a_0 + a_1 + \ldots + a_n \leqslant L_0\left(\frac{1}{k_0} - \frac{1}{k_n}\right) + U_0 \frac{1}{k_n},$$

whence it follows that if both the upper and the lower limit of the (a, k)-series, and therefore also U_0 and L_0, are finite, all the limits of the a-series (1) are finite. This proves that under the given circumstances at least one of the extreme limits of the (a, k)-series must be infinite. That it is unnecessary for both the extreme limits to be infinite, indeed that the (a, k)-series may, under the given circumstances, properly diverge, and not oscillate, is shewn by Ex. 2.

To prove IV we may, without loss of generality, assume that the upper limit of the a-series is $+\infty$, as we can, in the other case, ensure this by changing all the signs of the a's.

This being so we can determine an integer q such that for $m > q$, $s_m > s_0$, and therefore the lower bound say l' of the first m quantities s_i remains fixed as n increases.

But by the former argument we have

$$a_0 k_0 + a_1 k_1 + \ldots + a_n k_n \geqslant (k_0 - k_m)\, l' + k_m l_m.$$

Hence all the limits of the (a, k)-series are $\geqslant$ the right-hand side of this inequality, and this is true for each chosen value of m. Now make m increase indefinitely, then the first term has, since l' remains constant, a finite limit, while the second member has the limit $+\infty$, since l_m has the limit $+\infty$. This proves that the (a, k)-series, like the a-series, has the unique limit $+\infty$. Ex. 3 shews that when k is zero, a properly divergent a-series may even be converted into an (a, k)-series which oscillates finitely.

§ 5. Thus, if k is zero, finite oscillation is converted into convergence, and as we saw infinite oscillation may be converted into proper divergence. The following example shews that this latter may even be the case whether k has the value zero or not. If not converted into proper divergence, the oscillation remains, of course, infinite.

Ex. 1. Consider the series

$$-1-1, \quad +1+1, \quad -1-1-1-1, \quad +1+1+1+1, \quad -\ldots,$$

the number of positive and negative terms doubling in each successive group.

Suppose
$$k_1 = k_2 = k + \tfrac{1}{2},$$
$$k_3 = k_4 = k + \tfrac{1}{4},$$
$$k_5 = \ldots = k_8 = k + \tfrac{1}{4},$$
$$k_9 = \ldots = k_{16} = k + \tfrac{1}{8}, \text{ and so on.}$$

It is evident that the first series oscillates between $-\infty$ and 0, whereas the modified series diverges to $-\infty$, whatever be the value of k. In fact, when the first series is bracketed so as to have zero for unique limit, the second series, bracketed in the same way, has the value

$$-\tfrac{1}{2}, \ -\tfrac{1}{2}, \ -\tfrac{1}{2}, \text{ ad inf.},$$

that is $-\infty$, while any other mode of bracketing leads us still more rapidly down the negative scale.

Ex. 2. Let $a_1 = 1, \ a_2 = -2, \ldots, \ a_{2n+1} = (n+1)\,2^{2n+1}, \quad a_{2n+2} = -(n+1)\,2^{2n+1} - 1,$

$$k_1 = 1, \ k_2 = \tfrac{3}{4}, \ldots, \ k_{2n+1} = \tfrac{1}{2} + \frac{1}{2^{2n+1}}, \quad k_{2n+2} = \tfrac{1}{2} + \frac{1}{2^{2n+2}}.$$

The a-series then oscillates between $+\infty$ and $-\infty$, and $k = \tfrac{1}{2}$, but the second series diverges properly to $+\infty$, since $S_{2n+1} - S_{2n} > 1$, while

$$a_{2n+1}k_{2n+1} + a_{2n+2}k_{2n+2} = \tfrac{1}{2}(n+1) - \tfrac{1}{2} - \frac{1}{2^{2n+2}} > \tfrac{1}{2}(n-1),$$

so that, from and after $n = 3$, $S_{2n+2} - S_{2n+1}$ is greater than 1.

Ex. 3. Let the a-series be

$$3 - 2\tfrac{1}{2} + 9 - 8\tfrac{1}{2} + \ldots + (2^{2n+1} + 1) - (2^{2n+1} + \tfrac{1}{2}) + \ldots,$$

which diverges properly to $+\infty$.

Let the sequence of multipliers be

$$\frac{1}{2^2}, \ \frac{1}{2^2}, \ \frac{1}{2^4}, \ \frac{1}{2^4}, \ \ldots, \ \frac{1}{2^{2n+2}}, \ \frac{1}{2^{2n+2}}, \ \ldots.$$

The (a, k)-series is therefore

$$\tfrac{3}{4} - \tfrac{5}{8} + \tfrac{9}{16} - \tfrac{17}{32} + \ldots + \left(\frac{1}{2} + \frac{1}{2^{2n+2}}\right) - \left(\frac{1}{2} + \frac{1}{2^{2n+3}}\right) + \ldots,$$

whose oscillation is $\tfrac{1}{2}$, since it is obtained from the series $1 - 1 + 1 - 1 + \ldots$ by a sequence of multipliers whose limit is $\tfrac{1}{2}$.

The preceding theorem serves to shew that *any mode of bracketing the terms which gives the upper limit U of the a-series, serves also to give the upper limit of the (a, k)-series.* For the series

$$(k_0 - k)\,a_0 + (k_1 - k)\,a_1 + \dots$$

converges to a definite limit K, since

$$k_0 - k \geqslant k_1 - k \geqslant \dots$$

is a monotone decreasing sequence with zero as limit.

Hence, if we choose any mode of bracketing which gives for the a-series an unique limit v, this same mode of bracketing gives for the (a, k)-series the unique limit $K + kv$, and conversely any mode of bracketing which gives for the (a, k)-series the unique limit $K + kv$, gives for the a-series the unique limit v.

Hence it follows that the upper limit of the (a, k)-series is $K + ku$, and is given by any mode of bracketing which gives for the a-series its upper limit u.

§ 6. Theorem 3. *If*

$$a_0 + a_1 + \dots \qquad\qquad (1)$$

be a series of constant terms, the magnitude of whose oscillation is M, where M is finite, and

$$f_0(x) \geqslant f_1(x), \quad \geqslant f_2(x), \quad \geqslant \dots \qquad\qquad (2)$$

be a monotone decreasing sequence of continuous, nowhere negative, functions of any number of variables x, whose limiting function is $f(x)$, then the series

$$a_0 f_0(x) + a_1 f_1(x) + \dots \qquad\qquad (3)$$

oscillates uniformly both above and below, so that its upper and lower functions are at every point respectively upper and lower semi-continuous.

Moreover the oscillation function of the (a, f)-series (3) is $Mf(x)$, so that

(I) *if $f(x)$ is continuous at any point, the upper and lower functions are continuous at the same point;*

(II) *if $f(x)$ is zero at any point, the upper and lower functions agree at that point, and the series (3) converges uniformly (in the extended sense*) at that point, so that, once again, the upper and lower functions are continuous there, which is in accordance with the fact that such a zero of $f(x)$ is necessarily a point of continuity of $f(x)$, since this function is non-negative and upper semi-continuous.*

(III) *if M is zero, the series (3) converges uniformly everywhere, and therefore represents a continuous function.*

Let $U(x)$ and $L(x)$ denote the upper and lower functions of the series (3), and denote by $S_n(x)$ the sum of the first $\overline{n+1}$ terms of (3).

Then by Theorem 2 the oscillation function $U(x) - L(x)$ is $Mf(x)$.

Next to prove that the oscillation is uniform both above and below, we consider the peak

* Applicable to oscillating series.

and chasm functions $\Pi(x)$ and $X(x)$. As usual I denote by $M_{n,Q}$ the upper bound of $S_n(x)$ and by $L_{n,Q}$ the lower bound of the same, in a region Q having a chosen point P inside it. M_Q denotes the upper limit of $M_{n,Q}$ as n increase indefinitely, and L_Q the lower limit of $L_{n,Q}$.

It will be remembered that the peak function $\Pi(x)$ is at once the unique limit and the lower bound of all the M_Q's as the regions shrink up to the point P, and the chasm function is the unique limit and at the same time the upper bound of L_Q.

We have then as in the proof of Theorem 2, if l_0 and u_0 be the lower and upper bounds of the partial summations of the a-series, the inequalities

$$l_0 f_0(x) \leqslant s_n(x) \leqslant u_0 f_0(x).$$

Let x_n be the point of the region Q at which the continuous function $s_n(x)$ assumes its upper bound $M_{n,Q}$, it then follows that

$$M_{n,Q} \leqslant u_0 f_0(x_n).$$

If n describe any sequence of positive integers, the points x_n will have one or more limiting points, and, if y is one of these, we may so choose the sequence of n's that y is the unique limiting point. The point y is evidently a point of the closed region Q. Thus

$$M_Q \leqslant u_0 f_0(y),$$

whence also, letting the region Q shrink up to the point P,

$$\Pi(P) \leqslant u_0 f_0(P),$$

as f_0 is continuous at P.

Similarly $$X(P) \geqslant l_0 f_0(P),$$

and therefore $$\Pi(P) - X(P) \leqslant (u_0 - l_0) f_0(P).$$

But though the peak and chasm functions themselves will be affected by omitting any finite number of terms from the series (3), their difference will not. Thus omitting the first n terms of (1) and (3)

$$\Pi(P) - X(P) \leqslant (u_n - l_n) f_n(P).$$

This being true for all values of n, we get in the limit

$$\Pi(P) - X(P) \leqslant Mf(P).$$

But the upper and lower functions both lie between the peak and chasm functions, and we have just proved that their difference is precisely $Mf(P)$. Therefore not only must we in the inequality just obtained take the sign of equality, but further the peak function necessarily everywhere coincides with the upper function and the chasm function with the lower function, so that there is uniform oscillation everywhere both above and below.

The remaining statements in the enunciation of the theorem at once follow. For, by the theory of uniform oscillation above and below, the peak function is an upper semi-continuous function and the chasm function is a lower semi-continuous function, so that in this case the same is true respectively of the upper and lower functions.

Further at a point at which $f(x)$ is continuous,

$$U(x) = L(x) + Mf(x)$$

has necessarily the same kind of semi-continuity as $L(x)$, therefore it is lower semi-continuous; but it is upper semi-continuous everywhere, therefore it is continuous.

(II) follows at once from the theory of uniform oscillation, and (III) is obviously a special case of our theorem.

§ 7. Theorem 3*a*. *If in Theorem 3 only the upper (lower) limit of the a-series is finite, so that M is not finite, then the (a, f)-series oscillates uniformly above (below) so that its upper (lower) function is everywhere upper (lower) semi-continuous.*

As before we get

$$\Pi(P) \leqslant u_0 f_0(P).$$

We now put zero for all the a's before a_n. The peak function will then be obtained by adding to the peak function of this new series the sum of the first n terms of the (a, f)-series. We thus get

$$\Pi(P) \leqslant a_0 f_0 + a_1 f_1 + \dots + a_{n-1} f_{n-1} + (u_n - a_0 - a_1 - \dots - a_{n-1}) f_n,$$

where the coefficient of f_n is the upper bound of the partial summations of the new series of the a's. Thus u_n is the upper bound of all the summations after the nth of the original a-series (1), and hence if we make n describe such a sequence that we get the upper limit for series (1), the coefficient of f_n in the preceding inequality will have zero as limit. Hence we get

$$\Pi(P) \leqslant U(P),$$

whence the required result follows.

It should be noticed that it does not follow because the (a, f)-series oscillates everywhere uniformly above (for example) that it oscillates everywhere finitely. It might, as follows from Theorem 2, diverge properly to $-\infty$ above or more points.

§ 8. Theorem 4. *If*

$$v_0 + v_1 + \dots$$

be a series of continuous functions of any number of variables, denoted by x, and

$$f_0(x) \geqslant f_1(x) \geqslant \dots$$

be a monotone decreasing sequence of continuous, never negative, functions of the variables x, whose limiting function is $f(x)$, then at any point at which the v-series has a unique limiting function or oscillates uniformly above (below) the same is true of the series

$$v_0 f_0 + v_1 f_1 + \dots.$$

Indeed it follows at once from Theorem 1 that, denoting the upper, lower, peak and chasm functions of the v-series by u, l, π and χ, and using the corresponding large letters to refer to the (v, f)-series,

$$U - L = (u - l) f,$$

provided the right-hand side of this equation is definite. This proves the first of the above statements, since if $u = l$, we must have $U = L$. To prove the remaining statements we proceed to prove the inequality

$$\Pi - U \leqslant (\pi - u) f,$$

from which the corresponding inequality

$$L - X \leqslant (l - \chi) f$$

follows by analogy.

For, denoting by s_n and S_n the sums of the first $(n+1)$ terms of the v-series and the (v, f)-series respectively,

$$\begin{aligned} S_n &= v_0 f_0 + v_1 f_1 + \dots + v_n f_n \\ &= S_i + (s_{i+1} - s_i) f_{i+1} + \dots + (s_n - s_{n-1}) f_n \\ &= S_i - s_i f_{i+1} + s_{i+1} (f_{i+1} - f_{i+2}) + \dots + s_{n-1} (f_{n-1} - f_n) + s_n f_n. \end{aligned}$$

Now the f's and their successive differences are never negative; hence, denoting by b_{i+1} the upper bound in some closed region Q containing the point x in question, of all the partial summations $s_{i+1}, s_{i+2}, \dots$

$$S_n - S_i + s_i f_{i+1} \leqslant b_{i+1} f_{i+1}.$$

Now denoting by $M_{n, Q}$ the upper bound of S_n in the region Q, there is a point x_n of that region at which S_n attains its upper bound, since it is continuous. Hence

$$M_{n, Q} - S_i(x_n) + s_i(x_n) f_{i+1}(x_n) \leqslant b_{i+1} f_{i+1}(x_n).$$

Now if we let n increase without limit in such a way as to give the upper limit M_Q of $M_{n, Q}$, the points x_n have at least one limiting point y, which, since the points x_n all lie inside the region Q, also belongs to the region Q, and we may confine our attention to such a sequence of the values of n in question, that the point y is the unique limiting point of the points x_n. We thus get

$$M_Q - S_i(y) + s_i(y) f_{i+1}(y) \leqslant b_{i+1} f_{i+1}(y),$$

since the functions S_i, s_i, f_i and f_{i+1} are continuous.

Letting the region Q shrink up to the point x, we then have, by the definition of the peak function,

$$\Pi - S_i(x) + s_i(x) f_i(x) \leqslant b_{i+1} f_{i+1}(x),$$

in which b_{i+1} has been retained, since in the process of shrinkage, the corresponding coefficient never increases.

Now, by the definition of b_{i+1} it is the upper bound of $m_{n, Q}$ for all values of $n \geqslant i+1$, where $m_{n, Q}$ is the upper bound of s_n in the region Q. Hence, by what was pointed out in § 3 about upper limits, the quantities b_n form a monotone non-increasing sequence whose limit is the upper limit m_Q of the quantities $m_{n, Q}$. Hence, allowing the integer i to describe such a sequence that $s_i(x)$ has the unique limit $u(x)$, we get

$$\Pi - U(x) + u(x) f(x) \leqslant m_Q f(x),$$

since $S_i(x)$ will have a limit or limits less than $U(x)$.

Since this is true for all regions Q containing the point x,

$$\Pi(x) - U(x) + u(x) f(x) \leqslant \pi(x) f(x),$$

that is

$$\Pi(x) - U(x) \leqslant \{\pi(x) - u(x)\} f(x).$$

But at a point where the oscillation above is uniform the peak function is equal to the upper function, so that the preceding equation shews that where the u-series oscillates uniformly above, so does the (u, f)-series, since the peak function is never less than the upper function. Similarly the statement about the lower oscillation may be proved.

§ 9. The proof of Theorem 4 at once gives us the following additional theorem :

Theorem 4*a*. *At a point at which $f(x)$ has the value zero and the peak and chasm functions of the v-series are finite, the (v, f)-series converges uniformly* and therefore the upper and lower functions are equal and continuous there.*

The following theorem follows, by reasoning similar to that employed in proving the later statements of Theorem 2.

Theorem 4*b*. *If f is different from zero and the measure of the non-uniformity of the oscillation above (below) is finite, so is that of the (v, f)-series, and if the former is infinite so is the latter, provided one only of the oscillations, above or below, of the v-series is infinite. If both measures are infinite, so that $\pi = +\infty$, $\chi = -\infty$, either $\Pi = +\infty$ or $\mathrm{X} = -\infty$.*

The first statement is an immediate result of the inequality

$$\Pi - U \leqslant (\pi - u) f,$$

established in the course of the proof of Theorem 4, or the independent inequality

$$L - \mathrm{X} \leqslant (l - \chi) f.$$

To prove the remaining statements we take the case of infinite non-uniform oscillation above and write

$$s_n = \frac{v_0 f_0}{f_0} + \frac{v_1 f_1}{f_1} + \ldots + \frac{v_n f_n}{f_n} = \frac{S_0}{f_0} + \frac{S_1 - S_0}{f_1} + \ldots + \frac{S_n - S_{n-1}}{f_n}$$
$$= S_0 \left\{ \frac{1}{f_0} - \frac{1}{f_1} \right\} + \ldots + S_{n-1} \left\{ \frac{1}{f_{n-1}} - \frac{1}{f_n} \right\} + S_n \frac{1}{f_n}.$$

Let x_n be the point of a region Q containing the point P at which the continuous function s_n assumes its upper bound $m_{n,Q}$, and let $S_j(x_n)$ be the least of the quantities $S_r(x_n)$, for values of r less than n, or, if this is not unique, that one whose index r is the greatest. Then, since the coefficients of $S_0, \ldots S_{n-1}$ in the preceding equation are all negative or zero, we get

$$m_{n,Q} \leqslant S_j(x_n) \left(\frac{1}{f_0(x_n)} - \frac{1}{f_n(x_n)} \right) + S_n(x_n) \frac{1}{f_n(x_n)}$$
$$\leqslant - S_j(x_n) k_n + M_{n,Q} \frac{1}{f_n(x_n)},$$

where k_n lies between 0 and $\dfrac{1}{f(x_n)} - \dfrac{1}{f_0(x_n)}$.

Now if we let n describe such a sequence of integers that $m_{n,Q}$ has its upper limit m_Q, the points x_n will have one or more limiting points, and, if y be one of these, we may confine our attention to such a sequence of n's that y is the unique limiting point of the points x_n. The coefficient k_n will then have a limit or limits lying between 0 and

$$\frac{1}{f(y)} - \frac{1}{f_0(y)},$$

and we may confine our attention to such a sequence of integers n that k_n has a unique limit k.

* In the extended sense applicable to oscillating series.

As to the integer j two things may occur:

(i) it may have a finite upper bound J. In this case, from and after a certain value of n, $j=J$ always;

(ii) j may have no finite upper bound; $S_j(x_n)$ is then greater than or equal to its lower bound in the region Q, that is $L_{j,Q}$, and therefore has a limit greater than or equal to the lower limit L_Q of $L_{j,Q}$.

Thus we get either

$$m_Q \leqslant -S_j(y)\,k + M_Q\left(k+\frac{1}{f_0(y)}\right),$$

or

$$m_Q \leqslant -L_Q k + M_Q\left(k+\frac{1}{f_0(y)}\right).$$

Now if $\pi(P)$ is $+\infty$, so is m_Q, and therefore, if the former of these two inequalities holds, so is M_Q, since the remaining quantities involved are finite; in this case, therefore, $\Pi(P)=+\infty$. But if the second of these inequalities holds either $L_Q=-\infty$ or $M_Q=+\infty$, so that either $\mathrm{X}(P)=-\infty$ or $\Pi(P)=+\infty$.

But if the measure of the non-uniformity of the oscillation below is finite for the v-series, so it is for the (v,f)-series, and therefore $\mathrm{X}(P)$ is finite, so that in this case $\Pi(P)=+\infty$, as stated in the enunciation.

§ 10. The two theorems which follow are again an immediate consequence of the proof of Theorem 4.

Theorem 4*c*. *If any point of continuity of $f(x)$ is also a point of continuity of the oscillation function $(u-l)$ of the v-series, the upper and lower functions of the (v,f)-series are continuous at this point.*

Theorem 4*d*. *If the v-series converges everywhere uniformly, so does the (v,f)-series, and therefore represents a continuous function*.*

Attention is called to the special case of these theorems when the multipliers $f_0, f_1, \ldots$ are constants.

§ 11. As an illustration of the use of the results arrived at, we may notice the simple example of a power series which converges when $x<r$, and oscillates when $x=r$. Here

$$f_k(x)=\frac{x^k}{r^k},$$

so that $f(x)$ is zero when x is less than r, and unity when $x=r$.

It then follows from Theorem 4 that the upper function of the power series is upper semi-continuous and the lower function lower semi-continuous at the point $x=r$, while they of course coincide and are continuous elsewhere. Hence from the very definition of upper and lower semi-continuity, the upper and lower limits of the series of constant terms obtained by putting

* This is a generalisation of Abel's and Dirichlet's test for uniform convergence, see Bromwich, *Introduction to the Study of Infinite Series*, Macmillan (1908), pp. 113, 114.

$x=r$ are respectively $\geqslant$ and $\leqslant$ all the limits of the sum-function for all modes of approach of x to the value r.

This result is, of course, well-known, it is merely given here as shewing how a whole class of results become self-evident in the light of the theorems we have proved.

§ 12. I now pass to one of the obvious consequences of the results of the paper on "Oscillating Successions of Continuous Functions," referred to in § 19 of that paper, viz. to a particular case of the result given in Ex. 1, § 17, p. 317. We there found that *when the functions* $f_1(x), f_2(x), \ldots$ *which represent the successive sums of an oscillating series, are monotone, the peak function is the associated upper limiting function of the upper function and the chasm function is the associated lower limiting function of the lower function.* It of course immediately follows that *if the upper function is upper semi-continuous at a point* P *the oscillation above at* P *is uniform, and similarly for the lower function. In other words, there are no invisible points of non-uniform oscillation.*

As a very particular case of the example in question we have the following theorem:

Theorem 5. *If* $f_1(x), f_2(x), \ldots$ *form a sequence of continuous monotone increasing (or decreasing) functions, having as unique limit a function* $f(x)$, *then at any point at which the function* f *is continuous, the sequence necessarily converges uniformly. Moreover at a point at which it is continuous in the extended sense in which we distinguish* $+\infty$ *from* $-\infty$ *the sequence diverges uniformly.*

Anyone who has taken the trouble to master the ideas of the peak and chasm function will hardly, I think, fail to see the intuitiveness of this result. It is often, however, a satisfaction to have a result, obtained by a new method, confirmed by the use of an old one. I now give therefore a proof which depends on the use of the ϵ-machinery and the original definition of uniform convergence at a point.

Let a be a point of continuity of the limiting function $f(x)$, and choose the interval (a, b) so that

$$0 \leqslant |f(a)-f(b)| \leqslant \tfrac{1}{2}e \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and choose m so that for all integers $n \geqslant m$,

$$|f_n(a)-f(a)| < \tfrac{1}{2}e \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and also

$$|f_n(b)-f(b)| < \tfrac{1}{2}e \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also bearing in mind that $f(x)$ is itself, like $f_n(x)$, a monotone function, we see that $f_n(x)$ lies between $f_n(a)$ and $f_n(b)$, and $f(x)$ between $f(a)$ and $f(b)$, if x is any point of the interval (a, b), and accordingly that

$$f_n(x)-f(x)$$

lies between $f_n(b)-f(a)$ and $f_n(a)-f(b)$.

But from (1) and (3), $|f_n(b)-f(a)| < e$,

and from (1) and (2), $|f_n(a)-f(b)| < e$.

Hence $|f_n(x)-f(x)|$,

being necessarily less than the greater of these two moduli, is less than e. Thus, corresponding to the point a and the value e, we have found an interval d on one side of a, and an m, so that for all points x inside d and for all values of $n \geqslant m$

$$|R_n(x)| \leqslant e.$$

Taking similarly an interval on the other side, the two form a whole interval surrounding the point a for which this is true. Hence by the definition of uniform convergence at a point, the sequence converges uniformly at the point a. Q. E. D.

It will be noticed that the above proof, like all ϵ-proofs, requires suitable modification if a is a point of continuity in the extended sense.

§ 13. This theorem may be compared with the known theorem that *a monotone sequence of continuous functions has no invisible points of non-uniform convergence.*

The student who is at home with double limits will recognise that both these theorems are an immediate consequence of the following simple theorem:

Theorem 6. *If $f(x, y)$ is a monotone increasing (or decreasing) function of y at all points of a closed $(+, +)$-neighbourhood* of the point P, and is continuous with respect to x on the ordinate of P, and also with respect to y at P itself, it is a continuous† function of the ensemble (x, y) at the point P.*

For let R be any point on the ordinate of P (Fig. 1), and let the parallels to the x-axis through P and R meet the ordinate of any point T in Q and S.

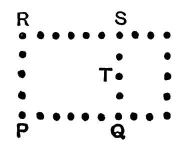

Fig. 1.

Since $f(x, y)$ is monotone with respect to y, $f(T)$ lies between $f(Q)$ and $f(S)$. But, as T moves up to P in any manner, Q moves up to P and S to R: therefore, since f is continuous with respect to x at the points P and R, $f(Q)$ and $f(S)$ have respectively the unique limits $f(P)$ and $f(R)$, so that all the limits of $f(T)$ lie between $f(P)$ and $f(R)$.

Since this is true for all positions of the point R on the ordinate of P, and f is continuous at P with respect to y, it follows that $f(T)$ has the unique limit $f(P)$, which proves the theorem‡.

* That is a neighbourhood bounded below and on the left by the axial cross through P.

† That is continuous in as far as the $(+, +)$-neighbourhood is concerned. There is no assumption and therefore no result as to continuity with respect to the other quadrants. Of course if f is defined all round P, and similar conditions hold in all four quadrants separately, or in a whole region with P as internal point, we shall get complete continuity at P. In the same way the continuity demanded with respect to x and to y is only one-sided, viz. on the right for x and above for y. Moreover continuity here does not include finitude, infinite values are allowed, and the two infinities, $+\infty$ and $-\infty$, are distinguished from one another.

‡ It will be noticed from the proof that it is not really necessary that f should be continuous with respect to x except at P itself, provided the upper and lower limits of $f(x, y)$ for constant y when x has for limit the abscissa of P, have, when the point R approaches P as limiting point, $f(P)$ as unique limit.

The corresponding theorem for double series is evidently as follows:

If $S_{m,n}$ is monotone for each fixed value of m as n increases, and if its repeated limits

$$\underset{m=\infty}{\mathrm{Lt}}\ \underset{n=\infty}{\mathrm{Llt}}\ S_{m,n} \text{ and } \underset{n=\infty}{\mathrm{Lt}}\ \underset{m=\infty}{\mathrm{Llt}}\ S_{m,n}$$

are all equal, then a unique double limit of $S_{m,n}$ exists, that is,

$$\underset{i=\infty}{\mathrm{Lt}}\ S_{m_i, n_i}$$

is unique, whatever sequence (m_i, n_i) of integers be chosen.

Since finishing this paper I found a statement of a theorem on double series by Bromwich, substantially the same as this, and differing from it only in so far that the condition of finiteness is required.

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THE DETERMINATION OF SOLUTIONS OF THE EQUATION OF WAVE MOTION INVOLVING AN ARBITRARY FUNCTION OF THREE VARIABLES WHICH SATISFIES A PARTIAL DIFFERENTIAL EQUATION.

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X. *The determination of solutions of the equation of wave motion involving an arbitrary function of three variables which satisfies a partial differential equation.*

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§ 1. THE transformations of coordinates which enable us to pass from any given solution of the equation of wave motion to another have already been investigated*, they form a very limited class and are characterized by the property that

$$(dx^2+dy^2+dz^2-c^2d\tau^2)=\lambda\,(dx'^2+dy'^2+dz'^2-c^2d\tau'^2).$$

Besides these, however, there are large classes of transformations which can be applied to solutions which satisfy certain linear conditions, but cannot be applied to an arbitrary solution of the equation. These transformations may be divided into two classes. A transformation of the first class may be illustrated by means of an identical relation of the type†

$$dx^2+dy^2+dz^2-c^2d\tau^2-(p\,dx+q\,dy+r\,dz+sc\,d\tau)^2$$
$$=dx'^2+dy'^2+dz'^2-c^2d\tau'^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

in which $\quad p^2+q^2+r^2=s^2.$

Such a transformation enables us to pass from a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial \tau^2},$$

satisfying the relation $\quad p\frac{\partial V}{\partial x}+q\frac{\partial V}{\partial y}+r\frac{\partial V}{\partial z}=\frac{s}{c}\frac{\partial V}{\partial \tau},$

to a solution of $\quad \frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial \tau'^2}.$

* *Proceedings of the London Mathematical Society*, Vol. 7, Ser. 2, p. 70 (1909).

† There are similar transformations which can be applied to solutions of Laplace's equation.

If

$$dx^2+dy^2+dz^2-(p\,dx+q\,dy+r\,dz)^2=dx'^2+dy'^2+dz'^2,$$

where $\quad p^2+q^2+r^2=0,$

then a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which satisfies the relation

$$p\frac{\partial V}{\partial x}+q\frac{\partial V}{\partial y}+r\frac{\partial V}{\partial z}=0,$$

is transformed into a solution of

$$\frac{\partial^2 V}{\partial x'^2}+\frac{\partial^2 V}{\partial y'^2}+\frac{\partial^2 V}{\partial z'^2}=0.$$

A more general result may be obtained by considering a transformation for which

$$dx^2+dy^2+dz^2-(p\,dx+q\,dy+r\,dz)^2$$
$$=\lambda\,(dx'^2+dy'^2+dz'^2).$$

A transformation of the second class is characterized by means of a relation of the type

$$(p^2+q^2+r^2+s^2)(dx^2+dy^2+dz^2+dt^2)-(p\,dx+q\,dy+r\,dz+s\,dt)^2$$
$$=dX^2+dY^2+dZ^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where t is written in place of $ic\tau$.

It can be shewn that if p, q, r, s satisfy certain conditions the transformation enables us to pass from a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0,$$

that satisfies a certain linear relation of the type

$$p\frac{\partial V}{\partial x}+q\frac{\partial V}{\partial y}+r\frac{\partial V}{\partial z}+s\frac{\partial V}{\partial t}+mV=0,$$

to a solution of

$$\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial^2 V}{\partial Z^2}=0,$$

and *vice versâ*.

It is the transformations of the second class which form the subject of this paper. The problem to be solved may be enunciated as follows.

To determine three functions X, Y, Z and a fourth function W, such that if $F(X, Y, Z)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2}+\frac{\partial^2 F}{\partial Y^2}+\frac{\partial^2 F}{\partial Z^2}=0,$$

the function $$V=WF(X, Y, Z),$$

may be a solution of $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

In attacking this problem it is convenient to consider a problem of a more general nature in which the object is to construct solutions of the form

$$V=WF(X, Y, Z),$$

where F is any solution of a certain partial differential equation having X, Y, Z as independent variables.

In solving this problem I have made use of some theorems on the multiplication and transformation of integral forms given in my paper on the transformation of the electrodynamical equations*.

Transformations possessing the property (2) are analogous to conformal transformations, for if we take elements through a point (x, y, z, t) which satisfy the relation

$$p\,dx+q\,dy+r\,dz+s\,dt=0,$$

they are evidently proportional to the corresponding elements in the (X, Y, Z) space.

§ 2. *The direct treatment of the problem and the fundamental equations.*

The conditions to be satisfied in order that the differential equation

$$\nabla_4^2 V \equiv \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

* *Proceedings of the London Mathematical Society* (1910).

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where a is an arbitrary constant, then

$$Z = f(u)$$

is a solution of the equations

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 = 0,$$

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \frac{\partial^2 Z}{\partial z^2} = 0.$$

A second solution of these equations is given by

$$V = \frac{g(u)}{a - xp'(u) - yq'(u) - zr'(u)},$$

so that in this case

$$W = \frac{1}{a - xp'(u) - yq'(u) - zr'(u)}.$$

§ 3. *The transformation of the differential equations.*

Let a fourth quantity T be associated with X, Y, Z in such a way that x, y, z, t can be expressed in terms of X, Y, Z, T; then we may write

$$\left.\begin{aligned}
\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial X}{\partial z}\right)^2 + \left(\frac{\partial X}{\partial t}\right)^2 &= a \\
\left(\frac{\partial Y}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial z}\right)^2 + \left(\frac{\partial Y}{\partial t}\right)^2 &= b \\
\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 + \left(\frac{\partial Z}{\partial z}\right)^2 + \left(\frac{\partial Z}{\partial t}\right)^2 &= c \\
\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2 + \left(\frac{\partial T}{\partial t}\right)^2 &= \theta \\
\frac{\partial Y}{\partial x}\frac{\partial Z}{\partial x} + \frac{\partial Y}{\partial y}\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\frac{\partial Z}{\partial z} + \frac{\partial Y}{\partial t}\frac{\partial Z}{\partial t} &= f \\
\frac{\partial Z}{\partial x}\frac{\partial X}{\partial x} + \frac{\partial Z}{\partial y}\frac{\partial X}{\partial y} + \frac{\partial Z}{\partial z}\frac{\partial X}{\partial z} + \frac{\partial Z}{\partial t}\frac{\partial X}{\partial t} &= g \\
\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial Y}{\partial z} + \frac{\partial X}{\partial t}\frac{\partial Y}{\partial t} &= h \\
\frac{\partial X}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial X}{\partial t}\frac{\partial T}{\partial t} &= u \\
\frac{\partial Y}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial Y}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial Y}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial Y}{\partial t}\frac{\partial T}{\partial t} &= v \\
\frac{\partial Z}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial Z}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial Z}{\partial z}\frac{\partial T}{\partial z} + \frac{\partial Z}{\partial t}\frac{\partial T}{\partial t} &= w
\end{aligned}\right\} \quad \text{..........................(3).}$$

Multiplying these equations by

$$\frac{\partial \xi}{\partial X}\frac{\partial \eta}{\partial X}, \quad \frac{\partial \xi}{\partial Y}\frac{\partial \eta}{\partial Y}, \ldots \quad \frac{\partial \xi}{\partial Y}\frac{\partial \eta}{\partial Z} + \frac{\partial \xi}{\partial Z}\frac{\partial \eta}{\partial Y}, \ldots$$

and adding we obtain the relation

$$\begin{aligned}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial z}+\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial t}&\\ =a\frac{\partial\xi}{\partial X}\frac{\partial\eta}{\partial X}+b\frac{\partial\xi}{\partial Y}\frac{\partial\eta}{\partial Y}+c\frac{\partial\xi}{\partial Z}\frac{\partial\eta}{\partial Z}+\theta\frac{\partial\xi}{\partial T}\frac{\partial\eta}{\partial T}&\\ +f\left(\frac{\partial\xi}{\partial Y}\frac{\partial\eta}{\partial Z}+\frac{\partial\xi}{\partial Z}\frac{\partial\eta}{\partial Y}\right)+\ldots\ldots&\\ +u\left(\frac{\partial\xi}{\partial X}\frac{\partial\eta}{\partial T}+\frac{\partial\xi}{\partial T}\frac{\partial\eta}{\partial X}\right)+\ldots\ldots&\quad\ldots\ldots\ldots\ldots(4).\end{aligned}$$

Now take six arbitrary functions ξ_1, ξ_2, ξ_3, η_1, η_2, η_3 and form a determinant with the elements

$$\frac{\partial\xi_r}{\partial x}\frac{\partial\eta_s}{\partial x}+\frac{\partial\xi_r}{\partial y}\frac{\partial\eta_s}{\partial y}+\frac{\partial\xi_r}{\partial z}\frac{\partial\eta_s}{\partial z}+\frac{\partial\xi_r}{\partial t}\frac{\partial\eta_s}{\partial t}.$$

The value of this determinant is easily seen to be

$$\begin{aligned}\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(y,z,t)}\cdot\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(y,z,t)}+\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(z,x,t)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(z,x,t)}&\\ +\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x,y,t)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(x,y,t)}+\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x,y,z)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(x,y,z)}&\ldots\ldots\ldots\ldots(5),\end{aligned}$$

while the value of the determinant formed by the quantities on the right-hand side is*

$$\begin{aligned}A\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(Y,Z,T)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(Y,Z,T)}+\ldots\ldots&\\ +F\left[\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(Z,X,T)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(X,Y,T)}+\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(X,Y,T)}\frac{\partial(\eta_1,\eta_2,\eta_3)}{\partial(Z,X,T)}\right]+\ldots\ldots&\ \ldots(6),\end{aligned}$$

where A, B, C, F, G, H, U, V, W, Θ are the cofactors of a, b, c, f, g, h, u, v, w, θ in the determinant

$$\begin{vmatrix} a & h & g & u\\ h & b & f & v\\ g & f & c & w\\ u & v & w & \theta\end{vmatrix}.$$

Now let dx, dy, dz, dt be increments of x, y, z, t when ξ_1, ξ_2, ξ_3 are kept constant, dX, dY, dZ, dT the corresponding increments in X, Y, Z, T. Since ξ_1, ξ_2, ξ_3 are arbitrary functions the increments dx, dy, dz, dt are really arbitrary. We have

$$\frac{\partial\xi_1}{\partial x}dx+\frac{\partial\xi_1}{\partial y}dy+\frac{\partial\xi_1}{\partial z}dz+\frac{\partial\xi_1}{\partial t}dt=0,$$

$$\frac{\partial\xi_2}{\partial x}dx+\frac{\partial\xi_2}{\partial y}dy+\frac{\partial\xi_2}{\partial z}dz+\frac{\partial\xi_2}{\partial t}dt=0,$$

$$\frac{\partial\xi_3}{\partial x}dx+\frac{\partial\xi_3}{\partial y}dy+\frac{\partial\xi_3}{\partial z}dz+\frac{\partial\xi_3}{\partial t}dt=0,$$

therefore
$$\frac{\dfrac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(y,z,t)}}{dx}=\frac{\dfrac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(z,x,t)}}{dy}=\frac{\dfrac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x,y,t)}}{dz}=\frac{\dfrac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x,y,z)}}{dt}=\kappa,\text{ say.}$$

* The terms containing U, V, W have a negative sign.

Similarly $$\frac{\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(Y, Z, T)}}{dX} = \frac{\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(Z, X, T)}}{dY} = \frac{\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(X, Y, T)}}{dZ} = \frac{\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(X, Y, Z)}}{dT} = K.$$

Again, if δx, δy, δz, δt; δX, δY, δZ, δT are increments in the variables when η_1, η_2, η_3 are kept constant, we can express them in terms of η_1, η_2, η_3 by a repetition of the above process.

Substituting κdx, $\kappa' \delta x$, $K dX$, $K' \delta X$, etc. for

$$\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(y, z, t)}, \quad \frac{\partial(\eta_1, \eta_2, \eta_3)}{\partial(y, z, t)}, \quad \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(Y, Z, T)}, \quad \frac{\partial(\eta_1, \eta_2, \eta_3)}{\partial(Y, Z, T)},$$

in the expressions (5) and (6) we obtain the relation

$$\begin{aligned}\frac{\kappa\kappa'}{KK'}[dx\,\delta x + dy\,\delta y + dz\,\delta z + dt\,\delta t] &= A\,dX\,\delta X + B\,dY\,\delta Y \\ &+ C\,dZ\,\delta Z + \Theta\,dT\,\delta T + F(dY\,\delta Z + dZ\,\delta Y) + G(dZ\,\delta X + dX\,\delta Z) \\ &+ H(dX\,\delta Y + dY\,\delta X) + U(dX\,\delta T + \delta X\,dT) + V(dY\,\delta T + \delta Y\,dT) \\ &\qquad + W(dZ\,\delta T + \delta Z\,dT) \qquad (7).\end{aligned}$$

Putting $dx = \delta x$, etc., $\frac{\kappa\kappa'}{KK'} = \lambda$, we get

$$\begin{aligned}\lambda[dx^2 + dy^2 + dz^2 + dt^2] &= A\,dX^2 + B\,dY^2 + C\,dZ^2 + \Theta\,dT^2 \\ &+ 2F\,dY\,dZ + 2G\,dZ\,dX + 2H\,dX\,dY + 2U\,dX\,dT \\ &+ 2V\,dY\,dT + 2W\,dZ\,dT \qquad (8).\end{aligned}$$

This is the characteristic relation of the transformation. The quantities A, B, C, F, G, H are linear functions of θ, thus

$$A = \theta(bc - f^2) - v^2c - w^2b + 2fvw;$$

the quadratic differential form on the right-hand side can therefore be written

$$\begin{aligned}&\theta[(bc - f^2)\,dX^2 + (ca - g^2)\,dY^2 + (ab - h^2)\,dZ^2 \\ &\qquad + 2(gh - af)\,dY\,dZ + 2(hf - bg)\,dZ\,dX + 2(fg - ch)\,dX\,dY] \\ &- [(v^2c + w^2b - 2fvw)\,dX^2 + (w^2a + u^2c - 2gwu)\,dY^2 \\ &\qquad + (u^2b + v^2a - 2huv)\,dZ^2 + (abc + 2fgh - af^2 - bg^2 - ch^2)\,dT^2 \\ &\qquad + 2(hw^2 + cuv - gvw - fwu)\,dX\,dY + \ldots\ldots \\ &\qquad + 2\{u(bc - f^2) + v(fg - ch) + w(hf - bg)\}\,dX\,dT].\end{aligned}$$

Now the second half of this expression is a perfect square if the single condition

$$\begin{aligned}&(bc - f^2)\,u^2 + (ca - g^2)\,v^2 + (ab - h^2)\,w^2 \\ &\qquad + 2(gh - af)\,vw + 2(hf - bg)\,wu + 2(fg - ch)\,uv = 0 \qquad (9),\end{aligned}$$

is satisfied. We shall suppose that this condition is satisfied* and that the expression is the square of

$$P\,dX + Q\,dY + R\,dZ + S\,dT.$$

Let the equivalent of this in terms of the variables x, y, z, t be $p\,dx + q\,dy + r\,dz + s\,dt$, then equation (8) may be replaced by the equation

$$\begin{aligned}&\lambda[dx^2 + dy^2 + dz^2 + dt^2] - (p\,dx + q\,dy + r\,dz + s\,dt)^2 \\ &\qquad = [(bc - f^2)\,dX^2 + (ca - g^2)\,dY^2 + (ab - h^2)\,dZ^2 \\ &\qquad + 2(gh - af)\,dY\,dZ + 2(hf - bg)\,dZ\,dX + 2(fg - ch)\,dX\,dY]\,\theta \ldots (10).\end{aligned}$$

* The condition may be regarded as a differential equation for determining T and by a suitable choice of T the equation may be satisfied.

Since the quantities a, b, c, f, g, h are all proportional to functions of X, Y, Z, the coefficients of the quadratic form on the right-hand side can be reduced to functions of X, Y, Z by multiplying them all by a certain factor ϕ. The quantity λ may be determined by the fact that the quadratic form on the left-hand side of the equation can be expressed as the sum of three squares, the discriminant is therefore zero and so

$$\lambda = p^2 + q^2 + r^2 + s^2.$$

Absorbing a suitable factor from the right-hand side of the equation into p, q, r, s, the meanings of these quantities being now slightly different from before, the equation takes the form

$$(p^2 + q^2 + r^2 + s^2)(dx^2 + dy^2 + dz^2 + dt^2) - (pdx + qdy + rdz + sdt)^2$$
$$= a'dX^2 + b'dY^2 + c'dZ^2 + 2f'dYdZ + 2g'dZdX + 2h'dXdY \quad\text{......(11)},$$

where the coefficients of the quadratic form on the right-hand side are functions of X, Y, Z alone. In the particular case when $a = b = c$, $f = g = h = 0$ the equation can be reduced to the simple form

$$(p^2 + q^2 + r^2 + s^2)(dx^2 + dy^2 + dz^2 + dt^2) - (pdx + qdy + rdz + sdt)^2$$
$$= dX^2 + dY^2 + dZ^2 \quad\text{..........(12)}.$$

In the general case we can find a transformation of the X, Y, Z coordinates which reduces the quadratic differential form on the right-hand side to the sum of three squares. The equation then becomes

$$(p^2 + q^2 + r^2 + s^2)(dx^2 + dy^2 + dz^2 + dt^2) - (pdx + qdy + rdz + sdt)^2$$
$$= ldX^2 + mdY^2 + ndZ^2 \quad\text{..........(13)},$$

where l, m, n are functions of X, Y, Z.

The problem of determining transformations of coordinates which can be applied to the equation of wave motion is thus reduced to that of finding functions p, q, r, s such that the quadratic form on the left-hand side can be expressed in the form given on the right.

§ 4. *The determination of the functions* p, q, r, s.

When equation (7) of § 3 is transformed in the same way as (8) it eventually takes the form

$$(p^2 + q^2 + r^2 + s^2)(dx\delta x + dy\delta y + dz\delta z + dt\delta t)$$
$$- (pdx + qdy + rdz + sdt)(p\delta x + q\delta y + r\delta z + s\delta t)$$
$$= ldX\delta X + mdY\delta Y + ndZ\delta Z \quad\text{..........(14)},$$

corresponding to (13).

We shall now have occasion to use the method of multiplying integral forms described in my paper on the transformation of the electrodynamical equations *.

In this method of multiplication the sign of a product such as $dxdydz$ depends upon the order of the terms so that $dxdydz = -dydxdz$. The product of two terms containing the same constituent, e.g. $Adxdy$ and $Bdxdz$, is consequently zero. On this account the product of a number of integral forms usually takes a very simple form.

* *Proceedings of the London Mathematical Society.*

Applying this method of multiplication to the integral forms occurring in equation (14) and cubing the equation we obtain

$$(p^2+q^2+r^2+s^2)^2\,[p\,dy\,dz\,dt+q\,dz\,dx\,dt+r\,dx\,dy\,dt-s\,dx\,dy\,dz]$$
$$[p\,\delta y\,\delta z\,\delta t+q\,\delta z\,\delta x\,\delta t+r\,\delta x\,\delta y\,\delta t-s\,\delta x\,\delta y\,\delta z]$$
$$=lmn\,dX\,dY\,dZ\,\delta X\,\delta Y\,\delta Z.$$

Putting $dx=\delta x$, etc. and extracting the square root we have

$$P\,dy\,dz\,dt+Q\,dz\,dx\,dt+R\,dx\,dy\,dt-S\,dx\,dy\,dz$$
$$=\sqrt{lmn}\,dX\,dY\,dZ \qquad \text{(15)},$$

where $$P=p\,(p^2+q^2+r^2+s^2), \qquad R=r\,(p^2+q^2+r^2+s^2),$$
$$Q=q\,(p^2+q^2+r^2+s^2), \qquad S=s\,(p^2+q^2+r^2+s^2).$$

Again, $$\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz+\frac{\partial}{\partial t}dt=\frac{\partial}{\partial X}dX+\frac{\partial}{\partial Y}dY+\frac{\partial}{\partial Z}dZ+\frac{\partial}{\partial T}dT,$$

and it is known that the law of multiplication holds also for integral forms of this type. Now $\sqrt{lmn}$ is independent of T, therefore

$$\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}+\frac{\partial S}{\partial t}\right)dx\,dy\,dz\,dt=-\frac{\partial}{\partial T}(\sqrt{lmn})\,dX\,dY\,dZ\,dT=0.$$

This gives the relation $$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}+\frac{\partial S}{\partial t}=0 \qquad \text{(16)}.$$

This is a condition which must be satisfied in order that the bilinear integral form on the left of (14) may be expressible in the form given on the right where l, m, n are independent of T.

Next let $F(X, Y, Z)$ be any function of X, Y, Z; then

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz+\frac{\partial F}{\partial t}dt=\frac{\partial F}{\partial X}dX+\frac{\partial F}{\partial Y}dY+\frac{\partial F}{\partial Z}dZ \qquad \text{(17)}.$$

Multiplying this equation by

$$P\,dy\,dz\,dt+Q\,dz\,dx\,dt+R\,dx\,dy\,dt-S\,dx\,dy\,dz=\sqrt{lmn}\,dX\,dY\,dZ,$$

according to the given rule, we obtain the relation

$$P\frac{\partial F}{\partial x}+Q\frac{\partial F}{\partial y}+R\frac{\partial F}{\partial z}+S\frac{\partial F}{\partial t}=0,$$

or $$p\frac{\partial F}{\partial x}+q\frac{\partial F}{\partial y}+r\frac{\partial F}{\partial z}+s\frac{\partial F}{\partial t}=0 \qquad \text{(18)}.$$

Again, on multiplying (17) by the square of (14) we get

$$\left[\left\{(p^2+s^2)\frac{\partial F}{\partial t}+rs\frac{\partial F}{\partial z}+qs\frac{\partial F}{\partial y}\right\}\delta y\,\delta z+\left(pq\frac{\partial F}{\partial t}-ps\frac{\partial F}{\partial y}\right)\delta z\,\delta x\right.$$
$$+\left(pr\frac{\partial F}{\partial t}-ps\frac{\partial F}{\partial z}\right)\delta x\,\delta y+\left(pq\frac{\partial F}{\partial z}-pr\frac{\partial F}{\partial y}\right)\delta x\,\delta t$$
$$\left.+\left\{-(r^2+p^2)\frac{\partial F}{\partial z}-rs\frac{\partial F}{\partial t}-qr\frac{\partial F}{\partial y}\right\}\delta y\,\delta t+\left\{qs\frac{\partial F}{\partial t}+qr\frac{\partial F}{\partial z}+(p^2+q^2)\frac{\partial F}{\partial z}\right\}\delta z\,\delta t\right]$$
$$[p^2+q^2+r^2+s^2]\,dy\,dz\,dt+\ldots=\left[mn\frac{\partial F}{\partial X}\delta Y\,\delta Z+nl\frac{\partial F}{\partial Y}\delta Z\,\delta X\right.$$
$$\left.+lm\frac{\partial F}{\partial Z}\delta X\,\delta Y\right]dX\,dY\,dZ \qquad \text{(19)}.$$

Simplifying this by means of equation (18), we obtain

$$\left[\left(p\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial x}\right)\delta y\,\delta z+\left(q\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial y}\right)\delta z\,\delta x+\left(r\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial z}\right)\delta x\,\delta y\right.$$
$$\left.+\left(q\frac{\partial F}{\partial z}-r\frac{\partial F}{\partial y}\right)\delta x\,\delta t+\left(r\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial z}\right)\delta y\,\delta t+\left(p\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial x}\right)\delta z\,\delta t\right]$$
$$[p\,dy\,dz\,dt+q\,dz\,dx\,dt+r\,dx\,dy\,dt-s\,dx\,dy\,dz]\,[p^2+q^2+r^2+s^2]$$
$$=\left[(mn)\frac{\partial F}{\partial X}\,\delta Y\delta Z+(nl)\frac{\partial F}{\partial Y}\,\delta Z\delta X+(lm)\frac{\partial F}{\partial Z}\,\delta X\delta Y\right]dX\,dY\,dZ.$$

Rejecting the factor $\sqrt{lmn}\,dX\,dY\,dZ$ and its equivalent the equation becomes

$$\left(p\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial x}\right)\delta y\,\delta z+\left(q\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial y}\right)\delta z\,\delta x+\left(r\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial z}\right)\delta x\,\delta y$$
$$+\left(q\frac{\partial F}{\partial z}-r\frac{\partial F}{\partial y}\right)\delta x\,\delta t+\left(r\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial z}\right)\delta y\,\delta t+\left(p\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial x}\right)\delta z\,\delta t$$
$$=\left(\sqrt{\frac{mn}{l}}\right)\frac{\partial F}{\partial X}\,\delta Y\delta Z+\left(\sqrt{\frac{nl}{m}}\right)\frac{\partial F}{\partial Y}\,\delta Z\delta X+\left(\sqrt{\frac{lm}{n}}\right)\frac{\partial F}{\partial Z}\,\delta X\,\delta Y \quad \text{......(20).}$$

Let this equation be multiplied by

$$\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz+\frac{\partial}{\partial t}dt=\frac{\partial}{\partial X}dX+\frac{\partial}{\partial Y}dY+\frac{\partial}{\partial Z}dZ+\frac{\partial}{\partial T}dT,$$

according to the rule explained above, then

$$\left[\frac{\partial}{\partial t}\left(p\frac{\partial F}{\partial t}-s\frac{\partial F}{\partial x}\right)+\frac{\partial}{\partial y}\left(p\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial x}\right)-\frac{\partial}{\partial z}\left(r\frac{\partial F}{\partial x}-p\frac{\partial F}{\partial z}\right)\right]\delta y\,\delta z\,\delta t+\ldots\ldots$$
$$=\left[\frac{\partial}{\partial X}\left(\sqrt{\frac{mn}{l}}\frac{\partial F}{\partial X}\right)+\frac{\partial}{\partial Y}\left(\sqrt{\frac{ln}{m}}\frac{\partial F}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\sqrt{\frac{lm}{n}}\frac{\partial F}{\partial Z}\right)\right]\delta X\,\delta Y\delta Z$$
$$+\frac{\partial F}{\partial X}\frac{\partial}{\partial T}\left(\sqrt{\frac{mn}{l}}\right)\delta Y\delta Z\delta T+\frac{\partial F}{\partial Y}\frac{\partial}{\partial T}\left(\sqrt{\frac{nl}{m}}\right)\delta Z\delta X\,\delta T$$
$$+\frac{\partial F}{\partial Z}\frac{\partial}{\partial T}\left(\sqrt{\frac{lm}{n}}\right)\delta X\,\delta Y\delta T \quad \text{......................................(21).}$$

If l, m, n are independent of T the coefficients of $\delta Y\delta Z\delta T$, $\delta Z\delta X\,\delta T$ and $\delta X\,\delta Y\delta T$ will disappear. *Hence the necessary and sufficient condition that l, m, n should be independent of T is that the integral form on the left-hand side should reduce to a multiple of $\delta X\,\delta Y\delta Z$, i.e. to a multiple of*

$$p\,\delta y\,\delta z\,\delta t+q\,\delta z\,\delta x\,\delta t+r\,\delta x\,\delta y\,\delta t-s\,\delta x\,\delta y\,\delta z.$$

Now the coefficient of $\delta y\,\delta z\,\delta t$ on the left-hand side is equal to

$$p\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 F}{\partial t^2}\right)-\left(p\frac{\partial^2 F}{\partial x^2}+q\frac{\partial^2 F}{\partial x\partial y}+r\frac{\partial^2 F}{\partial x\partial z}+s\frac{\partial^2 F}{\partial x\partial t}\right)$$
$$+\left(\frac{\partial p}{\partial x}\frac{\partial F}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial F}{\partial y}+\frac{\partial p}{\partial z}\frac{\partial F}{\partial z}+\frac{\partial p}{\partial t}\frac{\partial F}{\partial t}\right)-\frac{\partial F}{\partial x}\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right).$$

Since
$$p\frac{\partial F}{\partial x}+q\frac{\partial F}{\partial y}+r\frac{\partial F}{\partial z}+s\frac{\partial F}{\partial t}=0,$$

the second bracket may be replaced by

$$\frac{\partial p}{\partial x}\frac{\partial F}{\partial x}+\frac{\partial q}{\partial x}\frac{\partial F}{\partial y}+\frac{\partial r}{\partial x}\frac{\partial F}{\partial z}+\frac{\partial s}{\partial x}\frac{\partial F}{\partial t}.$$

We now require the conditions that the expression on the left-hand side of (21) may contain

$$p\,dy\,dz\,dt + q\,dz\,dx\,dt + r\,dx\,dy\,dt - s\,dx\,dy\,dz$$

as a factor. Let the other factor be

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + \frac{\partial^2 F}{\partial t^2} + \xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial z} + \tau\frac{\partial F}{\partial t} \quad\ldots\ldots\ldots\ldots(22),$$

then we must have

$$\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} - \frac{\partial r}{\partial z} - \frac{\partial s}{\partial t} = \xi p + \lambda p,$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = \eta p + \lambda q,$$

$$\frac{\partial p}{\partial z} + \frac{\partial r}{\partial x} = \zeta p + \lambda r,$$

$$\frac{\partial p}{\partial t} + \frac{\partial s}{\partial x} = \tau p + \lambda s;$$

the additional terms λp, λq, λr, λs being introduced because an arbitrary multiple of

$$p\frac{\partial F}{\partial x} + q\frac{\partial F}{\partial y} + r\frac{\partial F}{\partial z} + s\frac{\partial F}{\partial t}$$

can be added to the coefficient of $\delta y\,\delta z\,\delta t$.

The term $\delta z\,\delta x\,\delta t$ gives a similar set of equations

$$\frac{\partial q}{\partial y} - \frac{\partial r}{\partial z} - \frac{\partial s}{\partial t} - \frac{\partial p}{\partial x} = \eta q + \mu q,$$

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = \xi q + \mu p,$$

$$\frac{\partial q}{\partial z} + \frac{\partial r}{\partial y} = \zeta q + \mu r,$$

$$\frac{\partial q}{\partial t} + \frac{\partial s}{\partial y} = \tau q + \mu s,$$

and similarly for $\delta x\,\delta y\,\delta t$, $\delta x\,\delta y\,\delta z$.

Comparing the two expressions for $\dfrac{\partial p}{\partial y} + \dfrac{\partial q}{\partial x}$, we see that the four systems of equations are consistent with one another if

$$\lambda = \xi, \quad \mu = \eta, \quad \nu = \zeta, \quad \rho = \tau.$$

We thus obtain the system of equations

$$\left.\begin{array}{ll}
\dfrac{\partial q}{\partial z} + \dfrac{\partial r}{\partial y} = \zeta q + \eta r, & \dfrac{\partial s}{\partial x} + \dfrac{\partial p}{\partial t} = \tau p + \xi s \\
\dfrac{\partial r}{\partial x} + \dfrac{\partial p}{\partial z} = \xi r + \zeta p, & \dfrac{\partial s}{\partial y} + \dfrac{\partial q}{\partial t} = \tau q + \eta s \\
\dfrac{\partial p}{\partial y} + \dfrac{\partial q}{\partial x} = \eta p + \xi q, & \dfrac{\partial s}{\partial z} + \dfrac{\partial r}{\partial t} = \tau r + \zeta s \\
\multicolumn{2}{c}{\dfrac{\partial p}{\partial x} - \dfrac{\partial q}{\partial y} - \dfrac{\partial r}{\partial z} - \dfrac{\partial s}{\partial t} = 2\xi p} \\
\multicolumn{2}{c}{\dfrac{\partial q}{\partial y} - \dfrac{\partial r}{\partial z} - \dfrac{\partial s}{\partial t} - \dfrac{\partial p}{\partial x} = 2\eta q} \\
\multicolumn{2}{c}{\dfrac{\partial r}{\partial z} - \dfrac{\partial s}{\partial t} - \dfrac{\partial p}{\partial x} - \dfrac{\partial q}{\partial y} = 2\zeta r} \\
\multicolumn{2}{c}{\dfrac{\partial s}{\partial t} - \dfrac{\partial p}{\partial x} - \dfrac{\partial q}{\partial y} - \dfrac{\partial r}{\partial z} = 2\tau s}
\end{array}\right\} \quad\ldots\ldots\ldots\ldots(\mathrm{I}).$$

The expression

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 F}{\partial t^2}+\xi\frac{\partial F}{\partial x}+\eta\frac{\partial F}{\partial y}+\zeta\frac{\partial F}{\partial z}+\tau\frac{\partial F}{\partial t}$$

can be reduced to the form

$$\kappa\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}\right)$$

if

$$\xi=\frac{1}{\theta}\frac{\partial\theta}{\partial x}+\frac{\alpha p}{\theta},\qquad \zeta=\frac{1}{\theta}\frac{\partial\theta}{\partial z}+\frac{\alpha r}{\theta},$$

$$\eta=\frac{1}{\theta}\frac{\partial\theta}{\partial y}+\frac{\alpha q}{\theta},\qquad \tau=\frac{1}{\theta}\frac{\partial\theta}{\partial t}+\frac{\alpha s}{\theta},$$

$$V=\sqrt{\theta}\,F,\qquad \kappa=\sqrt{\theta},$$

where α is some function of x, y, z, t. The terms involving α in (22) disappear on account of the relation

$$p\frac{\partial F}{\partial x}+q\frac{\partial F}{\partial y}+r\frac{\partial F}{\partial z}+s\frac{\partial F}{\partial t}=0.$$

Making these substitutions the equations take the form

$$\left.\begin{array}{ll}
\dfrac{\partial}{\partial z}\left(\dfrac{q}{\theta}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{r}{\theta}\right)=2\alpha\dfrac{q}{\theta}\cdot\dfrac{r}{\theta}, & \dfrac{\partial}{\partial x}\left(\dfrac{s}{\theta}\right)+\dfrac{\partial}{\partial t}\left(\dfrac{p}{\theta}\right)=2\alpha\dfrac{s}{\theta}\cdot\dfrac{p}{\theta}\\[2ex]
\dfrac{\partial}{\partial x}\left(\dfrac{r}{\theta}\right)+\dfrac{\partial}{\partial z}\left(\dfrac{p}{\theta}\right)=2\alpha\dfrac{r}{\theta}\cdot\dfrac{p}{\theta}, & \dfrac{\partial}{\partial y}\left(\dfrac{s}{\theta}\right)+\dfrac{\partial}{\partial t}\left(\dfrac{q}{\theta}\right)=2\alpha\dfrac{s}{\theta}\cdot\dfrac{q}{\theta}\\[2ex]
\dfrac{\partial}{\partial y}\left(\dfrac{p}{\theta}\right)+\dfrac{\partial}{\partial x}\left(\dfrac{q}{\theta}\right)=2\alpha\dfrac{p}{\theta}\cdot\dfrac{q}{\theta}, & \dfrac{\partial}{\partial z}\left(\dfrac{s}{\theta}\right)+\dfrac{\partial}{\partial t}\left(\dfrac{r}{\theta}\right)=2\alpha\dfrac{s}{\theta}\cdot\dfrac{r}{\theta}\\[2ex]
\multicolumn{2}{c}{2\dfrac{\partial}{\partial x}\left(\dfrac{p}{\theta}\right)=2\alpha\left(\dfrac{p}{\theta}\right)^2+\dfrac{1}{\theta}\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial q}{\partial y}+\dfrac{\partial r}{\partial z}+\dfrac{\partial s}{\partial t}\right]}\\[2ex]
\multicolumn{2}{c}{2\dfrac{\partial}{\partial y}\left(\dfrac{q}{\theta}\right)=2\alpha\left(\dfrac{q}{\theta}\right)^2+\dfrac{1}{\theta}\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial q}{\partial y}+\dfrac{\partial r}{\partial z}+\dfrac{\partial s}{\partial t}\right]}\\[2ex]
\multicolumn{2}{c}{2\dfrac{\partial}{\partial z}\left(\dfrac{r}{\theta}\right)=2\alpha\left(\dfrac{r}{\theta}\right)^2+\dfrac{1}{\theta}\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial q}{\partial y}+\dfrac{\partial r}{\partial z}+\dfrac{\partial s}{\partial t}\right]}\\[2ex]
\multicolumn{2}{c}{2\dfrac{\partial}{\partial t}\left(\dfrac{s}{\theta}\right)=2\alpha\left(\dfrac{s}{\theta}\right)^2+\dfrac{1}{\theta}\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial q}{\partial y}+\dfrac{\partial r}{\partial z}+\dfrac{\partial s}{\partial t}\right]}
\end{array}\right\}\ldots\ldots\ldots\ldots\text{(II)}.$$

Adding the last four equations we obtain

$$\frac{\partial}{\partial x}(p\theta)+\frac{\partial}{\partial y}(q\theta)+\frac{\partial}{\partial z}(r\theta)+\frac{\partial}{\partial t}(s\theta)+\alpha(p^2+q^2+r^2+s^2)=0\ \ldots\ldots\ldots\ldots\text{(23)}.$$

If we put

$$\alpha(p^2+q^2+r^2+s^2)=\frac{\theta}{\phi}\left[p\frac{\partial\phi}{\partial x}+q\frac{\partial\phi}{\partial y}+r\frac{\partial\phi}{\partial z}+s\frac{\partial\phi}{\partial t}\right],$$

where ϕ is some function of x, y, z, t, equation (23) may be written

$$\frac{\partial}{\partial x}(p\theta\phi)+\frac{\partial}{\partial y}(q\theta\phi)+\frac{\partial}{\partial z}(r\theta\phi)+\frac{\partial}{\partial t}(s\theta\phi)=0,$$

and this is satisfied by taking

$$\theta\phi=(p^2+q^2+r^2+s^2)f(X, Y, Z),$$

for we have

$$\frac{\partial}{\partial x}[p(p^2+q^2+r^2+s^2)]+\frac{\partial}{\partial y}[q(p^2+q^2+r^2+s^2)]$$
$$+\frac{\partial}{\partial z}[r(p^2+q^2+r^2+s^2)]+\frac{\partial}{\partial t}[s(p^2+q^2+r^2+s^2)]=0,$$

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}+r\frac{\partial f}{\partial z}+s\frac{\partial f}{\partial t}=0.$$

When the equation (21) has been simplified by means of the identity

$$(pdydzdt+qdzdxdt+rdxdydt-sdxdydz)(p^2+q^2+r^2+s^2)=\sqrt{lmn}\,dXdYdZ,$$

it takes the form

$$\left[\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 F}{\partial t^2}+\xi\frac{\partial F}{\partial x}+\eta\frac{\partial F}{\partial y}+\zeta\frac{\partial F}{\partial z}+\tau\frac{\partial F}{\partial t}\right]\frac{1}{p^2+q^2+r^2+s^2}$$
$$=\frac{1}{\sqrt{lmn}}\left[\frac{\partial}{\partial X}\left(\sqrt{\frac{mn}{l}}\frac{\partial F}{\partial X}\right)+\frac{\partial}{\partial Y}\left(\sqrt{\frac{nl}{m}}\frac{\partial F}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\sqrt{\frac{lm}{n}}\frac{\partial F}{\partial Z}\right)\right]\ldots(24),$$

or

$$\frac{1}{(p^2+q^2+r^2+s^2)\sqrt{\theta}}\left[\frac{\partial^2}{\partial x^2}(F\sqrt{\theta})+\frac{\partial^2}{\partial y^2}(F\sqrt{\theta})+\frac{\partial^2}{\partial z^2}(F\sqrt{\theta})+\frac{\partial^2}{\partial t^2}(F\sqrt{\theta})\right]$$
$$=\frac{1}{\sqrt{lmn}}\left[\frac{\partial}{\partial X}\left(\sqrt{\frac{mn}{l}}\frac{\partial F}{\partial X}\right)+\frac{\partial}{\partial Y}\left(\sqrt{\frac{nl}{m}}\frac{\partial F}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\sqrt{\frac{lm}{n}}\frac{\partial F}{\partial Z}\right)\right]\ldots(25).$$

If by a transformation of the X, Y, Z coordinates the quadratic differential form

$$l\,dX^2+m\,dY^2+n\,dZ^2$$

can be reduced to the form

$$dX_1^2+dY_1^2+dZ_1^2,$$

the expression on the right-hand side of (25) reduces to

$$\frac{\partial^2 F}{\partial X_1^2}+\frac{\partial^2 F}{\partial Y_1^2}+\frac{\partial^2 F}{\partial Z_1^2}\equiv\nabla_1^2 F,$$

and we have the result that if $F(X_1, Y_1, Z_1)$ is a solution of this equation $\nabla_1^2 F=0$ the function

$$V=\sqrt{\theta}\,.\,F$$

is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

It should be remarked that the system of equations (I) provide us with the necessary and sufficient conditions that the quadratic differential form

$$(dx^2+dy^2+dz^2+dt^2)(p^2+q^2+r^2+s^2)-(pdx+qdy+rdz+sdt)^2$$

can be reduced to the form

$$a\,dX^2+b\,dY^2+c\,dZ^2+2f\,dYdZ+2g\,dZdX+2h\,dXdY,$$

where a, b, c, f, g, h are functions of X, Y, Z alone.

To prove this we take three independent functions X, Y, Z satisfying the equations*

$$p\frac{\partial X}{\partial x}+q\frac{\partial X}{\partial y}+r\frac{\partial X}{\partial z}+s\frac{\partial X}{\partial t}=0,$$
$$p\frac{\partial Y}{\partial x}+q\frac{\partial Y}{\partial y}+r\frac{\partial Y}{\partial z}+s\frac{\partial Y}{\partial t}=0,$$
$$p\frac{\partial Z}{\partial x}+q\frac{\partial Z}{\partial y}+r\frac{\partial Z}{\partial z}+s\frac{\partial Z}{\partial t}=0,$$

and a fourth function T which is independent of X, Y, Z. Expressing x, y, z, t in terms of the independent variables X, Y, Z, T; we may write

$$\begin{aligned}(dx^2+dy^2+dz^2+dt^2)(p^2+q^2+r^2+s^2)-(p\,dx+q\,dy+r\,dz+s\,dt)^2 \\ =a\,dX^2+b\,dY^2+c\,dZ^2+2f\,dY\,dZ+2g\,dZ\,dX+2h\,dX\,dY \\ +2u\,dX\,dT+2v\,dY\,dT+2w\,dZ\,dT+\theta\,dT^2,\end{aligned}$$

and polarising this equation we obtain

$$\begin{aligned}(dx\,\delta x+dy\,\delta y+dz\,\delta z+dt\,\delta t)(p^2+q^2+r^2+s^2) \\ -(p\,dx+q\,dy+r\,dz+s\,dt)(p\,\delta x+q\,\delta y+r\,\delta z+s\,\delta t) \\ =a\,dX\,\delta X+b\,dY\,\delta Y+c\,dZ\,\delta Z+f(dY\,\delta Z+dZ\,\delta Y) \\ +g(dZ\,\delta X+dX\,\delta Z)+h(dX\,\delta Y+dY\,\delta X) \\ +u(dX\,\delta T+\delta X\,dT)+v(dY\,\delta T+\delta Y\,dT) \\ +w(dZ\,\delta T+\delta Z\,dT)+\theta\,dT\,\delta T \quad \ldots\ldots(26).\end{aligned}$$

Cubing this equation according to our rule we obtain

$$\begin{aligned}(p\,dy\,dz\,dt+q\,dz\,dx\,dt+r\,dx\,dy\,dt-s\,dx\,dy\,dz) \\ (p\,\delta y\,\delta z\,\delta t+q\,\delta z\,\delta x\,\delta t+r\,\delta x\,\delta y\,\delta t-s\,\delta x\,\delta y\,\delta z)(p^2+q^2+r^2+s^2)^2 \\ =A\,dY\,dZ\,dT\,\delta Y\,\delta Z\,\delta T+B\,dZ\,dX\,dT\,\delta Z\,\delta X\,\delta T+\ldots\ldots \\ +F[\delta Z\,\delta X\,\delta T\,dX\,dY\,dT+dZ\,dX\,dT\,\delta X\,\delta Y\,\delta T]+\ldots\ldots \quad \ldots(27),\end{aligned}$$

where A, B, C, F, G, H, U, V, W, Θ are the minors of a, b, c, f, g, h, u, v, w, θ in the determinant

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & \theta \end{vmatrix}=\Delta.$$

Now since the quadratic form can be expressed as the sum of three squares $\Delta=0$ and we have

$$\left.\begin{aligned}BC=F^2,\quad CA=G^2,\quad AB=H^2 \\ A\Theta=U^2,\quad B\Theta=V^2,\quad C\Theta=W^2\end{aligned}\right\} \quad \ldots\ldots(28).$$

The bicubic integral form on the right-hand side of (27) may therefore be written

$$[\sqrt{A}\,dY\,dZ\,dT+\sqrt{B}\,dZ\,dX\,dT+\sqrt{C}\,dX\,dY\,dT-\sqrt{\Theta}\,dX\,dY\,dZ]$$
$$[\sqrt{A}\,\delta Y\,\delta Z\,\delta T+\sqrt{B}\,\delta Z\,\delta X\,\delta T+\sqrt{C}\,\delta X\,\delta Y\,\delta T-\sqrt{\Theta}\,\delta X\,\delta Y\,\delta Z].$$

* This is possible on account of a fundamental theorem in partial differential equations which states that a complete system of m linear equations in $m+n$ variables possesses n distinct solutions; see Forsyth's *Theory of Differential Equations*, Part IV. Vol. 5, pp. 53—83.

Putting $dx = \delta x$, etc. in (27) and extracting the square root we get

$$[p\,dy\,dz\,dt + q\,dz\,dx\,dt + r\,dx\,dy\,dt - s\,dx\,dy\,dz]\,(p^2+q^2+r^2+s^2)$$
$$= \sqrt{A}\,dY\,dZ\,dT + \sqrt{B}\,dZ\,dX\,dT + \sqrt{C}\,dX\,dY\,dT - \sqrt{\Theta}\,dX\,dY\,dZ.$$

This equation gives

$$\sqrt{A} = \left[p\frac{\partial(y,z,t)}{\partial(Y,Z,T)} + q\frac{\partial(z,x,t)}{\partial(Y,Z,T)} + r\frac{\partial(x,y,t)}{\partial(Y,Z,T)} - s\frac{\partial(x,y,z)}{\partial(Y,Z,T)}\right](p^2+q^2+r^2+s^2),$$

$$\sqrt{B} = \left[p\frac{\partial(y,z,t)}{\partial(Z,X,T)} + q\frac{\partial(z,x,t)}{\partial(Z,X,T)} + r\frac{\partial(x,y,t)}{\partial(Z,X,T)} - s\frac{\partial(x,y,z)}{\partial(Z,X,T)}\right](p^2+q^2+r^2+s^2),$$

$$\sqrt{C} = \left[p\frac{\partial(y,z,t)}{\partial(X,Y,T)} + q\frac{\partial(z,x,t)}{\partial(X,Y,T)} + r\frac{\partial(x,y,t)}{\partial(X,Y,T)} - s\frac{\partial(x,y,z)}{\partial(X,Y,T)}\right](p^2+q^2+r^2+s^2).$$

Now
$$\frac{\partial X}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial Y} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial Y} + \frac{\partial X}{\partial t}\frac{\partial t}{\partial Y} = 0,$$

$$\frac{\partial X}{\partial x}\frac{\partial x}{\partial Z} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial Z} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial Z} + \frac{\partial X}{\partial t}\frac{\partial t}{\partial Z} = 0,$$

$$\frac{\partial X}{\partial x}\frac{\partial x}{\partial T} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial T} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial T} + \frac{\partial X}{\partial t}\frac{\partial t}{\partial T} = 0,$$

therefore
$$\frac{\dfrac{\partial(y,z,t)}{\partial(Y,Z,T)}}{\dfrac{\partial X}{\partial x}} = \frac{\dfrac{\partial(z,x,t)}{\partial(Y,Z,T)}}{\dfrac{\partial X}{\partial y}} = \frac{\dfrac{\partial(x,y,t)}{\partial(Y,Z,T)}}{\dfrac{\partial X}{\partial z}} = \frac{-\dfrac{\partial(x,y,z)}{\partial(Y,Z,T)}}{\dfrac{\partial X}{\partial t}} = \mu, \text{ say.}$$

This gives
$$\sqrt{A} = \mu\,(p^2+q^2+r^2+s^2)\left[p\frac{\partial X}{\partial x} + q\frac{\partial X}{\partial y} + r\frac{\partial X}{\partial z} + s\frac{\partial X}{\partial t}\right] = 0.$$

Similarly $B = C = 0$ and equations (28) then give

$$F = G = H = U = V = W = A = B = C = 0.$$

Now on referring to the values of these minors in terms of $a, b, c, f, g, h, u, v, w, \theta$ it is easy to see that the above equations can only be satisfied if either

$$u = v = w = \theta = 0,$$

or
$$a = b = c = f = g = h = 0.$$

The second alternative may be rejected for it would imply that the quadratic form could be reduced to the sum of two squares. We may therefore put $u = v = w = \theta = 0$, hence

$$(p^2+q^2+r^2+s^2)(dx^2+dy^2+dz^2+dt^2) - (p\,dx + q\,dy + r\,dz + s\,dt)^2$$
$$= a\,dX^2 + b\,dY^2 + c\,dZ^2 + 2f\,dY\,dZ + 2g\,dZ\,dX + 2h\,dX\,dY \quad \text{.........(29).}$$

To prove that if p, q, r, s satisfy equations (I) the quantities a, b, c, f, g, h are independent of T we have only to repeat the process of pp. 265—266. The expression on the right-hand side of (20) however is now replaced by

$$\delta Y \delta Z \left[(bc - f^2)\frac{\partial F}{\partial X} + (fg - ch)\frac{\partial F}{\partial Y} + (hf - bg)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}}$$
$$+ \delta Z \delta X \left[(fg - ch)\frac{\partial F}{\partial X} + (ca - g^2)\frac{\partial F}{\partial Y} + (gh - af)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}}$$
$$+ \delta X \delta Y \left[(hf - bg)\frac{\partial F}{\partial X} + (gh - af)\frac{\partial F}{\partial Y} + (ab - h^2)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}},$$

where

$$M = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

and when we multiply by

$$\frac{\partial}{\partial X}\delta X + \frac{\partial}{\partial Y}\delta Y + \frac{\partial}{\partial Z}\delta Z + \frac{\partial}{\partial T}\delta T$$

the coefficients of $\delta Y \delta Z \delta T$, $\delta Z \delta X \delta T$, $\delta X \delta Y \delta T$ will vanish if

$$\left[(bc - f^2)\frac{\partial F}{\partial X} + (fg - ch)\frac{\partial F}{\partial Y} + (hf - bg)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}},$$
$$\left[(fg - ch)\frac{\partial F}{\partial X} + (ca - g^2)\frac{\partial F}{\partial Y} + (gh - af)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}},$$

and
$$\left[(hf - bg)\frac{\partial F}{\partial X} + (gh - af)\frac{\partial F}{\partial Y} + (ab - h^2)\frac{\partial F}{\partial Z}\right]\frac{1}{\sqrt{M}},$$

are independent of T.

Now these coefficients are known to vanish if the equations (I) are satisfied, for then the integral form on the left-hand side of (21) reduces to a multiple of

$$p\,\delta y\,\delta z\,\delta t + q\,\delta z\,\delta x\,\delta t + r\,\delta x\,\delta y\,\delta t - s\,\delta x\,\delta y\,\delta z,$$

i.e. to a multiple of $\delta X \delta Y \delta Z$. Moreover the function F is arbitrary, hence if equations (I) are satisfied

$$\frac{bc - f^2}{\sqrt{M}},\quad \frac{ca - g^2}{\sqrt{M}},\quad \frac{ab - h^2}{\sqrt{M}},\quad \frac{gh - af}{\sqrt{M}},\quad \frac{hf - bg}{\sqrt{M}},\quad \frac{fg - ch}{\sqrt{M}}$$

are independent of T. Since

$$\frac{(ca - g^2)(ab - h^2) - (gh - af)^2}{M} = a,$$

it follows that a, b, c, f, g, h are also independent of T. This establishes the theorem.

It can be shown that equation (16) is a consequence of the system of equations (I), for if we multiply the equations by $2qr$, $2rp$, $2pq$, $2ps$, $2qs$, $2rs$, p^2, q^2, r^2, s^2 respectively and add we obtain

$$p\frac{\partial}{\partial x}(p^2 + q^2 + r^2 + s^2) + q\frac{\partial}{\partial y}(p^2 + q^2 + r^2 + s^2) + r\frac{\partial}{\partial z}(p^2 + q^2 + r^2 + s^2)$$
$$+ s\frac{\partial}{\partial t}(p^2 + q^2 + r^2 + s^2) = 2(p^2 + q^2 + r^2 + s^2)(p\xi + q\eta + r\zeta + s\tau)$$
$$+ (p^2 + q^2 + r^2 + s^2)\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z} + \frac{\partial s}{\partial t}\right);$$

while on adding the last four equations we obtain

$$\xi p+\eta q+\zeta r+\tau s=-\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right).$$

These two equations combined give

$$\frac{\partial}{\partial x}[p(p^2+q^2+r^2+s^2)]+\frac{\partial}{\partial y}[q(p^2+q^2+r^2+s^2)] + \frac{\partial}{\partial z}[r(p^2+q^2+r^2+s^2)]+\frac{\partial}{\partial t}[s(p^2+q^2+r^2+s^2)]=0,$$

which is equation (16).

§ 5. *Particular solutions of the system of differential equations.*

It is easy to construct particular solutions of the system of equations

$$\frac{\partial}{\partial z}\left(\frac{q}{\theta}\right)+\frac{\partial}{\partial y}\left(\frac{r}{\theta}\right)=2\alpha\frac{q}{\theta}\cdot\frac{r}{\theta}, \qquad \frac{\partial}{\partial x}\left(\frac{s}{\theta}\right)+\frac{\partial}{\partial t}\left(\frac{p}{\theta}\right)=2\alpha\frac{s}{\theta}\cdot\frac{p}{\theta},$$

$$\frac{\partial}{\partial x}\left(\frac{r}{\theta}\right)+\frac{\partial}{\partial z}\left(\frac{p}{\theta}\right)=2\alpha\frac{r}{\theta}\cdot\frac{p}{\theta}, \qquad \frac{\partial}{\partial y}\left(\frac{s}{\theta}\right)+\frac{\partial}{\partial t}\left(\frac{q}{\theta}\right)=2\alpha\frac{s}{\theta}\cdot\frac{q}{\theta},$$

$$\frac{\partial}{\partial y}\left(\frac{p}{\theta}\right)+\frac{\partial}{\partial x}\left(\frac{q}{\theta}\right)=2\alpha\frac{p}{\theta}\cdot\frac{q}{\theta}, \qquad \frac{\partial}{\partial z}\left(\frac{s}{\theta}\right)+\frac{\partial}{\partial t}\left(\frac{r}{\theta}\right)=2\alpha\frac{s}{\theta}\cdot\frac{r}{\theta},$$

$$2\frac{\partial}{\partial x}\left(\frac{p}{\theta}\right)=2\alpha\left(\frac{p}{\theta}\right)^2+\frac{1}{\theta}\left[\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right],$$

$$2\frac{\partial}{\partial y}\left(\frac{q}{\theta}\right)=2\alpha\left(\frac{q}{\theta}\right)^2+\frac{1}{\theta}\left[\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right],$$

$$2\frac{\partial}{\partial z}\left(\frac{r}{\theta}\right)=2\alpha\left(\frac{r}{\theta}\right)^2+\frac{1}{\theta}\left[\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right],$$

$$2\frac{\partial}{\partial t}\left(\frac{s}{\theta}\right)=2\alpha\left(\frac{s}{\theta}\right)^2+\frac{1}{\theta}\left[\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}+\frac{\partial r}{\partial z}+\frac{\partial s}{\partial t}\right].$$

For instance, we may take

$$p=\theta x f(x^2+y^2+z^2+t^2)=\theta x f(R^2), \quad q=\theta y f(R^2), \quad r=\theta z f(R^2), \quad s=\theta t f(R^2).$$

The equations are then satisfied if

$$\alpha[f(R^2)]^2=2f'(R^2),$$

$$2f(R^2)=\frac{1}{\theta}\left[4\theta f(R^2)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+t\frac{\partial}{\partial t}\right)\theta f(R^2)\right],$$

i.e. if $\theta f(R^2)$ is a homogeneous function of degree -2.

In particular, if we take $f(R^2)=\frac{1}{R}$, $\theta=\frac{1}{R+t}$ the conditions are satisfied and $\sqrt{\theta}$ is a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

The problem is then reduced to that of expressing

$$\frac{1}{R^2(R+t)^2}[(x^2+y^2+z^2+t^2)(dx^2+dy^2+dz^2+dt^2)-(x\,dx+y\,dy+z\,dz+t\,dt)^2]$$

in the form $ldX^2+mdY^2+ndZ^2$. This may be solved by taking

$$X=\frac{x}{R+t}, \qquad Y=\frac{y}{R+t}, \qquad Z=\frac{z}{R+t};$$

the quadratic form then becomes $dX^2+dY^2+dZ^2$.

We have then the result that if $F(X, Y, Z)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2}+\frac{\partial^2 F}{\partial Y^2}+\frac{\partial^2 F}{\partial Z^2}=0,$$

the function
$$V=\frac{1}{\sqrt{R+t}}F\left(\frac{x}{R+t}, \frac{y}{R+t}, \frac{z}{R+t}\right)$$

is a solution of
$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

Another solution may be obtained from this by inversion, for if we replace

$$x \text{ by } \frac{x}{R^2}, \quad y \text{ by } \frac{y}{R^2}, \quad z \text{ by } \frac{z}{R^2}, \quad t \text{ by } \frac{t}{R^2} \text{ and } V \text{ by } \frac{V}{R^2},$$

we obtain the solution

$$V=\frac{1}{R\sqrt{R+t}}F\left(\frac{x}{R+t}, \frac{y}{R+t}, \frac{z}{R+t}\right).$$

It should be remarked that in general we cannot reduce the quadratic form to the simple form $dX^2+dY^2+dZ^2$. In order that this may be possible a number of additional equations must be satisfied by the functions p, q, r, s.

For instance, if we take

$$\theta=\frac{1}{R^4}, \quad f(R^2)=R^2, \quad X_1=\frac{x}{R}, \quad Y_1=\frac{y}{R}, \quad Z_1=\frac{z}{R},$$

we have

$$\frac{dx^2+dy^2+dz^2+dt^2-dR^2}{R^2}=dX_1^2+dY_1^2+dZ_1^2-\frac{1}{1-R_1^2}(X_1dX_1+Y_1dY_1+Z_1dZ_1)^2,$$

where
$$R_1^2=X_1^2+Y_1^2+Z_1^2.$$

This quadratic form however cannot be reduced to the form $d\xi^2+d\eta^2+d\zeta^2$; for if this were possible we should have

$$d\xi^2+d\eta^2+d\zeta^2=\left(\frac{R+t}{R}\right)^2[dX^2+dY^2+dZ^2],$$

where X, Y, Z are the functions defined in the previous example.

Now $\left(\dfrac{R+t}{R}\right)^2=\dfrac{4}{(X^2+Y^2+Z^2+1)^2}$ and so we should have

$$d\xi^2+d\eta^2+d\zeta^2=\frac{4}{(X^2+Y^2+Z^2+1)^2}(dX^2+dY^2+dZ^2).$$

This relation, however, would give a conformal transformation of the (ξ, η, ζ) space into the (X, Y, Z) space; but there is no transformation in which the multiplier is of the above form, for if

$$d\xi^2+d\eta^2+d\zeta^2=\lambda(dX^2+dY^2+dZ^2),$$

the equations $\lambda=0$, $\lambda=\infty$ must represent point-spheres or planes touching the circle at infinity.

It is easy to form a partial differential equation which must be satisfied by the function F in order that

$$V=\frac{1}{R^2}F(X_1,\ Y_1,\ Z_1)=\frac{1}{R^2}F\left(\frac{x}{R},\ \frac{y}{R},\ \frac{z}{R}\right)$$

may be a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

Another set of solutions of equations (II) are obtained by putting $\alpha=0$,

$$\frac{p}{\theta}=\quad ax+hy+gz+ut,$$

$$\frac{q}{\theta}=-hx+ay+fz+vt,$$

$$\frac{r}{\theta}=-gx-fy+az+wt,$$

$$\frac{s}{\theta}=-ux-vy-wz+at,$$

$$2a=\frac{1}{\theta}\left[4a\theta+(ax+hy+gz+ut)\frac{\partial\theta}{\partial x}+(-hx+ay+fz+vt)\frac{\partial\theta}{\partial y}\right.$$
$$\left.+(-gx-fy+az+wt)\frac{\partial\theta}{\partial y}+(-ux-vy-wz+at)\frac{\partial\theta}{\partial z}\right].$$

If $a=0$ this equation may be satisfied by taking $\theta=1$. We have then to express

$$[(hy+gz+ut)^2+(-hx+fz+vt)^2+(-gx-fy+wt)^2+(ux+vy+wz)^2]$$
$$[dx^2+dy^2+dz^2+dt^2]-[(hy+gz+ut)\,dx+(-hx+fz+vt)\,dy$$
$$+(-gx-fy+wt)\,dz-(ux+vy+wz)\,dt]^2$$

in terms of three variables. For the sake of simplicity we shall consider the case in which

$$f=u=1, \quad g=h=v=w=0,$$

the quadratic form then becomes

$$(x^2+y^2+z^2+t^2)(dx^2+dy^2+dz^2+dt^2)-(t\,dx+z\,dy-y\,dz-x\,dt)^2.$$

To express this in terms of three variables we must find three solutions of the equation

$$t\frac{\partial X}{\partial x}+z\frac{\partial X}{\partial y}-y\frac{\partial X}{\partial z}-x\frac{\partial X}{\partial t}=0.$$

Now this equation may be satisfied by taking

$$X=y^2+z^2-x^2-t^2, \quad Y=2(xy+zt), \quad Z=2(xz-yt),$$

and we easily find that

$$4(x^2+y^2+z^2+t^2)(dx^2+dy^2+dz^2+dt^2)-4(t\,dx+z\,dy-y\,dz-x\,dt)^2$$
$$=dX^2+dY^2+dZ^2.$$

Hence we have the result that if $F(X,\ Y,\ Z)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2}+\frac{\partial^2 F}{\partial Y^2}+\frac{\partial^2 F}{\partial Z^2}=0,$$

it is also a solution of

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 F}{\partial t^2}=0.$$

If we put

$$y = r\cos\theta\cos\phi, \quad z = r\cos\theta\sin\phi, \quad x = r\sin\theta\cos\psi, \quad t = r\sin\theta\sin\psi,$$

we have
$$X = r^2\cos 2\theta, \quad Y = r^2\sin 2\theta\cos(\phi-\psi), \quad Z = r^2\sin 2\theta\sin(\phi-\psi),$$

and if we change into polar coordinates by putting

$$X = R\cos\Theta, \quad Y = R\sin\Theta\cos\Phi, \quad Z = R\sin\Theta\sin\Phi,$$

we obtain the result that if $f(R, \Theta, \Phi)$ is a solution of Laplace's equation, then $f(r^2, 2\theta, \phi-\psi)$ is a solution of

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial t^2} = 0.$$

Other solutions of our problem may be derived from those already obtained by applying a conformal transformation to the variables x, y, z, t. For instance if we put*

$$x = \frac{x'}{z'-it'}, \quad y = \frac{y'}{z'-it'}, \quad z = \frac{r'^2 - a^2}{2a(z'-it')}, \quad t = \frac{r'^2+a^2}{2ia(z'-it')},$$

where
$$r'^2 = x'^2 + y'^2 + z'^2 + t'^2,$$

and apply the transformation

$$X = \frac{x}{r+t}, \quad Y = \frac{y}{r+t}, \quad Z = \frac{z}{r+t},$$

we obtain

$$X = \frac{iax'}{r'^2+a^2-2a\varpi'}, \quad Y = \frac{iay'}{r'^2+a^2-2a\varpi'}, \quad Z = \frac{i(r'^2-a^2)}{2(r'^2+a^2-2a\varpi')},$$

$$r + t = \frac{r'^2+a^2-2a\varpi'}{2ia(z'-it')},$$

where
$$\varpi'^2 = z'^2 + t'^2.$$

Now we know that if $V(\xi, \eta, \zeta, \tau)$ is a solution of

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial^2 V}{\partial \zeta^2} + \frac{\partial^2 V}{\partial \tau^2} = 0,$$

then
$$W = \frac{1}{z'-it'} V\left[\frac{x'}{z'-it'}, \frac{y'}{z'-it'}, \frac{r'^2-a^2}{2a(z'-it')}, \frac{r'^2+a^2}{2ia(z'-it')}\right]$$

is a solution of
$$\frac{\partial^2 W}{\partial x'^2} + \frac{\partial^2 W}{\partial y'^2} + \frac{\partial^2 W}{\partial z'^2} + \frac{\partial^2 W}{\partial t'^2} = 0.$$

Now
$$V = \frac{1}{\sqrt{r+t}} F\left(\frac{x}{r+t}, \frac{y}{r+t}, \frac{z}{r+t}\right)$$

is a solution of
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial t^2} = 0,$$

if $F(X, Y, Z)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} + \frac{\partial^2 F}{\partial Z^2} = 0.$$

Hence

$$\frac{1}{z'-it'}\sqrt{\frac{z'-it'}{r'^2+a^2-2a\varpi'}}\, F\left(\frac{2iax'}{r'^2+a^2-2a\varpi'}, \frac{2iay'}{r'^2+a^2-2a\varpi'}, \frac{i(r'^2-a^2)}{r'^2+a^2-2a\varpi'}\right)$$

is a solution of
$$\frac{\partial^2 W}{\partial x'^2} + \frac{\partial^2 W}{\partial y'^2} + \frac{\partial^2 W}{\partial z'^2} + \frac{\partial^2 W}{\partial t'^2} = 0.$$

* *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 7, p. 70 (1909).

Dropping the dashes and removing the factor i from each of the arguments of F, we obtain the result that if $F(X, Y, Z)$ is a solution of Laplace's equation

$$\frac{1}{\sqrt{(z-it)(r^2+a^2-2a\varpi)}} F\left(\frac{2ax}{r^2+a^2-2a\varpi}, \frac{2ay}{r^2+a^2-2a\varpi}, \frac{r^2-a^2}{r^2+a^2-2a\varpi}\right)$$

is a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+\frac{\partial^2 V}{\partial t^2}=0.$$

Putting $r^2=\rho^2+\varpi^2$, $z=\varpi\cos\phi$, $t=\varpi\sin\phi$, we obtain the two real solutions

$$\frac{\cos\frac{\phi}{2}}{\sqrt{\varpi}\sqrt{\rho^2+(\varpi-a)^2}} F\left[\frac{2ax}{\rho^2+(\varpi-a)^2}, \frac{2ay}{\rho^2+(\varpi-a)^2}, \frac{\rho^2+\varpi^2-a^2}{[\rho^2+(\varpi-a)^2]}\right],$$

$$\frac{\sin\frac{\phi}{2}}{\sqrt{\varpi}\sqrt{\rho^2+(\varpi-a)^2}} F\left[\frac{2ax}{\rho^2+(\varpi-a)^2}, \frac{2ay}{\rho^2+(\varpi-a)^2}, \frac{\rho^2+\varpi^2-a^2}{[\rho^2+(\varpi-a)^2]}\right].$$

Again, if we put

$$X=\frac{y}{r+x}, \quad Y=\frac{z}{r+x}, \quad Z=\frac{t}{r+x},$$

and perform the same transformation to the x', y', z', t' variables we obtain the solution

$$\frac{1}{\sqrt{(z-it)(x+i\varpi)}} F\left[\frac{y}{x+i\varpi}, \frac{r^2-a^2}{2a(x+i\varpi)}, \frac{r^2+a^2}{2ia(x+i\varpi)}\right].$$

Putting $t=ic\tau$ we obtain the following solution of the equation of wave motion

$$\frac{1}{\sqrt{(z+c\tau)\{x+\sqrt{c^2\tau^2-z^2}\}}} F\left[\frac{2ay}{x+\sqrt{c^2\tau^2-z^2}}, \frac{x^2+y^2+z^2-c^2\tau^2-a^2}{x+\sqrt{c^2\tau^2-z^2}}, \frac{x^2+y^2+z^2-c^2\tau^2+a^2}{ix+i\sqrt{c^2\tau^2-z^2}}\right],$$

$F(X, Y, Z)$ being a solution of Laplace's equation.

In particular, we have the solutions

$$\frac{1}{\sqrt{z+ct}} f[x+\sqrt{c^2\tau^2-z^2}],$$

$$\frac{1}{\sqrt{(z+c\tau)(x^2+y^2+z^2-c^2\tau^2)}} f\left[\frac{x+\sqrt{c^2\tau^2-z^2}}{x^2+y^2+z^2-c^2\tau^2}\right].$$

The first of these depends only on x, z, t, and indicates that if $\varpi=\sqrt{y^2+z^2}$,

$$V=\frac{1}{\sqrt{z\pm iy}} f(x\pm i\varpi)$$

is a solution of

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

§ 6. *The three-dimensional problem.*

The problem of constructing solutions of Laplace's equation having the form $WF(X, Y)$ may be solved in a similar way by considering the quadratic differential form

$$(p^2+q^2+r^2)(dx^2+dy^2+dz^2)-(p\,dx+q\,dy+r\,dz)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If this can be expressed in the form $dX^2 + dY^2$ we must have

$$(p^2+q^2+r^2)(dx\,\delta x + dy\,\delta y + dz\,\delta z) - (p\,dx + q\,dy + r\,dz)(p\,\delta x + q\,\delta y + r\,\delta z) = dX\,\delta X + dY\,\delta Y \quad\quad (2).$$

Squaring this we obtain

$$(p^2+q^2+r^2)(p\,dy\,dz + q\,dz\,dx + r\,dx\,dy)(p\,\delta y\,\delta z + q\,\delta z\,\delta x + r\,\delta x\,\delta y) = dX\,dY\,\delta X\,\delta Y,$$

therefore

$$(p\,dy\,dz + q\,dz\,dx + r\,dx\,dy)\sqrt{(p^2+q^2+r^2)} = dX\,dY \quad\quad (3).$$

Let $F(X, Y)$ be any function of X and Y, then

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial Y}dY \quad\quad (4).$$

Multiplying by (3) we obtain

$$p\frac{\partial F}{\partial x} + q\frac{\partial F}{\partial y} + r\frac{\partial F}{\partial z} = 0 \quad\quad (5).$$

Multiplying (4) by (2) we obtain

$$dy\,dz\left[\{(p^2+q^2)\,\delta z - rp\,\delta x - rq\,\delta y\}\frac{\partial F}{\partial y} - \{(r^2+p^2)\,\delta y - qp\,\delta x - qr\,\delta z\}\frac{\partial F}{\partial z}\right] + \dots$$
$$= dX\,dY\left[\frac{\partial F}{\partial X}\delta Y - \frac{\partial F}{\partial Y}\delta X\right].$$

Simplifying by means of (5)

$$(p\,dy\,dz + q\,dz\,dx + r\,dx\,dy)\left[\left(q\frac{\partial F}{\partial z} - r\frac{\partial F}{\partial y}\right)\delta x + \left(r\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial z}\right)\delta y + \left(p\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial x}\right)\delta z\right]$$
$$= dX\,dY\left[\frac{\partial F}{\partial X}\delta Y - \frac{\partial F}{\partial Y}\delta X\right],$$

and dividing out by (4) we get

$$\left[\left(q\frac{\partial F}{\partial z} - r\frac{\partial F}{\partial y}\right)\delta x + \left(r\frac{\partial F}{\partial x} - p\frac{\partial F}{\partial z}\right)\delta y + \left(p\frac{\partial F}{\partial y} - q\frac{\partial F}{\partial x}\right)\delta z\right](p^2+q^2+r^2)^{-\frac{1}{2}} = \frac{\partial F}{\partial X}\delta Y - \frac{\partial F}{\partial Y}\delta X.$$

Now let this equation be multiplied by

$$\frac{\partial}{\partial x}\delta x + \frac{\partial}{\partial y}\delta y + \frac{\partial}{\partial z}\delta z = \frac{\partial}{\partial X}\delta X + \frac{\partial}{\partial Y}\delta Y + \frac{\partial}{\partial Z}\delta Z.$$

If we write $P = \dfrac{p}{\sqrt{p^2+q^2+r^2}}$, etc., the resulting equation is

$$\delta y\,\delta z\left[\frac{\partial}{\partial y}\left(P\frac{\partial F}{\partial y} - Q\frac{\partial F}{\partial x}\right) - \frac{\partial}{\partial z}\left(R\frac{\partial F}{\partial x} - P\frac{\partial F}{\partial z}\right)\right] + \dots = \left(\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2}\right)dX\,dY.$$

The coefficient of $\delta y\,\delta z$ is

$$P\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}\right) - \frac{\partial F}{\partial x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) + \left(\frac{\partial P}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial P}{\partial y}\frac{\partial F}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial F}{\partial z}\right)$$
$$- \left(P\frac{\partial^2 F}{\partial x^2} + Q\frac{\partial^2 F}{\partial x\partial y} + R\frac{\partial^2 F}{\partial x\partial z}\right),$$

the last term of which may be replaced by

$$\frac{\partial P}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial Q}{\partial x}\frac{\partial F}{\partial y} + \frac{\partial R}{\partial x}\frac{\partial F}{\partial z}.$$

This coefficient is equal to

$$P\left[\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}+\xi\frac{\partial F}{\partial x}+\eta\frac{\partial F}{\partial y}+\zeta\frac{\partial F}{\partial z}+\kappa\left(P\frac{\partial F}{\partial x}+Q\frac{\partial F}{\partial y}+R\frac{\partial F}{\partial z}\right)\right],$$

if
$$2\frac{\partial P}{\partial x}-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=\xi P+\alpha P+\kappa P^2,$$

$$\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=\eta P+\alpha Q+\kappa PQ,$$

$$\frac{\partial P}{\partial z}+\frac{\partial R}{\partial x}=\zeta P+\alpha R+\kappa PR.$$

These lead as before to a system of equations

$$2\frac{\partial P}{\partial x}-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=2\xi P+\kappa P^2,$$

$$2\frac{\partial Q}{\partial y}-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=2\eta Q+\kappa Q^2,$$

$$2\frac{\partial R}{\partial z}-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=2\zeta R+\kappa R^2,$$

$$\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=\xi Q+\eta P+\kappa PQ,$$

$$\frac{\partial R}{\partial x}+\frac{\partial P}{\partial z}=\zeta P+\xi R+\kappa RP,$$

$$\frac{\partial Q}{\partial z}+\frac{\partial R}{\partial y}=\eta R+\zeta Q+\kappa QR.$$

If
$$\xi=\frac{1}{\theta}\frac{\partial\theta}{\partial x},\quad \eta=\frac{1}{\theta}\frac{\partial\theta}{\partial y},\quad \zeta=\frac{1}{\theta}\frac{\partial\theta}{\partial z},$$

we have the equation

$$\frac{1}{\sqrt{\theta}\,(p^2+q^2+r^2)}\left[\nabla^2(\sqrt{\theta}F)-F\nabla^2(\sqrt{\theta})\right]=\frac{\partial^2 F}{\partial X^2}+\frac{\partial^2 F}{\partial Y^2}.$$

Hence if $\sqrt{\theta}$ is a solution of Laplace's equation and F is a solution of

$$\frac{\partial^2 F}{\partial X^2}+\frac{\partial^2 F}{\partial Y^2}=0,$$

the function $\sqrt{\theta}F(X, Y)$ is a solution of Laplace's equation.

The equations for P, Q, R may now be written

$$2\frac{\partial}{\partial x}\left(\frac{P}{\theta}\right)-\frac{1}{\theta}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=\frac{\kappa}{\theta}P^2,$$

$$\frac{\partial}{\partial y}\left(\frac{R}{\theta}\right)+\frac{\partial}{\partial z}\left(\frac{Q}{\theta}\right)=\frac{\kappa}{\theta}QR,\ \text{etc.}$$

Putting
$$\frac{P}{\theta}=u,\quad \frac{Q}{\theta}=v,\quad \frac{R}{\theta}=w,\quad \kappa\theta=n,\quad \frac{1}{\theta}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)=l,$$

they take the simple form

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = nvw, \text{ etc.,}$$

$$2\frac{\partial u}{\partial x} - l = nu^2, \text{ etc.}$$

Particular solutions of these equations may be obtained without difficulty, for instance we may take

$$\theta = 1, \quad \kappa = 0, \quad P = hy - gz, \quad Q = fz - hx, \quad R = gx - fy.$$

Two independent solutions of the equation

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = 0$$

are given by $fx + gy + hz$ and $x^2 + y^2 + z^2$, hence we have the result that it is possible to find solutions of Laplace's equation of the form

$$F(fx + gy + hz, \quad x^2 + y^2 + z^2),$$

where $F(X, Y)$ satisfies a certain partial differential equation. This of course is well known.

Other solutions are given by $\theta = 1$, $P = \frac{x}{r}$, $Q = \frac{y}{r}$, $R = \frac{z}{r}$, $X = \frac{x}{z+r}$, $Y = \frac{y}{z+r}$, and we may deduce that if $F(X, Y)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = 0,$$

it is also a solution of

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0$$

and $\frac{1}{r}F(X, Y)$ is another solution of this equation.

Another solution which is not so well known is obtained by taking

$$\theta = \frac{1}{x + iy}, \quad X = z, \quad Y = \sqrt{x^2 + y^2};$$

we then have the result that if $F(X, Y)$ is a solution of

$$\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2} = 0,$$

the function

$$V = \frac{1}{\sqrt{x + iy}} F(z, \sqrt{x^2 + y^2})$$

is a solution of

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

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THE CONTINUATIONS OF FUNCTIONS DEFINED BY GENERALISED HYPERGEOMETRIC SERIES.

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XI. *The Continuations of Functions defined by Generalised Hypergeometric Series.*

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1. Let p be any complex quantity; and let $[n]$ denote $(p^n-1)/(p-1)$, so that when $p=1$, $[n]$ reduces to n. The quantity $[n]$ is termed a *basic number*, p being termed the base.

It has been shewn by Rev. F. H. Jackson in a series of papers * that various functions may be formed in which the quantities [1], [2], [3], ... occupy the place of the natural numbers in the ordinary functions of analysis.

I propose to develop, very briefly, the theory of the functions analogous to the hypergeometric function, by means of the theory of contour integration. The contour integrals which will be employed are very similar to those employed by Dr Barnes † in dealing with the theory of the ordinary hypergeometric functions.

The functions, which will be considered, do not exist when the modulus of the base is unity, unless the base is actually equal to unity ‡.

We shall assume that $|p|<1$, and when we wish to employ a base greater than unity, we shall denote the base by q where $q=p^{-1}$.

Further, we shall put

$$\log p = -\log q = -\omega = -(\omega_1 + i\omega_2)$$

where ω, ω_1, ω_2 are definite quantities, ω_1 and ω_2 being real; and since $|p|<1$, $\omega_1>0$.

We shall prove all the theorems, which will be obtained, for base p, and deduce from them the corresponding theorems when the modulus of the base is greater than unity by making use of the properties of the inverse base. We cannot employ the ordinary methods of analytic continuation, since the circle $|p|=1$ is a *barrier* for the analytic functions which will be introduced.

* Reference may be made especially to the following: *Proc. Roy. Soc.*, Vol. LXXIV. (1905), pp. 64—72; *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1 (1904), pp. 63—88; Vol. 2 (1905), pp. 192—220.

† *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 5 (1907), pp. 59—118; Vol. 6 (1908), pp. 141—177.

‡ When the functions reduce to the ordinary functions of analysis.

The basic hypergeometric function will be defined by the series

$$F([\alpha], [\beta]; [\gamma]; x) = 1 + \frac{[\alpha][\beta]}{[\gamma][1]} x + \frac{[\alpha][\alpha+1][\beta][\beta+1]}{[\gamma][\gamma+1][1][2]} x^2 + \dots \qquad \dots\dots\dots(1).$$

It converges when $|x| < 1$ if the base be p, but when $|x| < |q^{\gamma+1-\alpha-\beta}|$ if the base be q.

The basic gamma function is defined as a Weierstrassian product by the equation

$$\{\Gamma_p([x])\}^{-1} = [x] \exp(Qx) \prod_{s=1}^{\infty} \left\{ \left(1 + p^s \frac{[x]}{[s]}\right) \exp\left(-\frac{p^s x}{[s]}\right) \right\} \qquad \dots\dots\dots\dots\dots\dots(2)$$

$$= (1-p)^{x-1} \prod_{s=0}^{\infty} (1-p^{x+s}) \Big/ \prod_{s=1}^{\infty} (1-p^s),$$

where $$Q = p + \frac{p^2}{[2]} + \frac{p^3}{[3]} + \dots - \left\{ p + \frac{p^2}{2} + \frac{p^3}{3} + \dots \right\}$$

and $$|\arg(1-p)| < \tfrac{1}{2}\pi.$$

We define an associated function $G_p(x)$ by the equation

$$\{G_p(x)\}^{-1} = \prod_{s=0}^{\infty} (1-p^{x+s}),$$

so that $$G_p(x+1) = (1-p^x) G_p(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2a),$$

and $$G_p(x) = (1-p)^{x-1} \left\{ \prod_{s=1}^{\infty} (1-p^s) \right\}^{-1} \Gamma_p(x) \dots\dots\dots\dots\dots\dots\dots\dots(2b).$$

We shall need the asymptotic expansion of the function $G_p(x)$ for large values of $|x|$. It has been given by Littlewood*, as follows:

(i) When $R(\omega x)$ is large and positive, $G_p(x)$ tends uniformly to unity as $|x|$ tends to infinity.

(ii) When $R(\omega x)$ is large and negative, and $|x - x_0| > \epsilon$, where x_0 denotes any pole of $G_p(x)$, and ϵ is an assigned quantity which is not zero,

$$R \log G_p(x) = -\frac{1}{2\omega_1} \{R(\omega x)\}^2 - \tfrac{1}{2} R(\omega x) + J \qquad \dots\dots\dots\dots\dots\dots\dots\dots(3),$$

where $|J|$ does not exceed a finite quantity depending on ϵ.

We shall need the following formula due to Heine†, true when $|p^{\gamma-\alpha-\beta}| < 1$:

$$F([\alpha], [\beta]; [\gamma]; p^{\gamma-\alpha-\beta}) = \frac{G_p(\gamma) G_p(\gamma-\alpha-\beta)}{G_p(\gamma-\alpha) G_p(\gamma-\beta)} \qquad \dots\dots\dots\dots\dots\dots\dots(4a).$$

If β be a negative integer, the right-hand side is a rational function of p^α, p^γ, p; and so also is the left-hand side; the result must then be an identity for *all* values of p^α, p^γ, p. We may consequently write q in place of p in this case; and we have

$$F([\alpha], [\beta]; [\gamma]; q^{\gamma-\alpha-\beta}) = \prod_{n=1}^{-\beta} \left(\frac{1-q^{\gamma+n}}{1-q^{\gamma-\alpha+n}} \right) \qquad \text{(base } q\text{)}.$$

Putting $q = p^{-1}$, this becomes

$$F([\alpha], [\beta]; [\gamma]; p) = \frac{1}{p^{\alpha\beta}} \frac{G_p(\gamma) G_p(\gamma-\alpha-\beta)}{G_p(\gamma-\alpha) G_p(\gamma-\beta)} \qquad \text{(base } p\text{)}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots(4b),$$

* *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 5 (1907), pp. 395—398.

† Heine, *Kugelfunctionen*, Bd. I. (1878), p. 107.

provided that β (or α) is a negative integer. If neither α nor β be a negative integer the theorem is not true.

2. We shall need the following Lemma, which proves to be of considerable importance.

Lemma A. If there be any series

$$\Psi(z) = \sum_{n=0}^{\infty} u_0 u_1 \ldots u_n \kappa_n z^n = \sum_{n=0}^{\infty} \alpha_n z^n,$$

such that u_n and κ_n can be expanded in the convergent series

$$u_n = 1 + A_1 p^n + A_2 p^{2n} + \ldots$$
$$\kappa_n = B_0 p^{mn} + B_1 p^{(m+1)n} + \ldots$$

for all integer values of n greater than a certain finite value n_0, it being supposed that $A_1, A_2, \ldots, B_1, B_2, \ldots$, are independent of n and that m is not necessarily an integer, *the only possible singularities of the analytic function $\Psi(z)$ in the finite part of the plane are simple poles at the points $z = p^{-m}, p^{-m-1}, p^{-m-2}, \ldots$.*

To prove the lemma we observe that as $n \to \infty$ $\alpha_n/\alpha_{n+1} \to p^{-m}$; hence $\Psi(z)$ has a singularity* at $z = p^{-m}$; but it has no singularities inside the circle $|z| = |p^{-m}|$.

When $$|z| < |p^{-m}|,$$

$$(1 - zp^m)\Psi(z) = \alpha_0 + \sum_{n=0}^{\infty} (\alpha_{n+1} - p^m \alpha_n) z^{n+1},$$

and $$\alpha_{n+1} - p^m \alpha_n = u_0 u_1 \ldots u_n \kappa_n'$$

where we may write $$\kappa_n' = B_0' p^{(m+1)n} + B_1' p^{(m+2)n} + \ldots,$$

and some of the quantities $B_0', B_1', \ldots$, may vanish†; this expansion for κ_n' is valid when $n > n_0 + 1$.

Hence the function $(1 - zp^m)\Psi(z)$ is regular within the circle $|z| = |p^{-m-1}|$.

Consequently $\Psi(z)$ has a simple pole at $z = p^{-m}$.

Treating the functions $$(1 - zp^m)\Psi(z),$$
$$(1 - zp^m)(1 - zp^{m+1})\Psi(z), \ldots$$

in a similar manner, the lemma follows by induction.

PART I.

The theory of the ordinary basic hypergeometric function.

3. We shall now consider the analytic continuation of the function $F([\alpha], [\beta]; [\gamma]; z)$. From Lemma A, it follows that the only possible singularities of the function in the finite part of the plane are simple poles at $z = 1, p^{-1}, p^{-2}, \ldots$, and that the function is uniform over the whole plane.

* Hadamard, *La Série de Taylor* (Scientia), p. 19.

† If B_0' vanishes, the function has no singularity at $z = p^{-m-1}$.

We proceed to obtain an asymptotic expansion for $F([\alpha], [\beta]; [\gamma]; z)$ when $|z|$ is large.

Consider the contour integral

$$I = \frac{1}{2\pi i}\int_C \frac{G_p(\alpha+x)\,G_p(\beta+x)}{G_p(\gamma+x)\,G_p(1+x)}\,\frac{\pi(-z)^x}{\sin x\pi}\,dx \quad\text{.........................(5)}$$

taken along a contour C which is parallel to the line $R(\omega x)=0$ with a loop, if necessary to ensure that the points 0, 1, 2, ... lie to the right of the contour*, while the poles† of $G_p(\alpha+x)$ and $G_p(\beta+x)$ lie to the left of the contour.

The integral converges if $R[x\log(-z)-\log\sin(x\pi)]$ is negative for large values of $|x|$ on the contour; i.e. if

$$|\{\arg(-z)-\omega_2\omega_1^{-1}\log|z|\}|<\pi \quad\text{.............................(5a).}$$

[This restriction means that the z plane has a cross-cut in the shape of an equiangular spiral, the equation to which is $r=\exp(\omega_1\theta/\omega_2)$ in polar coordinates.]

Consider the integral (5) taken along a contour C' consisting of an arc of a large circle the centre of which is at the origin; the arc lies to the right of C and is terminated by C, and it does not pass infinitesimally near the poles of $\operatorname{cosec}(x\pi)$.

Consider also the integral taken along three contours A, A', B, the equations to which are

$$A;\ R(-ix)=m_0, \qquad A';\ R(-ix)=-m_1, \qquad B;\ R(\omega x)=-s_0\omega_1,$$

where m_0, m_1, s_0 are large positive quantities, so chosen that the lines A, A', B are at a distance, from each pole and zero of

$$\frac{G_p(\alpha+x)\,G_p(\beta+x)}{G_p(\gamma+x)\,G_p(1+x)}$$

at least equal to ϵ, where ϵ is some preassigned (non-zero) quantity.

The lines A, A', B are supposed to be terminated by each other and by C.

From the asymptotic expansion (3) of the function G_p it follows that when $|z|<1$, the value of the integral taken along C' can be made as small as we please, by taking the radius of the circle C' sufficiently large.

Hence, by Cauchy's theorem, when $|z|<1$, I is equal to the sum of the residues of the integrand at its poles on the right of C.

Evaluating these residues we get

$$I = \frac{G_p(\alpha)\,G_p(\beta)}{G_p(\gamma)\,G_p(1)}F([\alpha], [\beta]; [\gamma]; z) \text{ when } |z|<1 \quad\text{..................(5b).}$$

On the other hand, when $|z|>|p^{\gamma+1-\alpha-\beta}|$, the value of the integral taken along the contours A, A', B can be made as small as we please, by taking m_0, m_1, s_0 sufficiently large; this follows from (3).

Hence, by Cauchy's theorem, when $|z|>|p^{\gamma+1-\alpha-\beta}|$, I is equal to minus the sum of the residues of the integrand at its poles to the left of C.

* We shall always use the symbol C to denote a contour of this nature.

† These points are given by $x=-\alpha-s+2\pi i\omega^{-1}m$, $-\beta-s+2\pi i\omega^{-1}m$, where m is any integer, and s any positive integer (including zero).

The residue of the integrand at $-\alpha - s + 2\pi i m/\omega$ is

$$\frac{G_p(\beta-\alpha-s)\{G_p(1)\}^2}{G_p(\gamma-\alpha-s)\,G_p(1-\alpha-s)\,G_p(s+1)}\;\frac{\pi(-z)^{-\alpha-s}}{\omega p^{-\frac{1}{2}s(s+1)}}$$
$$\times \exp\{2m\pi i\omega^{-1}\log(-z)\}\operatorname{cosec}(2m\pi^2 i\omega^{-1}-\alpha\pi).$$

Hence, after some slight reductions we find that, when $|z| > |p^{\gamma+1-\alpha-\beta}|$,

$$I = \sum_{m=-\infty}^{\infty}\operatorname{cosec}(\alpha\pi - 2m\pi^2 i\omega^{-1})\exp\{2m\pi i\omega^{-1}\log(-z)\}$$
$$\times \frac{\pi G_p(\beta-\alpha)\,G_p(1)}{\omega G_p(\gamma-\alpha)\,G_p(1-\alpha)}(-z)^{-\alpha}F([\alpha],\,[\alpha-\gamma+1];\,[\alpha-\beta+1];\,z^{-1}p^{\gamma+1-\alpha-\beta})$$

+ a similar expression with α and β interchanged ....(5c).

We have assumed that $\alpha-\beta$ is not an integer; when $\alpha-\beta$ is an integer, the necessary modifications are easily made.

We now require the sum of series of the type

$$\sum_{m=-\infty}^{\infty}\operatorname{cosec}(\alpha\pi - 2m\pi^2 i\omega^{-1})\exp\{2m\pi i\omega^{-1}\log(-z)\}$$

which converge when the z plane has a cross-cut in accordance with the inequality

$$|\{\arg(-z) - \omega_2\omega_1^{-1}\log|z|\}| < \pi.$$

To sum such a series, we consider the integral of (5) when modified by putting $\beta=\gamma$.

Comparing (5b) and (5c), we see that the analytic continuation of

$$\frac{G_p(\alpha)}{G_p(1)}F([\alpha],\,[\beta];\,[\beta];\,z)$$

is
$$\sum_{m=-\infty}^{\infty}\operatorname{cosec}(\alpha\pi - 2m\pi^2 i\omega^{-1})\exp\{2m\pi i\omega^{-1}\log(-z)\}$$
$$\times\frac{\pi\,G_p(1)}{\omega G_p(1-\alpha)}(-z)^{-\alpha}F([\alpha],\,[\alpha-\beta+1];\,[\alpha-\beta+1];\,z^{-1}p^{1-\alpha}).$$

But, by a well known theorem *,

$$F([\alpha],\,[\beta];\,[\beta];\,z) = \prod_{n=0}^{\infty}(1-zp^{\alpha+n}) \div \prod_{n=0}^{\infty}(1-zp^n) \quad\text{.....................(6)},$$

and these products converge for all values of $|z|$, while their asymptotic expansions, when $|z|$ is large, have been obtained. Hence we find, without difficulty, that

$$\sum_{m=-\infty}^{\infty}\operatorname{cosec}(\alpha\pi - 2m\pi^2 i\omega^{-1})\,.\exp\{2m\pi i\omega^{-1}\log(-z)\}\,.\,(-z)^{-\alpha}$$
$$= \frac{\omega G_p(\alpha)\,G_p(1-\alpha)}{\pi\{G_p(1)\}^2}\prod_{n=0}^{\infty}\frac{(1-zp^{\alpha+n})(1-z^{-1}p^{1-\alpha+n})}{(1-zp^n)(1-z^{-1}p^{1+n})} \quad\text{..................(6a)}.$$

Making use of these results, we see that the analytic continuation of the series which defines $F([\alpha],\,[\beta];\,[\gamma];\,z)$ is given by the equation :

$$\frac{G_p(\alpha)\,G_p(\beta)}{G_p(\gamma)}\left\{\prod_{n=0}^{\infty}(1-zp^n)(1-z^{-1}p^{n+1})\right\}F([\alpha],\,[\beta];\,[\gamma];\,z)$$
$$= \frac{G_p(\alpha)\,G_p(\beta-\alpha)}{G_p(\gamma-\alpha)}\left\{\prod_{n=0}^{\infty}(1-zp^{\alpha+n})(1-z^{-1}p^{1-\alpha+n})\right\}F([\alpha],\,[\alpha-\gamma+1];\,[\alpha-\beta+1];\,z^{-1}p^{\gamma+1-\alpha-\beta})$$
$$+ \frac{G_p(\beta)\,G_p(\alpha-\beta)}{G_p(\gamma-\beta)}\left\{\prod_{n=0}^{\infty}(1-zp^{\beta+n})(1-z^{-1}p^{1-\beta+n})\right\}F([\beta],\,[\beta-\gamma+1];\,[\beta-\alpha+1];\,z^{-1}p^{\gamma+1-\alpha-\beta})$$
$$\text{..................(7)},$$

* Jackson, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2 (1905), p. 193.

where the symbol "=" is used in the sense "is the analytic continuation of." The functions involved in this result being uniform, the restriction (5a) may be removed.

Putting $q = p^{-1}$, we get the corresponding theorem for a base with modulus greater than unity:

$$\frac{\Gamma_q([\alpha])\,\Gamma_q([\beta])}{\Gamma_q([\gamma])}\left\{\prod_{n=0}^{\infty}(1 - zq^{\alpha+\beta-\gamma-1-n})\,(1 - z^{-1}q^{\gamma-\alpha-\beta-n})\right\} F([\alpha],\ [\beta];\ [\gamma];\ z)$$

$$= q^{\alpha(\beta-\gamma)}\,\frac{\Gamma_q([\alpha])\,\Gamma_q([\beta-\alpha])}{\Gamma_q([\gamma-\alpha])}\left\{\prod_{n=0}^{\infty}(1 - zq^{\beta-\gamma-1-n})(1 - z^{-1}q^{\gamma-\beta-n})\right\}$$

$$F([\alpha],\ [\alpha-\gamma+1];\ [\alpha-\beta+1];\ z^{-1}q^{\gamma+1-\alpha-\beta})$$

$$+ \text{a similar expression with } \alpha,\ \beta \text{ interchanged} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7a),$$

where Γ_q denotes Jackson's gamma function for a base whose modulus is > 1.

Putting $z = p^{\gamma-\alpha-\beta}$ in (7) we find, on using Heine's theorem, that when $|p^{\gamma-\alpha-\beta}| < 1$,

$$\frac{G_p(\alpha)\,G_p(\beta-\alpha)}{G_p(\gamma-\alpha)\,G_p(\gamma-\beta)\,G_p(1-\gamma+\beta)}\,F([\alpha],\ [\alpha-\gamma+1];\ [\alpha-\beta+1];\ p)$$

$$+ \text{a similar expression with } \alpha,\ \beta \text{ interchanged}$$

$$= \frac{G_p(\alpha)\,G_p(\beta)}{G_p(\gamma)\,G_p(\gamma-\alpha-\beta)\,G_p(1-\gamma+\alpha+\beta)}\,F([\alpha],\ [\beta];\ [\gamma];\ p^{\gamma-\alpha-\beta})$$

$$= \frac{G_p(\alpha)\,G_p(\beta)}{G_p(1-\gamma+\alpha+\beta)\,G_p(\gamma-\alpha)\,G_p(\gamma-\beta)}.$$

Putting $\alpha = \alpha_1 + \beta_1$, $\beta = \alpha_1 + \beta_2$, $\gamma = \alpha_1 - \alpha_2 + 1$, this may be expressed in the form

$$\frac{G_p(\alpha_1+\beta_1)\,G_p(\beta_2-\beta_1)}{G_p(\alpha_2+\beta_2)}\,F([\alpha_1+\beta_1],\ [\alpha_2+\beta_1];\ [\beta_1-\beta_2+1];\ p)$$

$$+ \text{a similar expression with } \beta_1,\ \beta_2 \text{ interchanged}$$

$$= \frac{G_p(\alpha_1+\beta_1)\,G_p(\alpha_1+\beta_2)}{G_p(\alpha_1+\alpha_2+\beta_1+\beta_2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

By the theory of analytic continuation this result holds for all values of α_1, α_2, β_1, β_2 which do not make the equation meaningless.

4. This result is of use in proving the following Lemma which provides another representation of the analytic continuation of $F([\alpha],\ [\beta];\ [\gamma];\ z)$.

Lemma B. If the contour of integration, D, be drawn parallel to the line $R(\omega x) = 0$ with loops, if necessary, to ensure that the poles of $G_p(\alpha_1 + x)\,G_p(\alpha_2 + x)$ lie to the left of the contour while the points $\beta_1,\ \beta_1+1,\ \beta_1+2,\ \ldots;\ \beta_2,\ \beta_2+1,\ \ldots$ lie to the right of the contour, then for all complex values of α_1, α_2, β_1, β_2 for which the contour can be drawn

$$I_1 = \frac{1}{2\pi i}\int_D \frac{G_p(\alpha_1+x)\,G_p(\alpha_2+x)}{G_p(1-\beta_1+x)\,G_p(1-\beta_2+x)}\,\frac{\pi^2 p^x\,dx}{\sin(\beta_1-x)\,\pi\,\sin(\beta_2-x)\,\pi}$$

$$= \frac{\pi p^{\beta_1+\beta_2}(1-p)^{\alpha_1+\alpha_2+\beta_1+\beta_2}}{\sin(\beta_1-\beta_2)\pi\,.\,(p^{\beta_1}-p^{\beta_2})}\,\frac{\Gamma_p([\alpha_1+\beta_1])\,\Gamma_p([\alpha_1+\beta_2])\,\Gamma_p([\alpha_2+\beta_1])\,\Gamma_p([\alpha_2+\beta_2])}{\Gamma_p([\beta_1-\beta_2])\,\Gamma_p([\beta_2-\beta_1])\,\Gamma_p([\alpha_1+\alpha_2+\beta_1+\beta_2])}.$$

For, by using the asymptotic expansion of the function G_p, it follows that the contour can be bent round to enclose the sequences of points $\beta_1,\ \beta_1+1,\ \beta_1+2,\ \ldots;\ \beta_2,\ \beta_2+1,\ \beta_2+2,\ \ldots;$

and, consequently, by Cauchy's theorem, I_1 is equal to the sum of the residues of the integrand at these points; hence

$$I_1 = \frac{\pi p^{\beta_1}}{\sin(\beta_2-\beta_1)\pi} \frac{G_p(\alpha_1+\beta_1)\,G_p(\alpha_2+\beta_1)}{G_p(1)\,G_p(1-\beta_2+\beta_1)} F([\alpha_1+\beta_1],\ [\alpha_2+\beta_1];\ [1-\beta_2+\beta_1];\ p)$$

+ a similar expression with β_1, β_2 interchanged.

On employing the theorem (8), and changing the functions G_p to Jackson's gamma functions, we obtain the required result immediately. This proves the Lemma.

We now consider the integral

$$I_2 = \frac{1}{2\pi i}\int_E \frac{G_p(\gamma-\alpha-\beta+s)\,G_p(s)}{G_p(1-\alpha+s)\,G_p(1-\beta+s)} \frac{\pi^2 p^s}{\sin(\alpha-s)\pi\sin(\beta-s)\pi}\left\{\prod_{n=0}^{\infty}\left(\frac{1-zp^{s+n}}{1-zp^n}\right)\right\} ds;$$

E, the contour of integration being parallel to the line $R(\omega s)=0$ with loops, if necessary to ensure that the poles of $G_p(\gamma-\alpha-\beta+s)\,G_p(s)$ lie to the left of the contour, while the points $\alpha,\ \alpha+1,\ \alpha+2,\ \dots;\ \beta,\ \beta+1,\ \beta+2;\ \dots$ lie to the right of the contour.

We shall shew that

$$I_2 = \frac{\pi p^{\alpha+\beta}}{\sin(\alpha-\beta)\pi} \frac{(1-p)^\gamma}{p^\alpha - p^\beta} \frac{\Gamma_p([\alpha])\,\Gamma_p([\beta])\,\Gamma_p([\gamma-\alpha])\,\Gamma_p([\gamma-\beta])}{\Gamma_p([\gamma])\,\Gamma_p([\alpha-\beta])\,\Gamma_p([\beta-\alpha])} F([\alpha],\ [\beta];\ [\gamma];\ z) \quad \dots(9).$$

On making use of the asymptotic expansion of the function G_p, it is readily seen that the integral exists and that it will be legitimate to bend round the contour until it encloses the poles of $\Gamma(\alpha-s)\,\Gamma(\beta-s)$.

Evaluating the integral, thus modified, by Cauchy's theorem, we get

$$I_2 = \frac{\pi(1-p)^{\gamma-2}}{\sin(\alpha-\beta)\pi}\sum_{n=0}^{\infty}\frac{\Gamma_p([\gamma-\beta-n])\,\Gamma_p([\alpha+n])}{\Gamma_p([n+1])\,\Gamma_p([1+\alpha-\beta+n])}\left\{\prod_{m=0}^{\infty}\left(\frac{1-zp^{\alpha+n+m}}{1-zp^m}\right)\right\}$$

+ a similar expression with α, β interchanged............(9a).

(These two series always converge unless $z=1,\ p^{-1},\ p^{-2}\dots$.)

Also, making use of the result (a particular case of formula 5b)

$$(1-p)^{s-1}\,\Gamma_p([s])\prod_{n=0}^{\infty}\left(\frac{1-zp^{s+n}}{1-zp^n}\right) = \frac{1}{2\pi i}\int_C \frac{G_p(s-x)}{G_p(1-x)}\frac{\pi(-z)^{-x}}{\sin x\pi}\,dx,$$

we get

$$I_2 = \left(\frac{1}{2\pi i}\right)^2\iint \frac{G_p(1)\,G_p(\gamma-\alpha-\beta+s)}{G_p(1-\alpha+s)\,G_p(1-\beta+s)}\frac{\pi^2 p^s}{\sin(\alpha-s)\pi\sin(\beta-s)\pi}\frac{G_p(s-x)}{G_p(1-x)}\frac{\pi(-z)^{-x}}{\sin x\pi}\,dx\,ds,$$

where the s contour lies to the right of the x contour both being parallel to $R(\omega x)=0$, with loops to ensure that both contours are to the right of the poles of $\Gamma(x)\,G_p(\gamma-\alpha-\beta+x)$ and to the left of the poles of $\Gamma(\alpha-x)\,\Gamma(\beta-x)$.

An easy, but slightly tedious, investigation shews that we may invert the order of integration, and, employing Lemma B, we then get

$$I_2 = \frac{1}{2\pi i}\int \frac{G_p(1)}{G_p(1-x)}\frac{\pi(-z)^{-x}}{\sin x\pi}\frac{\pi p^{\alpha+\beta}(1-p)^{\gamma-x}\,\Gamma_p([\alpha-x])\,\Gamma_p([\beta-x])\,\Gamma_p([\gamma-\beta])\,\Gamma_p([\gamma-\alpha])}{\sin(\alpha-\beta)\pi\,.\,(p^\alpha-p^\beta)\,\Gamma_p([\alpha-\beta])\,\Gamma_p([\beta-\alpha])\,\Gamma_p([\gamma-x])}\,dx$$

along a contour parallel to $R(\omega x)=0$.

Modifying the contour when $|z|<1$ to enclose the poles of $\Gamma(x)$ we find the result (9) stated above.

A comparison of (9) and (9a) gives the theorem, already known, that $F([\alpha], [\beta]; [\gamma]; z)$ is a uniform function with simple poles at the points $1, p^{-1}, p^{-2}, \ldots\ldots$; while if $|p^{\gamma+1-\alpha-\beta}|>1$ it affords an indication of the nature of $F([\alpha], [\beta]; [\gamma]; z)$ in the ring-shaped region given by $1<|z|<|p^{\gamma+1-\alpha-\beta}|$ which is not given by formula (7).

This completes the theory of the ordinary basic hypergeometric function.

PART II.

The theory of generalised basic hypergeometric functions.

5. We define two hypergeometric functions of order (r, s) by the equations

$$\frac{\Gamma_p([\alpha_1])\,\Gamma_p([\alpha_2])\ldots\Gamma_p([\alpha_r])}{\Gamma_p([\rho_1])\,\Gamma_p([\rho_2])\ldots\Gamma_p([\rho_s])}\,{}_rF_s([\alpha_1],\ldots[\alpha_r];\,[\rho_1],\ldots[\rho_s];\,z)$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma_p([\alpha_1+n])\,\Gamma_p([\alpha_2+n])\ldots\Gamma_p([\alpha_r+n])}{\Gamma_p([\rho_1+n])\,\Gamma_p([\rho_2+n])\ldots\Gamma_p([\rho_s+n])\,\Gamma_p([1+n])}\,z^n$$
$$={}_r\mathfrak{F}_s([\alpha_1],\ldots[\alpha_r];\,[\rho_1],\ldots[\rho_s];\,z)\ldots\ldots\ldots\ldots\ldots(10);$$

and we write, for the sake of brevity,

$$s+1-r=\mu.$$

The above series converges when $|z|<|(1-p)|^{-\mu}$.

The theory is simplest when μ is zero or a negative integer.

To the consideration of this case we now proceed.

We put
$$y=z(1-p)^{\mu}$$
and suppose that
$$|\{\arg(-y)-\omega_2\,\omega_1^{-1}\log|y|\}|<\pi.$$

We then consider the integral

$$I_3=\frac{1}{2\pi i}\int_C\prod_{t=1}^{r,s}\left\{\frac{G_p(\alpha_t+x)}{G_p(\rho_t+x)}\right\}\frac{\pi(-y)^x\,dx}{\sin(x\pi)\,.\,G_p(1+x)}\ldots\ldots\ldots\ldots(11)$$

along a contour similar to the contour of § 3; where

$$\prod_{t=1}^{r,s}\frac{G_p(\alpha_t+x)}{G_p(\rho_t+x)}$$

is written in place of

$$\left\{\prod_{t=1}^{r}G_p(\alpha_t+x)\right\}\div\left\{\prod_{t=1}^{s}G_p(\rho_t+x)\right\}.$$

The integral is convergent; when $|y|<1$, we may bend round the contour so as to embrace the line on which the poles of $\Gamma(-x)$ lie, and we find, by Cauchy's theorem,

$$I_3={}_r\mathfrak{F}_s([\alpha_1],\ldots[\alpha_r];\,[\rho_1],\ldots[\rho_s];\,z)\times(1-p)^{\Sigma\alpha-\Sigma\rho-1+\mu}\{G_p(1)\}^{-\mu}.$$

Also, if either (i) $|z|>|p^{1+\Sigma\rho-\Sigma\alpha}|$ when $\mu=0$, or (ii) if $|z|$ be finite when μ is a negative integer, we may shew that I_3 is equal to minus the sum of the residues of the integrand at those poles of the integrand which lie to the left of the contour (just as in § 3).

Putting $\Gamma_p([\alpha])\,\Gamma_p([1-\alpha]) = \{S(\alpha)\}^{-1}$, we get, for the range of values of $|z|$ under consideration

$$I_3 = \sum_{t=1}^{r} \frac{\pi \exp\{(\mu+\mu\alpha_t+\Sigma\alpha-\Sigma\rho)\log(1-p)-\alpha_t\log(-y)\}}{\omega\,\{G_p(1)\}^{\mu}}$$
$$\sum_{m=-\infty}^{\infty}\left\{\exp\left(\frac{2m\pi i\log(-y)}{\omega}\right)\operatorname{cosec}\left(\alpha_t\pi-\frac{2m\pi^2 i}{\omega}\right)\right\}$$
$$\times\frac{S(1-\rho_1+\alpha_t)\dots S(1-\rho_s+\alpha_t)\,S(\alpha_t)}{S(\alpha_1-\alpha_t)\dots * \dots S(\alpha_r-\alpha_t)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma_p([1-\rho_1+\alpha_t+n])\dots\Gamma_p([1-\rho_s+\alpha_t+n])}{\Gamma_p([1-\alpha_1+\alpha_t+n])\dots\Gamma_p([1-\alpha_r+\alpha_t+n])}(-)^{n\mu}z^{-n}p^{-n(\mu\alpha_t+\Sigma\alpha-\Sigma\rho-1)-\frac{1}{2}\mu n(n+1)}$$

the asterisk denoting the term $S(\alpha_t-\alpha_t)$ is to be omitted.

Consequently, substituting for the cosecant-exponential series, we see that the analytic continuation of the series denoted by

$${}_r\mathfrak{F}_s([\alpha_1],\dots[\alpha_r];\,[\rho_1],\dots[\rho_s];\,z)$$

is given by the aggregate of series

$$\sum_{t=1}^{r}(1-p)^{\mu\alpha_t}\left\{\prod_{n=0}^{\infty}\frac{(1-yp^{\alpha_t+n})(1-y^{-1}p^{1-\alpha_t+n})}{(1-yp^n)(1-y^{-1}p^{1+n})}\right\}\left\{\prod_{m=1}^{r,s}{}^{*}\,\frac{S(1-\rho_m+\alpha_t)}{S(\alpha_m-\alpha_t)}\right\}$$
$$\times\sum_{n=0}^{\infty}\left\{\prod_{m=1}^{r,s}\frac{\Gamma_p([1-\rho_m+\alpha_t+n])}{\Gamma_p([1-\alpha_m+\alpha_t+n])}\right\}\Gamma_p([\alpha_t+n])\,(-)^{n\mu}z^{-n}p^{-n(\mu\alpha_t+\Sigma\alpha-\Sigma\rho-1)-\frac{1}{2}\mu n(n+1)}\quad\dots(11a)$$

where $y = z(1-p)^{\mu}$.

We observe that each member of the aggregate of series is an integral function of z^{-1} when $\mu < 0$.

6. The theory when μ is a positive integer (zero excluded) is more complicated. We shall not follow Barnes† by considering separately the cases when $\mu = 1$ and when $\mu > 1$, as the case when $\mu = 1$ is not appreciably simpler than the case when $\mu > 1$.

Our procedure will be as follows: we shall show that r linear combinations of $s+1$ functions of the type ${}_r\mathfrak{F}_s$ possess asymptotic expansions, which can easily be obtained. We shall then show, by rather elaborate arguments (similar to those in Barnes' memoir just quoted) that μ other linear combinations of the same functions possess analytic continuations involving known functions and convergent series whose coefficients may be calculated with sufficient labour.

To obtain the r formulae of the first type, we consider the integral

$$I_4 = \frac{1}{2\pi i}\int_C G_p(x)\prod_{t=1}^{r,s}\frac{G_p(x-\rho_t+1)}{G_p(x-\alpha_t+1)}\,\frac{\pi y_1^{-x}}{\sin(x-\alpha_1)\pi}\,p^{x(1+\Sigma\rho-\Sigma\alpha)-\frac{1}{2}\mu x(x+1)}dx\quad\dots(12)$$

where $y_1 = (-)^{\mu-1}y = (-)^{\mu-1}(1-p)^{\mu}z$.

The integral converges for all finite values of y_1; for convenience we suppose that y_1 has a definite argument.

† See Parts II. and III. of his memoir, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 5 (1907), pp. 59—118.

When $|y|<1$, the integral is equal to the sum of the residues at the poles of the integrand which lie to the left of the contour; i.e. at the poles of

$$G_p(x) \prod_{t=1}^{s} G_p(1-\rho_t+x).$$

Consequently when $|y|<1$,

$$I_4 = \frac{\pi}{\omega} G_p(1) \prod_{t=1}^{r,s} \frac{G_p(1-\rho_t)}{G_p(1-\alpha_t)} \, {}_rF_s([\alpha_1], \ldots [\alpha_r]; [\rho_1], \ldots [\rho_s]; z)$$

$$\times \sum_{k=-\infty}^{\infty} \operatorname{cosec}(\alpha_1 + 2k\pi i \omega^{-1})\pi \,.\, \exp\{-2\mu k^2\pi^2\omega^{-1} - 2k\pi i(\Sigma\alpha - \Sigma\rho + \tfrac{1}{2}\mu) + 2k\pi i\omega^{-1}\log y_1\}$$

$$+ \sum_{n=1}^{s} \frac{\pi}{\omega} G_p(1) \left\{\prod_{t=1}^{r,s}{}^{*} \frac{G_p(\rho_n-\rho_t)}{G_p(\rho_n-\alpha_t)}\right\} G_p(\rho_n-1) \exp\{(1-\rho_n)\log y_1\}$$

$$\times {}_rF_s([\alpha_1-\rho_n+1], \ldots [\alpha_r-\rho_n+1]; [\rho_1-\rho_n+1] \ldots * \ldots [\rho_s-\rho_n+1], [2-\rho_n]; z)$$

$$\times \sum_{k=-\infty}^{\infty} \operatorname{cosec}(\alpha_1-\rho_n+1+2k\pi i\omega^{-1})\pi \,.\, \exp\{-2\mu k^2\pi^2\omega^{-1} - 2k\pi i(\Sigma\alpha - \Sigma\rho - \tfrac{1}{2}\mu + \mu\rho_n) + 2k\pi i\omega^{-1}\log y_1\}$$

$$\times \exp\{\omega(\rho_n-1)(\Sigma\alpha - \Sigma\rho - 1 + \tfrac{1}{2}\mu\rho_n)\} \quad \ldots\ldots\ldots\ldots(12a).$$

The cosecant-exponential series are very rapidly convergent for all finite values of $|y_1|$. It is not possible to express them in terms of functions more elementary than functions of the type ${}_1F_1$.

We can now obtain the asymptotic expansion of I_4 for large values of $|z|$ by considering the integrand of I_4 integrated along the lines (i) C, (ii) $I(x)=-s_1$, (iii) $R(\omega x)=l\omega_1$, (iv) $I(x)=s_2$ where s_1, s_2 are large positive quantities and $\omega_1^{-1}R(\omega l-\omega\alpha_1)$ is not an integer.

It may be shown without difficulty that the integral along the line (iii) is of order y^{-l}.

Hence by Cauchy's theorem, if k be the greatest integer satisfying the inequality

$$R\{\omega(k-\alpha_1)\} < l\omega_1,$$

we get

$$I_4 = \sum_{n=0}^{k} y_1^{-\alpha_1} G_p(\alpha_1+n) \prod_{t=1}^{r,s} \frac{G_p(\alpha_1-\rho_t+n+1)}{G_p(\alpha_1-\alpha_t+n+1)} (-y_1)^{-n} p^{(\alpha_1+n)(\Sigma\rho-\Sigma\alpha+1)-\frac{1}{2}\mu(\alpha_1+n)(\alpha_1+n+1)} + J \quad \ldots\ldots(12b)$$

where $|y_1^{\alpha_1+k}J|$ tends uniformly to zero as $|y_1|$ tends to infinity.

A comparison of the results ($12a$) and ($12b$) on writing $\alpha_2, \alpha_3 \ldots \alpha_r$ in turn for α_1 gives the r asymptotic expansions of the first type.

7. We now require the $s+1-r\ (=\mu)$ formulae of analytic continuation of the second type.

To obtain them we consider the function ${}_rS_s(x)$ defined by the following equation:

$${}_rS_s(x) = \sum_{n=0}^{\infty} \frac{G_p(n\mu^{-1}-x)}{\prod_{t=1}^{\mu} G_p(n\mu^{-1}+t\mu^{-1})} \prod_{t=1}^{r,s} \frac{G_p(1-\rho_t+n\mu^{-1}-x)}{G_p(1-\alpha_t+n\mu^{-1}-x)} p^{-\frac{1}{2}\mu x(x-1)+nx-\frac{1}{2}(n+x)-(x-n/\mu)(\frac{1}{2}+\Sigma\rho-\Sigma\alpha)} \quad \ldots\ldots(13).$$

We shall shew that ${}_rS_s(x)$ possesses the following properties:

(I) The only singularities of ${}_rS_s(x)$ in the finite part of the plane, besides simple poles at the poles of $G_p(n\mu^{-1}-x)\prod_{t=1}^{s} G_p(1-\rho_t+n\mu^{-1}-x)$, (where $n=0, 1, \ldots \mu-1$), are simple poles at each of the poles of

$$G_{\varpi}(\mu x - \tfrac{1}{2}\mu + \tfrac{1}{2} + \Sigma\rho - \Sigma\alpha)$$

where the base $\varpi = p^{1/\mu}$; from this result, it will follow that ${}_rS_s(x)$ is uniform over the whole of the x plane.

(II) If $x = r_2 + r_1(\omega_2 + \omega_1 i)$, r_1 and r_2 being real, then the asymptotic expansions of ${}_rS_s(x)$ are the following:

(i) when $|r_1|$ is large and r_2 finite and x is such that its least distance from any of the poles of ${}_rS_s(x)$ is greater than an assigned quantity ϵ,

$$|{}_rS_s(x)| < J \exp\{-\tfrac{1}{2}\mu(r_1-\beta)^2 \omega_1(\omega_1^2+\omega_2^2)\};$$

(ii) when r_2 is large and positive, and

(iii) when r_2 is large and negative,

the same asymptotic expansion holds; where J, β are finite quantities depending on ϵ.

(III) The residues of ${}_rS_s(x)$ at the poles of $G_{\varpi}(\mu x - \tfrac{1}{2}\mu + \tfrac{1}{2} + \Sigma\rho - \Sigma\alpha)$ can be calculated with sufficient labour, should necessity arise.

These statements we proceed to prove.

(I) Let

$$\begin{aligned}{}_rS_s(x,y) &= p^{-\frac{1}{2}\mu x(x-1) - x(1+\Sigma\rho-\Sigma\alpha)} \frac{G_p(-x)}{\prod_{t=1}^{\mu} G_p(t/\mu)} \prod_{t=1}^{r,s} \frac{G_p(1-\rho_t-x)}{G_p(1-\alpha_t-x)}\, {}_rT_s(x,y) \\ &= \sum_{n=0}^{\infty} \frac{G_p(n\mu^{-1}-x)}{\prod_{t=1}^{\mu} G_p(n\mu^{-1}+t\mu^{-1})} \prod_{t=1}^{r,s} \frac{G_p(1-\rho_t+n\mu^{-1}-x)}{G_p(1-\alpha_t+n\mu^{-1}-x)}\, y^n p^{-\frac{1}{2}\mu x(x-1)+nx-\frac{1}{2}(n+x)-(x-n/\mu)(\frac{1}{2}+\Sigma\rho-\Sigma\alpha)} \end{aligned} \qquad \text{........................(14).}$$

This definition of ${}_rS_s(x,y)$ is valid when $|yp^{x-\frac{1}{2}+(\frac{1}{2}+\Sigma\rho-\Sigma\alpha)/\mu}| < 1$; draw circles of radius ϵ with centres at the poles of $\prod_{t=1}^{r} G_p(1-\alpha_t+n/\mu-x)$ (where $n=0,1,2,\ldots\mu-1$); let us exclude the interior of these circles from consideration.

By Lemma A, ${}_rT_s(x,y)$ *qua* function of y is a uniform function of y over the finite part of the y plane, with simple poles at the points* $y = p^{-\lambda-x}, p^{-\lambda-x-1/\mu}, p^{-\lambda-x-2/\mu}, \ldots$; and if we form the product

$$\prod_{m=0}^{l} (1-yp^{\lambda+x+m/\mu})\, {}_rT_s(x,y),$$

and arrange it in powers of y, so that it is equal to $\sum_{n=0}^{\infty} \Phi_{l,n}(x)\,.\,y^n$, then for all the values of x under consideration $|\Phi_{l,n}(x)/\Phi_{l,n+1}(x)| \to |p^{-\lambda-x-(l+1)/\mu}|$ as $n \to \infty$.

* We put $-\tfrac{1}{2}+(\tfrac{1}{2}+\Sigma\rho-\Sigma\alpha)/\mu = \lambda$, for brevity.

Now when $|p^{\lambda+x}|<1$, we see by putting $y=1$ that

$$\prod_{m=0}^{l}(1-p^{\lambda+x+m/\mu})\,{}_rS_s(x)$$
$$=p^{-\frac{1}{2}\mu x(x-1)-x(1+\Sigma\rho-\Sigma\alpha)}\frac{G_p(-x)}{\prod\limits_{t=1}^{\mu}G_p(t/\mu)}\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t-x)}{G_p(1-\alpha_t-x)}\sum_{n=0}^{\infty}\Phi_{l,n}(x)\ \ldots\ldots\ldots\ldots(15).$$

But $\sum\limits_{n=0}^{\infty}\Phi_{l,n}(x)$ is absolutely and uniformly convergent when $|p^{\lambda+x}|<|p^{1/\mu}|^{-l-1}$.

Consequently equation (15) *gives the analytic continuation of* ${}_rS_s(x)$ *over the region* $1\leqslant|p^{\lambda+x}|<|p^{1/\mu}|^{-l-1}$.

Now in this region, $\prod\limits_{m=0}^{l}(1-p^{\lambda+x+m/\mu})^{-1}$ has simple poles (but no other singularities) at the points given by

$$p^{\lambda+x}=1,\ p^{-1/\mu},\ p^{-2/\mu},\ \ldots,\ p^{-l/\mu}.$$

That is to say, the only singularities of ${}_rS_s(x)$ within the region are simple poles at the poles of $G_{\varpi}(\mu x+\mu\lambda)$; but l is any finite integer; hence the theorem (I) is true.

(II) To prove (i) we notice that if k be the integer which satisfies the conditions

$$-1<r_1\pi^{-1}(\omega_1^2+\omega_2^2)-2k\leqslant 1,$$

and if
$$\xi=x-2k\pi(\omega_2-\omega_1 i)^{-1},$$

so that ξ is finite when r_2 is finite, and $p^x=p^\xi$, then, when $|p^{\lambda+x}|<1$,

$${}_rS_s(x)={}_rS_s(\xi)\times p^{\frac{1}{2}\mu(\xi-x)(\xi+x-1)+(\xi-x)(1+\Sigma\rho-\Sigma\alpha)}.$$

By the theory of analytic continuation this result is true when $|p^{\lambda+x}|\geqslant 1$.

Now ${}_rS_s(\xi)$ is less than a definite finite quantity; and hence we can deduce the asymptotic expansion (i) by use of some elementary algebra.

The proofs of (ii) and (iii) are more difficult. We first shew that if

$${}_rQ_s(x)=\frac{\Omega_\mu}{\omega}\sum_{m=-\infty}^{\infty}\sin\pi\left(\frac{2m\pi i}{\omega}-x\right).\exp\left\{-\frac{2\mu m^2\pi^2}{\omega}-2m\pi i\left(x+\Sigma\rho-\Sigma\alpha-\tfrac{1}{2}\right)\right\}$$

where
$$\Omega_\mu=\left\{G_p\left(\frac{1}{\mu}\right)G_p\left(\frac{2}{\mu}\right)\ldots G_p\left(\frac{\mu}{\mu}\right)\right\}^2,$$

then

$${}_rS_s(x).{}_rQ_s(x)=-\frac{1}{2\pi i}\int_C G_p(\phi)\prod_{t=1}^{\mu}G_p\left(\frac{t-1}{\mu}-x-\phi\right)\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t+\phi)}{G_p(1-\alpha_t+\phi)}$$
$$\times p^{-\frac{1}{2}\mu\phi(\phi+1)+(\Sigma\rho-\Sigma\alpha)\phi-x}\sin(\mu\pi\phi)\,d\phi\ \ldots\ldots(16)$$

along a contour C parallel to $R(\omega x)=0$ with loops, if necessary, to ensure that the poles of $G_p(\phi)\prod\limits_{t=1}^{s}G_p(1-\rho_t+\phi)$ lie to the left of the contour, while the other poles of the integrand lie to the right of the contour.

The integral converges for all finite values of $|x|$, certain isolated points excepted.

If $R\{\omega(\mu x-\frac{1}{2}\mu+\frac{1}{2}+\Sigma\rho-\Sigma\alpha)\}>0$, i.e. if the series defining ${}_rS_s(x)$ is convergent, we may shew in the usual manner that the integral is equal to minus the sum of the residues of the integrand at those poles which lie to the right of the contour. By writing down the residue at any one of these poles, we at once obtain the theorem stated. We may also shew that, under the same limitation as regards x,

$$-\frac{1}{2\pi i}\int_C \prod_{t=1}^{\mu} G_p\left(\frac{t-1}{\mu}-x-\phi\right)\prod_{t=1}^{\mu} G_p\left(\frac{t}{\mu}-\sigma+\phi\right) p^{-\frac{1}{2}\mu\phi(\phi+1)+(\Sigma\rho-\Sigma\alpha)\phi-x}\sin(\mu\pi\phi)\,d\phi$$

$$= {}_rQ_s(x)\sum_{n=0}^{\infty} p^{-\frac{1}{2}\mu x(x-1)+nx-\frac{1}{2}(n+x)-(x-n/\mu)(\frac{1}{2}+\Sigma\rho-\Sigma\alpha)} \times \prod_{t=1}^{\mu}\frac{G_p\left(\frac{t+n}{\mu}-\sigma-x\right)}{G_p\left(\frac{t+n}{\mu}\right)}$$

$$= {}_rQ_s(x)p^{-\frac{1}{2}\mu x(x-1)-x(1+\Sigma\rho-\Sigma\alpha)}\frac{G_\varpi(1-\mu\sigma-\mu x)\,G_\varpi(\mu x-\frac{1}{2}\mu+\frac{1}{2}+\Sigma\rho-\Sigma\alpha)}{G_\varpi(1)\,G_\varpi(\frac{3}{2}-\mu\sigma-\frac{1}{2}\mu+\Sigma\rho-\Sigma\alpha)} \quad\text{.........(16a)}$$

on transforming to base $\varpi\,(=p^{1/\mu})$ and making use of the basic analogue of the binomial theorem †.

We may also shew that for all finite values of $|x|$, the integrals in equations (16) and (16a) are equal to the sums of the residues of the integrands at the poles to the left of the contour. From the first we get ‡:

$$-{}_rS_s(x)\,{}_rQ_s(x)\,.\,p^x$$

$$=\frac{\Omega_\mu}{\omega}\sum_{n=0}^{\infty} G_\varpi(n\mu-\mu x)\prod_{t=1}^{r,s}\frac{S_p'(\alpha_t)}{S_p'(\rho_t)}\frac{G_p(\alpha_t+n)}{G_p(\rho_t+n)}\frac{p^n}{G_p(1+n)}$$

$$\times\sum_{m=-\infty}^{\infty}\sin\left(\frac{2m\mu\pi^2 i}{\omega}\right)\exp\left\{-\frac{2m^2\mu\pi^2}{\omega}+2m\pi i(\Sigma\rho-\Sigma\alpha-\tfrac{1}{2}\mu)\right\}$$

$$+\sum_{k=1}^{s}\frac{\Omega_\mu}{\omega}\sum_{n=0}^{\infty} G_\varpi(n\mu-\mu\rho_k+\mu-\mu x)\prod_{t=0}^{r,s}{}^{*}\left\{\frac{S_p'(\alpha_t-\alpha_k+1)}{S_p'(\rho_t-\rho_k+1)}\frac{G_p(\alpha_t-\rho_k+n+1)}{G_p(\rho_t-\rho_k+n+1)}\right.$$

$$\left.\frac{p^n}{S_p'(2-\rho_k)\,G_p(n+1)}G_p(2-\rho_k+n)\right\}$$

$$\times\sum_{m=-\infty}^{\infty}\sin\left(1-\rho_k+\frac{2m\pi i}{\omega}\right)\mu\pi\,.\,\exp\left\{-\frac{2m^2\mu\pi^2}{\omega}+2m\pi i(\Sigma\rho-\Sigma\alpha+\tfrac{1}{2}\mu-\mu\rho_k)\right.$$

$$\left.+\tfrac{1}{2}\omega(\rho_k-1)(2\Sigma\alpha-2\Sigma\rho+\mu\rho_k)\right\}\quad\text{...(16b).}$$

From the second, we find, after some algebra:

$$-\frac{\Omega_\mu}{\omega}\sum_{m=-\infty}^{\infty}\sin\left(1-\mu\sigma+\frac{2m\pi i\mu}{\omega}\right)\exp\left\{-\frac{2\mu m^2\pi^2}{\omega}+2m\pi i(\Sigma\rho-\Sigma\alpha-\tfrac{1}{2}\mu-\mu\sigma)\right\}$$

$$\times\exp\{\tfrac{1}{2}\omega(\sigma-\mu^{-1})(2\Sigma\alpha-2\Sigma\rho+\mu\sigma+\mu-1)\}$$

$$={}_rQ_s(x)\,.\,p^{-\frac{1}{2}\mu x(x-1)-x(\Sigma\rho-\Sigma\alpha)}\frac{S_\varpi'(\frac{1}{2}\mu-\frac{1}{2}+\mu\sigma-\Sigma\rho+\Sigma\alpha)}{S_\varpi'(\frac{1}{2}\mu+\frac{1}{2}-\mu x-\Sigma\rho+\Sigma\alpha)}\quad\text{..................................(16c),}$$

where $$S_\varpi'(x)=\{G_\varpi(x)\,G_\varpi(1-x)\}^{-1}.$$

In this result put, in turn,

$$\sigma=\mu^{-1},\ \mu^{-1}-1+\rho_1,\ \ldots,\ \mu^{-1}-1+\rho_s,$$

† See equation (6).

‡ We put $G_p(x)\,G_p(1-x)=\{S_p'(x)\}^{-1}$.

and from the results so obtained substitute for the sine-exponential series in (16*b*). We then get

$$
\begin{aligned}
&{}_rS_s(x)\,.\,p^{\frac{1}{2}\mu x(x-1)+x(1+\Sigma\rho-\Sigma\alpha)}\,S_{\varpi}'\left(\tfrac{1}{2}\mu+\tfrac{1}{2}-\mu x-\Sigma\rho+\Sigma\alpha\right)\\
&=\sum_{n=0}^{\infty} G_{\varpi}(\mu n-\mu x)\left\{\prod_{t=1}^{r,s}\frac{S_p'(\alpha_t)}{S_p'(\rho_t)}\,\frac{G_p(\alpha_t+n)}{G_p(\rho_t+n)}\right\}\frac{p^n}{G_p(n+1)}\,S_{\varpi}'\left(\tfrac{1}{2}\mu+\tfrac{1}{2}-\Sigma\rho+\Sigma\alpha\right)\\
&+\sum_{k=1}^{s}\sum_{n=0}^{\infty} G_{\varpi}(\mu n-\mu\rho_k+\mu-\mu x)\left\{\prod_{t=1}^{r,s}{}^{*}\,\frac{S_p'(\alpha_t-\rho_k+1)}{S_p'(\rho_t-\rho_k+1)}\,\frac{G_p(\alpha_t-\rho_k+n+1)}{G_p(\rho_t-\rho_k+n+1)}\right\}\\
&\frac{p^n}{G_p(n+1)\,S_p'(2-\rho_k)\,G_p(2-\rho_k+n)}\times S_{\varpi}'\left(\tfrac{1}{2}-\tfrac{1}{2}\mu+\mu\rho_k-\Sigma\rho+\Sigma\alpha\right)\ \ldots\ldots\ldots\ldots\ldots\ldots(16d).
\end{aligned}
$$

We shall write this in the form

$$
{}_rS_s(x)=p^{-\frac{1}{2}\mu x(x-1)-x(1+\Sigma\rho-\Sigma\alpha)}\left\{S_{\varpi}'\left(\tfrac{1}{2}\mu+\tfrac{1}{2}-\mu x-\Sigma\rho+\Sigma\alpha\right)\right\}^{-1}\sum_{k=0}^{s}T_k(x)\quad\ldots(16e).
$$

This formula gives the analytic continuation of ${}_rS_s(x)$ over the whole x plane; we observe that as $r_2\to-\infty$, $T_k(x)\to$ a finite limit. Using the asymptotic expansion of the function S_{ϖ}', we obtain theorem II (iii) without difficulty; while the theorem II (ii) is obtainable from the same formula on observing that, when r_2 is positive, $|G_{\varpi}(\mu n-\mu x)|$ does not exceed a finite quantity depending on ϵ.

Theorem (III) may be proved as follows: put

$$
\frac{G_p(n\mu^{-1}-x)}{G_{\varpi}(n+1)}\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t+n\mu^{-1}-x)}{G_p(1-\alpha_t+n\mu^{-1}-x)}=\chi_n(x).
$$

Then if n be sufficiently large one of the values of $\log\chi_n(x)$ may be expanded in the form

$$
\log\chi_n(x)=\sum_{m=1}^{\infty}p^{nm/\mu}f_m(x),
$$

where the functions $f_m(x)$ may be calculated with sufficient labour; this result follows from the expansion †

$$
\log G_p(n\mu^{-1}+\alpha)=\sum_{m=1}^{\infty}\frac{p^{nm/\mu}\,p^{\alpha m}}{m(1-p^m)}.
$$

Now
$$
\prod_{m=0}^{l}\left(1-yp^{x+\lambda+m/\mu}\right)\sum_{n=0}^{\infty}\chi_n(x)\,y^n p^{n(x+\lambda)}\equiv\sum_{n=0}^{\infty}\Psi_{l,n}(x)\,.\,y^n\quad\ldots\ldots\ldots\ldots\ldots\ldots(17)
$$

when $|y|$ is sufficiently small; where

$$
\begin{aligned}
\Psi_{l,n}(x)=p^{n(x+\lambda)}\Big\{\chi_n(x)-\frac{[l+1]}{[1]}\chi_{n-1}(x)+\ldots&\\
+(-)^r\frac{[l+1]\,[l]\ldots[l-r+2]}{[1]\ldots[r]}\,\varpi^{\frac{1}{2}r(r-1)}\,\chi_{n-r}(x)+\ldots\Big\}&,
\end{aligned}
$$

the coefficients of the functions χ being the basic binomial coefficients *with base* ϖ.

Put $x_0=-\lambda-l/\mu$, and we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\Psi_{l,n}(-\lambda-l/\mu)=\operatorname*{Lt}_{n=\infty}p^{-n\lambda/\mu}\Big\{\chi_n(x_0)-\frac{[l]}{[1]}\chi_{n-1}(x_0)+\ldots&\\
+(-)^r\frac{[l]\ldots[l-r+1]}{[1]\ldots[r]}\,\varpi^{\frac{1}{2}r(r-1)}\,\chi_{n-r}(x_0)+\ldots\Big\}&,
\end{aligned}
$$

the base being ϖ.

† Littlewood, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 5 (1907), p. 395.

Putting $\chi_n(x) = \sum_{m=0}^{\infty} p^{nm/\mu} f_m'(x)$ we find without difficulty from this formula that

$$\sum_{n=0}^{\infty} \Psi_{l,n}(-\lambda - l/\mu) = (1-\varpi^{-1})(1-\varpi^{-2})\dots(1-\varpi^{-l}) . f_l'\left(\tfrac{1}{2} - \frac{1}{2\mu} + \frac{1}{\mu}\Sigma\alpha - \frac{1}{\mu}\Sigma\rho - \frac{l}{\mu}\right),$$

where the values of the functions $f_0', f_1', f_2', \dots$ may be determined successively.

Putting $y=1$ in (17) we observe that the residues of $\sum_{n=0}^{\infty} \chi_n(x) p^{n(x+\lambda)}$ at the points where $p^{x+\lambda} = p^{-m/\mu}$ may be calculated with sufficient labour; that is to say that the residues of ${}_rS_s(x)$ at the poles of $G_\varpi(\mu x - \tfrac{1}{2}\mu + \tfrac{1}{2} + \Sigma\rho - \Sigma\alpha)$ may be calculated with sufficient labour should necessity arise.

Let the residue of ${}_rS_s(x)$ at $x = -\lambda - l\mu^{-1}$ be $U(l)$.

Now consider the integral

$$I_5 = \frac{1}{2\pi i}\int_C {}_rS_s(x)(-y_1)^x\, dx.$$

By theorem (II) the integral converges for all finite values of $|y_1|$ and $\arg(-y_1)$. When $|y_1| < 1$ we may shew, using theorem II (ii), that the integral is equal to the sum of the residues of the integrand at those poles which lie to the right of the contour; and consequently

$$-I_5 = \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \frac{G_p(1)}{\omega\, G_\varpi(n+1)} (-y_1)^{n/\mu}\, \varpi^{\frac{1}{2}n(n-1)} \prod_{t=1}^{r,s} \frac{G_p(1-\rho_t)}{G_p(1-\alpha_t)}\, {}_rF_s([\alpha_1], \dots [\alpha_r]; [\rho_1], \dots [\rho_s]; z)$$

$$\times \exp\left\{-\frac{2\mu k^2\pi^2}{\omega} + 2k\pi i(\Sigma\rho - \Sigma\alpha + 1 - \tfrac{1}{2}\mu) + \frac{2k\pi i}{\omega}\log(-y_1)\right\}$$

$$+ \sum_{m=1}^{s}\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \frac{G_p(1)}{\omega\, G_\varpi(n+1)} (-y_1)^{n/\mu}\, \varpi^{\frac{1}{2}n(n-1)} (-y_1)^{1-\rho_m} \prod_{t=1}^{r,s}{}^{*} \frac{G_p(\rho_m - \rho_t)}{G_p(\rho_m - \alpha_t)} G_p(\rho_m - 1)$$

$${}_rF_s([\alpha_1 - \rho_m + 1], \dots [\alpha_r - \rho_m + 1]; [\rho_1 - \rho_m + 1] \dots {*} \dots [\rho_s - \rho_m + 1], [2 - \rho_m]; z)$$

$$\times \exp\left\{-\frac{2\mu k^2\pi^2}{\omega} + 2k\pi i(\Sigma\rho - \Sigma\alpha + 1 + \tfrac{1}{2}\mu - \mu\rho_m) + \frac{2k\pi i}{\omega}\log(-y_1) + \omega(1-\rho_m)\right.$$
$$\left.(\Sigma\rho - \Sigma\alpha + 1 - \tfrac{1}{2}\mu\rho_m)\right\}.$$

On the other hand, using theorem II (iii), we may shew that when $|y_1| > 1$, I_5 is equal to minus the sum of the residues of the integrand at the poles of $G_\varpi(\mu x - \tfrac{1}{2}\mu + \tfrac{1}{2} + \Sigma\rho - \Sigma\alpha)$; so that

$$I_5 = -(-y_1)^{\frac{1}{2} - (\frac{1}{2} - \Sigma\alpha + \Sigma\rho)/\mu} \sum_{n=0}^{\infty} U(n) . (-y_1)^{-n/\mu}$$
$$\times \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{2\mu k^2\pi^2}{\omega} - k\pi i - \frac{2k\pi i}{\omega}\log(-y_1)\right\}.$$

To get rid of the exponential series, we consider the integral

$$-\frac{1}{2\pi i}\int_C G_\varpi(\alpha - x)\, G_\varpi(\rho + x) . \varpi^{-\frac{1}{2}\mu x(x+1) + \lambda x}\, (-y_1)^{x/\mu}\, dx.$$

When $|y_1| < |p^{\alpha-\lambda}|$, the integral is equal to minus the sum of the residues of the integrand at poles to the right of the contour; when $|y_1| > |p^{1-\rho-\lambda}|$, the integral is equal to the sum of the residues of the integrand at poles to the left of the contour; hence, using the basic analogue of the binomial theorem, we find that the analytic continuation of

$$(-y_1)^{\alpha/\mu} \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{2\mu k^2\pi^2}{\omega} + \frac{2k\pi i}{\omega}\log(-y_1) + k\pi i(2\alpha+1-2\lambda) - \frac{\omega\alpha}{2\mu}(2\lambda-\alpha-1)\right\}$$

is $$(-y_1)^{-\rho/\mu} \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{2\mu k^2\pi^2}{\omega} + k\pi i(2\rho+2\lambda-1) - \frac{2k\pi i}{\omega}\log(-y_1) + \frac{\omega\rho}{2\mu}(2\lambda+\rho-1)\right\}$$

$$\times \prod_{n=0}^{\infty} \frac{\{1+(-y_1)^{1/\mu}\varpi^{\lambda-\alpha+n}\}\{1+(-y_1)^{-1/\mu}\varpi^{1+\alpha-\lambda+n}\}}{\{1+(-y_1)^{1/\mu}\varpi^{\rho+\lambda+n}\}\{1+(-y_1)^{-1/\mu}\varpi^{1-\rho-\lambda+n}\}}$$

In this result put, in turn,

$$\alpha = 0,\ \mu(1-\rho_1),\ \mu(1-\rho_2) \ldots \mu(1-\rho_s)$$

$$\lambda = -\rho = \Sigma\alpha - \Sigma\rho + \tfrac{1}{2}\mu - \tfrac{1}{2},$$

and taking note of the theorem (the basic analogue of the exponential theorem) that

$$\sum_{n=0}^{\infty} \frac{(-y_1)^{n/\mu}\varpi^{\frac{1}{2}n(n-1)}}{G_\varpi(n+1)} = \frac{1}{G_\varpi(1)} \prod_{n=0}^{\infty}\{1+(-y_1)^{1/\mu}\varpi^n\}$$

we find on comparing the two values of I_5 that the analytic continuation of

$$\prod_{t=1}^{r,s} \frac{G_p(1-\rho_t)}{G_p(1-\alpha_t)}\, {}_rF_s([\alpha_1], \ldots [\alpha_r]; [\rho_1], \ldots [\rho_s]; z)$$

$$\times \prod_{n=0}^{\infty}\{1+(-y_1)^{1/\mu}\varpi^{\Sigma\alpha-\Sigma\rho-\frac{1}{2}+\frac{1}{2}\mu+n}\}\{1+(-y_1)^{-1/\mu}\varpi^{\frac{3}{2}+\Sigma\rho-\Sigma\alpha-\frac{1}{2}\mu+n}\}$$

$$+ \sum_{m=1}^{s} \prod_{t=1}^{r,s}{}^{*} \frac{G_p(\rho_m-\rho_t)}{G_p(\rho_m-\alpha_t)} G_p(\rho_m-1)\, {}_rF_s([\alpha_1-\rho_m+1], \ldots [\alpha_r-\rho_m+1]; [\rho_1-\rho_m+1],$$
$$\ldots * \ldots [\rho_s-\rho_m+1], [2-\rho_m]; z)$$

$$\times \prod_{n=0}^{\infty}\{1+(-y_1)^{1/\mu}\varpi^{\Sigma\alpha-\Sigma\rho-\frac{1}{2}\mu-\frac{1}{2}+\mu\rho_m+n}\}\{1+(-y_1)^{-1/\mu}\varpi^{\frac{3}{2}+\Sigma\rho-\Sigma\alpha+\frac{1}{2}\mu-\mu\rho_m+n}\}$$

is

$$\frac{\omega\, G_\varpi(1)}{G_p(1)} \exp\left\{\frac{\omega}{2\mu}(\Sigma\rho-\Sigma\alpha-\tfrac{1}{2}\mu+\tfrac{1}{2})(\Sigma\rho-\Sigma\alpha-\tfrac{1}{2}\mu+\tfrac{3}{2})\right\} \prod_{n=0}^{\infty}\{1+(-y_1)^{-1/\mu}\varpi^{n+1}\}$$
$$\times \sum_{n=0}^{\infty} U(n)\,.\,(-y_1)^{-n/\mu} \quad \ldots\ldots\ldots(18).$$

There are μ different results concerning the same $s+1$ basic hypergeometric functions contained in this theorem. It was assumed that $-y_1$ had a definite argument; if we write in turn $(-y_1)e^{2\pi i}$, $(-y_1)e^{4\pi i}$, … $(-y_1)e^{2(\mu-1)\pi i}$ in place of $(-y_1)$, the basic hypergeometric functions are unaltered, but their coefficients are altered, giving the μ formulae required to complete this portion of the theory.

8. The basic hypergeometric function ${}_rF_s$ (base q) does not exist when $\mu<0$; if $p=q^{-1}$ and $\mu\geqslant 0$, we have

$$
\begin{aligned}
&{}_rF_s([\alpha_1], \dots [\alpha_r]; [\rho_1], \dots [\rho_s]; z) \quad (\text{base } q)\\
&= G_p(1)\prod_{t=1}^{r,s}\frac{G_p(\rho_t)}{G_p(\alpha_t)}\sum_{n=0}^{\infty}\frac{G_p(\alpha_t+n)}{G_p(\rho_t+n)}\frac{(1-p)^{\mu n}}{G_p(1+n)}z^n p^{\frac{1}{2}\mu n(n-3)+n(1+\Sigma\rho-\Sigma\alpha)}\\
&= {}_r\mathfrak{P}_s([\alpha_1], \dots [\alpha_r]; [\rho_1], \dots [\rho_s]; z), \quad (\text{base } p), \text{ say.}
\end{aligned}
$$

When $\mu=0$, ${}_r\mathfrak{P}_s$ is substantially ${}_rF_s$, and has radius of convergence equal to $|p^{\Sigma\alpha-\Sigma\rho-1}|$; when $\mu>0$, ${}_r\mathfrak{P}_s$ is an integral function of z.

We shall shew that r linear combinations of $s+1$ functions of the type ${}_r\mathfrak{P}_s$ admit of (convergent) asymptotic expansions of a simple type; and we shall shew how the (divergent) asymptotic expansion of any one of these functions may be obtained.

Consider the integral

$$I_6=-\frac{1}{2\pi i}\int_C\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t-x)}{G_p(1-\alpha_t-x)}\frac{\pi G_p(-x)}{\sin\pi(x+\alpha_1)}y_2^x\,dx,$$

where

$$y_2=(-)^{\mu-1}p^{-\mu}(1-p)^\mu z.$$

The integral converges if $|\{\arg(y_2)-\omega_2\omega_1^{-1}\log|y_2|\}|<\pi$.

For all finite values of $|y_2|$ we may shew that the integral is equal to minus the sum of the residues at the poles of the integrand which lie to the right of the contour.

Hence

$$
\begin{aligned}
I_6 &= \frac{\pi}{\omega}G_p(1)\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t)}{G_p(1-\alpha_t)}\,{}_r\mathfrak{P}_s([\alpha_1], \dots [\alpha_r]; [\rho_1], \dots [\rho_s]; z)\\
&\times\sum_{k=-\infty}^{\infty}\operatorname{cosec}\left(\alpha_1+\frac{2k\pi i}{\omega}\right)\pi\,.\exp\left\{\frac{2k\pi i}{\omega}\log y_2\right\}\\
&+\sum_{m=1}^{s}\frac{\pi}{\omega}G_p(1)\prod_{t=1}^{r,s}{}^{*}\frac{G_p(\rho_m-\rho_t)}{G_p(\rho_m-\alpha_t)}G_p(\rho_m-1)\,y_2^{1-\rho_m}\,{}_r\mathfrak{P}_s([\alpha_1-\rho_m+1], \dots [\alpha_r-\rho_m+1];\\
&\qquad\qquad [\rho_1-\rho_m+1], \dots * \dots [\rho_s-\rho_m+1], [2-\rho_m]; z)\\
&\times\sum_{k=-\infty}^{\infty}\operatorname{cosec}\left(\alpha_1+1-\rho_m+\frac{2k\pi i}{\omega}\right)\pi\,.\exp\left\{\frac{2k\pi i}{\omega}\log y_2\right\}.
\end{aligned}
$$

When $|y_2|<1$, I_6 is equal to the sum of the residues at the poles of the integrand which lie to the left of the contour; i.e.

$$
\begin{aligned}
I_6 &= y_2^{-\alpha_1}G_p(\alpha_1)\prod_{t=1}^{r,s}\frac{G_p(1-\rho_t+\alpha_1)}{G_p(1-\alpha_t+\alpha_1)}\,{}_{s+1}F_{r-1}([\alpha_1], [1-\rho_1+\alpha_1], \dots [1-\rho_s+\alpha_1];\\
&\qquad\qquad [1-\alpha_2+1]\dots[1-\alpha_r+\alpha_1]; (-)^\mu p^\mu z^{-1}).
\end{aligned}
$$

Getting rid of the cosecant-exponential series by formula ($6a$), we see that the (convergent) asymptotic expansion of

$$\prod_{t=1}^{r,s} \frac{G_p(1-\rho_t)}{G_p(1-\alpha_t)} G_p(\alpha_1)\, {}_r\mathfrak{P}_s([\alpha_1], \ldots [\alpha_r]; [\rho_1], \ldots [\rho_s]; z) \prod_{m=0}^{\infty} (1+y_2 p^{1-\alpha_1+m})(1+y_2^{-1}p^{\alpha_1+m})$$

$$+ \sum_{n=1}^{s} \prod_{t=1}^{r,s}{}^{*} \frac{G_p(\rho_n-\rho_t)}{G_p(\rho_n-\alpha_t)} G_p(\alpha_1-\rho_n+1)\, G_p(\rho_n-\alpha_1) \prod_{m=0}^{\infty} (1+y_2 p^{\rho_n-\alpha_1+m})(1+y_2^{-1}p^{1-\rho_n+\alpha_1+m})$$

$${}_r\mathfrak{P}_s([\alpha_1-\rho_n+1], \ldots [\alpha_r-\rho_n+1]; [\rho_1-\rho_n+1] \ldots * \ldots [\rho_s-\rho_n+1], [2-\rho_n]; z)$$

is $\displaystyle\prod_{t=1}^{r,s} \frac{G_p(1-\rho_t+\alpha_1)}{G_p(1-\alpha_t+\alpha_1)} G_p(\alpha_1)\, G_p(1) \prod_{m=0}^{\infty} (1+y_2 p^{m+1})(1+y_2^{-1}p^m)$

$$\times\, {}_{s+1}F_{r-1}([1-\rho_1+\alpha_1], \ldots [1-\rho_s+\alpha_1], [\alpha_1]; [1-\alpha_2+\alpha_1] \ldots [1-\alpha_r+\alpha_1]; (-)^{\mu} p^{\mu} z^{-1}).$$

On interchanging $\alpha_1, \alpha_2, \ldots \alpha_r$ cyclically in this result, we get the (convergent) asymptotic expansions as stated.

9. To obtain the asymptotic expansion of any one of the functions ${}_r\mathfrak{P}_s$ we proceed as follows:

Let $$ {}_rP_s(x) = \frac{S_{\varpi}'(\frac{1}{2}\mu+\frac{1}{2}-\mu x-\Sigma\rho+\Sigma\alpha)}{\sin(\frac{1}{2}\mu+\frac{1}{2}-\mu x-\Sigma\rho+\Sigma\alpha)\pi} p^{-x(\frac{1}{2}+\frac{1}{2}\mu-\Sigma\rho+\Sigma\alpha)}$$
$$\times \sum_{n=0}^{\infty} \frac{G_{\varpi}(\mu n-\mu x)}{G_p(n+1)} \prod_{t=1}^{r,s} \frac{G_p(\alpha_t+n)}{G_p(\rho_t+n)} p^{n(\frac{3}{2}+\Sigma\rho-\Sigma\alpha+\mu x-\frac{1}{2}\mu)}.$$

We shall denote the general term of the infinite series by $M_n(x)$ and put

$$\sum_{n=0}^{\infty} M_n(x) = M(x).$$

Then we may shew: (I) that the only singularities of $M(x)$ apart from simple poles at the poles of $G_{\varpi}(-\mu x)$ are simple poles at the poles of $G_p(\frac{3}{2}+\Sigma\rho-\Sigma\alpha+\mu x-\frac{1}{2}\mu)$; and the residues of $M(x)$ at these points may be calculated with sufficient labour.

(II) That if $x = r_2 + r_1(\omega_2+\omega_1 i)$, r_2 and r_1 being real, then the asymptotic expansions of $M(x)$ are the following:

(i) when $|r_1|$ is large and $|r_2|$ finite, $|M(x)| < J$;

(ii) when r_2 is large and positive

$$|M(x)| < J \exp(-\tfrac{1}{2}\mu\omega_1 r_2^2 + \tfrac{1}{2}\omega_1 r_2)$$

where J is a finite quantity.

Theorem (I) and Theorem II (i) may be proved in precisely the same manner as Theorems (I), (III) and II (i) of § 7. We shall not, therefore, give proofs of these theorems.

We denote the residue of ${}_rP_s(x)$ at the point

$$x = \tfrac{1}{2} + \tfrac{1}{2}\mu^{-1} - \mu^{-1}\Sigma\rho + \mu^{-1}\Sigma\alpha - n + \frac{2k\pi i}{\omega} + \frac{2g\pi i}{\mu\omega}$$

where g is one of the integers, 0, 1, ... $\mu-1$, by the symbol $U(n, g)$; and we note that $U(n, g)$ is independent of k.

Theorem II (ii) may be proved as follows:

Let N be the integer such that $N \leqslant r_2 < N+1$.

Then $$|M(x)| < \sum_{n=0}^{N} |M_n(x)| + \sum_{n=N+1}^{\infty} |M_n(x)|.$$

When $n \leqslant N$ we can find a finite quantity J independent of x such that

$$|G_{\varpi}(\mu n - \mu x)| < J \exp\left(-\frac{\mu}{2\omega_1}\{R(\omega n - \omega x)\}^2 - \tfrac{1}{2}R(\omega n - \omega x)\right).$$

Hence

$$|M_n(x)| < J' \exp\left(-\frac{\mu}{2\omega_1}\{R(\omega x)\}^2 + \tfrac{1}{2}R(\omega x) - \frac{\mu\omega_1}{2}n^2 - nR\{\omega(2 + \Sigma\rho - \Sigma\alpha - \tfrac{1}{2}\mu)\}\right).$$

Consequently $$\sum_{n=0}^{N} |M_n(x)| < J'' \exp\left(-\frac{\mu}{2\omega_1}\{R(\omega x)\}^2 + \tfrac{1}{2}R(\omega x)\right).$$

When $n > N$, $|G_{\varpi}(\mu n - \mu x)| < J$, and, taking into account the power of p involved in $M_n(x)$, we obtain the required theorem without difficulty.

We may then shew that, by II (i), the integral

$$I_7 = -\frac{1}{2\pi i}\int_C {}_rP_s(x)\, y^x\, dx$$

{where $y = z(1-p)^{\mu} p^{-\mu}$} converges if

$$|\{\arg y - \omega_2\omega_1^{-1}\log|y| + (\omega_1^2 + \omega_2^2)\,\omega_1^{-1} R(i\Sigma\rho - i\Sigma\alpha)\}| < \mu\pi \qquad (19).$$

Also, the integral is equal to minus the sum of the residues of the integrand at those poles which lie to the right of the contour, giving, after some algebra and the use of formula (6a), the result that

$$I_7 = \sum_{m=0}^{\infty} \frac{(-)^m y^{m/\mu}}{\pi\, G_{\varpi}(m+1)} \prod_{n=0}^{\infty} \left\{\frac{(1 + y^{1/\mu}\varpi^n)(1 + y^{-1/\mu}\varpi^{n+1})}{(1 + y^{1/\mu}\varpi^{n-\frac{1}{2}-\frac{1}{2}\mu+\Sigma\rho-\Sigma\alpha})(1 + y^{-1/\mu}\varpi^{n+\frac{3}{2}+\frac{1}{2}\mu-\Sigma\rho+\Sigma\alpha})}\right\}$$
$$\times \frac{1}{G_p(1)} \prod_{t=1}^{r,s} \frac{G_p(\alpha_t)}{G_p(\rho_t)}\, {}_r\mathfrak{P}_s([\alpha_1], \ldots [\alpha_r]; [\rho_1], \ldots [\rho_s]; z).$$

Also it may be shewn that the integral, when taken along the line $R(\omega x) = -l\omega_1$, is of order y^{-l}; where l is finite and positive, and m is the largest integer such that $m < l$; we deduce an asymptotic expansion, by Cauchy's theorem, in the form:

$I_7 + Jy^{-l} =$ the sum of the residues at the poles of the integrand which lie between the contours

$$= \frac{\omega}{\mu\pi\Omega_\mu}\exp\left\{\frac{\omega}{\mu}(\tfrac{1}{2} + \tfrac{1}{2}\mu - \Sigma\rho + \Sigma\alpha)^2\right\} y^{(\frac{1}{2}+\frac{1}{2}\mu-\Sigma\rho+\Sigma\alpha)/\mu} \sum_{n=0}^{m}\sum_{g=0}^{\mu-1} U(n, g)\, y^{-n} p^{-\frac{1}{2}\mu n(n-1) - n(\Sigma\rho - \Sigma\alpha)}$$
$$\times \prod_{k=0}^{\infty} \frac{(1 + y^{1/\mu}\varpi^{-\frac{1}{2}-\frac{1}{2}\mu+\Sigma\rho-\Sigma\alpha+k}\exp 2g\pi i\mu^{-1})(1 + y^{-1/\mu}\varpi^{\frac{3}{2}+\frac{1}{2}\mu+\Sigma\alpha-\Sigma\rho+k}\exp -2g\pi i\mu^{-1})}{(1 + y^{1/\mu}\varpi^{k-\frac{1}{2}-\frac{1}{2}\mu+\Sigma\rho-\Sigma\alpha})(1 + y^{-1/\mu}\varpi^{k+\frac{3}{2}-\frac{1}{2}\mu-\Sigma\rho+\Sigma\alpha})};$$

whence we obtain the asymptotic expansion of ${}_r\mathfrak{P}_s$ in terms of known functions.

This result completes the theory of generalised basic hypergeometric functions.

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XII. *On a Class of Integral Functions.*

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Introduction.

In the theory of elliptic ϑ-functions, the formulae of transformation from $\vartheta(\nu \mid \tau)$ to $\vartheta(\tau^{-1}\nu \mid -\tau^{-1})$ enable us, in the case when τ is small (or when $q = e^{\iota\pi\tau}$ is nearly unity), to find for the function $\vartheta(\nu \mid \tau)$ a simple expression which represents it with a very high degree of approximation. In fact, in the notation which will be employed throughout the present paper, if we write

$$\psi(\phi, \omega) = \prod_{s=1}^{\infty}(1 + e^{\phi\iota - s\omega}) \prod_{s=0}^{\infty}(1 + e^{-\phi\iota - s\omega}),$$

where $R\omega > 0$, and where ω is supposed to be small, we easily obtain from the known transformation formulae the equation,

$$\psi(\phi, \omega) = [1 + \chi(\omega)] \exp\left[\left(\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2\right)\omega^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega\right] \quad \text{.................(1)},$$

where $\chi(\omega)$ is comparable in order with $\exp(-|\omega|^{-1})$. Since $\exp(|\omega|^{-1})$ is of a higher type of order than $|\omega|^{-1}$, the approximation given by (1) is extremely close.

The following considerations suggest that this formula may play an important part in the problem of approximating for integral functions which satisfy appropriate conditions. Suppose that we have an integral function, of order not greater than unity, defined by the product-form $\Pi(1 + z/a_s)$, where the a's are arranged (as usual) in order of increasing moduli, and where a_s is a naturally constructed function of s. Suppose z is large and equal to $a_n e^{\iota\phi}$ (where we suppose for simplicity that ϕ is real). The number n is evidently large. We have

$$\log F(z) - \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] = \sum_{s=1}^{\infty} \log\left(1 + \frac{z}{a_{n+s}}\right) + \sum_{s=0}^{n-1} \log\left(1 + \frac{a_{n-s}}{z}\right) \quad \text{...........(2)}.$$

Now if $\log(a_{n+s}/a_n)$ is expansible in a power-series in s,

$$\log(a_{n+s}/a_n) = s\omega(n) + s^2\omega_1(n) + \ldots,$$

where

$$\omega_r(n) = \frac{1}{(r+1)!}\left(\frac{d}{dn}\right)^{r+1}(\log a_n).$$

If a_n is of less order than $\exp(\epsilon n)$ *for all values of* ϵ, $\omega(n)$ will be small, and $\omega_r(n)$ will be of order not greater than $n^{-r}\omega(n)$. If now we assume that within a certain limit for s we may replace $a_{n\pm s}$ by $a_n \exp[\pm s\omega(n)]$, neglecting the terms (of less order in n) $s^2\omega_1(n) + s^3\omega_2(n) + \dots$, and that beyond this limit for s we may neglect the terms $\log(1 + z/a_{n+s})$ and $\log(1 + a_{n-s}/z)$ altogether, the following expression suggests itself for the right-hand side of (2):

$$\sum_{s=1}^{\infty} \log(1 + e^{\iota\phi - s\omega}) + \sum_{s=0}^{\infty} \log(1 + e^{-\iota\phi - s\omega}), \quad \text{or} \quad \log \psi(\phi, \omega).$$

On using the formula (1) for $\psi(\phi, \omega)$ we obtain for $\log F(z)$ the approximation

$$\log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] + \left(\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2\right)[\omega(n)]^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega(n),$$

which is expressed in terms of z, and the n and ϕ depending on z.

When we investigate the assumptions made in this method we find that, in order that any of the terms furnished by $\log\psi(\phi, \omega)$ shall be relevant (i.e. of higher order than the error) we must suppose that

$$\lim_{n\to\infty} |n\omega(n)| = \infty.$$

We must confine ourselves, then, to the functions $F(z)$ for which $\omega(n)$ satisfies the two conditions $\lim |n\omega(n)| = \infty$ and $\lim |\omega(n)| = 0$. The first condition requires the "order" of $F(z)$ to be zero, and the second requires that the nth zero of $F(z)$, for all sufficiently great values of n, is less than $\exp(\epsilon n)$, however small the positive number ϵ may be. It is evident that further conditions for the a's must be satisfied in order that our method may be possible. It will, however, appear that we can find a set of conditions which will be satisfied at any rate if $F(z)$ is any integral function of zero order whose nth zero is a naturally constructed function of n of less order than $\exp(\epsilon n)$ for all positive values of ϵ*. Moreover, these conditions being assumed satisfied, a further development of the idea outlined above leads to a general formula of approximation for $\log F(z)$ in which the order of the terms descends as low as that of $[\omega(n)]^{-1}[n\omega(n)]^{-p}$, where p is any positive integer†. The remainder term in the expansion is the sum of two terms, one comparable in order with $[\omega(n)]^{-1}[n\omega(n)]^{-p-1}$, and the other comparable with $\exp\{-[\omega(n)]^{-\delta}\}$ $(\delta > 0)$. The former expression decreases, while the latter increases, as the rate of decrease of $\omega(n)$ becomes more and more slow, and the necessity of the two conditions imposed on $\omega(n)$ is evident from the forms of the two remainder terms. It is further evident that the approximation for $\log F(z)$ is best when the order of $\omega(n)$ is neither high nor low in the range of order permitted by the conditions. In fact, if $\omega(n)$ is of order comparable with n^{k-1} $(0 < k < 1)$ [which is roughly equivalent to saying that a_n is comparable in order with $\exp(n^k)$] it will be seen that the error term in the formula for $\log F(z)$ is of order $(\log z)^{-r}$, where r is an arbitrary positive constant. If, however, we

* For example

$$\exp[n^{\frac{2}{3}} \exp\{(\log n)^{\frac{3}{4}} (\log\log n)^2\} + \tfrac{1}{2} n^{\frac{1}{2}} (\log n)^3 (\log\log n)^{-4}].$$

† It is found possible to determine the effect on the expression (2) of any finite number of terms of the series

$$s^2\omega_1(n) + s^3\omega_2(n) + \dots$$

occurring in the expansion of $\log(a_{n+s}/a_n)$.

have, for example, on the one hand $\omega(n) = (\log n)^{-1}$, or on the other hand $\omega(n) = n^{-1}\log n$, the approximation is of a different character, and is not so good.

In Part I of the paper, Section I is devoted to the consideration of the asymptotic formula for $\log F(z)$. After the formal proof (which occupies §§ 5—12) it is shown (§ 13) that the conditions of the result are satisfied when $F(z)$ is a naturally constructed product-form in which the order of a_n is restricted in the manner explained above. A number of particular functions $F(z)$ are then discussed. The discussion serves incidentally to illustrate the methods by which, in any particular case, the general formula involving z, n, and ϕ is reduced, when this is possible, to an expression in terms of z alone. In certain of the examples [those for which $\omega(n)$ decreases very slowly as n increases] the error term of the approximation assumes a somewhat remarkable form. For a discussion of this point the reader is referred to § 22.

In Section II of Part I we consider the problem of finding an approximation for the coefficient c_n of z^n in the Taylor-series of a function $F(z)$ defined by a product-form. It will be seen that provided $F(z)$ belongs to a certain sub-class of the class of functions considered in Section I, we can find a formula for c_n of the type

$$c_n = [1 + \epsilon(n)] \frac{f(n)}{a_1 a_2 \dots a_n},$$

where $\epsilon(n)$ tends to zero as n tends to ∞, and where $f(n)$ is expressed in finite terms as an explicit function of n. The main idea of the proof is to take for the contour C in the equation

$$2\pi\iota c_n = \int_C F(z) z^{-n-1} dz,$$

the circle $|z| = a_n$, and to use the approximation for $F(z)$ in order to approximate for the integral. It is necessary to add to the conditions of Section I, a further restriction on the a's, which is roughly equivalent to the condition that for *some* positive value of η, $|a_n|$ is of greater order than $\exp(n^\eta)$. [The new class of functions is thus characterised by the fact that $|a_n|$ is intermediate in order between $\exp(n^\eta)$ and $\exp(\epsilon n)$, for some value of η, and for all values of ϵ.] It should be mentioned that the original form of the approximate formula for $\log F(z)$ is not, by itself, sufficient for our new purpose. The necessary modification is established in § 24.

The solubility of the two problems concerning product-forms, and especially that of the second, suggests that it may be possible to develop some kind of theory for integral functions $F(z)$ defined by Taylor-series $\Sigma c_n z^n$ whose coefficients satisfy appropriate conditions. The two corresponding problems presented by Taylor-series are to find approximate formulae for $F(z)$ when z is large, and for the nth zero of $F(z)$ when n is large. It will be seen in Part II that these problems are soluble provided the c's satisfy a certain set of conditions. These conditions are satisfied, at any rate, by all naturally constructed Taylor-series for which $|c_n|^{-1}$ is of less order than $\exp(\epsilon n^2)$ for all values of ϵ, and of greater order than $\exp(n^{1+\eta})$ for some value of η. The functions of the class determined by the conditions are substantially the same as those of the class considered in Part I, Section II*, and the degree of the approximation attained for each different type of Taylor-series is the same as that attained for the corresponding type of product-form, although the result is expressed in a different way.

With regard to the zeros of $F(z)$, it is shown that we can find a set of circles Γ_s, of radii so small that (after a certain value of s) they are all external to each other, such that when s is greater than a certain constant, the sth zero of $F(z)$ (in order of increasing moduli) lies inside Γ_s. This result leads incidentally to the interesting consequence that in the case when the coefficients of $F(z)$ are real, the integral function has only a finite number of zeros which are not real.

NOTATION.—It will be found convenient to denote by a unique functional form any function of x which tends to zero with x (or $1/x$) more rapidly than any power of x (or $1/x$), just as we use $\epsilon(x)$ to denote indiscriminately any function of x which tends to zero. We shall, when it is convenient, write $\chi(x)$ for any function $f(x)$ of x such that $\lim_{x\to 0} |x^{-p} f(x)| = 0$ in the case when $x \longrightarrow 0$, or such that $\lim_{x\to\infty} |x^p f(x)| = 0$ in the case when $x \longrightarrow \infty$.

We shall write $O[f(x)]$ for any function $\phi(x)$ such that $|\phi(x)/f(x)|$ is less than a finite constant for all values of x considered.

When we wish to express that $|f(x)|$ remains less than some finite constant, we shall write $|f(x)| < K$, always using the unique symbol K. We shall, moreover, extend the use of this symbol. Thus we write

$$(1 + x^{-1}) \exp [(x + x^{\frac{1}{2}})^2] < K \exp (Kx^3), \quad \exp x > K^{-1} \exp (x^{K-1}).$$

Either inequality is certainly satisfied (when $x > \frac{1}{2}$, say) *provided the constant K is chosen sufficiently large.* Moreover if it is satisfied for any particular value of K, *it will remain satisfied when K is increased.* Any particular K always makes its *first* appearance in an inequality for which this is the case. Finally, in a sequence of inequalities, such as

$$(1 + x) \exp (x) < K_1 \exp (K_1 x) < K_2 \exp (K_2 x^2)$$

(where x may assume all positive values), we shall omit the suffixes. For the second inequality to hold, K_2 must evidently depend on K_1: thus the K at the right-hand end of a sequence of inequalities may depend on all previous K's. The convention of dropping the suffixes is, however, unlikely to lead to ambiguity†.

* It will be found that the product-forms of Part I, Section II (which are *included* among those of Section I) and the Taylor-series of Part II agree in the common ground of their order in z. In fact all these functions are of order in z comparable with that of $\exp[(\log z)^{2+k}]$ (where k may have any positive or zero value), that is, for some value of k, and for sufficiently large values of r,

$$\exp[(\log r)^{2+k-\epsilon}] < M(r) < \exp[(\log r)^{2+k+\epsilon}],$$

where $M(r)$ is the maximum modulus of $F(z)$ on the circle $|z| = r$, and ϵ is arbitrarily small.

† In exceptional cases different K's are distinguished by suffixes.

PRELIMINARY FORMULAE.

§ 1. The theory of Part I, Section I is essentially based on two formulae of approximation, both more or less connected with the theory of elliptic ϑ-functions. The first of these is a simple deduction from the formulae of transformation from $\vartheta(\nu \mid \tau)$ to $\vartheta(\tau^{-1}\nu \mid -\tau^{-1})$.

I. *Let* $\psi(\phi, \omega) = \prod_{s=1}^{\infty}(1+e^{\iota\phi - s\omega})\prod_{s=0}^{\infty}(1+e^{-\iota\phi - s\omega})$, *where* $\mathfrak{R}\omega > 0$, *and where* ω *may tend to zero in any manner subject to the condition*

$$\mathfrak{R}\omega > K^{-1}\mid\omega\mid.$$

Then
$$\log\psi(\phi, \omega) = (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega + \chi(\omega),$$

provided ϕ *is subject to the following condition:—*

Let ϕ *be put in the form* $\theta + \iota\alpha\omega$, *where* θ *and* α *are real*. Then we are to have*

$$\pi - \mid\theta\mid > \mid\omega\mid^{1-\delta},$$

where δ *is an arbitrarily small positive constant.*

We have
$$\psi(\phi, \omega) = e^{-\frac{1}{2}\iota\phi}\,2\cos\tfrac{1}{2}\phi\prod_{s=1}^{\infty}[(1+e^{\iota\phi - s\omega})(1+e^{-\iota\phi - s\omega})]$$
$$= e^{-\frac{1}{2}\iota\phi}(q^{\frac{1}{4}}q_0)^{-1}\vartheta_2(\nu \mid \tau)\dagger \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where
$$q = e^{\tau\pi\iota} = e^{-\frac{1}{2}\omega},\quad 2\pi\nu = \phi,\quad q_0 = \prod_{s=1}^{\infty}(1-q^{2s}).$$

Now,

$$\vartheta_2(\nu \mid \tau) = \iota^{\frac{1}{2}}\tau^{-\frac{1}{2}}e^{-\pi\iota\tau^{-1}\nu^2}\vartheta_4(\tau^{-1}\nu \mid -\tau^{-1})\ddagger$$
$$= (2\pi\omega^{-1})^{\frac{1}{2}}e^{-\frac{1}{2}\omega^{-1}\phi^2}[1 - 2e^{-\pi\iota\tau^{-1}}\cos(2\pi\nu\tau^{-1}) + \ldots + (-)^n 2e^{-n^2\pi\iota\tau^{-1}}\cos(2n\pi\nu\tau^{-1}) + \ldots]\S \;\ldots\ldots\ldots (2).$$

Now when $n \geqslant 1$,

$$\mid e^{-n^2\pi\iota\tau^{-1}}\cos(2n\pi\nu\tau^{-1})\mid = \mid e^{-2\pi^2n^2\omega^{-1}}\cos(2\pi n\iota\phi\omega^{-1})\mid = \mid e^{-2\pi^2n^2\omega^{-1}}\cos(2\pi n\iota\theta\omega^{-1} - 2n\pi\alpha)\mid$$
$$\leqslant \tfrac{1}{2}\mid\exp(-2\pi^2n^2\omega^{-1} + 2\pi n\theta\omega^{-1})\mid + \tfrac{1}{2}\mid\exp(-2\pi^2n^2\omega^{-1} - 2\pi n\theta\omega^{-1})\mid$$
$$< \mid\exp[-2\pi^2(n^2-n)\,\omega^{-1}]\mid . \mid\exp[-2\pi\mid\omega\mid^{1-\delta}\omega^{-1}]\mid,$$

since $\pi - \mid\theta\mid > \mid\omega\mid^{1-\delta}$,

$$< \mid\exp[-(n^2-n)\,\omega^{-1}]\mid . \exp[-K^{-1}\mid\omega\mid^{-\delta}],$$

since, by the condition for ω,

$$\mathfrak{R}2\pi\mid\omega\mid^{1-\delta}\omega^{-1} = 2\pi\mid\omega\mid^{1-\delta}.\mid\omega\mid^{-2}.\mathfrak{R}\omega > 2\pi\mid\omega\mid^{1-\delta}.\mid\omega\mid^{-2}.K^{-1}\mid\omega\mid.$$

Then

$$\left|\sum_{n=1}^{\infty}(-)^n 2e^{-2\pi^2n^2\omega^{-1}}\cos(2\pi n\iota\phi\omega^{-1})\right| < \exp[-K^{-1}\mid\omega\mid^{-\delta}]\sum_{n=1}^{\infty}\mid\exp[-(n^2-n)\,\omega^{-1}]\mid$$
$$< \exp[-K^{-1}\mid\omega\mid^{-\delta}].K$$

* ϕ is, of course, not restricted to be real.

† Tannery and Molk, *Éléments de la Théorie des Fonctions Elliptiques*, t. II. p. 252, formula (6).

‡ *ibid.* p. 263, (8).

§ p. 252, (4).

(since $\sum_{n=1}^{\infty} | \exp[-(n^2-n)\,\omega^{-1}] | < \sum_{n=1}^{\infty} \exp[-(n^2-n)\,K^{-1}] < K$),

$$= \chi(\omega) \quad \text{.......................................}(3).$$

Now we clearly have $\log[1+\chi(\omega)] = \chi(\omega)$.

Therefore, from (2),

$$\vartheta_2(\nu \mid \tau) = (2\pi\omega^{-1})^{\frac{1}{2}} \exp[-\tfrac{1}{2}\phi^2\omega^{-1} + \chi(\omega)] \quad \text{..........................}(4).$$

Again, we have the formulae

$$\begin{aligned} 2\pi q_0{}^3 q^{\frac{1}{4}} &= \vartheta_1{}'(0 \mid \tau)^* \\ &= [-\iota^{\frac{1}{2}}.\tau.\tau^{\frac{1}{2}}]^{-1}\vartheta_1{}'(0 \mid -\tau^{-1})\dagger \\ &= [-\iota^{\frac{1}{2}}\tau^{\frac{3}{2}}]^{-1} 2\pi e^{\frac{1}{4}(-\pi\iota\tau^{-1})}[1 + \sum_{n=1}^{\infty}(-)^n(2n+1)\,e^{-n(n+1)\pi\iota\tau^{-1}}]\ddagger \\ &= (2\pi\omega^{-1})^{\frac{3}{2}}.2\pi e^{-\frac{1}{2}\pi^2\omega^{-1}}[1 + \sum_{n=1}^{\infty}(-)^n(2n+1)\,e^{-n(n+1)2\pi^2\omega^{-1}}] \\ &= (2\pi\omega^{-1})^{\frac{3}{2}}.2\pi e^{-\frac{1}{2}\pi^2\omega^{-1}}[1+\chi(\omega)] \quad \text{..}(5), \end{aligned}$$

as is easily seen by a slight modification of the reasoning employed to establish (3). Hence

$$\begin{aligned} q_0 q^{\frac{1}{4}} &= (2\pi\omega^{-1})^{\frac{1}{2}} q^{\frac{1}{6}} \exp[-\tfrac{1}{6}\pi^2\omega^{-1} + \chi(\omega)] \\ &= (2\pi\omega^{-1})^{\frac{1}{2}} \exp[-\tfrac{1}{6}\pi^2\omega^{-1} - \tfrac{1}{12}\omega + \chi(\omega)] \quad \text{....................}(6). \end{aligned}$$

From (1), (4), and (6) we obtain the desired result.

§ 2. II. *Provided that* $|\alpha| < \frac{3}{4}$§, *where, as in* § 1, $\phi = \theta + \iota\alpha\omega$, *we have for the double series*

$$S_{p,q} = \sum_{n=1}^{\infty}\sum_{s=1}^{\infty}(-)^{n-1}[e^{n\phi\iota} - (-)^q e^{-n\phi\iota}]\,n^p s^{p+q} e^{-ns\omega},$$

where p and q are positive integers, and $p \geqslant 0$, $q \geqslant 2$, the formula

$$S_{p,q} = (p+q)!\,[\omega^{-p-q-1}A_{q+1}(\phi) + B_{p,q}] + \chi(\omega),$$

where $A_m(\phi)$ is the polynomial in ϕ which is the coefficient of x^m in the expansion of

$$\pi x \operatorname{cosec}(\pi x)\,e^{x\phi\iota},$$

and where $B_{p,q} = 0$ if $p > 0$, and $B_{o,q}$ is the coefficient of x^{q+1} in the expansion of

$$xe^x(1-e^x)^{-1}.$$

Since $|e^{n\phi\iota}e^{-ns\omega}| = |e^{n\theta\iota - n\alpha\omega - ns\omega}| \leqslant |e^{-n(s-\frac{3}{4})\omega}|$,

it is easily seen that, whether p and q are positive or negative, the series $S_{p,q}$ is absolutely convergent, and uniformly convergent in the variable ω. We may, then, differentiate the series $S_{-q,q}$ term by term any number of times with respect to ω|| (since the resulting series is uniformly convergent), and we obtain

$$\begin{aligned} S_{p,q} &= \left(-\frac{d}{d\omega}\right)^{p+q} S_{-q,q} = \left(-\frac{d}{d\omega}\right)^{p+q} \sum_{n=1}^{\infty}\sum_{s=1}^{\infty}(-)^{n-1}[e^{n\phi\iota} - (-)^q e^{-n\phi\iota}]\,n^{-q}e^{-ns\omega} \\ &= \left(-\frac{d}{d\omega}\right)^{p+q} \sum_{n=1}^{\infty}(-)^{n-1}[e^{n\phi\iota} - (-)^q e^{-n\phi\iota}]\,n^{-q}\frac{e^{-n\omega}}{1-e^{-n\omega}} \quad \text{.........................}(1). \end{aligned}$$

* Tannery and Molk, *Éléments de la Théorie des Fonctions Elliptiques*, t. II. p. 257, (2).

† p. 263, (3).

‡ p. 257, (1).

§ The number $\frac{3}{4}$ might be replaced by any number between $\frac{1}{2}$ and 1, both exclusive.

|| Regarding ϕ as a constant.

First let us suppose that $\alpha \leqslant 0$. Let

$$f(z) = z^{-q} \frac{e^{-z\omega}}{1-e^{-z\omega}} e^{z\phi\iota}.$$

Consider the integral $\frac{1}{2\iota}\int_C \frac{f(z)dz}{\sin \pi z}$ taken round a large circle $|z| = R$, denoted by C. Since θ is real and $|\theta|$ is less than π, it is possible to make the circle C avoid the poles of cosec πz in such a manner that we have, on the circle,

$$|e^{\iota\theta z} \operatorname{cosec} \pi z| < K \text{ ....(2).}$$

Again, it is possible to make C avoid the poles of $(1-e^{-z\omega})^{-1}$ in such a manner that, when $\mathfrak{R}(z\omega) \geqslant 0$, we have

$$\left|\frac{e^{-z\omega}}{1-e^{-z\omega}} \cdot e^{-\alpha z\omega}\right| < K\,|e^{-z(1+\alpha)\omega}|$$

$$< K, \text{ since } \alpha \geqslant -\tfrac{3}{4}\text{*},$$

and when $\mathfrak{R}(z\omega) \leqslant 0$,

$$\left|\frac{e^{-z\omega}}{1-e^{-z\omega}}\, e^{-\alpha z\omega}\right| = \left|\frac{e^{-\alpha z\omega}}{e^{z\omega}-1}\right| < K\,|e^{-\alpha z\omega}|$$

$$< K, \text{ since } \alpha \leqslant 0\text{* ....(3),}$$

so that this inequality holds on the whole contour C. It is, moreover, easily proved that C can be chosen to avoid the poles both of cosec πz and of $(1-e^{-z\omega})^{-1}$, in such a manner that (2) and (3) hold simultaneously for all points of the contour C†. Since $\iota\phi = \iota\theta - \alpha\omega$, it follows from (2) and (3) that for all points z of C

$$|f(z) \operatorname{cosec} \pi z| < K\,|z|^{-q},$$

and therefore, since K is independent of R and since we are supposing that $q > 1$,

$$\lim_{R\to\infty} \int_C f(z) \operatorname{cosec} \pi z\, dz = 0 \text{ ....(4).}$$

The singularities of $(2\iota)^{-1} f(z)$ cosec πz within C are at the points 0, $\pm m$, $\pm 2m\pi\iota\omega^{-1}$. All except the first are simple poles. The sum of the residues at m and $-m$, when multiplied by $2\pi\iota$, is

$$(-)^m \left[e^{m\phi\iota} m^{-q} \frac{e^{-m\omega}}{1-e^{-m\omega}} + e^{-m\phi\iota}(-m)^{-q} \frac{e^{m\omega}}{1-e^{m\omega}}\right]$$

$$= (-)^m m^{-q} \left[e^{m\phi\iota} \frac{e^{-m\omega}}{1-e^{-m\omega}} - (-)^q e^{-m\phi\iota} \frac{e^{-m\omega}}{1-e^{-m\omega}}\right] + (-)^{q+m} m^{-q} e^{-m\phi\iota} \text{ ......(5).}$$

The sum of the residues at $\pm 2m\pi\iota\omega^{-1}$, when multiplied by $2\pi\iota$, is

$$\pi\omega^{-1}(2m\pi\iota\omega^{-1})^{-q} \operatorname{cosec}(2m\pi^2\omega^{-1})\,[e^{-2m\pi\phi\omega^{-1}} - (-)^q e^{2m\pi\phi\omega^{-1}}] \text{ ....(6).}$$

From (4), (5), and (6) we have

$$\epsilon(R) = \sum_{1\leqslant m<R} (-)^m m^{-q} [e^{m\phi\iota} - (-)^q e^{-m\phi\iota}] \frac{e^{-m\omega}}{1-e^{-m\omega}} + (-)^q \sum_{1\leqslant m<R} (-)^m m^{-q} e^{-m\phi\iota}$$

$$+ \pi\omega^{-1} \cdot \sum_{|2\pi m\omega^{-1}|<R} (2m\pi\iota\omega^{-1})^{-q} [e^{-2m\pi\phi\omega^{-1}} - (-)^q e^{2m\pi\phi\omega^{-1}}] \operatorname{cosech}(2m\pi^2\omega^{-1}) + R_0 \text{...(7),}$$

where R_0 is the residue of π cosec $\pi z . f(z)$ at $z = 0$.

* K is here independent of R, but not of ω.

† I omit the detailed proofs of this result and of (2), as the procedure is fairly well-known, and a little tedious.

Now the first series in (7), when taken to infinity, is the series for $-S_{-q,q}$, and is convergent. The second series, taken to infinity, converges since $m^{-q} \leqslant m^{-2}$ and $|e^{-m\phi\iota}| = |e^{m\alpha}| \leqslant 1$ (α being negative or zero). Making R tend to infinity in (7) we see that the third series converges (when taken to infinity) and that

$$S_{-q,q} = R_0 + (-)^q \sum_{m=1}^{\infty} (-)^m m^{-q} e^{-m\phi\iota}$$
$$+ \pi\omega^{-1} \sum_{m=1}^{\infty} (2m\pi\iota\omega^{-1})^{-q} \operatorname{cosech}(2m\pi^2\omega^{-1}) [e^{-2m\pi\phi\omega^{-1}} - (-)^q e^{2m\pi\phi\omega^{-1}}] \quad \text{......(8).}$$

It may be shown without any special difficulty that the series resulting from the differentiation term by term $(p+q)$ times with respect to ω, of the second series in (8), is uniformly convergent. It is further obvious on a little consideration that the modulus of the general term of the derived series is less than

$$K|\omega|^{-K}.m^K.|\exp(-2m\pi^2\omega^{-1})|.[|e^{-2m\pi\phi\omega^{-1}}| + |e^{2m\pi\phi\omega^{-1}}|],$$

and therefore (since $e^{2m\pi\phi\omega^{-1}} = e^{2m\pi\theta\omega^{-1} + 2m\pi\alpha\iota}$) that it is less than

$$K|\omega|^{-K}.m^K.2|\exp[-2\pi m\omega^{-1}(\pi - |\theta|)]|$$
$$< K|\omega|^{-K}.m^K.2|\exp(-2\pi m|\omega|^{1-\delta}\omega^{-1})|, \text{ since } \pi - |\theta| > |\omega|^{1-\delta},$$
$$< K|\omega|^{-K}.m^K.2\exp[-mK^{-1}|\omega|^{-\delta}], \text{ since } \mathfrak{R}\omega^{-1} = \mathfrak{R}\omega.|\omega|^{-2} < K^{-1}|\omega|^{-1}.$$

Hence, since $\sum_{m=1}^{\infty}(-)^m m^{-q} e^{-m\phi\iota}$ is independent of ω, we have, on differentiating (8) $(p+q)$ times,

$$\left|S_{p,q} - \left(-\frac{d}{d\omega}\right)^{p+q} R_0\right| < |\omega|^{-K} \sum_{m=1}^{\infty} (Km^K) \exp[-mK^{-1}|\omega|^{-\delta}]$$
$$< |\omega|^{-K} \sum_{m=1}^{\infty} (Ke^m) \exp[-mK^{-1}|\omega|^{-\delta}]$$

(where ω is supposed so small that $K^{-1}|\omega|^{-\delta} > 1$)

$$< K.|\omega|^{-K}.\exp(-K^{-1}|\omega|^{-\delta}+1)/[1 - \exp(-K^{-1}|\omega|^{-\delta}+1)].$$

This expression is evidently of the form $\chi(\omega)$. Therefore

$$S_{p,q} = \left(-\frac{d}{d\omega}\right)^{p+q} R_0 + \chi(\omega) \quad \text{..................................(9).}$$

Now R_0 is the residue at $z=0$ of

$$\omega^{-1} z^{-(q+2)} \cdot \frac{\pi z e^{z\phi\iota}}{\sin \pi z} \cdot \frac{z\omega e^{-z\omega}}{1 - e^{-z\omega}},$$

or of $\qquad \omega^{-1} z^{-(q+2)} [1 + A_1(\phi) z + A_2(\phi) z^2 + \dots] [1 + c_1 \omega z + c_2 \omega^2 z^2 + \dots]$

[where $\Sigma c_n(-x)^n$ is the expansion of $(-x)e^{-(-x)}[1 - e^{-(-x)}]^{-1}$ or $-xe^x(1-e^x)^{-1}$],

and is therefore equal to

$$\omega^{-1} A_{q+1}(\phi) + \omega^q c_{q+1} + [\text{a polynomial in } \omega \text{ and } \phi \text{ of degree } q-1 \text{ in } \omega].$$

Hence, if $p>0$, we have

$$\left(-\frac{d}{d\omega}\right)^{p+q} R_0=(p+q)!\,\omega^{-p-q-1}A_{q+1}(\phi) \qquad\qquad (10)_1,$$

and if $p=0$,

$$\left(-\frac{d}{d\omega}\right)^{p+q} R_0=(p+q)!\,\omega^{-p-q-1}A_{q+1}(\phi)-(-)^{q+1}q!\,c_{q+1} \qquad\qquad (10)_2.$$

From (9) and $(10)_1$, $(10)_2$, we obtain the desired result for $S_{p,q}$. This result, however, has so far been proved only for the case when $\alpha \leqslant 0$. If $\alpha>0$ we take

$$f(z)=e^{-z\phi\iota}\frac{e^{z\omega}}{1-e^{z\omega}}.$$

Treating $f(z)$ as before we obtain

$$(-)^q S_{p,q}=\left(-\frac{d}{d\omega}\right)^{p+q} R_0'+\chi(\omega),$$

where R_0' is the residue at $z=0$ of

$$z^{-q}\frac{e^{-z\phi\iota}}{\sin\pi z}\cdot\frac{e^{z\omega}}{1-e^{z\omega}},$$

and is therefore equal to the residue at $y=0$ of

$$-(-y)^q\frac{e^{y\phi\iota}}{\sin(-\pi y)}\cdot\frac{e^{-y\omega}}{1-e^{-y\omega}}.$$

R_0' is thus equal to $(-)^q R_0$, and we obtain the same expression for $S_{p,q}$ as before.

PART I. PRODUCT-FORMS.

SECTION I. ASYMPTOTIC EXPANSIONS.

§ 3. We shall now attack the problem of approximating for $\log F(z)$ in the case when $F(z)$ is defined by a product-form.

Let $F(z)=\prod\limits_{s=1}^{\infty}\left(1+\dfrac{z}{a_s}\right)$ be an integral function of zero order. We shall suppose that the a's satisfy the following conditions:

1°. *When n is large and $s\leqslant\mu(n)$, and when N is less than some (arbitrary) constant,*

$$\log(a_{n\pm s}/a_n)=(\pm s)\,\omega(n)+(\pm s)^2\omega_1(n)+\ldots+(\pm s)^N\omega_{N-1}(n)+O[s^{N+1}n^{-N}\omega(n)],$$

where $$\mu(n)=[\nu(n)]^{\frac{1}{3}}\,|\,\omega(n)\,|^{-1},\quad \nu(n)=|\,n\omega(n)\,|.$$

2°. $\lim\limits_{n\to\infty}|\,\omega(n)\,|=0.$

3°. $\lim\limits_{n\to\infty}\nu(n)=\infty.$

4°. $\mathfrak{R}\,\omega(n)>K^{-1}|\,\omega(n)\,|$, *algebraically.*

5°. *When r is less than some (arbitrary) constant,*

$$|\,\omega_r(n)\,|<Kn^{-r}\,|\,\omega(n)\,|.$$

6°. $|a_n| \sum_{s=\mu(n)+1}^{\infty} |a_{n+s}|^{-1} = |\omega(n)|^{-1} \chi[\nu(n)],$

and $|a_n|^{-1} \sum_{s=\mu(n)+1}^{n-1} |a_{n-s}| = |\omega(n)|^{-1} \chi[\nu(n)]$*.

Some remarks on the conditions may be found useful at this point.

The most important of the conditions are undoubtedly 2° and 3°, which limit the 'order' of Q_n. The remaining conditions are satisfied in natural cases when these two are, and express the fact that the function Q_n behaves like a naturally constructed function of n.

Writing ω, μ, ν, etc. for $\omega(n)$, $\mu(n)$, $\nu(n)$, etc., we have $\mu/n = \nu^{-\frac{2}{3}}$. (Thus μ is less than n when n is large.) In the series in condition 1°, the modulus of the ratio of the term $s^r \omega_{r-1}(n)$ to the first term $s\omega$ is (by condition 5°) less than

$$Ks^{r-1} n^{-(r-1)} < K(\mu/n)^{r-1} < K\nu^{-\frac{2}{3}(r-1)}.$$

Thus the first term is, for all values of s considered, of higher order than all succeeding terms (including, as is easily seen, the remainder-term), and, in general, the succeeding terms decrease in a ratio which is certainly not greater than $K\nu^{-\frac{2}{3}}$.

It is supposed that, given any constant integer h, the functions $\omega_1(n)$, $\omega_2(n)$, ... $\omega_h(n)$ exist. It is evident, by what is said above, that if the equation of 1° is true for any particular value of N (independent of n) it must remain true when N is decreased, and it is in fact supposed in the statement of 1° that we may give any value we please to N, *provided we do not make N depend on n.*

The index $\frac{1}{3}$ which occurs in the definition of $\mu(n)$ might be replaced by any number between 0 and $\frac{1}{2}$. Practically no generality, however, is to be gained by allowing a choice of this index.

Condition 6° may appear somewhat unsatisfactory, but I believe it to be the simplest available. It will be shown in § 13 that it (as well as the other five conditions) is satisfied in all natural cases.

Although $\lim |\omega(n)|^{-1} = \infty$, the function $|\omega(n)|^{-1}$ may increase† more slowly than any assigned function which tends to infinity. When $|\omega(n)|^{-1}$ increases very slowly $\nu(n)$ increases nearly as fast as n, since $\nu(n) = |n\omega(n)|$. In the same way $\nu(n)$ may increase very slowly, and $|\omega(n)|^{-1}$ will then increase nearly as fast as n. Thus it is possible for $|\omega(n)|^{-1}$ to increase more rapidly than $f[\nu(n)]$ or more slowly than $\phi[\nu(n)]$, however rapidly $f(x)$, or however slowly $\phi(x)$ may increase with x. In particular we must be careful not to assume that the expressions $\omega^{-1}\chi(\nu)$ occurring in 6° tend to zero as n tends to infinity. The independence in the orders of ν and $|\omega|^{-1}$ is the main cause of the complexity of the proof which follows, and should be borne in mind throughout.

§ 4. Let n be the integer, depending on z, which is such that $|(\log|z/a_n|)|$ is a minimum‡, and let $z = a_n e^{\iota\phi}$. Let ϕ be put in the form $\phi = \theta + \iota\alpha\omega$, where θ and α are real. It is then easily shown that we have $|\alpha| < \frac{3}{4}$ when n is large§.

* If u is not an integer it is supposed to be replaced, in each of the symbols $\sum_{s=1}^{u}$, $\sum_{s=u}^{\infty}$, by the greatest integer contained in u.

† It is not assumed in the conditions that $|\omega|^{-1}$ and ν increase *steadily* with n; they may decrease for certain ranges of n provided only that they tend to infinity with n.

‡ If two values of n give equal values of this expression, either may be taken.

§ It is clear that $|z|$ must lie between $|a_{n-1}|$ and $|a_{n+1}|$. If we suppose

$$|a_n| \leqslant |z| < |a_{n+1}|,$$

we have, by the definition of n,

$$0 \leqslant \log\left|\frac{z}{a_n}\right| < \log\left|\frac{a_{n+1}}{z}\right| < \log\left|\frac{a_{n+1}}{a_n}\cdot\frac{a_n}{z}\right|$$

or $$0 < \mathfrak{R}\iota\phi < \mathfrak{R}\log(a_{n+1}/a_n) - \mathfrak{R}\iota\phi,$$

or, taking $N=1$ in condition 1°,

$$0 < \mathfrak{R}\iota\phi < \mathfrak{R}[\omega + O(n^{-1}\omega) - \iota\phi],$$

or $$0 \leqslant \mathfrak{R}(-\alpha\omega) < \mathfrak{R}[(1+\alpha)\omega + O(n^{-1}\omega)].$$

The first inequality gives $\alpha \leqslant 0$ (since $\mathfrak{R}\omega > 0$), and the second

$$\mathfrak{R}(1+2\alpha)\omega > O(n^{-1}\omega),$$

or (since $\mathfrak{R}\omega > K^{-1}|\omega|$)

$$1 + 2\alpha > O(n^{-1}|\omega|)$$
$$> -\tfrac{1}{2}, \text{ when } n \text{ is large,}$$

so that $$\alpha > -\tfrac{3}{4}.$$

In the same way, if

$$|a_{n-1}| < |z| < |a_n|,$$

we find that $$\tfrac{3}{4} > \alpha > 0.$$

We proceed in the next eight articles to establish the following theorem:

Under the conditions 1° *to* 6° *for the a's, and with the condition for* ϕ *that*

$$\pi - |\theta| > |\omega(n)|^{1-\delta} + [\nu(n)]^{\delta-1},$$

where δ *is an arbitrarily small positive constant,*

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] + (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)[\omega(n)]^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega(n) + \chi[\omega(n)]$$
$$+ [\omega(n)]^{-1}\left[\sum_{p+q\leqslant\lambda} \beta_{p,q}(n)\{[\omega(n)]^{-p-q} A_{q+2}(\phi) + \omega(n) B_{p-1,q+1}\} + O(\nu^{-\frac{1}{2}\lambda})\right],$$

where λ *is an arbitrarily large integer, and where*

$$\Sigma\beta_{p,q}(n)\frac{x^p y^{p+q}}{(p+q)!}$$

is the formal expansion of

$$\exp[-x\{y^2\omega_1(n) + y^3\omega_2(n) + \ldots\}] - 1,$$

so that $\beta_{p,q}(n)$ *is a polynomial in* $\omega_1(n), \omega_2(n) \ldots \omega_{p+q-1}(n)$.

No expansion for $\log F(z)$ can be valid in the whole extent of any region containing zeros of $F(z)$. When z is equal to the zero $-a_n$, $\phi = \theta = \pi$. The condition for ϕ prevents the near approach of z to any zero.

Although a negative power of ω occurs in the terms of the series $\sum_{p+q\leqslant\lambda}$, the term $\beta_{p,q}(n)\,\omega^{-p-q}$ is less than $K\nu^{-t}$ $(t>0)$ in modulus. (This will appear later, but is easily seen if we remember that

$$|\omega_r| < Kn^{-r}|\omega| < K\nu^{-r}|\omega|^{r+1}.)$$

The factor ω^{-1} outside the large bracket, however, is not neutralised, and may remain large (in the case when $\nu(n)$ increase slowly) even when multiplied by ν^{-t}, where t is arbitrary.

§ 5. We have

$$\log F(z) - \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] - \log\psi(\phi, \omega)$$
$$= \sum_{s=1}^{\infty}\log\left(1 + \frac{z}{a_{n+s}}\right) + \sum_{s=0}^{n-1}\log\left(1 + \frac{a_{n-s}}{z}\right) - \sum_{s=1}^{\infty}\log(1 + e^{\iota\phi - s\omega}) - \sum_{s=0}^{\infty}\log(1 + e^{-\phi\iota - s\omega})$$
$$= \sum_{s=1}^{\mu}\left[\log\left(1 + \frac{z}{a_{n+s}}\right) - \log(1 + e^{\phi\iota - s\omega})\right] + \sum_{s=1}^{\mu}\left[\log\left(1 + \frac{a_{n-s}}{z}\right) - \log(1 + e^{-\iota\phi - s\omega})\right]$$
$$+ R_1 + R_2 + R_1' + R_2'^{*},$$

where

$$\left.\begin{aligned} R_1 &= \sum_{s=\mu+1}^{\infty}\log\left(1 + \frac{z}{a_{n+s}}\right), & R_1' &= \sum_{s=\mu+1}^{n-1}\log\left(1 + \frac{a_{n-s}}{z}\right) \\ R_2 &= -\sum_{s=\mu+1}^{\infty}\log(1 + e^{\iota\phi - s\omega}), & R_2' &= -\sum_{s=\mu+1}^{\infty}\log(1 + e^{-\phi\iota - s\omega}) \end{aligned}\right\} \quad \ldots\ldots\ldots(1),$$

We shall first dispose of the R's by showing that

$$|R_1| + |R_1'| + |R_2| + |R_2'| = |\omega|^{-1}\chi(\nu) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Consider R_1, and let μ' be the greatest integer contained in μ. Then, by condition 1°,

$$|a_n/a_{n+\mu'}| = |\exp[-\mu'\omega + O(\mu'^2 n^{-1}\omega)]|$$
$$< K|\exp(-\mu'\omega)|, \text{ since } \mu'^2 n^{-1}|\omega| \leqslant \mu^2 n^{-1}|\omega| \leqslant \nu^{-\frac{2}{3}} < K,$$
$$< K\exp(-K^{-1}\mu'|\omega|), \text{ since } \mathfrak{R}\omega > K^{-1}|\omega|,$$
$$< \tfrac{1}{2}, \text{ when } n \text{ is large,}$$

* The term in the second summation corresponding to $s=0$ is zero, since $a_n/z = e^{-\iota\phi}$.

since $$\mu' \mid \omega \mid \geqslant (\mu - 1) \mid \omega \mid > \nu^{\frac{1}{3}} - K.$$

Hence, when $s \geqslant \mu + 1$, we have

$$\left|\frac{z}{a_{n+s}}\right| = \left|\frac{z}{a_{n+1}} \cdot \frac{a_{n+1}}{a_{n+\mu'}} \cdot \frac{a_{n+\mu'}}{a_{n+s}}\right| < \left|\frac{a_{n+1}}{a_{n+\mu'}}\right| < \tfrac{1}{2}.$$

It follows that* $$\left|\log\left(1 + \frac{z}{a_{n+s}}\right)\right| < 2\left|\frac{z}{a_{n+s}}\right|.$$

Hence

$$|R_1| \leqslant \sum_{s=\mu+1}^{\infty}\left|\log\left(1+\frac{z}{a_{n+s}}\right)\right| < 2\sum_{s=\mu+1}^{\infty}\left|\frac{z}{a_{n+s}}\right| < 2\,|a_n e^{\iota\phi}|\sum_{s=\mu+1}^{\infty}|a_{n+s}|^{-1}$$

$$= |\omega|^{-1}\chi(\nu), \text{ by the first part of } 6^\circ, \text{ since } |e^{\iota\phi}| < K\dagger.$$

In a similar manner it follows from the second part of 6° that

$$|R_1'| = |\omega|^{-1}\chi(\nu).$$

Again, it is easily seen after the above that when $s \geqslant \mu + 1$,

$$|e^{-s\omega}| < \tfrac{1}{2}.$$

Therefore $$|R_2| < K\sum_{s=\mu+1}^{\infty}|e^{-s\omega}| < K\sum_{s=\mu+1}^{\infty} e^{-s\Re\omega}$$

$$< K\sum_{s=\mu+1}^{\infty} e^{-sK^{-1}|\omega|} < K\frac{e^{-\mu K^{-1}|\omega|}}{1-e^{-K^{-1}|\omega|}}$$

$$= \frac{\chi(\nu)}{1-e^{-K^{-1}|\omega|}}, \text{ since } \mu|\omega| = \nu^{\frac{1}{3}},$$

$$= |\omega|^{-1}\chi(\nu), \text{ since } 1 - e^{-x} > K^{-1}x \text{ when } x \text{ is small.}$$

We have a similar formula for R_2', and the result (2) is established.

From (1) and (2) we obtain

$$G(z) \equiv \log F(z) - \log\psi(\phi, \omega)$$

$$= \sum_{s=1}^{\mu}\left[\left\{\log\left(1+\frac{z}{a_{n+s}}\right) - \log\left(1+e^{\iota\phi-s\omega}\right)\right\}\right.$$

$$\left.+\left\{\log\left(1+\frac{a_{n-s}}{z}\right) - \log\left(1+e^{-\iota\phi-s\omega}\right)\right\}\right] + \omega^{-1}\chi(\nu) \quad \text{.........(A)}\ddagger.$$

Up to the present we have made no use of the condition $\pi - |\theta| > |\omega|^{1-\delta} + \nu^{\delta-1}$; the result (A) therefore holds for all real values of θ. We shall, in Section II, have occasion to return to this point.

§ 6. When $s \geqslant 1$, $|z/a_{n+s}|$ and $|a_{n-s}/z|$ are less than unity. Therefore

$$\log\left(1+\frac{z}{a_{n+s}}\right) = \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m}\left(\frac{z}{a_{n+s}}\right)^m,$$

and $$\log\left(1+\frac{a_{n-s}}{z}\right) = \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m}\left(\frac{a_{n-s}}{z}\right)^m.$$

* It is a well-known result that if
$|x| < \frac{1}{2}$, then $|\log(1+x)| < 2|x|$.
More generally, if
$|x| < 1 - K_1^{-1}$, then $|\log(1+x)| < K_2|x|$.

† For $|e^{\iota\phi}| = |e^{-\alpha\omega}|$, where $|\alpha| < \frac{3}{4}$, and ω is small.

‡ Capital letters are used to mark prominent stages in the proof.

Substituting these expansions in (A) we obtain

$$G(z)=\sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m}\sum_{s=1}^{\mu}\left[\left(\frac{z}{a_{n+s}}\right)^{m}-e^{m\phi\iota-ms\omega}+\left(\frac{a_{n-s}}{z}\right)^{m}-e^{-m\phi\iota-ms\omega}\right]+\omega^{-1}\chi(\nu)$$

$$=\sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m}\sum_{s=1}^{\mu}\Big[e^{m\phi\iota-ms\omega}\{\exp[-m(s^2\omega_1+s^3\omega_2+\dots)]-1\}$$

$$+e^{-m\phi\iota-ms\omega}\{\exp[m(s^2\omega_1-s^3\omega_2+\dots)]-1\}\Big]+\omega^{-1}\chi(\nu)\ \dots\dots\text{(B)}.$$

We shall show that *if we replace, in (B), the expression* $\exp[-m(s^2\omega_1+s^3\omega_2+\dots)]-1$ *by the polynomial* $P_\lambda(m,s)$ *which consists of those terms of the formal expansion of that expression which are of degree in s not greater than* λ, *and if we replace*

$$\exp[m(s^2\omega_1-s^3\omega_2+\dots)]-1$$

by the corresponding polynomial, which is evidently $P_\lambda(-m,-s)$, *the error will be of the form* $O(\omega^{-1}\nu^{-\lambda_1})$, *where* λ_1 *is a positive number tending to infinity with* λ; *so that*

$$G(z)=\sum_{m=1}^{\infty}\frac{(-)^{m-1}}{m}\sum_{s=1}^{\mu}[e^{m\phi\iota-ms\omega}P_\lambda(m,s)+e^{-m\phi\iota-ms\omega}P_\lambda(-m,-s)]+O(\omega^{-1}\nu^{-\lambda_1})\quad\dots\text{(C)}.$$

This result exacts a proof of some length, and will be approached by three stages.

I. We shall determine a number $\rho(n)$ such that the sum of all the terms of (B) for which $ms<\rho$ differs from the sum of the corresponding terms of (C) by an expression of the form $O(\omega^{-1}\nu^{-\lambda_1})$.

II. It will be shown that the sum of the remaining terms of (B), i.e. those for which $ms\geqslant\rho$, is of the form $O(\omega^{-1}\nu^{-\lambda_1})$.

III. The terms of (C) for which $ms\geqslant\rho$ will be shown to give a sum $O(\omega^{-1}\nu^{-\lambda_1})$.

These three results evidently suffice to establish the truth of (C).

Let $\rho(n)=\nu^{\frac{1}{2}}|\omega|^{-1}$. Then we shall show that

$$\sum_{ms<\rho}^{s\leqslant\mu}\frac{(-)^{m-1}}{m}\Big[e^{m\phi\iota-ms\omega}(\exp[-m(s^2\omega_1+s^3\omega_2+\dots)]-1-P_\lambda(m,s))$$

$$+e^{-m\phi\iota-ms\omega}(\exp[m(s^2\omega_1-s^3\omega_2+\dots)]-1-P_\lambda(-m,-s))\Big]=O(\omega^{-1}\nu^{-\lambda_1})\ \dots\text{(C)}_1.$$

$$\sum_{ms\geqslant\rho}^{s\leqslant\mu}\frac{(-)^{m-1}}{m}\Big[e^{m\phi\iota-ms\omega}(\exp[-m(s^2\omega_1+s^3\omega_2+\dots)]-1)+e^{-m\phi\iota-ms\omega}(\exp[m(s^2\omega_1-s^3\omega_2+\dots)]-1)\Big]$$

$$=O(\omega^{-1}\nu^{-\lambda_1})\ \dots\dots\dots\dots\text{(C)}_2,$$

$$\sum_{ms\geqslant\rho}^{s\leqslant\mu}\frac{(-)^{m-1}}{m}\Big[e^{m\phi\iota-ms\omega}P_\lambda(m,s)+e^{-m\phi\iota-ms\omega}P_\lambda(-m,-s)\Big]=O(\omega^{-1}\nu^{-\lambda_1})\ \dots\dots\dots\text{(C)}_3.$$

§ 7*. In the case of (C)_1, we have $ms<\rho$, $s\leqslant\mu$. We shall first show that

$$\exp[-m(s^2\omega_1+s^3\omega_2+\dots)]=1+P_\lambda(m,s)+O\{(\rho\omega s n^{-1})^{\frac{1}{2}\lambda}\}\ \dots\dots\dots\dots\dots(1).$$

* The following list may be found useful in what follows:

$$\mu=\nu^{\frac{1}{3}}\,.\,|\omega|^{-1}=\nu^{-\frac{1}{6}}\,.\,\rho=\nu^{-\frac{2}{3}}\,.\,n,$$

$$\rho=\nu^{\frac{1}{2}}\,.\,|\omega|^{-1}=\nu^{\frac{1}{6}}\,.\,\mu=\nu^{-\frac{1}{2}}\,.\,n,$$

$|\omega|^{-1}$, μ, ρ, n, are arranged in increasing order. $\nu=n|\omega|$ comes before n, but apart from this may have any place in the series.

We have, by 1°,

$$|u| \equiv |-m(s^2\omega_1 + \ldots)| = O\,|ms^2\omega n^{-1}|$$
$$< Kms\,.\,\mu\,|\omega|\,n^{-1} \qquad \text{since } s \leqslant \mu,$$
$$< K\,.\,\rho\,.\,\mu\,.\,|\omega|\,.\,n^{-1} < K\,.\,n\nu^{-\frac{1}{2}}\,.\,n\nu^{-\frac{2}{3}}\,.\,\nu n^{-1}\,.\,n^{-1} < K\nu^{-\frac{1}{6}}$$
$$= \epsilon(n) \quad \ldots\ldots\ldots\ldots(2).$$

Therefore

$$\exp(u) = 1 + \sum_{r=1}^{\lambda} \frac{u^r}{r!} + O(u^{r+1})$$
$$= 1 + \sum_{r=1}^{\lambda} \frac{1}{r!}[-m\{s^2\omega_1 + s^3\omega_2 + \ldots + s^\lambda\omega_{\lambda-1} + s^{\lambda+1}\,O(\omega n^{-\lambda})\}]^r + O(ms^2\omega n^{-1})^{\lambda+1} \ldots\ldots(3)$$

(by a double application of 1°).

In (3) the remainder term $O(ms^2\omega^{-1}n^{-1})^{\lambda+1}$ is of the form

$$O(\rho s\omega n^{-1})^{\lambda+1} = O(\rho s\omega n^{-1})^{\frac{1}{2}\lambda} \ldots\ldots\ldots\ldots(4),$$

since $\qquad \rho s\omega n^{-1} = O(\rho\mu\omega n^{-1}) = \epsilon(n).$

Again, any term of the expansion in powers of s of

$$[-m\{s^2\omega_1 + \ldots + s^{\lambda+1}\,O(\omega n^{-\lambda})\}]^r,$$

whose degree in s is greater than λ, is of the form

$$O[(ms^2\omega_1)^{\alpha_1}(ms^3\omega_2)^{\alpha_2}\ldots(ms^{\lambda+1}\omega n^{-\lambda})^{\alpha_\lambda}],$$

where $\qquad 2\alpha_1 + 3\alpha_2 + \ldots + (\lambda+1)\alpha_\lambda > \lambda.$

Since $\omega_r = O(\omega n^{-r})$ this expression is equal to

$$O[(ms\omega)^{\alpha_1+\alpha_2+\ldots}(sn^{-1})^{\alpha_1+2\alpha_2+3\alpha_3+\ldots}] = O[(\rho\omega)^{\alpha_1+\alpha_2+\ldots}(sn^{-1})^{\alpha_1+2\alpha_2+\ldots}] \ldots\ldots\ldots\ldots(5).$$

Now $|\rho\omega| > 1$ (when n is large), and $\alpha_1 + \alpha_2 + \ldots \leqslant \frac{1}{2}(2\alpha_1 + 3\alpha_2 + \ldots) < \frac{1}{2}\lambda$; also $sn^{-1} < \mu n^{-1} < 1$, and $\alpha_1 + 2\alpha_2 + \ldots \geqslant \frac{1}{2}(2\alpha_1 + 3\alpha_2 + \ldots) \geqslant \frac{1}{2}\lambda$. Hence the right-hand side of (5) is of the form

$$O[(\rho\omega)^{\frac{1}{2}\lambda}(sn^{-1})^{\frac{1}{2}\lambda}] \ldots\ldots\ldots\ldots(6).$$

From this result, from (4), and from the definition of $P_\lambda(m, s)$, the equation (1) follows immediately*.

To establish $(C)_1$, it is evidently sufficient to establish the relation

$$T \equiv \sum_{ms<\rho}^{s\leqslant\mu} \left|\frac{(-)^{m-1}}{m}[\exp\{-m(s^2\omega_1 + \ldots)\} - 1 - P_\lambda(m, s)]\,e^{m\phi_\iota - ms\omega}\right| = O(\omega^{-1}\nu^{-\lambda_1}),$$

together with an analogous result containing $P_\lambda(-m, -s)$. Now from (1) we have

$$T < \sum_{ms<\rho}^{s\leqslant\mu} K(\rho\omega n^{-1})^{\frac{1}{2}\lambda} s^{\frac{1}{2}\lambda}\,|e^{m\phi_\iota - ms\omega}|$$
$$< K(\rho\omega n^{-1})^{\frac{1}{2}\lambda} \sum_{ms<\rho}^{s\leqslant\mu} s^{\frac{1}{2}\lambda}\,e^{m(-\alpha\omega' - s\omega')}, \qquad \text{where } \omega' = R\omega,$$
$$< K(\rho\omega n^{-1})^{\frac{1}{2}\lambda} \sum_{ms<\rho}^{s\leqslant\mu} s^{\frac{1}{2}\lambda}\,e^{-K^{-1}ms|\omega|},$$

* For all the terms of $P_\lambda(m, s)$ are evidently included among the terms in the expansion of (3).

since $\omega' > K^{-1}|\omega|$, and since $s+\alpha > s - \frac{3}{4} > \frac{1}{4}s$,

$$< K(\rho\omega n^{-1})^{\frac{1}{2}\lambda} \sum_{s=1}^{\infty} s^{\frac{1}{2}\lambda} \frac{e^{-s\omega''}}{1-e^{-s\omega''}} \qquad \text{.................................(7)},$$

where we have written ω'' for $K^{-1}|\omega|$, and performed the summation with respect to m. Now it may be shown that*

$$\sum_{s=1}^{\infty} s^{\frac{1}{2}\lambda} \frac{e^{-s\omega''}}{1-e^{-s\omega''}} < K[(\omega'')^{-\frac{1}{2}\lambda-1}] \qquad \text{................................(8)}$$

$$< K|\omega|^{-\frac{1}{2}\lambda-1}.$$

Hence from (7), we have

$$T < K(\rho n^{-1})^{-\frac{1}{2}\lambda}\omega^{-1} < K\nu^{-\frac{1}{4}\lambda}\omega^{-1} \qquad \text{................................(9)}.$$

In a similar manner we obtain

$$T' \equiv \sum_{ms<\rho}^{s\leqslant\mu} \left|\frac{(-)^{m-1}}{m}[\exp\{m(s^2\omega_1 - s^3\omega_2 + \ldots)\} - 1 - P_\lambda(-m,-s)]e^{-m\phi\iota - ms\omega}\right| < K\nu^{-\frac{1}{4}\lambda}\omega^{-1} \text{...(10)}.$$

From (9) and (10) the result $(\mathrm{C})_1$ follows at once, provided we take $\lambda_1 = \frac{1}{4}\lambda$.

§ 8. We must now enter upon the rather intricate analysis which is required to establish $(\mathrm{C})_2$. We shall not begin with the left-hand side of that equation, but shall show that if $0 \leqslant x \leqslant 1$, then

$$|f(x)| \equiv \left|\sum_{ms\geqslant\rho}^{s\leqslant\mu}(-)^{m-1}x^m e^{-ms\omega}\left[e^{m\phi\iota}\exp\{-m(s^2\omega_1+\ldots)\} - e^{m\phi\iota} + e^{-m\phi\iota}\exp\{m(s^2\omega_1 - \ldots)\} - e^{-m\phi\iota}\right]\right|$$

$$< |\omega|^{-1}\chi(\nu) \qquad \text{..}(\mathrm{C})_2',$$

where the $\chi(\nu)$ is independent of x. The left-hand side differs from that of $(\mathrm{C})_2$ by the introduction of x^m and the suppression of $1/m$ in the general term of the latter formula. The result will be established by separating the series into four series corresponding to the four terms inside the large brackets, and by showing that each series satisfies an inequality similar to $(\mathrm{C})_2'$.

We have

$$\left|\sum_{ms\geqslant\rho}^{s\leqslant\mu} x^m(-)^{m-1}e^{m\phi\iota - ms\omega}\exp[-m(s^2\omega_1+\ldots)]\right| = \left|\sum_{s=1}^{\mu}\sum_{m\geqslant\rho/s} x^m(-)^{m-1}e^{m\phi\iota-ms\omega}\exp[-m(s^2\omega_1+\ldots)]\right|$$

$$= \left|\sum_{s=1}^{\mu}\frac{\pm x^{\{\rho/s\}}\exp[\{\rho/s\}(\phi\iota - s\omega - s^2\omega_1 - \ldots)]}{1 + x\exp[\phi\iota - s\omega - s^2\omega_1 - \ldots]}\right|,$$

$$\leqslant \sum_{s=1}^{\mu}\frac{|\exp[\{\rho/s\}(\phi\iota - s\omega - s^2\omega_1 - \ldots)]|}{|1 + x\exp[\phi\iota - s\omega - s^2\omega_1 - \ldots]|} \qquad \text{..................................(1)},$$

where $\{\rho/s\}$ denotes the least integer containing ρ/s.

* When $0<x<1$, we have $(1-e^{-x})^{-1} < Kx^{-1}$, and when $x \geqslant 1$ we have $(1-e^{-x})^{-1} < K$, so that for all values of x,

$$(1-e^{-x})^{-1} < K + Kx^{-1}.$$

Hence

$$\sum_{s=1}^{\infty} s^{\frac{1}{2}\lambda}\frac{e^{-s\omega''}}{1-e^{-s\omega''}} < K\sum_{s=1}^{\infty} s^{\frac{1}{2}\lambda}e^{-s\omega''} + K(\omega'')^{-1}\sum_{s=1}^{\infty} s^{\frac{1}{2}\lambda-1}e^{-s\omega''}$$

$$< K\sum_{s=1}^{\infty}\int_s^{s+1} x^{\frac{1}{2}\lambda}e^{-(x-1)\omega''}dx$$

$$+ K(\omega'')^{-1}\sum_{s=1}^{\infty}\int_s^{s+1} x^{\frac{1}{2}\lambda-1}e^{-(x-1)\omega''}dx$$

$$< Ke^{\omega''}\int_0^{\infty} x^{\frac{1}{2}\lambda}e^{-x\omega''}dx + Ke^{\omega''}.(\omega'')^{-1}\int_0^{\infty} x^{\frac{1}{2}\lambda-1}e^{-x\omega''}dx$$

$$< Ke^{\omega''}\Gamma(\tfrac{1}{2}\lambda+1)(\omega'')^{-\frac{1}{2}\lambda-1} + Ke^{\omega''}\Gamma(\tfrac{1}{2}\lambda)(\omega'')^{-1-\frac{1}{2}\lambda}$$

$$< K(\omega'')^{-\frac{1}{2}\lambda-1}, \quad \text{since } e^{\omega''} < K.$$

First consider the numerator of the general term of this series. We have

$$\mathfrak{R}\,[\{\rho/s\}(\phi\iota - s\omega - s^2\omega_1 - \ldots)] = \mathfrak{R}\,[\{\rho/s\}(\theta\iota - \alpha\omega - s\omega + s^2 O[\omega n^{-1}])]$$
$$= \mathfrak{R}\,[-\{\rho/s\}(\alpha + s)\,\omega + \{\rho/s\}\,s^2 O(\omega n^{-1})] \quad \ldots\ldots\ldots\ldots(2).$$

Now $\{\rho/s\} \geqslant \rho/s$; also $\alpha > -\frac{3}{4}$, so that $\alpha + s > \frac{1}{4}s$. Therefore (since $\mathfrak{R}\omega > 0$)

$$\mathfrak{R}\,[-\{\rho/s\}(\alpha+s)\,\omega] < \mathfrak{R}\,(-\tfrac{1}{4}\rho\omega) < -K_1^{-1}\rho\,|\,\omega\,| \quad \ldots\ldots\ldots\ldots(3).$$

Again, $\rho/s > \rho/\mu$, and is large, so that $\{\rho/s\} = O(\rho/s)$: therefore,

$$\mathfrak{R}\,[\{\rho/s\}\,s^2 O(\omega n^{-1})] = O(\rho\mu\,|\,\omega\,|\,n^{-1}) < \tfrac{1}{2}K_1^{-1}\rho\,|\,\omega\,| \quad \ldots\ldots\ldots\ldots(4),$$

since $\mu n^{-1} < \frac{1}{2}K_1^{-1}$ when n is large. From (3) and (4) we see that the right-hand side of (2) is less than $-\frac{1}{2}K_1^{-1}\rho\,|\,\omega\,|$, or $-\frac{1}{2}K_1^{-1}\nu^{\frac{1}{2}}$. Therefore

$$|\exp[\{\rho/s\}(\phi\iota - s\omega - s^2\omega_1 - \ldots)]| < \exp(-\tfrac{1}{2}K_1^{-1}\nu^{\frac{1}{2}}) < \chi_1(\nu) \quad \ldots\ldots\ldots\ldots(5),$$

where $\chi_1(\nu)$ is independent of x and s.

Considering now the denominator of the general term in (1), we shall show that

$$|\,1 + x\exp[\phi\iota - s\omega - s^2\omega_1 - \ldots]\,| > K^{-1}\nu^{-1} \quad \ldots\ldots\ldots\ldots(6),$$

where K is independent of x.

Let us suppose that θ is positive or zero. We have

$$\exp(-s^2\omega_1 - s^3\omega_2 - \ldots) = \exp[s^2 O(\omega n^{-1})]$$
$$= 1 + s^2 O(\omega n^{-1}),$$

since $\qquad |\,s^2 O(\omega n^{-1})\,| = O(\mu^2\omega\nu^{-1}) = \epsilon(n).$

Therefore $\qquad 1 + x\exp(\phi\iota - s\omega - s^2\omega_1 - \ldots) = 1 - xe^{-s\omega - (\pi-\phi)\iota}[1 + s^2 O(\omega n^{-1})] \quad \ldots\ldots\ldots\ldots(7).$

Now $|\,e^{-s\omega}s^2\,| = e^{-s\mathfrak{R}\omega}s^2$ is a maximum when $s = 2(\mathfrak{R}\omega)^{-1}$, and is of the form $O[(\mathfrak{R}\omega)^{-2}]$ or $O(\omega^{-2})$. Hence, from (7),

$$1 + x\exp(\phi\iota - s\omega - s^2\omega_1 - \ldots) = 1 - xe^{-s\omega - (\pi-\phi)\iota} + O(\omega^{-2}\omega n^{-1})$$
$$= 1 - xe^{-(s+\alpha)\omega - (\pi-\theta)\iota} + O(\nu^{-1}) \quad \ldots\ldots\ldots\ldots(8).$$

Our next step in the proof of (6) is to show that

$$|\,1 - xe^{-(s+\alpha)\omega - (\pi-\theta)\iota}\,| > K^{-1}(\pi - \theta) \quad \ldots\ldots\ldots\ldots(9).$$

First suppose that $\qquad |\,(s+\alpha)\,\omega\,| \leqslant \frac{1}{2}(\pi - \theta).$

Then $\qquad |$ the imaginary part of $(s+\alpha)\,\omega\,| < \frac{1}{2}(\pi - \theta)$,

and therefore $\qquad |\,1 - e^{-(s+\alpha)\omega - (\pi-\theta)\iota}\,| = |\,1 - qe^{-(\pi-\theta)\iota + \tau\iota}\,|,$

where q is real and less than unity [since $x \leqslant 1$, $\mathfrak{R}(s+\alpha)\,\omega > 0$], and $|\,\tau\,| < \frac{1}{2}(\pi - \theta)$. Now

$$|\,1 - qe^{-(\pi-\theta)\iota + \tau\iota}\,| > |\,1 - qe^{-(\pi-\theta)\iota + \frac{1}{2}(\pi-\theta)\iota}\,|$$

[for, as is evident from a figure, $(1 - qe^{\iota\psi})$ is a minimum for any real range of ψ when ψ differs from zero or a multiple of 2π by as little as possible],

$$> K^{-1}(\pi - \theta),$$

(as is also evident from a figure).

If we suppose, on the other hand, that $|\,(s+\alpha)\,\omega\,| \geqslant (\pi - \theta)$, we have

$$\mathfrak{R}\,[-(s+\alpha)\,\omega] > \mathfrak{R}\,[-(s+\alpha)\,K^{-1}|\,\omega\,|] > K^{-1}(\pi - \theta),$$

and therefore
$$|1 - xe^{-(s+\alpha)\omega - (\pi-\theta)\iota}| > K^{-1}|1 - \exp[-K^{-1}(\pi-\theta)]|$$
$$> K^{-1}(\pi - \theta).$$

The inequality (9) is therefore established.

From (8) and (9) we have
$$|1 + x\exp[\phi\iota - s\omega - s^2\omega_1 - \ldots]| > K_1^{-1}(\pi-\theta) - K_2\nu^{-1} \quad\ldots\ldots(10).$$

Now since
$$\pi - \theta > |\omega|^{1-\delta} + \nu^{\delta-1} > \nu^{\delta-1},$$
$$K_2\nu^{-1} < \tfrac{1}{2}K_1^{-1}(\pi-\theta), \quad \text{when } n \text{ is large.}$$

Therefore, from (10),
$$|1 + x\exp(\phi\iota - s\omega - s^2\omega_1 - \ldots)| > \tfrac{1}{2}K_1^{-1}(\pi-\theta) \quad\ldots\ldots(11)_1$$
$$> K^{-1}\nu^{-1} \quad\ldots\ldots(11)_2.$$

The number θ has been supposed positive or zero; if it is negative it is easily seen that we obtain $(11)_1$ with θ replaced by $|\theta|$*, and therefore $(11)_2$. We have therefore established the result (6).

Returning now to equation (1), we have, from (5) and (6),
$$\left|\sum_{ms\geqslant\rho}^{s\leqslant\mu} x^m(-)^{m-1}e^{m\phi\iota - ms\omega}\exp[-m(s^2\omega_1 + \ldots)]\right| < \sum_{s=1}^{\mu}\frac{\chi_1(\nu)}{K^{-1}\nu^{-1}}$$
$$< K\mu\nu\chi_1(\nu)\dagger$$
$$< |\omega|^{-1}\chi_2(\nu) \quad\ldots\ldots(12)_1,$$
where $\chi_2(\nu)$ is independent of x.

It is evident that the reasoning of the whole article is unaffected if we write $\omega_1 = \omega_2 = \ldots = 0$. We therefore have
$$\left|\sum_{ms\geqslant\rho}^{s\leqslant\mu} x^m(-)^{m-1}e^{m\phi\iota - ms\omega}\right| < |\omega|^{-1}\chi(\nu) \quad\ldots\ldots(12)_2.$$

It is also evident that the proof of $(12)_1$ applies, mutatis mutandis, to the series
$$\sum_{ms\geqslant\rho}^{s\leqslant\mu} x^m(-)^{m-1}e^{-m\phi\iota - ms\omega}\exp[m(s^2\omega_1 - s^3\omega_2 + \ldots)].$$

We thus have results $(12)_3$ and $(12)_4$ corresponding to $(12)_1$ and $(12)_2$. From these four results we obtain the equation $(C)_2'$.

§ 9. From $(C)_2'$ the result $(C)_2$ follows without difficulty. The series $f(x)$ is uniformly convergent in x, and may be integrated term by term. We therefore have
$$\left|\sum_{ms\geqslant\rho}^{s\leqslant\mu}\frac{(-)^{m-1}}{m}e^{-ms\omega}\left[e^{m\phi\iota}\exp\{-m(s^2\omega_1 + \ldots)\} - e^{m\phi\iota} + e^{-m\phi\iota}\exp\{m(s^2\omega_1 - \ldots)\} - e^{-m\phi\iota}\right]\right|$$
$$= \left|\int_0^1 f(x)\,dx\right| \leqslant \int_0^1 |f(x)|\,dx$$
$$< \int_0^1 |\omega|^{-1}\chi(\nu)\,dx, \quad \text{where } \chi(\nu) \text{ is independent of } x,$$
$$< |\omega|^{-1}\chi(\nu),$$
$$< O(|\omega|^{-1}\nu^{-\lambda_1}) \quad \text{(whatever value may be given to the constant } \lambda_1),$$
which is the desired result $(C)_2$.

* $(\pi + \phi)\,\iota$ is written for $-(\pi - \phi)\,\iota$ in equation (7). † For $\chi_1(\nu)$ is independent of s.

§ 10. There remains to be established the third of the results (C), viz.,

$$\sum_{ms \geqslant \rho}^{s \leqslant \mu} \frac{(-)^{m-1}}{m} e^{-ms\omega} [e^{m\phi\iota} P_\lambda(m, s) + e^{-m\phi\iota} P_\lambda(-m, -s)] = O(\omega^{-1} \nu^{-\lambda_1}).$$

Let $a_{p,q} m^p s^{p+q}$ be any term of $m^{-1} P_\lambda(m, s)$, so that $p \geqslant 0$, $q \geqslant 2$. Consider the expression

$$\sigma_{p,q} = \sum_{ms \geqslant \rho}^{s \leqslant \mu} (-)^{m-1} m^p s^{p+q} e^{m\phi\iota - ms\omega} \quad \text{.................................(1)}.$$

The right-hand side is the sum of $\{\mu\}$ series each uniformly convergent in ω, and is therefore equal to

$$\left(-\frac{d}{d\omega}\right)^p \sum_{s=1}^{\mu} \sum_{ms \geqslant \rho} (-)^{m-1} s^q e^{m\phi\iota - ms\omega}$$

$$= \left(-\frac{d}{d\omega}\right)^p \sum_{s=1}^{\mu} \pm s^q \frac{\exp[\{\rho/s\}(\phi\iota - s\omega)]}{1 + \exp(\phi\iota - s\omega)} \quad \text{...........................(2)}.$$

It will be seen on a little consideration that

$$\left(\frac{d}{d\omega}\right)^p \frac{\exp[\{\rho/s\}(\phi\iota - s\omega)]}{1 + \exp(\phi\iota - s\omega)} = \sum_{p_1=0}^{p} O\left\{\left[\left\{\frac{\rho}{s}\right\} s\right]^{p_1} \cdot \frac{s^{p_2}}{[1 + \exp(\phi\iota - s\omega)]^{p_2}} \cdot \exp[\{\rho/s\}(\phi\iota - s\omega)]\right\} \text{...(3)},$$

where $p_1 + p_2 = p$. Now by (9) of § 9*, with x equal to unity,

$$|1 + \exp(\phi\iota - s\omega)| > K^{-1}(\pi - |\theta|) > K^{-1}\nu^{-1}.$$

Also, by (5) of the same article (with $\omega_1 = \omega_2 = \ldots = 0$),

$$|\exp[\{\rho/s\}(\phi\iota - s\omega)]| = \chi(\nu).$$

Hence, from (3),

$$\left(\frac{d}{d\omega}\right)^p \frac{\exp[\{\rho/s\}(\phi\iota - s\omega)]}{1 + \exp(\phi\iota - s\omega)} = \sum_{p_1=0}^{p} \rho^{p_1} s^{p_2} \nu^p \chi(\nu)$$

$$= \sum_{p_1=0}^{p} \rho^p \chi(\nu), \qquad \text{since } s \leqslant \mu < K\rho,$$

$$= \rho^p \chi(\nu).$$

Therefore, from (2),

$$|\sigma_{p,q}| \leqslant \sum_{s=1}^{\mu} s^q \rho^p \chi(\nu) = \mu^{1+q} \rho^p \chi(\nu)$$

$$= |\omega|^{-(p+q+1)} \chi(\nu) \quad \text{..........................(4)}$$

(since μ and ρ are of the form $\nu^a |\omega|^{-1}$).

Now we have

$$\sum_{ms \geqslant \rho}^{s \leqslant \mu} \frac{(-)^{m-1}}{m} e^{-ms\omega} [e^{m\phi\iota} P_\lambda(m, s)] = \sum_{p+q \leqslant \lambda} a_{p,q} \sigma_{p,q} \quad \text{........................(5)}.$$

Therefore, by (4), if we can show that

$$a_{p,q} = O(\omega^{p+q}) \quad \text{..(6)},$$

it will follow that the left-hand side of (5) is of the form

$$O[\omega^{-1} \chi(\nu)],$$

* This inequality, proved in the text for $\theta \geqslant 0$, is easily seen to be true when θ is negative if $\pi - \theta$ is replaced by $\pi - |\theta|$.

and therefore of the form $\quad O(\omega^{-1}\nu^{-\lambda_1})$ ....(7),

whatever constant value may be given to λ_1.

We proceed, then, to establish (6). Now $\alpha_{p,q}\, m^p s^{p+q}$ is a term of the expansion in powers of s of

$$m^{-1}\exp\left[-m\left(s^2\omega_1 + s^3\omega_2 + \ldots\right)\right],$$

and is therefore of the form

$$O\left[m^{-1}(ms^2\omega n^{-1})^{\alpha_1}(ms^3\omega n^{-2})^{\alpha_2}\ldots\right],$$

where $\quad 2\alpha_1 + 3\alpha_2 + \ldots = p + q, \qquad -1 + \alpha_1 + \alpha_2 + \ldots = p.$

Hence $\alpha_{p,q}$ is of the form

$$O\left[(\omega n^{-1})^{\alpha_1}(\omega n^{-2})^{\alpha_2}\ldots\right] = O\left[(\omega n)^{-\alpha_1 - 2\alpha_2 - \ldots}\,.\,\omega^{2\alpha_1+3\alpha_2+\ldots}\right] = O\left[(\omega n)^{-\frac{1}{2}(2\alpha_1+3\alpha_2+\ldots)}\,.\,\omega^{2\alpha_1+3\alpha_2+\ldots}\right]$$
$$= O\left(\omega^{p+q}\nu^{-\frac{1}{2}\lambda}\right) \quad \ldots\ldots\ldots\ldots(8),$$

a result which incidentally proves (6), and to which we shall have occasion to return later. We have seen that (7) follows from (6). The result, analogous to (7), concerning $e^{-m\phi\iota}P_\lambda(-m, -s)$ evidently follows by a similar proof, and $(\mathrm{C})_3$ then follows immediately.

§ 11. We have now the formula (C), viz.

$$G(z) = \sum_{m=1}^{\infty}\sum_{s=1}^{\mu}\frac{(-)^{m-1}}{m}\left[e^{m\phi\iota}P_\lambda(m, s) + e^{-m\phi\iota}P_\lambda(-m, -s)\right]e^{-sm\omega} + O(\omega^{-1}\nu^{-\lambda_1}),$$

where λ_1 tends to infinity with λ (and may be taken to be $\frac{1}{4}\lambda$). We shall next show that we may replace $\sum_{s=1}^{\mu}$ by $\sum_{s=1}^{\infty}$ in this formula.

If $\alpha_{p,q}\, m^p s^{p+q}$ be any term of $m^{-1}P_\lambda(m, s)$, we have, by the equation (8) above,

$$|\alpha_{p,q}| < K.$$

It is therefore sufficient to prove that

$$\sigma'_{p,q} = \sum_{m=1}^{\infty}\sum_{s=\mu+1}^{\infty} m^p s^{p+q}\left|e^{\pm m\phi\iota - ms\omega}\right| = |\omega|^{-p-q-1}\chi(\nu).$$

Now $\quad \left|e^{\pm m\phi\iota - ms\omega}\right| \leqslant \left|e^{-m(s-|\alpha|)\omega}\right| < e^{-ms\omega'},$

where $\quad \omega' = K^{-1}|\omega|$*.

Then $\quad \sigma'_{p,q} = \sum_{s=\mu+1}^{\infty} s^{p+q}\sum_{m=1}^{\infty} m^p e^{-ms\omega'}$ ....(1).

Now, if $0 < x < 1$, $\quad \sum_{m=1}^{\infty} m^p x^m = \sum_{r=0}^{p}\frac{A_r x}{(1-x)^r}$ ....(2).

When $x = e^{-s\omega'}$, $s \geqslant \mu$, we have

$$x \leqslant e^{-\mu\omega'} < e^{-\nu^{\frac{1}{3}}K^{-1}} < \tfrac{1}{2},$$

and therefore $\quad |(1-x)^{-r}| < K.$

* For $\Re\omega > K^{-1}|\omega|$; $s - |\alpha| \geqslant s - \frac{3}{4} > \frac{1}{4}s$.

Hence, from (1) and (2),

$$\begin{aligned}\sigma'_{p,q} &< K \sum_{s=\mu+1}^{\infty} s^{p+q} e^{-s\omega'} \\ &< K \sum_{s=\mu+1}^{\infty} \int_s^{s+1} x^{p+q} e^{-(x-1)\omega'} dx < Ke^{\omega'} \int_{\mu}^{\infty} x^{p+q} e^{-x\omega'} dx \\ &< (Ke^{\omega'}) (\omega')^{-p-q-1} \int_{\mu(\omega')^{-1}}^{\infty} y^{p+q} . e^{-y} dy \\ &< (K) (\omega')^{-p-q-1} . \int_{\mu(\omega')^{-1}}^{\infty} Ke^{\frac{1}{2}y} . e^{-y} dy < K (\omega')^{-p-q-1} . 2Ke^{-\frac{1}{2}\mu(\omega')^{-1}} \\ &< K \mid \omega \mid^{-p-q-1} e^{-K^{-1}\mid\omega\mid^{-1}\nu^{\frac{1}{3}}\mid\omega\mid} \\ &= \mid \omega \mid^{-p-q-1} \chi(\nu),\end{aligned}$$

which is the desired result.

§ 12. We now have

$$G(z) = \sum_{m=1}^{\infty} \sum_{s=1}^{1} \frac{(-)^{m-1}}{m} \left[P_\lambda(m, s) e^{m\phi\iota} + P_\lambda(-m, -s) e^{-m\phi\iota}\right] e^{-ms\omega} + O(\omega^{-1}\nu^{-\lambda_1}) \;\ldots\text{(D)}.$$

Now $$P_\lambda(m, s) = \sum_{p+q \leqslant \lambda} \beta_{p,q}(n) \frac{m^p s^{p+q}}{(p+q)!},$$

where $$\Sigma \beta_{p,q}(n) \frac{x^p y^{p+q}}{(p+q)!}$$

is the expansion of $$\exp[-x(y^2\omega_1 + y^3\omega_2 + \ldots)] - 1.$$

(We evidently have $p \geqslant 1$, $q \geqslant 1$.) Hence

$$G(z) = \sum_{p+q<\lambda} \frac{\beta_{p,q}(n)}{(p+q)!} \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} (-)^{m-1} m^{(p-1)} s^{(p-1)+(q+1)} [e^{m\phi\iota} - (-)^{q+1} e^{-m\phi\iota}] e^{-ms\omega} + O(\omega^{-1}\nu^{-\lambda_1}).$$

Now the double series is $S_{p-1,q+1}$ of § 2. The conditions for ϕ in the result II of that article are satisfied; therefore

$$\begin{aligned}G(z) &= \sum_{p+q \leqslant \lambda} \beta_{p,q}(n) [\omega^{-p-q-1} A_{q+2}(\phi) + B_{p-1,q+1} + \chi(\omega)] + O(\omega^{-1}\nu^{-\lambda_1}) \\ &= \sum_{p+q \leqslant \lambda} \beta_{p,q}(n) [\omega^{-p-q-1} A_{q+2}(\phi) + B_{p-1,q+1}] + \chi(\omega) + O(\omega^{-1}\nu^{-\lambda_1}) \;\ldots\ldots\ldots\ldots\text{(1)},\end{aligned}$$

since, by (8) of § 10, we have $$|\beta_{p,q}(n)| < K.$$

If now, in the formula

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] + \log \psi(\phi, \omega) + G(z),$$

we substitute for $G(z)$ from (1), and for $\log \psi(\phi, \omega)$ from the result I of § 1 (for which the condition is satisfied), we obtain

$$\begin{aligned}\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] &+ (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\omega^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega \\ &+ \sum_{p+q \leqslant \lambda} \beta_{p,q}(n) [A_{q+2}(\phi) . \omega^{-p-q-1} + B_{p-1,q+1}] + \chi(\omega) + O(\omega^{-1}\nu^{-\lambda_1}) \;\ldots\ldots\text{(2)}.\end{aligned}$$

The number λ_1 tends to infinity with λ. To obtain the complete form of our theorem it remains only to prove that we may take $\lambda_1 = \frac{1}{2}\lambda$. Let any number λ_0 be assigned. Let us choose the λ of (2) in such a manner that $\lambda_1 > \frac{1}{2}\lambda_0$. Then

$$\log F(z) - \log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] - (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1+\frac{1}{2}\phi\iota} - \tfrac{1}{12}\omega - \sum_{p+q\leqslant\lambda_0} \beta_{p,q}\left[A_{q+2}(\phi)\,\omega^{-p-q-1} + B_{p-1,q+1}\right]$$

$$= \Big(\sum_{p+q\leqslant\lambda} - \sum_{p+q\leqslant\lambda_0}\Big)\,\beta_{p,q}\left[A_{q+2}(\phi)\,\omega^{-p-q-1} + B_{p-1,q+1}\right] + \chi(\omega) + O(\omega^{-1}\nu^{-\frac{1}{2}\lambda_0}) \quad \text{......(3).}$$

Now, by the result (8) of § 10, if $p+q > \lambda_0$,

$$\beta_{p,q} = O(\omega^{p+q}\,\nu^{-\frac{1}{2}\lambda_0}).$$

Hence the right hand of (3) is of the form $\chi(\omega) + O(\omega^{-1}\nu^{-\frac{1}{2}\lambda_0})$. Since λ_0 is arbitrary the desired result is established.

§ 13. We shall now give the promised proof that the conditions for the a's are at any rate satisfied if a_n is a naturally constructed function of n, of less order than $\exp(\epsilon n)$*, and of greater order than $n^{1/\epsilon}$ for all values of ϵ, however small. The proof is necessarily of a rough character, but in any particular case it is easy to make it rigorous.

We have $$\omega(n) = \frac{d}{dn}(\log a_n).$$

From the restrictions on the order of n it is seen that $\omega(n)$ is intermediate in order between $\frac{d}{dn}(\epsilon^{-1}\log n)$ and $\frac{d}{dn}(\epsilon n)$, or between $\epsilon^{-1}n^{-1}$ and ϵ. Hence $\lim|\omega(n)| = 0$, $\lim n|\omega(n)| = \infty$, the conditions 2° and 3°.

Condition 4° will evidently be satisfied unless a_n is of the form $\exp[f_1(n) + \iota f_2(n)]$, where $f_2(n)$ is of higher order in n than $f_1(n)$.

For a function $f(x)$ which decreases with x, but not more rapidly than x^{-1}, we have

$$f^{(r)}(x) = O[f(x)\,x^{-r}].$$

Thus, since $\omega(n)$ is intermediate in order between 1 and n^{-1}, $\omega_r(n)$, which is of the order of $\left(\frac{d}{dn}\right)^r \omega(n)$, is of the form $O[\omega(n)\,n^{-r}]$. Condition 5° is thus satisfied.

There remain to be considered conditions 1° and 6°.

We have (s being positive or negative) $\log a_{n+s} = f(n+s)$, where $f(x)$ is intermediate in order between $\epsilon^{-1}\log x$ and ϵx. If a_n is a naturally constructed function $f(x)$ will have only a finite number of singularities. Thus the Taylor-series in powers of s for $f(n+s)$ will have a radius of convergence which differs from n by a finite number, and which is therefore greater than $\frac{1}{2}n$ when n is large. Then, since $\mu = n\nu^{-\frac{2}{5}}$ is of less order than n, we have when $|s| \leqslant \mu$,

$$f(n+s) = f(n) + sf'(n) + \dots + \frac{s^N}{N!}f^{(N)}(n) + \frac{1}{2\pi\iota}\int \frac{s^{N+1}}{x^N(x-s)}\,f(n+x)\,dx \;\text{......................(i)},$$

where the integral is taken round the circle $|x| = \frac{1}{2}n$.

* It should, perhaps, be mentioned in this connexion that if we require a_n to be an analytic function of n of order less than $\exp(\epsilon n)$, we exclude certain forms of a_n which might appear at first sight to be admissible. The form $\exp(n^k)$ $(0<k<1)$, is obviously admissible, and it might be concluded that $(-1)^n \exp(n^k)$ is equally so. This form, however, is excluded by condition 4°. The fact is that $(-1)^n$, regarded as an *analytic* function of n, must be considered to be of some such form as $\exp(\pi\iota n)$, which falls in the same category as $\exp(n)$.

Now since, for functions $f(z)$ of less order than z, $|f(z)|$ varies only by a finite factor when $|z|$ remains constant, and since the maximum modulus $M(|x|)$ of $|f(n+x)|$ on the circle $|x|=$constant is an increasing function of $|x|$

$$|f(n+x)|<Kf(|n+x|)<Kf(\tfrac{3}{2}n)$$
$$<Kf(n) \quad\text{.............(ii)},$$

since, when $|f(z)|<z$, we have $\qquad |f(Kz)|<K|f(z)|$.

From (ii) we see that the remainder-term in (i) is of the form

$$O\left[\frac{s^{N+1}}{n^N(n-\mu)}f(n)\right]=O\left[s^{N+1}n^{-(N+1)}f(n)\right],$$

or $\qquad O\left[s^{N+1}n^{-N}\omega(n)\right]$.

This is what is assumed in condition 1°*.

We have finally to consider condition 6°. Since the integral function $F(z)$ is of zero order, we have

$$|a_n|=n^{\phi(n)},$$

where $\qquad \lim \phi(n)=\infty$.

In cases arising naturally $\phi(n)$ is an increasing function. Also, with our assumptions as to the order of a_n, $\phi(n)<\epsilon n$ when n is large. We have

$$|a_n|\cdot\sum_{s=\mu+1}^{\infty}|a_{n+s}|^{-1}=n^{\phi(n)}\cdot\sum_{s=\mu+1}^{\infty}(n+s)^{-\phi(n+s)}<n^{\phi(n)}\cdot\sum_{s=\mu+1}^{\infty}(n+s)^{-\phi(n+\mu)}$$
$$<n^{\phi(n)}\left[(n+\mu)^{-\phi(n+\mu)}+\int_{\mu}^{\infty}(n+x)^{-\phi(n+\mu)}dx\right]<n^{\phi(n)}\left[(n+\mu)^{-\phi(n+\mu)}+\frac{(n+\mu)^{1-\phi(n+\mu)}}{\phi(n+\mu)-1}\right]$$
$$<Kn^{\phi(n)}\cdot\frac{n+\mu}{\phi(n+\mu)}(n+\mu)^{-\phi(n+\mu)},$$

since $\qquad \phi(n+\mu)-1>K^{-1}\phi(n+\mu)$, and $\phi(n+\mu)\cdot(n+\mu)^{-1}\to 0$,

$$<Kn[\phi(n)]^{-1}n^{\phi(n)}\cdot(n+\mu)^{-\phi(n+\mu)} \quad\text{.............(1)}.$$

Now
$$\mathfrak{R}\omega=\frac{d}{dn}[\phi(n)\log n]=n^{-1}\phi(n)+\log n\,\phi'(n) \quad\text{.............(2)}.$$

In the cases we are considering, one of the two following inequalities will hold for *all* large values of n:—

$$an^{-1}\phi(n)>\log n\,\phi'(n) \qquad (a>3-\epsilon) \quad\text{.............}(3)_1,$$
$$3n^{-1}\phi(n)<\log n\,\phi'(n) \quad\text{.............}(3)_2.$$

If $(3)_1$ holds we have, from (2)

$$\mathfrak{R}\omega=a'n^{-1}\phi(n) \quad\text{.............}(4)_1,$$

and if $(3)_2$ holds we have $\qquad \mathfrak{R}\omega=b'\log n\,\phi'(n)$ ....$(4)_2$,

where a' and b' lie between K^{-1} and K.

Since $|\omega|\geqslant\mathfrak{R}\omega>K^{-1}|\omega|$, we have in either case an equation for $|\omega|$ similar to that which holds for $\mathfrak{R}\omega$.

First let us assume $(3)_1$ to hold. We then have, from (1),

$$|a_n|\cdot\sum_{s=\mu+1}^{\infty}|a_{n+s}|^{-1}<K|\omega|^{-1}n^{\phi(n)}(n+\mu)^{-\phi(n)}$$
$$<K|\omega|^{-1}\left[(1+\mu n^{-1})^{-n\mu^{-1}}\right]^{\mu n^{-1}\phi(n)}$$
$$<K|\omega|^{-1}\left[\exp(-K^{-1})\right]^{K^{-1}\mu|\omega|} \qquad (\text{since } \mu n^{-1}\to 0)$$
$$<K|\omega|^{-1}\exp\left[-K^{-1}\mu|\omega|\right]$$
$$=|\omega|^{-1}\chi(\nu).$$

* It is interesting to notice that, as a matter of fact, 1° does not assume that the series $s\omega+s^2\omega_1+\ldots$ is *convergent*. All that is assumed is that $\log(a_{n+s}/a_n)$ is equal to any (constant) number of terms of the series, with a remainder of the order of the succeeding term.

Now let us assume $(3)_2$ to hold. That inequality may be written

$$\frac{d}{dn}[\log\phi(n)] > 3(n\log n)^{-1} > \frac{d}{dn}\{\log[(\log n)^3]\},$$

so that we shall have (when n is large)

$$\phi(n) > (\log n)^3,$$

and therefore

$$\phi'(n) > 3(\log n)^2 n^{-1} \qquad \text{(5)}.$$

Then

$$\nu = |n\omega| > nK^{-1}\log n\,\phi'(n), \qquad \text{by } (4)_2,$$
$$> nK^{-1}(\log n)^3 n^{-1} > K^{-1}(\log n)^3 \qquad \text{(6)}.$$

Now

$$\phi(n) - \phi(n+\mu) = -\mu\phi'(n+\mu) + \tfrac{1}{2}\mu^2\phi''[n+\mu+\theta(-\mu)],$$

(where $0<\theta<1$)

$$< -\mu\phi'(n+\mu),$$

for $\phi'(n)$ is a decreasing function [since $\phi(n)=\epsilon(n)n$], and therefore $\phi''(x)<0$. But $\phi'(n+\mu)/\phi'(n)\rightarrow 1$ for decreasing functions ϕ' satisfying (5). Therefore

$$\phi(n)-\phi(n+\mu) < -K^{-1}\mu\phi'(n)$$
$$< -K^{-1}\mu|\omega|(\log n)^{-1}, \qquad \text{from } (4)_2,$$
$$< -K^{-1}\nu^{\frac{1}{2}}(K\nu^{\frac{1}{3}})^{-1}, \qquad \text{from (6)},$$
$$< -K^{-1}\nu^{\frac{1}{6}} \qquad \text{(7)}.$$

Again

$$n[\phi(n)]^{-1} = [\phi'(n)]^{-1} n \frac{d}{dn}[\log\phi(n)]$$
$$< [\phi'(n)]^{-1} n \frac{d}{dn}[\log n] < [\phi'(n)]^{-1}$$
$$< K|\omega|^{-1}\log n \ \text{[by (4)]} \qquad \text{(8)}.$$

Hence from (1) we have

$$|a_n|\sum_{s=\mu+1}^{\infty}|a_{n+s}|^{-1} < Kn[\phi(n)]^{-1}\,.\,n^{\phi(n)-\phi(n+\mu)}$$
$$< K|\omega|^{-1}\log n\,.\,n^{-K^{-1}\nu^{\frac{1}{6}}}, \qquad \text{by (7) and (8)},$$
$$= |\omega|^{-1}\chi(\nu).$$

This equation, the first part of condition 6°, therefore holds in any case. The second part of the condition may be established by a similar line of argument.

§ 14. We have seen that $|\omega|^{-1}$ and ν may be of the same or of different types of order in n. The different possibilities may be classified roughly as follows. We may have either

$$(a) \quad \nu^K > |\omega|^{-1} > \nu^{-K^{-1}},$$

or

$$(b) \quad \omega = \chi(\nu),$$

or

$$(c) \quad \nu^{-1} = \chi(\omega),$$

where each inequality is supposed to hold (if at all) for *all* sufficiently great values of n*. We shall discuss the different forms which the remainder-term of our general expansion assumes, in the three cases, for functions $F(z)$ arising naturally.

The simplest of the three cases is (a), as we have there only one type of order to consider. The function $|\omega|^{-1}$ is, for some constant† value of k, comparable with n^{1-k} $(0<k<1)$, ν is thus comparable with n^k, and $|a_n|$ with $\exp(n^k)$. From the last fact it follows that $|\log z|$ is comparable with n^k, so that $|\omega|^{-1}$, ν, n, and $|\log z|$ are all of the same type of order, and, in particular,

$$\chi(\omega)=\chi(\log z), \quad \omega^{-1}\chi(\nu)=\chi(\log z), \quad \omega^{-1}\nu^{-\lambda_1}=O(\nu^{-\lambda_1'}).$$

* The classification is not complete, for it is quite possible (under the conditions) for $|\omega|^{-1}$ and ν to be of "irregular growth," in which case one of the above inequalities may hold for an infinite number of ranges of n, while another of the inequalities holds for another infinite number of ranges.

† That is, in natural cases. It is possible for k to vary in different ranges of n.

The remainder term of the general formula for $\log F(z)$ is therefore of the form $O[(\log z)^{-p}]$, where p is any positive constant.

§ 15. As an example of case (a) let us take the function

$$F(z)=\prod_{s=1}^{\infty}[1+z/\exp\{\rho s^{k}+\rho_1 s^{k_1}+\ldots+\rho_l s^{k_l}\}],$$

where k is a real number between 0 and 1, $\mathfrak{R}\rho>0$, $\rho_1, \rho_2, \ldots$ are any complex constants, and $1>k>\mathfrak{R}k_1\geqslant\mathfrak{R}k_2\geqslant\ldots$.

By a rigorous proof on the lines of § 13 it may be shown that the conditions for the a's are satisfied.

We have
$$\omega=\sum_{r=0}^{l}k_r\rho_r n^{k_r-1}\qquad(k_0=k,\ \rho_0=\rho),$$

$$\omega_p=\sum_{r=0}^{l}\rho_r k_r(k_r-1)\ldots(k_r-p)\,n^{k_r-p-1}.$$

The expression ω^{-t} may be put in the form

$$\omega^{-t}=(k\rho)^{-t}n^{t(1-k)}[1+b_1 n^{-\alpha_1}+b_2 n^{-\alpha_2}+\ldots]+O(n^{-\lambda_1}),$$

where λ_1 is an arbitrary positive number, the α's are rational functions of the k's arranged in order of increasing real parts, and the b's are rational functions of the k's and ρ's.

Since $\beta_{p,q}(n)$ is a polynomial in $\omega_1, \omega_2, \ldots$, it follows that the expression

$$(\tfrac{1}{6}\pi^2-\tfrac{1}{2}\phi^2)\,\omega^{-1}-\tfrac{1}{2}\phi\iota+\tfrac{1}{12}\omega+\sum_{p+q\leqslant\lambda}\beta_{p,q}(n)\,[A_{q+2}(\phi)\,\omega^{-p-q-1}+B_{p-1,q+1}]$$

is of the form*
$$n^{1-k}[b_0+b_1 n^{-\alpha_1}+b_2 n^{-\alpha_2}+\ldots]+O(n^{-\lambda_1})\quad\ldots\ldots\ldots\ldots(1),$$

where the b's are rational algebraic functions of ϕ and the k's and ρ's, the α's are rational algebraic functions of the k's arranged in order of increasing real parts, and λ_1 is any positive constant†.

Now consider the term
$$\log\left[\frac{z^n}{a_1 a_2\ldots a_n}\right]=n\phi\iota+\log\left[\frac{a_n{}^n}{a_1 a_2\ldots a_n}\right].$$

We have
$$\log\left[\frac{a_n{}^n}{a_1 a_2\ldots a_n}\right]=n\sum_{r=0}^{l}\rho_r n^{k_r}-\sum_{s=1}^{n}\left[\sum_{r=0}^{l}\rho_r s^{k_r}\right].$$

Now
$$\sum_{s=1}^{n}s^p=\frac{n^{p+1}}{p+1}+\tfrac{1}{2}n^p+\sum_{r=1}^{\lambda_1+p}C_r n^{p-r}+\zeta(-p)+O(n^{-\lambda_1});$$

therefore
$$\log\left[\frac{z^n}{a_1 a_2\ldots a_n}\right]=n\phi\iota+n^{k+1}[b_0+b_1 n^{-\alpha_1}+b_2 n^{-\alpha_2}+\ldots]-\sum_0^l\rho_r\zeta(-k_r)+O(n^{-\lambda_1})\ \ldots\ldots(2).$$

Again we have
$$\chi(\omega)=\chi(n),\quad \omega^{-1}\nu^{-\frac{1}{2}\lambda}=O(n^{-\lambda_1})\ldots\ldots\ldots\ldots(3).$$

From (1), (2), (3), and the general formula,

$$\log F(z)=n^{k+1}[b_0+b_1 n^{-\alpha_1}+\ldots]-\sum_0^l\rho_r\zeta(-k_r)+O(n^{-\lambda_1})\ \ldots\ldots\ldots\ldots(4).$$

Now we have
$$\sum_{r=0}^{l}\rho_r n^{k_r}=\log z-\phi\iota.$$

From this relation we can deduce that‡

$$n=(\log z-\phi\iota)^{k-1}[b_0+b_1(\log z-\phi\iota)^{-\alpha_1}+\ldots]+O[(\log z-\phi\iota)^{-\lambda_1}]$$

and thence that
$$n=(\log z)^{k-1}[b_0+b_1(\log z)^{-\alpha_1}+\ldots]+O[(\log z)^{-\lambda_1}].$$

* We know that
$$|\beta_{p,q}(n)\,\omega^{-p-q}|<K\nu^{-\frac{1}{2}(p+q)}.$$
The expression $(\frac{1}{6}\pi^2-\frac{1}{2}\phi^2)\,\omega^{-1}$ therefore gives the dominant term.

† In what follows we shall write $b_0, b_1\ldots, \alpha_1, \alpha_2\ldots, \lambda_1$ for any set of numbers such as the above.

‡ The problem of finding (in the general case) an approximation for $\log F(z)$ in terms of z alone turns on the possibility or otherwise of obtaining, from the equation $a_n=ze^{-\phi\iota}$, an approximate formula $n=f(z,\phi)$ of a sufficient degree of accuracy.

The proof of the inversion formula for the particular case above is based on the fact that k is greater than $\mathfrak{R}k_1$, etc.

A similar formula is easily shown to hold for any positive or negative power of n. We therefore obtain from (4),

$$\log F(z)=(\log z)^{1+k^{-1}}[b_0+b_1(\log z)^{-a_1}+\ldots]-\sum_{r=0}^{l}\rho_r\zeta(-k_r)+O[(\log z)^{-\lambda_1}] \quad \ldots\ldots\ldots\ldots(5).$$

The indices a and coefficients b may be calculated when the k's and λ_1 are given numerically. If this be done in any particular case, it will be found that ϕ disappears from the b's, so that we obtain an approximation in terms of z alone.

It is, as a matter of fact, possible to show that the ϕ's must disappear from the general formula (5), and, further, to determine the values of the a's and b's. The proof of this, it should be said, depends on results derived from a totally different theory, and has no application to the other functions which we shall consider.

If we can in any manner obtain for $\log F(z)$ an expression of the form of the right-hand side of (5), in a case where the k's, ρ's, and ϕ, although subject to restrictions, may yet each assume an infinity of values, the a's and b's in this expression and in (5) must be identical. For otherwise we obtain, by subtraction, an expression of the form of the right-hand side of (5), which is equal to a multiple of $2\pi\iota$ for all the values of the k's, ρ's, and ϕ considered. Since the a's are rational algebraic functions of the k's, and the b's are similar functions of the k's, ρ's, and ϕ, this is easily shown to be impossible.

I have shown elsewhere* that $\log F(z)$ possesses an asymptotic expansion of the form (5) in the case when the k's are real, $\rho_r\rho^{-1}$ is real, and $0<k<\frac{1}{2}$†. This expansion, which is now shown to hold under the conditions $0<k<1$, $k>\mathfrak{R}k_1\geqslant\mathfrak{R}k_2\ldots$, $\mathfrak{R}\rho>0$, and the general condition for ϕ, is as follows:—

$$\log F(z)=-\sum_{r=0}^{l}\rho_r\zeta(-k_r)-\tfrac{1}{2}\log z+\frac{1}{\pi}\sum_{m=0}C_m\sin(\pi k^{-1}-\pi\theta_m)(\rho^{-1}\log z)^{k^{-1}-\theta_m+1}$$
$$\times\left[\Gamma(\theta_m-k^{-1}-1)+2\sum_{s=1}\Gamma(2s+\theta_m-k^{-1}-1)\frac{2^{2s-1}-1}{2s\,!}B_{2s}(\log z)^{-2s}\right],$$

where the B's are Bernoulli's numbers, and where $\Sigma C_n x^{\theta_n}$ is the expansion

$$k^{-1}\sum_{m=0}^{\infty}\sum_{a_1+a_2+\ldots+a_l=m}\left[\prod_{r=1}^{l}\left\{\frac{\rho_r{}^{a_r}}{\Gamma(a_r+1)}\right\}\Gamma\left\{\sum_{r=1}^{l}k_rk^{-1}a_r+k^{-1}\right\}\rho^{-m}x^{\sum_{r=1}^{l}(1-k_rk^{-1})a_r}\right]$$

arranged in order of increasing indices of x‡.

§ 16. We must examine the nature of the restriction on z implied by the restriction on ϕ. Let us suppose, in the completely general case of our theorem, a circle C_s drawn with each zero $-a_s$ as centre, of radius

$$\rho_s=|a_s|\{|\omega(s)|^{1-\delta}+[\nu(s)]^{\delta-1}\}.$$

It may then be shown that if z is excluded from all the circles, the condition for ϕ is satisfied. For, for some value of n, the point z (when $|z|$ is large) lies in the annulus formed by the circles

$$|z|=|a_n\exp[\pm\tfrac{3}{4}\omega(n)]|.$$

The breadth of the annulus is less than $K|\omega(n)a_n|$, and is therefore of the form $\epsilon(n)\rho_n$. It is then evident from a figure, after a little consideration, that when z is excluded from C_n, we have

$$\pi-|\theta|>K^{-1}\{|\omega(n)|^{1-\delta}+[\nu(n)]^{\delta-1}\}$$
$$>|\omega(n)|^{1-\frac{1}{2}\delta}+[\nu(n)]^{\frac{1}{2}\delta-1}, \qquad \text{when } n \text{ is large.}$$

It may be shown that the circles overlap, and in such a manner that the space from which z is excluded is, roughly speaking, a spiral strip.

In the particular problem before us, we have

$$|\omega|^{1-\delta}+\nu^{\delta-1}<Kn^{-(1-k)(1-\delta)}+Kn^{-k(1-\delta)}.$$

Hence if $k\geqslant\frac{1}{2}$, we may take for the radius of the circle C_s,

$$\rho_s=s^{-(1-k)+\delta}|a_s|,$$

* *Proc. London Math. Soc.* Ser. 2, Vol. VII. p. 248. (The notation is different from that of the present paper, k^{-1}, k_r^{-1} ($k>1$) being written in place of k, k_r.)

† The case $1>k>\frac{1}{2}$ presents grave difficulties in this treatment of the problem.

‡ i.e. if p_n, p_{n+1} are consecutive indices, we are to have $\mathfrak{R}p_{n+1}\geqslant\mathfrak{R}p_n$.

where δ is an arbitrarily small positive constant; and if $k \leqslant \frac{1}{2}$, we may take

$$\rho_s = s^{-k+\delta} |a_s|.$$

As regards the extent of the region from which z is excluded, it is easily seen that if we take a large circle $|z| = R$, the ratio of the area of the part of the strip included within the circle, to the area of the whole circle, is of the order of

$$(\log R)^{-h+\delta},$$

where h is the lesser of k and $(1-k)$. Thus the strip shrinks to its least dimensions when $k = \frac{1}{2}$, and becomes larger when k either increases or decreases from this value*.

§ 17. As a second example, let us take

$$F(z) = \prod_{s=1}^{\infty} [1 + z/\{s^\sigma \exp(\rho s^k)\}],$$

where $1 > k > \frac{1}{2}$, and let us find an approximation for $\log F(z)$ with a remainder-term of the form $\epsilon(z)$.

We may neglect in the general formula the terms

$$\tfrac{1}{12}\omega + \Sigma \beta_{p,q}(n) [A_{q+2}(\phi)\, \omega^{-p-q-1} + B_{p-1,q+1}].$$

(It is easily seen that these terms are all of the form $\epsilon(z)$ when $k > \frac{1}{2}$.)

We expand the remaining terms of the formula in descending powers of n† on the lines of the last article, employing in the reduction of $\log(a_1 a_2 \ldots a_n)$ the known asymptotic formulae for $\log(n!)$ and $\sum_{s=1}^{n} s^p$. We have then to substitute for n in terms of z and ϕ by means of the equation

$$\rho [n^k + \sigma \log n] = \log z - \iota\phi.$$

This equation gives

$$n = f(z, \phi) + O[(\log z)^{k^{-1}-3+\delta}],$$

where $f(z, \phi)$ is an algebraic function of ϕ, $\log z$ and $\log(\rho^{-1}\log z)$. On effecting the substitution we find that ϕ disappears from the final result, which may be written

$$\begin{aligned}\log F(z) = \frac{k\rho}{k+1} (\rho^{-1}\log z)^{k^{-1}+1} - \sigma k^{-1} \log(\rho^{-1}\log z) \,.\, (\rho^{-1}\log z)^{k^{-1}} + \sigma (\rho^{-1}\log z)^{k^{-1}} - \tfrac{1}{2}\log z \\ + \left[\tfrac{1}{2}\sigma^2 \rho^{-1} k^{-3} \{\log(\rho^{-1}\log z)\}^2 + \tfrac{1}{6}\pi^2 k^{-1}\rho^{-1}\right] (\rho^{-1}\log z)^{k^{-1}-1} \\ - \rho\zeta(-k) - \tfrac{1}{2}\sigma \log 2\pi + O\{(\log z)^{-h+\delta}\},\end{aligned}$$

where h is the lesser of $(k^{-1}-2)$ and $(1-k^{-1})$.

§ 18. We shall next consider case (b), in which $\omega = \chi(\nu)$. In this case, however large the constant λ may be taken, $\omega^{-1}\nu^{-\frac{1}{2}\lambda}$ cannot be a small term. The function $\log|a_n|$ will be of the form $\log n \,.\, \theta(n)$, where $\theta(n)$ tends to infinity, but more slowly than any power of n.

If we have $\theta(n) > K^{-1}(\log n)^{K^{-1}}$, ν will be of the order of some positive power of $\log n \,.\, \theta(n)$, or of $\log z$. The general formula will therefore give for $\log F(z)$ an expression of the form

$$\log \left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] + \omega^{-1} P(z),$$

where $P(z)$ is a series in which the ratio of any term to the preceding is of less order than $(\log z)^{-h}$, where h is some positive constant.

* It is interesting to notice that $k = \frac{1}{2}$ is a critical value for k in the analytic theory of the function $F(z)$ developed in the paper cited above.

† In this case terms in $\log n$ appear in addition to simple powers of n.

If, on the other hand, $\theta(n)$ is of lower order than $(\log n)^\epsilon$ for all positive values of ϵ, ν will be of lower order than $(\log z)^\epsilon$, and we shall have

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] + \omega^{-1} P(z),$$

where $P(z)$ is a series in which the ratio of any term to the preceding is of less order than $[t(z)]^{-1}$, where $t(z)$ is a function of less order than any positive power of $\log z$.

As an example let us take the typical function of the first of the two cases,

$$F(z) = \prod_{s=1}^{\infty} [1 + z/\exp\{\rho(\log s)^{(1+k)}\}] \qquad (k>0).$$

The function ω is $\rho(1+k)n^{-1}(\log n)^k$, and ω^{-1} is of order comparable with $\exp[(\rho^{-1}\log z)^{(1+k)^{-1}}]$. Let us suppose that we wish to find an approximation for $\log F(z)$ with a remainder-term of the form

$$R(z) = \epsilon(z)(\log z)^{-2k(1+k)^{-1}} \exp[(\rho^{-1}\log z)^{(1+k)^{-1}}].$$

The terms
$$-\tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega + \Sigma\beta_{p,q}(n) B_{p-1,q+1}$$

may be neglected, and we have

$$\log F(z) = \rho n(\log n)^{1+k} - \rho\sum_{s=1}^{n}(\log s)^{1+k} + n\phi\iota + (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\omega^{-1} - \omega_1\omega^{-3}A_3(\phi) + R(z) \;\dots\dots\dots\dots(1).$$

We have

$$n = \exp[(\rho^{-1}\log z - \rho^{-1}\iota\phi)^{(1+k)^{-1}}] \sim \exp(t^{k'})[1 - k'\rho^{-1}\phi\iota t^{k'-1} - {}_{k'}C_2\rho^{-2}\phi^2 t^{k'-2} - \tfrac{1}{2}k'^2\rho^{-2}\phi^2 t^{2k'-2}] \;\dots(2),$$

where
$$k' = (1+k)^{-1}, \quad t = \rho^{-1}\log z.$$

On performing the substitution (2) for n in (1) we obtain*

$$\log F(z) = \exp[(\rho^{-1}\log z)^{(1+k)^{-1}}]$$
$$\times\left[\begin{array}{l}\rho(1+k)\{(\rho^{-1}\log z)^{k(1+k)^{-1}} - k(\rho^{-1}\log z)^{(k-1)(1+k)^{-1}} + k(k-1)(\rho^{-1}\log z)^{(k-2)(1+k)^{-1}} - \dots\} \\ \qquad + \tfrac{1}{6}\pi^2\rho^{-1}(1+k)^{-1}(\rho^{-1}\log z)^{-k(1+k)^{-1}} + \epsilon(z)(\log z)^{-2k(1+k)^{-1}}\end{array}\right],$$

where the series within crooked brackets is carried to the last term such that the index of $(\log z)$ is not less than $-2k(1+k)^{-1}$.

It is possible, when k is given numerically, to carry the series in large brackets as far as any numerically assigned negative power of $\log z$, i.e. it is possible to obtain any number of terms of a certain infinite series Q. This infinite series is certainly divergent as it stands (on account of the divergence of the series in crooked brackets), and is almost certainly divergent when the series in crooked brackets is suppressed, or replaced by the equivalent integral

$$\exp[-(\rho^{-1}\log z)^{(1+k)^{-1}}]\int_0^{(\rho^{-1}\log z)^{(1+k)^{-1}}} x^k e^x\,dx.$$

The formula
$$\log F(z) = \exp[(\rho^{-1}\log z)^{(1+k)^{-1}}][Q] \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

is therefore the only asymptotic expansion for $\log F(z)$ (at any rate of a simple character) which is possible at all. We cannot, for example, obtain a formula such as

$$\log F(z) = f(z) + \epsilon(z)$$

(where $f(z)$ is expressed in finite terms), on account of the exponential factor in (1).

* We may replace, for our purposes,
$$n(\log n)^{1+k} - \sum_{s=1}^{n}(\log s)^{1+k}$$
by
$$(1+k)\int_1^{\log n} x^k e^x\,dx.$$
The reduction then presents no difficulties.

§ 19. We shall finally discuss the case (c), in which $\nu^{-1} = \chi(\omega)$.

In this case all the terms

$$\Sigma \beta_{p,q}(n) [A_{q+2}(\phi)\, \omega^{-p-q-1} + B_{p-1,\, q+1}]$$

are *small*. They give rise to a series in which the ratio of each term to the preceding is of the order of a negative power of ν, while the function ν is of the form $O(\log z)$. But it is not necessarily true that the $\chi(\omega)$ which occurs in the general formula is a smaller term than $\beta_{1,1}(n) A_{q+2}(\phi)\, \omega^{-3}$, the most important term of the β-series. Unless this is the case, of course, the terms of the β-series are irrelevant.

It may be shown that the $\chi(\omega)$ of the general theorem is, to a first approximation, the $\chi(\omega)$ occurring in the formula

$$\log \psi(\phi, \omega) = (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\, \omega^{-1} - \tfrac{1}{2}\iota\phi + \tfrac{1}{12}\omega + \chi(\omega) \quad \text{.....................(1).}$$

It is easy to carry the approximation of § 1 one step further, and for the $\chi(\omega)$ of (1) we have

$$\chi(\omega) = -2\cos(2\pi\phi\iota\omega^{-1})\, e^{-2\pi^2\omega^{-1}} + \text{lower terms} \quad \text{........................(2).}$$

The general formula therefore becomes*

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] + (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\, \omega^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega + \sum_{p+q \leqslant \lambda} \beta_{p,q}(n) [A_{q+2}(\phi)\, \omega^{-p-q-1} + B_{p-1,\, q+1}]$$
$$- 2\cos(2\pi\phi\iota\omega^{-1})\, e^{-2\pi^2\omega^{-1}} [1 + \epsilon(z)] + O(\omega^{-1}\nu^{-\frac{1}{2}\lambda}) \quad \text{.....................(3).}$$

We have now to consider the relative importance of the last two terms in this expression.

In what follows we shall suppose that the condition for ϕ is replaced by the more stringent condition $\pi - |\theta| > \delta$.

Since $\qquad \cos(2\pi\phi\iota\omega^{-1}) = \cos(2\pi\theta\iota\omega^{-1} + 2\pi\alpha\iota),$

it is seen that $\qquad \chi_1(\omega) = [1 + \epsilon(z)]\, [-2\cos(2\pi\phi\iota\omega^{-1})\, e^{-2\pi^2\omega^{-1}}]$

is of the order of $\qquad \exp[-2\pi(\pi - |\theta|)\, \omega^{-1}]$.

Since $\mathfrak{R}\omega > K^{-1}|\omega| > K^{-1}\mathfrak{R}\omega$, and since $\pi - |\theta| > \delta$, we therefore have the following results:

(i) If
$$\lim_{n\to\infty} |\omega| \log \nu = 0,$$
$\chi_1(\omega)$ is of less order than $\omega^{-1}\nu^{-p}$ for all (constant) values of p, however large.

(ii) If
$$\lim_{n\to\infty} |\omega| \log \nu = \infty,$$
$\chi_1(\omega)$ is of greater order than $\omega^{-1}\nu^{-p}$ for all values of p, however small.

(iii) If
$$\lim_{n\to\infty} \log \nu / (\mathfrak{R}\omega^{-1}) = c,$$
where c is a (positive) non-zero constant, $\chi_1(\omega)$ is of greater or less order than $\omega^{-1}\nu^{-p}$ according as

$$p \gtrless \pi(\pi - |\theta|)\, c^{-1},$$

and, in particular, $\chi_1(\omega)$ is of greater order than $\omega^{-1}\nu^{-p}$ for all values of θ if $p > \pi^2 c^{-1}$.

* The $\chi(\omega)$ occurring in the formula II for $S_{p,q}$ is of the order of the first term in (2), multiplied by some power of ω. The remainder-term furnished by $\beta_{p,q}(n)\, S_{p-1,q+1}$, on account of the factor $\nu^{-\frac{1}{2}(p+q)}$ in $\beta_{p,q}(n)$, is therefore of less order than the first term in (2).

In case (i) the term $\chi_1(\omega)$ in (1) may be absorbed into the $O(\omega^{-1}\nu^{-\frac{1}{2}\lambda})$. The terms of the β-series are all relevant, and the expansion obtained is of the same type as that obtained for the functions of case (a).

For example if $$F(z)=\prod_{s=1}^{\infty}[1+z/\exp(\rho x_s)], \qquad (\Re\rho>0),$$
where x_s is the real root of the equation $$x_s(\log x_s)^2=s,$$
we can obtain for $\log F(z)$ an expansion in terms of z with a remainder of the form $O[(\log z)^{-p}]$, where p is arbitrary*.

§ 20. In case (ii) the β-series in (1) of § 18 may be absorbed into the term
$$\epsilon(z)\left[-2\cos(2\pi\phi\iota\omega^{-1})\,e^{-2\pi^2\omega^{-1}}\right],$$
and we obtain
$$\log F(z)=\log\left[\frac{z^n}{a_1a_2\ldots a_n}\right]+(\tfrac{1}{6}\pi^2-\tfrac{1}{2}\phi^2)\,\omega^{-1}-\tfrac{1}{2}\phi\iota+\tfrac{1}{12}\omega-2\,[1+\epsilon(z)]\cos(2\pi\phi\iota\omega^{-1})\,e^{-2\pi^2\omega^{-1}}.$$

As an example we may take
$$F(z)=\prod_{s=1}^{\infty}[1+z/\exp(\rho x_s)],$$
where† $$x_s(\log x_s)^{\frac{1}{2}}=s.$$
The conditions 1° to 6° for the a's may be verified without difficulty.

We have $$\omega=\rho\,[(\log x_n)^{\frac{1}{2}}+\tfrac{1}{2}(\log x_n)^{-\frac{1}{2}}]^{-1},\quad \log\left[\frac{z^n}{a_1a_2\ldots a_n}\right]=n\phi\iota+\rho\left[nx_n-\sum_{s=1}^{n}x_s\right] \qquad\ldots\ldots\ldots\ldots(1).$$

It may be shown that the Maclaurin-Bernoulli expansion is valid for the sum $\sum_{s=1}^{n}x_s$, and we have
$$\sum_{s=1}^{n}x_n=A+\int_0^{x_n}x[(\log x)^{\frac{1}{2}}+\tfrac{1}{2}(\log x)^{-\frac{1}{2}}]\,dx+\tfrac{1}{2}x_n+\tfrac{1}{12}[(\log x_n)^{\frac{1}{2}}+\tfrac{1}{2}(\log x_n)^{-\frac{1}{2}}]^{-1}+O(n^{-1}),$$
where A is a numerical constant. When we substitute $x_n(\log x_n)^{\frac{1}{2}}$ for n, and $\rho^{-1}(\log z-\iota\phi)$ for x_n, we obtain
$$\log F(z)=\rho\int_1^{\rho^{-1}\log z}x(\log x)^{\frac{1}{2}}\,dx-\tfrac{1}{2}\log z+\tfrac{1}{6}\pi^2\rho^{-1}[\log(\rho^{-1}\log z)]^{\frac{1}{2}}-\rho B$$
$$+\tfrac{1}{12}\pi^2\rho^{-1}[\log(\rho^{-1}\log z)]^{-\frac{1}{2}}-2\cosh\{2\pi\rho^{-1}\phi[\log(\rho^{-1}\log z)]^{\frac{1}{2}}\}\exp\{-2\pi^2\rho^{-1}[\log(\rho^{-1}\log z)]^{\frac{1}{2}}\}$$
$$+\text{lower terms},$$
where B is a numerical constant.

§ 21. As an example of case (iii), let us consider the function
$$F(z)=\prod_{s=1}^{\infty}[1+z/\exp(\rho x_s)],$$
where $$x_s\log x_s=s.$$
We obtain
$$\log F(z)=\tfrac{1}{4}\rho^{-1}\{2\log(\rho^{-1}\log z)-1\}(\log z)^2-\tfrac{1}{2}\log z-\rho A+\tfrac{1}{6}\pi^2\rho^{-1}\{\log(\rho^{-1}\log z)+1\}$$
$$+t_1(\log z)^{-2}+t_2(\log z)^{-4}+\ldots+t_p(\log z)^{-2p}$$
$$-[(e\rho^{-1}\log z)^{-2\pi\rho^{-1}(\pi+\phi)}+(e\rho^{-1}\log z)^{-2\pi\rho^{-1}(\pi-\phi)}]+\text{lower terms},$$

* In the absence of any proof to the contrary we must admit the possibility that ϕ may occur in this expansion in the form of a polynomial, but such occurrence is extremely unlikely.

For the methods to be adopted in the reduction, compare § 20.

† It might appear that a simpler function would be
$$\Pi\,[1+z/\exp\{s(\log s)^{-\frac{1}{2}}\}].$$
But in this case, the nature of the inversion formula corresponding to $a_n=e^{-\iota\phi}z$ is such that we cannot obtain a formula of the type
$$\log F(z)=f(z)+\epsilon(z),$$
where $f(z)$ is expressed in finite terms.

where p is the greatest integer contained in $\mathfrak{R}\pi^2\rho^{-1}$, A is a numerical constant, and the t's are rational algebraic functions of $\log(\rho^{-1}\log z)$ and (possibly) ϕ. In this expression, as $|\theta|$ increases from 0 to $\pi - \delta$, the term in square brackets becomes of greater order than the terms $t_p(\log z)^{-2p}$, $t_{p-1}(\log z)^{-2p+2}$, ... in succession, and as each such term is thus overtaken, it may be absorbed into the remainder-term.

§ 22. In the examples just given of cases (ii) and (iii) a term $\chi_1(\omega)$ occurs in which the number ϕ appears in the index. We shall try to account in a general manner for this peculiarity.

Let us return to the general theory of §§ 5—12. Since $\log F(z)$ becomes infinite when $\phi = \pm\pi$ (or $z = -a_n$), the order of remainder-term in the general formula for $\log F(z)$ must increase rapidly as ϕ approaches the value π. Now in cases when n can be eliminated from the final expression for $\log F(z)$*, the remainder-term consists of two parts, (1) a term whose order is independent of ϕ, corresponding roughly with the term $O(\omega^{-1}\nu^{-\frac{1}{2}\lambda})$ in the general formula, and (2) the disturbing term $\chi_1(\omega)$ [corresponding with $\chi(\omega)$] which becomes infinite at $\phi = \pm\pi$. The latter term, being supposed not to contain n, must necessarily involve ϕ. Now it is found that the order of $\chi_1(\omega)$ increases with the order of a_n. If we subject ϕ to the condition $\pi - |\theta| > \delta$†, then, when a_n is of less order than $\exp[\epsilon n(\log n)^{-1}]$, $\chi_1(\omega)$ is of less importance than the terms furnished by the β-series, and is absorbed into the other remainder-term, while when the order of a_n is increased beyond this limit, $\chi_1(\omega)$ becomes of greater importance than terms of the β-series, and thus forces itself upon our notice.

In the functions considered in this paper $\chi_1(\omega)$ is always small (when $\pi - |\theta| > \delta$). The increase of the order of $\chi_1(\omega)$ with the order of a_n, however, continues when a_n increases beyond the limit $\exp(\epsilon n)$ imposed here. It is possible to find asymptotic expressions in terms of z and ϕ for the logarithms of various functions for which a_n is of higher order than $\exp(\epsilon^{-1}n)$ (for all values of ϵ). In every such expression there appears a term involving ϕ (although not necessarily as an index)‡.

Section II. Taylor Coefficients.

§ 23. Let $F(z)$ be one of the class of functions considered in the general theorem of the last section, and let $\sum_{m=0}^{\infty} c_m z^m$ be the Taylor-series for $F(z)$. Then we have

$$c_m = \frac{1}{2\pi\iota}\int_C \frac{F(z)\,dz}{z^{m+1}},$$

where C is a circle about the origin. It is natural to inquire whether, by taking C of large radius, we can approximate for the integral when m is large by means of our asymptotic formula for $F(z)$. It will be seen that this is possible, not indeed in the general case, but when $F(z)$ belongs to one of the classes (a) and (c) of § 14, i.e. when a_n is of higher order than $\exp(n^\eta)$ for some positive value of η.

* In the present discussion we are, of course, only concerned with such cases.

† If $|\theta|$ is permitted to approach more nearly to the value π (as in the condition $\pi - |\theta| > |\omega|^{1-\delta} + \nu^{\delta-1}$), $\chi_1(\omega)$ disturbs the expression for $\log F(z)$ with a lower order of a_n. It is, however, a disturbance which takes place under the condition $\pi - |\theta| > \delta$ that is likely to be regarded as unusual.

‡ For a theory of functions whose nth zero is not of lower order than $\exp(\epsilon n)$, see *Proc. London Math. Soc.* Ser. 2, Vol. v. pp. 361—410.

To avoid the complication introduced by the difference of behaviour of $F(z)$ in the cases (a) and (c), we shall content ourselves with finding an approximation for c_m of the form $[1+\epsilon(m)]f(m)$. With a remainder-term of this order it is possible to work the two cases together.

In the theory of Section I the approximation for $\log F(z)$ is expressed in terms of n and ϕ, and is a polynomial in the latter variable. It is therefore obviously desirable to be able to keep the n depending on z constant in integrating $F(z)z^{-n-1}$ round a contour, thus replacing the z-contour by a ϕ-contour (not necessarily closed). Now it is true that for all points z of a circle C, with the origin as centre, the number n corresponding to z is constant. But in pursuing the method outlined above we shall find that we must deform the contour C in such a manner that the number n, as originally defined, does not remain constant. To meet this difficulty we shall establish the following modification of the general theorem of the last Section, in which n has a slightly different meaning.

Let us suppose that the a's satisfy all the conditions of § 3 *with the addition that* 3° *be replaced by the more stringent condition**

$$(3^\circ)' \qquad |\omega(n)| > K^{-1}n^{-K^{-1}}.$$

Then there exists a positive constant η, *such that when*

$$z = a_n e^{\iota\phi}, \quad \textit{and} \quad |\phi| < 2\eta,$$

we have

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] + (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1} - \tfrac{1}{2}\phi\iota + \tfrac{1}{12}\omega$$
$$+ \sum_{p+q<\lambda} \beta_{p,q}(n)\,[\omega^{-p-q-1}A_{q+2}(\phi) + B_{p-1,q+1}] + \chi(\omega) + O(\nu^{-\lambda_1}),$$

where λ_1 *is arbitrary, and* λ *depends on* λ_1.

§ 24. The proof of this theorem requires a different line of argument from that adopted in Section I. Since $\phi = \theta + \iota\alpha\omega$, the number α, under the new condition for ϕ, may be of the order of $|\omega|^{-1}$. $|z/a_{n+s}|$ and $|a_{n-s}/z|$ are not necessarily less than unity, and we cannot use the expansions for $\log(1+z/a_{n+s})$ and $\log(1+a_{n-s}/z)$. Further, the series $S_{p,q}$ may be divergent. The details of the proof, however, are very similar to those of the proof of the original theorem, and certain simplifications may be effected by means of the new condition $(3^\circ)'$. [For example, $\omega^{-1}\chi(\nu)$ is now of the form $\chi(\nu)$.] We shall, therefore, do no more than indicate the main lines to be followed.

Since $|e^{\iota\phi}| < K$ when $|\phi| < 2\eta$, whatever value may be given to the constant η, it is easily seen that we still have the result (A) of § 5, viz.,

$$G(z) \equiv \log F(z) - \log\left[\frac{z_n}{a_1 a_2 \ldots a_n}\right] - \log\psi(\phi, \omega)$$
$$= \sum_{s=1}^{\mu}\left[\log\left(1+\frac{z}{a_{n+s}}\right) - \log(1+e^{\iota\phi - s\omega})\right] + \sum_{s=1}^{\mu}\left[\log\left(1+\frac{a_{n-s}}{z}\right) - \log(1+e^{-\iota\phi - s\omega})\right] + \omega^{-1}\chi(\nu) \qquad \text{.......................(1).}$$

* This condition introduces the restriction that a_n is of greater order than $\exp(n^{\eta'})$ for some value of η'.

Since $|\omega| > K^{-1} n^{-K^{-1}}$, we have $|\omega| > K^{-1}\nu^{-K_1}$, and the ω^{-1} in the remainder-term of (1) may be omitted.

It may be shown that, for any assigned constant λ_2, when $s \leqslant \mu$,

$$z/a_{n+s} = e^{\iota\phi - s\omega} \exp[-s^2\omega_1 - \dots]$$

differs from

$$e^{\iota\phi - s\omega} \exp[sP(s)]$$

by a number of the form $O(\nu^{-\lambda_2})$, where $P(s)$ is the sum of a number (depending on λ_2) of terms of the expansion $-s\omega_1 - s^2\omega_2 \dots$. Again, it may be shown that, for all values of s,

$$|1 + e^{\pm\iota\phi - s\omega}| > K^{-1} \qquad\text{...(2)},$$

*provided the constant η is chosen sufficiently small**.

Now $|sP(s)| < K\nu^{-K^{-1}} = \epsilon(n)$, when $s \leqslant \mu$. It therefore follows from (2) that

$$|\log[1 + z/a_{n+s}] - \log[1 + \exp\{\phi\iota - s\omega + sP(s)\}]|$$

$$= \left|\log\left[1 + \frac{z/a_{n+s} - \exp[\phi\iota - s\omega + sP(s)]}{1 + \exp[\phi\iota - s\omega + P(s)]}\right]\right| < \left|\log\left[1 - \frac{K\nu^{-\lambda_2}}{K^{-1}}\right]\right|$$

$$= O(\nu^{-\lambda_2}).$$

A similar result holds when a_{n-s}/z is substituted for z/a_{n+s}, $-\phi$ for ϕ, and $P(-s)$ for $P(s)$. We therefore have

$$\sum_{s=1}^{\mu} [\log(1 + z/a_{n+s}) - \log(1 + e^{\iota\phi - s\omega}) + \log(1 + a_{n-s}/z) - \log(1 + e^{-\iota\phi - s\omega})]$$

$$= \sum_{s=1}^{\mu} \Big[\log[1 + \exp\{\phi\iota - s\omega + sP(s)\}] - \log(1 + e^{\iota\phi - s\omega})$$

$$+ \log[1 + \exp\{-\phi\iota - s\omega + sP(-s)\}] - \log(1 + e^{-\iota\phi - s\omega})\Big] + O(\mu\nu^{-\lambda_2}) \qquad\text{.........(3)}.$$

Now since $|1 + e^{\iota\phi - s\omega}| > K^{-1}$, the radius of convergence of the Taylor-series about the point $x = \iota\phi - s\omega$ of the analytic function $\log(1 + e^x)$, is greater than K^{-1}. Since

$$|sP(\pm s)| = \epsilon(n),$$

and since $|\log(1 + e^{\iota\phi - s\omega})| < K$, we therefore have

$$\log[1 + \exp\{\phi\iota - s\omega + sP(s)\}] - \log[1 + \exp(\phi\iota - s\omega)]$$

$$= \sum_{\rho=1}^{\lambda_3} \frac{\{sP(s)\}^p}{p!} \left\{\frac{d}{d(-s\omega)}\right\}^p \log(1 + e^{\phi\iota - s\omega}) + O'[\{sP(s)\}^{\lambda_3+1}],$$

* We have $\mathfrak{R}\omega^{-1} > K_1^{-1}|\omega|$.

Let η be so chosen that $2\eta < \frac{1}{4}\pi$ and

$$4\eta/(\tfrac{1}{2}\pi - 2\eta) < K_1^{-1}.$$

Then, when $s|\omega| \leqslant \frac{1}{2}\pi - 2\eta$,

the imaginary part of $\iota\phi - s\omega$ has a modulus less than

$$2\eta + \tfrac{1}{2}\pi - 2\eta, \quad\text{or } \tfrac{1}{2}\pi,$$

so that $|1 + e^{\iota\phi - s\omega}| > 1$;

while, on the other hand, when

$$s|\omega| > \tfrac{1}{2}\pi - 2\eta,$$

we have

$$\mathfrak{R}(\iota\phi - s\omega) < 2\eta - sK_1^{-1}|\omega$$

$$< \tfrac{1}{2}(\tfrac{1}{2}\pi - 2\eta)K_1^{-1} - (\tfrac{1}{2}\pi - 2\eta)K_1^{-1}$$

$$< -\tfrac{1}{2}(\tfrac{1}{2}\pi - 2\eta)K_1^{-1},$$

and therefore $|1 + e^{\iota\phi - s\omega}| > K^{-1}$.

for any value of the constant integer λ_3;

$$= -\sum_{p=1}^{\lambda_3} \frac{\{-sP(s)\}^p}{p!} \cdot s^{1-p} \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{\phi\iota - s\omega}}{1+e^{\phi\iota - s\omega}} + O(\nu^{-\lambda_2}) \ldots\ldots\ldots\ldots\ldots\ldots (4)_1,$$

when λ_3 is suitably chosen. We have similarly

$$\log[1+\exp\{-\phi\iota - s\omega - sP(s)\}] - \log[1+\exp(-\phi\iota - s\omega)]$$

$$= -\sum_{p=1}^{\lambda_3} \frac{\{-sP(-s)\}^p}{p!} s^{1-p} \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{-\phi\iota - s\omega}}{1+e^{-\phi\iota - s\omega}} + O(\nu^{-\lambda_2}) \ldots\ldots\ldots\ldots\ldots\ldots (4)_2.$$

On expanding $\{-sP(\pm s)\}^p$ in the form of a polynomial, and summing the right-hand sides of $(4)_1$ and $(4)_2$ from $s=1$ to μ, we therefore obtain, by means of (3),

$$\sum_{s=1}^{\mu}\left[\log\left(1+\frac{z}{a_{n+s}}\right) - \log(1+e^{\iota\phi - s\omega}) + \log\left(1+\frac{a_{n-s}}{z}\right) - \log(1+e^{-\phi\iota - s\omega})\right]$$

$$= \sum_{p,q}\left[\gamma_{p,q}(n) \sum_{s=1}^{\mu}\left\{s^{p+q} \cdot s \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{\iota\phi - s\omega}}{1+e^{\iota\phi - s\omega}} + (-s)^{p+q} \cdot s \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{-\iota\phi - s\omega}}{1+e^{-\iota\phi - s\omega}}\right\}\right] + O(\mu\nu^{-\lambda_2}) \ldots (5),$$

where $\gamma_{p,q}$ is a polynomial in $\omega_1, \omega_2 \ldots$, the summation with respect to p, q is taken over a finite number of terms, and where, since $P(s)$ has no term independent of s, we have $q > 0$.

Now it may be shown that

$$\sum_{s=\mu}^{\infty}\left[s^{p+q} \cdot s \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{\iota\phi - s\omega}}{1+e^{\iota\phi - s\omega}} + (-s)^{p+q} \cdot s \left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{-\iota\phi - s\omega}}{1+e^{-\iota\phi - s\omega}}\right] = \chi(\nu).$$

Therefore, since $\gamma_{p,q}$ is of the form $\epsilon(n)$ (being a polynomial in $\omega_1, \omega_2 \ldots$), we may replace $\sum_{s=1}^{\mu}$ by $\sum_{s=1}^{\infty}$ in (5). Since $O(\mu\nu^{-\lambda_2}) = O(\nu^{-\lambda_1})$ when λ_2 is suitably chosen, we now have, from (1) and (5),

$$G(z) = \sum_{p,q}\left[\gamma_{p,q}(n) \sum_{s=1}^{\infty} s^{p+q+1}\left\{\left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{\iota\phi - s\omega}}{1+e^{\iota\phi - s\omega}} + (-)^{p+q}\left(\frac{d}{d\omega}\right)^{p-1} \frac{e^{-\iota\phi - s\omega}}{1+e^{-\iota\phi - s\omega}}\right\}\right] + O(\nu^{-\lambda_1})$$

$$= \sum_{p,q}\left[\gamma_{p,q}(n)\left(\frac{d}{d\omega}\right)^{p-1} \sum_{s=1}^{\infty} s^{p+q+1}\left\{\frac{e^{\iota\phi - s\omega}}{1+e^{\iota\phi - s\omega}} - (-)^{p+q+1} \frac{e^{-\iota\phi - s\omega}}{1+e^{-\iota\phi - s\omega}}\right\}\right] + O(\nu^{-\lambda_1}) \ldots\ldots (6).$$

§ 25. We next proceed to show that, when $|\phi| < 2\eta$,

$$f(\phi) \equiv \left(\frac{d}{d\omega}\right)^{p-1} \sum_{s=1}^{\infty} s^{p+q+1}\left\{\frac{e^{\iota\phi - s\omega}}{1+e^{\iota\phi - s\omega}} - (-)^{p+q+1} \frac{e^{-\iota\phi - s\omega}}{1+e^{-\iota\phi - s\omega}}\right\}$$

is equal to

$$\omega^{-(2p+q+1)} Q(\phi, \omega) + \chi(\omega),$$

where $Q(\phi, \omega)$ is a polynomial in ϕ and ω.

It is easily shown that $f(\phi)$ is an analytic function of ϕ in the region $|\phi| < 2\eta$. We shall show that

$$f(\phi) = \omega^{-(2p+q+1)} Q(\phi, \omega) + f_1(\phi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $f_1(\phi)$ is a certain function which must evidently be analytic in $|\phi| < 2\eta$. Let us suppose that ϕ is real and $|\phi| < 2\eta$. We then have $|e^{\iota\phi - s\omega}| < 1$ for all values of s. Consequently*

$$f(\phi) = -\left(\frac{d}{d\omega}\right)^{p-1} \sum_{s=1}^{\infty} s^{p+q+1} \left\{ \sum_{r=1}^{\infty} (-)^r e^{r\phi\iota - rs\omega} - (-)^{p+q+1} \sum_{r=1}^{\infty} (-)^r e^{-r\phi\iota - rs\omega} \right\}$$

$$= (-)^{p+q} \left(\frac{d}{d\omega}\right)^{2p+q} \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} (-)^r \{e^{r\phi\iota - rs\omega} . r^{-(p+q+1)} - e^{-r\phi\iota - rs\omega} . (-r)^{-(p+q+1)}\} \quad \text{.........(2)},$$

or, changing the order of summation,

$$= (-)^{p+q} \left(\frac{d}{d\omega}\right)^{2p+q} \left[\sum_{r=1}^{\infty} (-)^r \frac{e^{-r\omega}}{1 - e^{-r\omega}} \{r^{-(p+q+1)} e^{r\phi\iota} - (-r)^{-(p+q+1)} e^{-r\phi\iota}\} \right] \quad \text{.........(3)}.$$

Now in the notation of § 2, the series within square brackets is $S_{-(p+q+1),\, p+q+1}$. By the result (8) of that article we therefore have

$$f(\phi) = (-)^{p+q} \left(\frac{d}{d\omega}\right)^{2p+q} R_0 + f_1(\phi) \quad \text{.............................(4)},$$

where R_0 is the residue of $\dfrac{e^{z\phi\iota}\, z^{-p-q-1}}{\sin \pi z} \dfrac{e^{-z\omega}}{1 - e^{-z\omega}}$ at $z = 0$, and where (since $q > 0$)

$$f_1(\phi) = (-)^{p+q} \left(\frac{d}{d\omega}\right)^{2p+q} \left[\pi\omega^{-1} \sum_{m=1}^{\infty} (2m\pi\iota\omega^{-1})^{-(p+q+1)} \operatorname{cosech}(2m\pi^2\omega^{-1}) \{e^{-2m\pi\phi\omega^{-1}} - (-)^{p+q+1} e^{2m\pi\phi\omega^{-1}}\} \right] \quad \text{...............(5)}.$$

The first term on the right-hand side of (4) is easily seen to be of the form $\omega^{-(2p+q+1)} Q(\phi, \omega)$. $f_1(\phi)$ also is an analytic function of ϕ in the region $|\phi| < 2\eta$. Since, then, the equation (4) holds for all real values of ϕ in the region $|\phi| < 2\eta$, it must hold throughout the whole of that region, and the result (1) is established, $f_1(\phi)$ being given by (5).

§ 26. Now it may be shown that when $|\phi| < 2\eta$, we have

$$f_1(\phi) = \chi(\omega).$$

It therefore follows from (6) of § 24, and (1) of § 25, that

$$G(z) = \sum_{r=0} \vartheta_r(n)\, \phi^r + O(\nu^{-\lambda_1}) + \chi(\omega) \quad \text{.............................(1)},$$

where $\vartheta_r(n)$ is a rational algebraic function of $\omega, \omega_1, \omega_2 \ldots$, and where the number of terms in the summation is finite (and depends on λ). Now in § 12 an expression was found for $G(z)$, under the conditions for ϕ, $\pi - |\theta| > |\omega|^{1-\delta} + \nu^{\delta-1}$, and $|\alpha| < \frac{3}{4}$, which is similar in form to (7) except that $O(\omega^{-1}\nu^{-\frac{1}{2}\lambda})$ occurs in the place of $O(\nu^{-\lambda_1})$. But since, when λ is suitably chosen, $O(\omega^{-1}\nu^{-\frac{1}{2}\lambda})$ is of the form $O(\nu^{-\lambda'})$, where λ' is arbitrary this expression may be written

$$G(z) = \sum_{r=0} \vartheta_r'(n)\, \phi^r + O(\nu^{-\lambda_1}) + \chi(\omega) \quad \text{.............................(2)}.$$

The forms (7) and (8) must be identical when ϕ is real, and $|\phi| < 2\eta$. From this fact it is easily proved that

$$\vartheta_r'(n) = \vartheta_r(n) + O(\nu^{-\lambda_1}) + \chi(\omega) \quad \text{..................................(3)}.$$

* The legitimacy of the next three steps is established without difficulty.

Now on referring to the theorem I of § 1 it will be seen that, under the condition $|\phi| < 2\eta$, we still have

$$\log \psi(\phi, \omega) = (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1} - \tfrac{1}{2}\iota\phi + \tfrac{1}{12}\omega + \chi(\omega).$$

Since $$G(z) = \log F(z) - \log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] - \log \psi(\phi, \omega),$$

our present theorem follows immediately from (2) and (3).

§ 27. Let $F(z)$ be a function satisfying the conditions of Section I, and the modified condition $(3°)'$. We shall establish for c_n a formula of the type

$$c_n = \frac{1+\epsilon(n)}{a_1 a_2 \dots a_n} f(n).$$

The expressions $\frac{1}{12}\omega$ and $\beta_{p,q}(n)$ are of the form $\epsilon(n)$. Hence, by a suitable choice of λ, we can ensure that

$$\log F(z) = \log\left[\frac{z^n}{a_1 a_2 \dots a_n}\right] + (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1} - \tfrac{1}{2}\iota\phi + \sum_{p+q<\lambda} \beta_{p,q}(n)\, A_{q+2}(\phi)\, \omega^{-p-q-1} + \epsilon(n) \quad \text{...(1)},$$

under *either* of the two conditions for ϕ;

$$(a) \quad \alpha = 0, \quad \pi - |\theta| > \eta,$$

$$(b) \quad |\phi| < 2\eta.$$

Now we have

$$2\pi c_n \,.\, a_1 a_2 \dots a_n = -\iota \int_C \frac{a_1 a_2 \dots a_n \,.\, F(z)\, dz}{z^{n+1}},$$

taken round any contour C enclosing the origin. Let us take for C the circle $|z| = |a_n|$. Writing $z = a_n e^{\iota\phi}$, where ϕ is real, we then have

$$2\pi c_n \,.\, a_1 a_2 \dots a_n = \int_{-\pi}^{\pi} F(z) \left[\frac{z^n}{a_1 a_2 \dots a_n}\right]^{-1} d\phi \quad \text{..........................(2)}.$$

We divide this last integral into the two parts,

$$I = \int_{-\eta}^{\eta} F(z) \left[\frac{z^n}{a_1 a_2 \dots a_n}\right]^{-1} d\phi \quad \text{..................................}(3)_1,$$

$$I' = \left(\int_{-\pi}^{-\eta} + \int_{\eta}^{\pi}\right) F(z) \left[\frac{z^n}{a_1 a_2 \dots a_n}\right]^{-1} d\phi \quad \text{..........................}(3)_2,$$

where η is the number of the condition $|\phi| < 2\eta$. It will be found that $I' = \epsilon(n) I$, so that I gives the important factor of c_n. We proceed first to consider this integral.

The integrand is a regular function of ϕ within the region $|\phi| < 2\eta$: the integration from $-\eta$ to η may therefore be performed along any path for ϕ lying within this region.

Now $$(\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\phi^2)\,\omega^{-1} - \tfrac{1}{2}\iota\phi + \sum_{p+q<\lambda} \beta_{p,q}(n)\, A_{q+2}(\phi)\, \omega^{-p-q-1}$$

is a polynomial in ϕ. Let this polynomial be written*

$$\gamma_0 + \gamma_1\phi + \gamma_2\phi^2 + \dots \quad \text{..................................(4)}.$$

* We have

$\gamma_0 = \frac{1}{6}\pi^2\omega^{-1} - 6\omega_2\omega^{-4}A_4(0) +$ lower terms,

$\gamma_1 = -\frac{1}{2}\iota - \frac{1}{3}\iota\pi^2\omega_1\omega^{-3} +$ lower terms,

$\gamma_2 = -\frac{1}{2}\omega^{-1} +$ lower terms,

$\gamma_3 = \frac{1}{3}\iota\pi^2\omega_1\omega^{-3} +$ lower terms,

$\gamma_4 = a\omega_2\omega^{-4} +$ lower terms, etc.

Of the γ's, γ_0 and γ_2 are of highest order, γ_1 and γ_3 are of next highest order*, and the remainder are of lower order than these four. Let us substitute $\phi = \phi' - \phi_0$ in (4), where ϕ_0 is the root of least modulus of the equation

$$\gamma_1 - 2\gamma_2\phi + 3\gamma_3\phi^2 + \ldots = 0 \quad \text{.................................(5).}$$

The resulting polynomial in ϕ' has then no term in ϕ'; let it be

$$\gamma_0' + \gamma_2'\phi'^2 + \gamma_3'\phi'^3 + \ldots \quad \text{..................................(6)}$$

[γ_0', γ_1', ... are, of course, polynomials in ϕ_0]. Since γ_2 is of higher order than the other γ's occurring in (5), it may be shown that ϕ_0 tends to zero as n tends to infinity, and that as a first approximation†

$$\phi_0 = \tfrac{1}{2}\gamma_1\gamma_2^{-1} \quad \text{..(7).}$$

We note also that

$$\gamma_2' = \gamma_2 + 3\gamma_3\phi_0 = -\tfrac{1}{2}\omega^{-1}[1 + \epsilon(n)] \quad \text{.............................(8),}$$

and that γ_3', γ_4', ... are of the form

$$O(\omega n^{-1}\omega^{-3}) \text{ or } O(\omega^{-2}n^{-1}) \quad \text{..................................(9).}$$

From (1), (4), (6), and $(3)_1$, we have

$$F(z)\left[\frac{z^n}{a_1a_2\ldots a_n}\right]^{-1} = \exp[\gamma_0' + \gamma_2'\phi'^2 + \gamma_3'\phi'^3 + \ldots + \epsilon(n)],$$

$$I = \int_{-\eta}^{\eta} \exp[\gamma_0' + \gamma_2'\phi'^2 + \ldots + \epsilon(n)]\,d\phi$$

$$= \int_{-\eta+\phi_0}^{\eta+\phi_0} \exp[\gamma_0' + \gamma_2'\phi'^2 + \ldots + \epsilon(n)]\,d\phi' \quad \text{......................(10).}$$

§ 28. Let $\tau = |\omega|^{\frac{1}{2}-h}$, where h is a small positive constant. The path of integration in the last integral may be taken to be the straight lines $-\eta + \phi_0$ to $-\tau$ to $+\tau$ to $\eta + \phi_0$, since the corresponding path $-\eta$ to $-\tau - \phi_0$ to $\tau - \phi_0$ to η for ϕ lies within the region $|\phi| < 2\eta$. Thus

$$I = I_1 + I_2 + I_2' \quad \text{...}(1)_1,$$

where

$$I_1 = \int_{-\tau}^{\tau} \exp[\gamma_0' + \gamma_2'\psi^2 + \gamma_3'\psi^3 + \ldots + \epsilon(n)]\,d\psi \quad \text{....................}(1)_2,$$

$$I_2 = \int_{-\eta+\phi_0}^{\tau} \exp[\gamma_0' + \gamma_2'\psi^2 + \ldots + \epsilon(n)]\,d\psi \quad \text{.........................}(1)_3,$$

$$I_3 = \int_{\tau}^{\eta+\phi_0} \exp[\gamma_0' + \gamma_2'\psi^2 + \ldots + \epsilon(n)]\,d\psi \quad \text{.........................}(1)_4.$$

Let us first consider I_1, which will be found to give the important part of I. When $|\psi| < \tau$, we see from (9) of the last article that

$$\begin{aligned}\gamma_3'\psi^3 + \gamma_4'\psi^4 + \ldots &= O(\tau^3\omega^{-2}n^{-1}) = O(\omega^{-\frac{1}{2}-3h}n^{-1})\\ &= O(\nu^{-1}), \text{ if } \tfrac{1}{2} + 3h < 1,\\ &= \epsilon(n),\end{aligned}$$

so that

$$\begin{aligned}I_1 &= \int_{-\tau}^{\tau} \exp[\gamma_0' + \gamma_2'\psi^2 + \epsilon(n)]\,d\psi\\ &= \int_{-\tau}^{\tau} \exp(\gamma_0' + \gamma_2'\psi^2)\,d\psi + \int_{-\tau}^{\tau} \epsilon(n)\exp(\gamma_0' + \gamma_2'\psi^2)\,d\psi\\ &= \exp(\gamma_0')\left[\int_{-\tau}^{\tau} \exp(\gamma_2'\psi^2)\,d\psi + \epsilon(n)\int_{-\tau}^{\tau} \exp(\mathfrak{R}\gamma_2'\psi^2)\,d\psi\right] \quad \text{.................(2).}\end{aligned}$$

* When $\omega_1\omega^{-3} = \epsilon(n)$, γ_1 is of higher order than γ_3, otherwise γ_1 and γ_3 are of the same order.

† A rigorous proof of a similar point will be found in § 54.

Now* $$\int_{-\tau}^{\tau} \exp(\gamma_2' \psi^2)\, d\psi = 2\int_0^{\infty} \exp(\gamma_2'\psi^2)\, d\psi - 2\int_{\tau}^{\infty} \exp(\gamma_2'\psi^2)\, d\psi$$
$$= \pi^{\frac{1}{2}} (-\gamma_2')^{-\frac{1}{2}} + O\left[\int_0^{\infty} \exp\{\mathfrak{R}\gamma_2'(\tau^2 + 2\tau\psi' + \psi'^2)\}\, d\psi'\right]$$
$$= \pi^{\frac{1}{2}} (-\gamma_2')^{-\frac{1}{2}} + O\left[\exp(\mathfrak{R}\gamma_2'\tau^2)\int_0^{\infty} \exp\{\mathfrak{R}\gamma_2'\psi'^2\} d\psi'\right] \quad \text{......(3).}$$

From (2) and (3) we have
$$I_1 = \exp(\gamma_0')\left[\pi^{\frac{1}{2}}(-\gamma_2')^{\frac{1}{2}} + \{\epsilon(n) + O[\exp(\mathfrak{R}\gamma_2'\tau^2)]\}\int_0^{\infty}\exp(\mathfrak{R}\gamma_2'\psi^2)\, d\psi\right] \quad \text{......(4).}$$

Now from (8) of last article,
$$\mathfrak{R}\gamma_2' = \mathfrak{R}\{-\tfrac{1}{2}\omega^{-1}[1+\epsilon(n)]\} < -K^{-1}|\gamma_2'|,$$
since $\mathfrak{R}\omega > K^{-1}|\omega|$. It follows that
$$\int_0^{\infty} \exp(\mathfrak{R}\gamma_2'\psi^2) d\psi = \tfrac{1}{2}\pi^{\frac{1}{2}}(-\mathfrak{R}\gamma_2')^{-\frac{1}{2}} = O(-\gamma_2')^{-\frac{1}{2}}$$
and
$$\exp(\mathfrak{R}\gamma_2'\tau^2) < \exp[-K^{-1}|\omega|^{-1}|\omega|^{1-2h}] = \chi(\omega) = \epsilon(n).$$

Hence, from (3) and (4), $$I_1 = \exp(\gamma_0')[\pi^{\frac{1}{2}}(-\gamma_2')^{-\frac{1}{2}}\{1+\epsilon(n)\}]$$
$$= \exp(\gamma_0')[\pi^{\frac{1}{2}}(\tfrac{1}{2}\omega^{-1})^{-\frac{1}{2}}\{1+\epsilon(n)\}], \text{ from (8) of § 27,}$$
$$= (2\pi\omega)^{\frac{1}{2}}\exp(\gamma_0').[1+\epsilon(n)] \quad \text{......(5).}$$

We shall now show that $I_2 = \chi(\omega) I_1$, $I_3 = \chi(\omega) I_1$ ......(6).

Since τ is real and $\phi_0 = \epsilon(n)$, the angle which the path of the integral $\int_{-\eta+\phi_0}^{-\tau}$ makes with the real axis is of the form $\epsilon(n)$. Thus for any point ϕ' of the path we have
$$\phi' = |\phi'|[1+\epsilon(n)] \quad \text{......(7).}$$

Again, if $p > 0$, we have
$$|\gamma'_{2+p}\phi'^{2+p}|/|\gamma_2'\phi'^2| < K(\omega^{-2}n^{-1})/|-\tfrac{1}{2}\omega^{-1}[1+\epsilon(n)]| = \epsilon(n).$$

Also, $$|\gamma_2'\phi'^2| \geqslant |-\tfrac{1}{2}\omega^{-1}[1+\epsilon(n)].\tau^2|$$
$$> K^{-1}|\omega|^{-1}|\omega|^{1-2h} > K^{-1}.$$

Therefore $$[|\gamma_3'\phi'^3| + |\gamma_4'\phi'^4| + \ldots + \epsilon(n)]/|\gamma_2'\phi'^2| = \epsilon(n) \quad \text{......(8).}$$

Therefore, from (7) and (8),
$$\mathfrak{R}[\gamma_2'\phi'^2 + \gamma_3'\phi'^3 + \ldots + \epsilon(n)] = \mathfrak{R}\{\gamma_2'\phi'^2[1+\epsilon(n)]\}$$
$$= \mathfrak{R}\{-\tfrac{1}{2}\omega^{-1}|\phi'|^2[1+\epsilon(n)]\}$$
$$< -K^{-1}|\omega|^{-1}\tau^2 < -K^{-1}|\omega|^{-2h}.$$

Consequently we have in $(1)_3$,
$$|I_2| \leqslant \int_{-\eta+\phi_0}^{-\tau} \exp[\mathfrak{R}\{\gamma_0' + \gamma_2'\phi'^2 + \ldots + \epsilon(n)\}]\,|d\phi'|$$
$$< |\exp(\gamma_0')|.\int_{-\eta+\phi_0}^{-\tau} \exp[-K^{-1}|\omega|^{-2h}]\,|d\phi'|$$
$$= \chi(\omega)\exp(\gamma_0')$$
(since the length of the path of integration is less than K)
$$= \chi(\omega)(2\pi\omega)^{\frac{1}{2}}\exp(\gamma_0')$$
$$= \chi(\omega) I_1.$$

* It should be remembered that $\mathfrak{R}\gamma_2' < 0$.

The second part of (6) is proved in the same way. From (1), (5), and (6) we now obtain

$$I = [1 + \epsilon(n)](2\pi\omega)^{\frac{1}{2}} \exp(\gamma_0') \quad \text{.................................(E).}$$

§ 29. Returning to the integral I' of § 27, we shall show that

$$I' \equiv \left(\int_{-\pi}^{-\eta} + \int_{\pi}^{\eta}\right) F(z) \left[\frac{z_n}{a_1 a_2 \ldots a_n}\right]^{-1} d\phi = \epsilon(n) I \quad \text{.....................(F)}_1.$$

We have
$$\gamma_0' = \gamma_0 - \phi_0\gamma_1 + \phi_0^2\gamma_2' \ldots$$
$$= \gamma_0 [1 + \epsilon(n)],$$

since $\gamma_1, \gamma_2, \ldots$ are not of higher order than γ_0, and $\phi_0 = \epsilon(n)$,

$$= \tfrac{1}{6}\pi^2\omega^{-1} [1 + \epsilon(n)].$$

It therefore follows from the form (E) for I, that $(F)_1$ will be established if we can prove that

$$| I' | < \exp(\tfrac{1}{6}\pi^2\omega^{-1}) \exp[-K^{-1} | \omega |^{-1}],$$

and therefore if we can prove a similar inequality for the integrands in I', i.e. if we can prove that

$$\mathfrak{R}\left\{\log F(z) - \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] - \tfrac{1}{6}\pi^2\omega^{-1}\right\} < -K^{-1} | \omega |^{-1} \quad \text{..................(1),}$$

when ϕ is real, and $\pi \geqslant | \phi | \geqslant \eta$. We proceed then to establish the result (1).

Since $| \phi |$ may now increase up to the limit π, we cannot use the formula of the general theorem of Section I. The result (A) of that section, however, does not involve any restriction on θ (which is here the value of ϕ). We thus have

$$\log F(z) - \log\left[\frac{z^n}{a_1 a_2 \ldots a_n}\right] - \log\psi(\phi, \omega)$$

$$= \sum_{s=1}^{\mu}\left[\log\left(1 + \frac{z}{a_{n+s}}\right) - \log(1 + e^{\iota\phi - s\omega}) + \log\left(1 + \frac{a_{n-s}}{z}\right) - \log(1 + e^{-\iota\phi - s\omega})\right] + \omega^{-1}\chi(\nu) \ldots (2).$$

Now when $s \leqslant \mu$, we have

$$\left|\left(1 + \frac{z}{a_{n+s}}\right)\Big/(1 + e^{\iota\phi - s\omega}) - 1\right| = \left|\frac{z/a_{n+s} - e^{\iota\phi - s\omega}}{1 + e^{\iota\phi - s\omega}}\right|$$

$$= \left|\frac{e^{\iota\phi - s\omega}\{\exp[O(s^2\omega n^{-1})] - 1\}}{1 + e^{\iota\phi - s\omega}}\right|$$

$$= \frac{e^{-s\omega'} . | \{O(s^2\omega n^{-1})\} |}{1 - e^{-s\omega'}}, \quad (\omega' = \mathfrak{R}\omega),$$

since $| O(s^2\omega n^{-1}) | < K\mu^2 | \omega | n^{-1} < K$. It follows that

$$\left|\sum_{s=1}^{\mu}\left[\log\left(1 + \frac{z}{a_{n+s}}\right) - \log(1 + e^{\iota\phi - s\omega})\right]\right| < K | \omega | n^{-1} . \sum_{s=1}^{\infty} \frac{s^2 e^{-s\omega'}}{1 - e^{-s\omega'}}$$

$$< K | \omega | n^{-1} . | \omega' |^{-3},$$

by the proof of (8) of § 7,
$$< K | \omega |^{-2} n^{-1}$$
$$= \epsilon(n) | \omega |^{-1} \quad \text{............................(3).}$$

We have similarly

$$\left|\sum_{s=1}^{\mu}\left[\log\left(1+\frac{a_{n-s}}{z}\right)-\log\left(1+e^{-\iota\phi-s\omega}\right)\right]\right|=\epsilon(n).|\omega|^{-1} \quad \text{..................(4)}.$$

From (2), (3), and (4) we now obtain

$$\log F(z)-\log\left[\frac{z^n}{a_1 a_2\ldots a_n}\right]=\log\psi(\phi,\omega)+\epsilon(n).\omega^{-1} \quad \text{..................(5)}.$$

Now on referring to § 1, we have

$$\psi(\phi,\omega)=e^{-\frac{1}{2}\iota\phi}(q^{\frac{1}{4}}q_0)^{-1}\vartheta_2(\nu\,|\,\tau) \qquad \text{[by (1)]}$$

$$=e^{-\frac{1}{2}\iota\phi}(q^{\frac{1}{4}}q_0)^{-1}(2\pi\omega^{-1})^{\frac{1}{2}}e^{-\frac{1}{2}\phi^2\omega^{-1}}.\vartheta_4(\tau^{-1}\nu\,|-\tau^{-1})$$

$$=e^{-\frac{1}{2}\iota\phi}(q^{\frac{1}{4}}q_0)^{-1}(2\pi\omega^{-1})^{\frac{1}{2}}e^{-\frac{1}{2}\phi^2\omega^{-1}}\left[1-2\sum_{n=1}^{\infty}e^{-2\pi^2n^2\omega^{-1}}\cos(2\pi n\phi\iota\omega^{-1})\right] \qquad \text{[by (2)]}$$

$$=O\left[(q^{\frac{1}{4}}q_0)^{-1}\omega^{-\frac{1}{2}}e^{-\frac{1}{2}\phi^2\omega^{-1}}\right] \quad \text{..(6)},$$

since the modulus of the sum of the infinite series is less than

$$1+2\sum_{n=1}^{\infty}\left|e^{-2\pi^2(n^2-n)\omega^{-1}}\right|,$$

which is evidently less than K. Again, by (6) of § 1, we have

$$q_0 q^{\frac{1}{4}}=(2\pi\omega^{-1})^{\frac{1}{2}}\exp\left[-\tfrac{1}{6}\pi^2\omega^{-1}+\epsilon(n)\right].$$

Therefore from (6) above

$$\psi(\phi,\omega)=O\left[\exp\left(\tfrac{1}{6}\pi^2\omega^{-1}\right).\exp\left(-\tfrac{1}{2}\phi^2\omega^{-1}\right)\right]$$

$$=O\left[\exp\left(\tfrac{1}{6}\pi^2\omega^{-1}\right).\exp\left(-\tfrac{1}{2}\eta^2\omega^{-1}\right)\right] \quad \text{..........................(7)},$$

since $\phi^2 \geqslant \eta^2$.

We now have from (2) and (7)

$$\mathfrak{R}\left\{\log F(z)-\log\left[\frac{z^n}{a_1a_2\ldots a_n}\right]-\tfrac{1}{6}\pi^2\omega^{-1}\right\}$$

$$<\mathfrak{R}\left\{-\tfrac{1}{2}\eta^2\omega^{-1}+\epsilon(n)\,\omega^{-1}\right\}$$

$$<-K^{-1}\mathfrak{R}\,\omega^{-1}$$

$$<-K^{-1}|\omega|^{-1}.$$

This is the desired result (1).

§ 30. From (E) and (F), we now obtain

$$2\pi c_n.a_1a_2\ldots a_n\equiv I+I'=[1+\epsilon(n)](2\pi\omega)^{\frac{1}{2}}\exp(\gamma_0')$$

$$=[1+\epsilon(n)](2\pi\omega)^{\frac{1}{2}}\exp\left[\gamma_0-\phi_0\gamma_1+\phi_0{}^2\gamma^2-\ldots\right] \quad \text{...........(F)}.$$

The right-hand side contains the number ϕ_0, and we must show how to obtain for ϕ_0, and its positive powers, approximations which may be substituted in (F) without altering the form of that equation.

ϕ_0 is the root of the equation

$$\gamma_1 - 2\gamma_2\phi + 3\gamma_3\phi^2 - \ldots = 0 \quad \ldots\ldots\ldots\ldots(1)$$

which has the least modulus. We have seen that a first approximation is $\phi_1 = \frac{1}{2}\gamma_1\gamma_2^{-1}$. Substituting $\phi = \phi_1 + \phi'$ in (1), we have for ϕ' an equation

$$\gamma_1'' + \gamma_2''\phi' + \gamma_3''\phi'^2 + \ldots = 0 \quad \ldots\ldots\ldots\ldots(2),$$

where $\quad \gamma_1'' = 3\gamma_3(\frac{1}{2}\gamma_1\gamma_2^{-1})^2 + \text{lower terms} = O(n^{-1}\omega^{-2}),$

$$\gamma_2'' = -2\gamma_2 + \text{lower terms} = \omega^{-1} + \text{lower terms},$$

and where γ_3'', $\gamma_4''\ldots$ are of order not greater than γ_1''. The first approximation for ϕ' is

$$\phi_2 = -\gamma_1''(\gamma_2'')^{-1}.$$

Now $|\phi_2| < Kn^{-1}|\omega|^{-2}.|\omega| < K\nu^{-1}$. This result enables us to calculate ϕ', by successive approximation, with an error of the form $O(\nu^{-\lambda_1})$, where λ_1 is arbitrary. We have, in fact, from (2),

$$\phi' = \phi_2 - (\gamma_2'')^{-1}[\gamma_3''\phi'^2 + \gamma_4''\phi'^3 + \ldots] \quad \ldots\ldots\ldots\ldots(3),$$

and therefore (since ϕ_2 is the first approximation for ϕ')

$$\phi' - \phi_2 = O[(\omega^{-1})^{-1}.\, n^{-1}\omega^{-2}.\, \nu^{-2}] = O(\nu^{-3}).$$

If, therefore, we substitute ϕ_2 for ϕ' in the right-hand side of (3) we commit an error in ϕ' of the form $O[(-\gamma_2'')^{-1}\gamma_3''.\,\phi'(\phi_2 - \phi')]$, or of the form $O[(\omega^{-1})^{-1}n^{-1}\omega^{-2}.\,\nu^{-1}.\,\nu^{-3}]$ or $O(\nu^{-5})$. Let the approximation obtained be ϕ_3. We then substitute ϕ_3 for ϕ'* in the right-hand side of (3), and obtain ϕ_4 for ϕ', with an error $O[(-\gamma_2'')^{-1}\gamma_3''\phi'(\phi_3 - \phi')]$ or $O(\nu^{-7})$. Continuing this process we can obtain for ϕ', and therefore for ϕ_0, an approximation with an error $O(\nu^{-\lambda_1})$.

Let us denote by ψ the approximation for ϕ_0. Then by choosing λ_1 suitably we can ensure that

$$(\gamma_0 - \gamma_1\phi_0 + \gamma_2\phi_0^2 - \ldots) = (\gamma_0 - \gamma_1\psi + \gamma_2\psi^2 - \ldots) + \epsilon(n),$$

and consequently

$$\exp(\gamma_0 - \gamma_1\phi_0 + \ldots) = [1 + \epsilon(n)]\exp(\gamma_0 - \gamma_1\psi + \ldots).$$

We may therefore replace ϕ_0 by ψ in the equation (F).

§ 31. In the expression

$$[1 + \epsilon(n)]\exp(\gamma_0 - \gamma_1\psi + \ldots)$$

we may evidently suppress all terms $\pm\gamma_r\psi^r$ which are of the form $\epsilon(n)$, for $\exp[\epsilon(n)]$ is of the form $1 + \epsilon(n)$. There is a convenient rule to determine which γ's may be thus suppressed†, which may be employed before the calculation of ψ. We shall show that if $\gamma_r\nu^{-r} = \epsilon(n)$, then $\gamma_r\psi^r = \epsilon(n)$.

We have
$$\psi = O(\phi_1) = O[\gamma_1\gamma_2^{-1}]$$
$$= O[\{1 + n^{-1}|\omega|^{-2}\}|\omega|] = O(|\omega|) + O(\nu^{-1}) \quad \ldots\ldots\ldots\ldots(1).$$

Now if $r = 1, 3, 4, \ldots$

$$\omega\gamma_r = O[\omega\{1 + n^{-1}|\omega|^{-2}\}] = \epsilon(n).$$

Therefore, from (1),

$$\gamma_r\psi^r = \gamma_r[O\{|\omega|^r\} + O\{|\omega|^{r-1}\nu^{-1}\} + \ldots + O\{\nu^{-r}\}]$$
$$= \epsilon(n) + O(\gamma_r\nu^{-r}) \quad \ldots\ldots\ldots\ldots(2).$$

* But not, of course, for ϕ_2.

† It should be remembered that these γ's must not be suppressed *in the equation for* ϕ_0.

For the case omitted, $r=2$, we have

$$\begin{aligned}\gamma_2\psi^2 &= O(\omega^{-1})\,[O\{|\omega|^2\}+O\{|\omega|\,\nu^{-1}\}+O\{\nu^{-2}\}]\\ &= \epsilon(n)+O(\omega^{-1}\nu^{-2})\\ &= \epsilon(n)+O(\gamma_2\nu^{-2}),\end{aligned}$$

since
$$|\gamma_2| = |[1+\epsilon(n)](-\tfrac{1}{2}\omega^{-1})| > K^{-1}|\omega|^{-1}.$$

The equation (2) therefore holds for all values of r. Our rule then follows immediately.

§ 32. Summing up, we have the following theorem:

If the a's satisfy the conditions of § 23, *we have*

$$c_n = \frac{1+\epsilon(n)}{a_1a_2\dots a_n}\left(\frac{\omega}{2\pi}\right)^{\frac{1}{2}}\exp\,[\gamma_0-\gamma_1\psi+\dots\pm\gamma_p\psi^p],$$

where all the terms $\gamma_r\psi^r$ within the square brackets may be omitted for which we have

$$\gamma_r\nu^{-r} = \epsilon(n),$$

and where ψ is determined from the equation

$$\gamma_1 - 2\gamma_2\phi + 3\gamma_3\phi^2 \dots \mp p\gamma_p\phi^{p-1} = 0,$$

by the method of successive approximation explained in § 31, *to such a point that*

$$(\gamma_0-\gamma_1\psi+\dots\pm\gamma_p\psi^p) = (\gamma_0-\gamma_1\phi+\dots\pm\gamma_p\phi^p)+\epsilon(n).$$

The numbers $\gamma_0, \gamma_1, \dots \gamma_p$ are the coefficients of $\phi^0, \phi^1, \dots$ in the polynomial

$$-\tfrac{1}{2}\iota\phi + (\tfrac{1}{6}\pi^2-\tfrac{1}{2}\phi^2)\,\omega^{-1} + \sum_{p+q\leqslant\lambda}\beta_{p,q}(n)\,\omega^{-p-q-1}A_{q+2}(\phi),$$

where λ is chosen so large that

$$\omega^{-1}\nu^{-\frac{1}{2}\lambda} = \epsilon(n).$$

§ 33. The formula for c_n involves the expression $a_1a_2\dots a_n$. In cases arising naturally we can find an expression of the form $[1+\epsilon(n)]\,f(n)$ for $a_1a_2\dots a_n$, where $f(n)$ is expressed in terms of known functions and one indefinite integral. In fact, we have in all natural cases*

$$\log\,[a_1a_2\dots a_n] = \sum_{s=1}^{n}\log a_s = \int^n \log a_s\,ds + A + \tfrac{1}{2}\log a_n + \epsilon(n),$$

where A is a constant. When the integral cannot be evaluated, it is sometimes possible to obtain for it an approximation with a remainder-term of the form $\epsilon(n)$.

§ 34. As an example let us consider the function

$$F(z) = \prod_{s=1}^{\infty}\,[1+z/\{s^\sigma\exp(\rho s^k)\}],$$

where
$$\mathfrak{R}\rho>0, \quad 1>k>\tfrac{1}{5}.$$

We may take

$$\gamma_0 = \tfrac{1}{6}\pi^2\omega^{-1} - 6\omega_2\omega^{-4}A_4(0), \quad \gamma_1 = -\tfrac{1}{2}\iota - \tfrac{1}{3}\pi^2\iota\omega_1\omega^{-3}, \quad \gamma_2 = -\tfrac{1}{2}\omega^{-1},$$
$$\psi = \iota\omega\,.\,\tfrac{1}{3}\pi^2\omega_1\omega^{-3} = \tfrac{1}{3}\iota\pi^2\omega_1\omega^{-2}.$$

In the expression
$$\exp\,(\gamma_0-\gamma_1\psi+\gamma_2\psi^2-\dots) \quad\dots\dots(1),$$

* This formula is a particular case of the Maclaurin-Bernoulli sum-formula. The $\epsilon(n)$ is of the order of $\frac{d}{dn}(\log a_n)$, or $\omega(n)$. The formula is easily established in any particular case.

we may suppress the terms $\gamma_3\psi^3$, $\gamma_4\psi^4$, ..., and may take

$$-\gamma_1\psi+\gamma_2\psi^2=-\tfrac{1}{18}\pi^2\omega_1^2\omega^{-5}.$$

We have then to substitute

$$\omega=\rho k n^{k-1}+\sigma n^{-1},\quad \omega_1=\rho_k C_2 n^{k-2}-\tfrac{1}{2}\sigma n^{-2},\quad \omega_2=\rho_k C_3 n^{k-3}+\tfrac{1}{6}\sigma n^{-3},$$

and we obtain a formula $c_n=\dfrac{1+\epsilon(n)}{a_1 a_2 \dots a_n} f(n)$. A number of terms may be suppressed in $f(n)$, and absorbed into the factor $1+\epsilon(n)$, but the expression remains somewhat complicated.

If, however, we suppose that $k>\frac{1}{4}$, we may suppress more terms in $f(n)$, and obtain

$$c_n=\frac{1+\epsilon(n)}{a_1 a_2 \dots a_n}(2\pi\rho^{-1}k^{-1}n^{1-k})^{-\frac{1}{2}}$$
$$\times\exp\left[\tfrac{1}{6}\pi^2(\rho k)^{-1}n^{1-k}-\tfrac{1}{6}\pi^2\sigma(\rho k)^{-2}n^{1-2k}+\pi^2(\rho k)^{-3}\left\{\tfrac{1}{6}\sigma^2-\tfrac{1}{72}(k-1)^2-\tfrac{7}{90}\pi^2(k-1)(k-2)\right\}n^{1-3k}\right].$$

In this expression we may write for $(a_1 a_2 \dots a_n)^{-1}$,

$$(2\pi n)^{\frac{1}{2}\sigma}\exp\{-\rho(k+1)^{-1}n^{k+1}-\sigma n(\log n-1)-\tfrac{1}{2}\rho n^k\}.$$

§ 35. If a_n is of higher order than $\exp(\epsilon^{-1}n^{\frac{1}{3}})$, the general formula for c_n assumes a very simple form. If, in fact, we assume, instead of (3)′, the still more stringent condition

$$|\,\omega\,|=\epsilon(n)\,\nu^{-2},$$

it may be shown without difficulty that

$$-\gamma_1\psi+\gamma_2\psi^2=O(n^{-2}\omega^{-3})=\epsilon(n),\qquad -\gamma_3\psi^3+\gamma_4\psi^4-\dots=\epsilon(n),$$

and

$$\gamma_0=\tfrac{1}{6}\pi^2\omega^{-1}+\epsilon(n).$$

The general formula therefore becomes

$$c_n=\frac{1+\epsilon(n)}{a_1 a_2 \dots a_n}\cdot\left(\frac{\omega}{2\pi}\right)^{\frac{1}{2}}\exp\left(\tfrac{1}{6}\pi^2\omega^{-1}\right)^*.$$

As an example we may take the function considered in § 34, with $k>\frac{1}{3}$. The term in n^{1-3k} may be omitted in the square brackets in the expression for c_n.

As another example, consider the function

$$F(z)=\prod_{s=1}^{\infty}\{1+z/\exp[\rho(s+1)\{\log(s+1)\}^{-1}]\}.$$

We obtain

$$c_n=\frac{1+\epsilon(n)}{a_1 a_2 \dots a_n}\left(\frac{\rho}{2\pi\log n}\right)^{\frac{1}{2}}\exp\left(\tfrac{1}{6}\pi^2\rho^{-1}\right).\,n^{\frac{1}{6}\pi^2\rho^{-1}}.$$

For $(a_1 a_2 \dots a_n)^{-1}$ we may write

$$\exp\left[-n^2\int_0^\infty\left\{\frac{n^{-x}}{2-x}+\frac{1-\zeta(x-1)}{n^2}\right\}dx+C-\tfrac{1}{2}n(\log n)^{-1}\right],$$

where C is a numerical constant. The first term within the square brackets may be shown to be equal to

$$-n^2\left[\sum_{r=1}^{p-1}r!\,(2\log n)^{-r-1}+O(\log n)^{-p-1}\right],$$

where p is an arbitrary (constant) integer. It is evidently impossible to replace this term by elementary functions together with a remainder-term of order $\epsilon(n)$.

* This formula is appropriate to the case when ω decreases slowly. It may be worth while to note the form of a_n when $\log(a_n/a_{n+1})$ and $\omega(n)\rightarrow -v$ (v a finite constant). We then have

$$c_n=[1+\epsilon(n)]\frac{A^{-1}}{a_1 a_2 \dots a_n},$$

where

$$A=\prod_{s=1}^{\infty}(1-e^{-sv}).$$

This result follows without difficulty from the asymptotic form for $\Pi(1+z/a_n)$ given in my paper cited on p. 330.

§ 36. In the case of functions $F(z)$ for which a_n is of less order than $\exp(n^\epsilon)$ for arbitrarily small values of ϵ, our methods do not lead to formulae for c_n of any particular interest. The asymptotic formula for $\log F(z)$ gives

$$F(z) = f(z) \exp[R(z)],$$

where all that is known concerning $R(z)$ is that it is less than some known function of z, which function tends in all cases to infinity with z. Our methods will certainly give an upper limit for $|c_n|$, but they cannot give a lower limit. In fact, for anything that we can show to the contrary, c_n may be zero. For since $|R(z)|$ is large its imaginary part may vary with $\arg z$ in such a manner that

$$\int_C f(z) \exp[R(z)] z^{-n-1} dz = 0.$$

PART II. TAYLOR SERIES.

SECTION I. ASYMPTOTIC EXPANSIONS.

§ 37. In Part II we shall consider a class of integral functions $F(z)$ defined by a Taylor-series $\Sigma c_n z^n$, in which the coefficients c satisfy a set of conditions analogous to the conditions imposed on the a's in Part I, and shall find asymptotic formulae for $F(z)$ when z is large, and for the nth zero of $F(z)$ when n is large. The conditions will be satisfied, at any rate, if c_n is a naturally constructed function of n such that $|c_n^{-1}|$ is of less order than $\exp(\epsilon n^2)$ and of greater order than $\exp(n^{1+\eta})$, for all values of ϵ and for some value of η. It will appear that the nth zero of $F(z)$ is of less order than $\exp(\epsilon n)$ and of greater order than $\exp(n^\eta)$, so that this range of order is the same as that implied by the conditions of Part I, Section II. The class of functions corresponding to the class (b) of § 14 is excluded from consideration, as class (b) was excluded in the preceding Section, and for somewhat similar reasons.

§ 38. The conditions which we shall impose on the c's are the following:

1°. *Given any positive ϵ, then, when n is sufficiently large, there exists an analytic function $f_n(x)$, regular in the region $|x| < \epsilon^{-1}\mu$, such that*

$$f_n(s) = \log(c_n/c_{n+s}),$$

where
$$\mu = [\omega(n)]^{-1} \nu^{\frac{1}{3}}, \quad \nu = n\,|\omega(n)|.$$

Further, when $|x| < \epsilon^{-1}\mu$, we have, for any value of the constant integer N,

$$f_n(x) = x\Omega(n) + \tfrac{1}{2}x^2\omega(n) + x^3\omega_1(n) + \ldots + x^{N-1}\omega_{N-2}(n) + x^N O[n^{-N+1}\omega(n)].$$

2°. $\lim\limits_{n\to\infty} |\omega(n)| = 0.$

3°. $|\omega(n)| > K^{-1}[\nu(n)]^{-K}.$

4°. $\mathfrak{R}\omega(n) > K^{-1}|\omega(n)|$, *algebraically.*

5°. *For any value of the constant integer r,*

$$|\omega_r(n)| < Kn^{-r}|\omega(n)|.$$

6°. *For any value of the constant* h,

$$\sum_{s=\mu}^{\infty}\left|\frac{c_{n+s}}{c_n}\exp\left[s\left\{\Omega(n)+h\omega(n)\right\}\right]\right|=O\left[\exp\left\{-K^{-1}\mu^2|\omega|\right\}\right],$$

and

$$\sum_{s=\mu}^{n}\left|\frac{c_{n-s}}{c_n}\exp\left[-s\left\{\Omega(n)-h\omega(n)\right\}\right]\right|=O\left[\exp\left\{-K^{-1}\mu^2|\omega|\right\}\right].$$

From 1°, 3°, and 4° it follows that*

$$\lim_{n\to\infty}\mathfrak{R}\Omega(n)=\infty \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, by condition 1°, we have

$$\log(c_{n+1}^{-1}c_n)=\Omega(n)+\tfrac{1}{2}\omega(n)+O\left[n^{-1}\omega(n)\right]=\Omega(n)+\left[\tfrac{1}{2}+\epsilon(n)\right]\omega(n),$$

and similarly $\quad \log(c_n^{-1}c_{n+1})=-\Omega(n+1)+\left[\tfrac{1}{2}+\epsilon(n)\right]\omega(n+1),$

so that $\quad \Omega(n+1)-\left[\tfrac{1}{2}+\epsilon_1(n+1)\right]\omega(n+1)=\Omega(n)+\left[\tfrac{1}{2}+\epsilon_2(n)\right]\omega(n)\ldots\ldots\ldots\ldots\ldots(2),$

where $\epsilon_1(n)$, $\epsilon_2(n)$ tend to zero as n tends to infinity.

From (1) and (2) it follows that if z be any complex number, it is possible to choose n so that†

$$\left.\begin{array}{l} z=e^{\iota\phi}\exp\left[\Omega(n)\right],\quad \phi=\theta+\iota\alpha\omega(n), \\ \text{where } \theta \text{ and } \alpha \text{ are real, and} \\ |\theta|\leqslant\pi,\quad |\alpha|<\tfrac{1}{2}+\epsilon(n) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For the present, however, we shall assume only that

$$z=e^{\iota\phi}\exp\left[\Omega(n)\right],\quad |\theta|<K_1,\quad |\alpha\omega|<K_2,$$

where K_1, K_2 are arbitrary positive constants.

§ 39. We have

$$\frac{F(z)}{c_n z^n}=\left[\sum_{s=0}^{\infty}c_s z^s\right]/\left[c_n z^n\right]$$

$$=\sum_{s=-\mu}^{\mu}c_n^{-1}c_{n+s}z^s+\sum_{s=\mu}^{\infty}c_n^{-1}c_{n+s}z^s+\sum_{s=\mu}^{n}c_n^{-1}c_{n-s}z^{-s} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

* For from the equation (2) below it follows that

$|\mathfrak{R}\Omega(n+1)-\mathfrak{R}\Omega(n)$

$=\mathfrak{R}\,\tfrac{1}{2}\left[\omega(n)+\omega(n+1)\right]+\epsilon(n)\,\omega(n)+\epsilon(n)\,\omega(n+1)$

$>K^{-1}|\omega(n)|+K^{-1}|\omega(n+1)|$

$+\epsilon(n)|\omega(n)|+\epsilon(n)|\omega(n+1)|,\quad$ by 4°,

$>K^{-1}|\omega(n)|.$

Now from (3°), $\quad |\omega(n)|>K^{-1}\left[n|\omega(n)|\right]^{-K},$

and therefore $\quad |\omega(n)|>K^{-1}n^{-1+K^{-1}}.$

We therefore have

$$\mathfrak{R}\Omega(n+1)>K^{-1}\sum_{r=1}^{n}\omega(r)>K^{-1}\sum_{r=1}^{n}r^{-1+K^{-1}},$$

which tends to infinity like $n^{K^{-1}}$.

† Since $\Omega(n)\to\infty$ there is, when z is large, a number m such that

$$|\exp\left[\Omega(m+1)\right]|>|z|>|\exp\left[\Omega(m)\right]|.$$

It follows from (2) that we must have either

$$|\exp\Omega(m+1)-\left\{\tfrac{1}{2}+\epsilon_1(m+1)\right\}\omega(m+1)|\leqslant|z|,$$

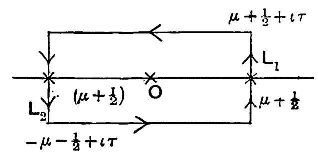
or $\quad |\exp\left[\Omega(m)+\left\{\tfrac{1}{2}+\epsilon_1(m)\right\}\omega(m)\right]|>|z|.$

In the former event we take $n=m+1$, and in the latter $n=m$. Since $\mathfrak{R}\omega(n)>0$, the result (3) follows immediately.

Since $|z|^{\pm s} = |\exp[\pm s(\Omega - \alpha\omega)]|$, it is seen from condition 6° that the second and third terms of (1) are of the form $O[\exp\{-K^{-1}\mu^2|\omega|\}]$. We therefore have

$$\left.\begin{aligned} &\frac{F(z)}{c_n z^n} = T + O\{\exp[-K^{-1}\mu^2|\omega|]\},\\ \text{where}\qquad &T = \sum_{s=-\mu}^{\mu} c_n^{-1} c_{n+s} z^s = \sum_{s=-\mu}^{\mu} \exp(s\phi\iota - \tfrac{1}{2}s^2\omega - s^3\omega_1 - s^4\omega_2 - \ldots)\end{aligned}\right\} \text{...............(G).}$$

§ 40. Consider the rectangular contour formed by the straight lines parallel to the real axis and distant τ from it, and the straight lines parallel to the imaginary axis and distant $\mu + \frac{1}{2}$ from it. Let the contour be described counter-clockwise and let L_1, L_2 be the parts respectively above and below the real axis. Let $\delta\mu$ be taken for the number τ, where δ is a small positive constant. Then if n is large, and if ϵ is chosen sufficiently small, the contour lies within the region $|x| < \epsilon^{-1}\mu$. Now within this region the function represented by



$$\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - x^3\omega_1 - \ldots)$$

is regular [since it is equal to $f_n(x)\exp(x\phi\iota + x\Omega)$]. Cauchy's theory of residues therefore gives

$$T = \int_{L_1} \exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\frac{dx}{e^{2\pi x\iota} - 1} + \int_{L_2} \exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\frac{dx}{e^{2\pi x\iota} - 1} \text{ ......(1).}$$

Writing
$$\frac{1}{e^{2\pi x\iota} - 1} = -1 - e^{2\pi x\iota} + \frac{e^{4\pi x\iota}}{e^{2\pi x\iota} - 1} \text{ or } e^{-2\pi x\iota} + \frac{e^{-2\pi x\iota}}{e^{2\pi x\iota} - 1},$$

according as x is a point of the contour L_1 or L_2, we obtain from (1),

$$\left.\begin{aligned} &T = T_1 + T_2,\\ \text{where}\qquad &T_1 = -\int_{L_1}(1 + e^{2\pi x\iota})\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx + \int_{L_2} e^{-2\pi x\iota}\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx\\ &T_2 = \int_{L_1}\frac{e^{4\pi x\iota}}{e^{2\pi x\iota} - 1}\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx + \int_{L_2}\frac{e^{-2\pi x\iota}}{e^{2\pi x\iota} - 1}\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx\\ &\quad = T_2' + T_2''\end{aligned}\right\} \text{...(H).}$$

§ 41. We shall show that when $|\theta| < 2\pi - \epsilon$, where ϵ is an arbitrarily small positive number,

$$T_2 = O[\exp\{-\tfrac{1}{2}\omega^{-1}\phi^2 - K^{-1}|\omega|^{-1}\}] \text{.............................(K).}$$

Consider first the part T_2'. Let C_1, C_1' be the parts of L_1 parallel to the imaginary axis, C_1'' the part parallel to the real axis. Then

$$T_2' = \int_{C^1}\frac{e^{4\pi x\iota}}{e^{2\pi x\iota} - 1}\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx = \left(\int_{C_1} + \int_{C_1'} + \int_{C_1''}\right)\frac{e^{4\pi x\iota}}{e^{2\pi x\iota} - 1}\exp(x\phi\iota - \tfrac{1}{2}x^2\omega - \ldots)\,dx$$
$$\text{.........(1).}$$

It is easily shown* that, for points x of C_1 and C_1',

$$\left|\frac{e^{4\pi x\iota}}{e^{2\pi x\iota}-1}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\right|<K\exp[-K^{-1}\mu^2|\omega|] \quad \text{.................(2).}$$

Hence
$$\left|\left(\int_{C_1}+\int_{C_1'}\right)\frac{e^{4\pi x\iota}}{e^{2\pi x\iota}-1}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,dx\right|<2\delta\mu\,.\exp[-K^{-1}\mu^2|\omega|]$$
$$<K\mu^2|\omega|\,.\exp[-K^{-1}\mu^2|\omega|]$$
$$<K\exp[-K^{-1}\mu^2|\omega|] \quad \text{...........(K)}_1.$$

§ 42. We have now to consider the remaining integral in T_2', viz.

$$\int_{C_1''}\frac{e^{4\pi x\iota}}{e^{2\pi x\iota}-1}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,dx=\sum_{r=2}^{\infty}\int_{C_1''}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,e^{2r\pi x\iota}\,dx\dagger.$$

We shall divide the right-hand series into two parts, corresponding to $\sum_{r=2}^{h}$ and $\sum_{r=h+1}^{\infty}$, and shall show that these parts are severally of the form $O[\exp\{-\tfrac{1}{2}\omega^{-1}\phi^2-K^{-1}|\omega|^{-1}\}]$.

For points x of C_1'' we may write $x=y+\iota\tau$. Then

$$|\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,.\,e^{2r\pi x\iota}|<|\exp[K\mu-\tfrac{1}{2}y^2\omega-\iota\tau y\omega+K\mu^3n^{-1}|\omega|]|\,.\,e^{-2r\pi\tau}$$
$$<\exp[K_1\mu^2|\omega|]\,.\,e^{-2r\pi\tau} \quad \text{................................(1),}$$

where K_1 is independent of r. Let us choose h so that

$$h\pi\tau=K_1\mu^2|\omega|,\quad\text{or}\quad h=K_1(\pi\delta)^{-1}.\,\nu^{\frac{1}{3}} \quad \text{..........................(2).}$$

Then, if $r\geqslant h+1$, we have

$$|\exp(x\phi\iota-\dots)\,e^{2r\pi x\iota}|<e^{-r\pi\tau},$$

and therefore

$$\left|\sum_{r=h+1}^{\infty}\int_{C_1''}\exp(x\phi\iota-\dots)\,e^{2r\pi x\iota}\,dx\right|<\sum_{r=h+1}^{\infty}2(\mu+\tfrac{1}{2})\,.\,e^{-r\pi\tau}$$
$$<K\mu\,.\,e^{-h\pi\tau}<K\mu^2|\omega|\,.\exp[-K_1\mu^2|\omega|]$$
$$=O\{\exp[-K^{-1}\mu^2|\omega|]\} \quad \text{.......................(K)}_2.$$

§ 43. We shall now show that, when $2\leqslant r<h$, we have

$$\int_{C_1''}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,e^{2r\pi x\iota}dx=O\{\exp[-\tfrac{1}{2}\omega^{-1}\phi^2-K^{-1}|\omega|^{-1}]\} \quad \text{...........(1).}$$

* We have $\quad |e^{4\pi x\iota}/(e^{2\pi x\iota}-1)|<K.$

Also $\quad x=\pm(\mu+\tfrac{1}{2})+\iota y,$

where $y\leqslant\delta\mu$, so that $|x|<K\mu$, and therefore

$$\mathfrak{R}(-x^3\omega_1-x^4\omega_2-\dots)<K\mu^3n^{-1}|\omega|=\epsilon(n)\mu^2|\omega|\ \dots\text{(i).}$$

Also

$$\mathfrak{R}(x\phi\iota-\tfrac{1}{2}x^2\omega)<K\mu-\mathfrak{R}[\tfrac{1}{2}\omega(\mu+\tfrac{1}{2})^2\pm\iota\omega(\mu+\tfrac{1}{2})\,y+\tfrac{1}{2}y^2\omega]$$
$$<K\mu-(\mathfrak{R}\tfrac{1}{2}\omega)(\mu+\tfrac{1}{2})^2+|\omega|(\mu+\tfrac{1}{2})\,.\,\delta\mu$$
$$<K\mu-K_1^{-1}\mu^2|\omega|+2\delta\mu^2|\omega| \quad \text{.........(ii).}$$

Now when n is large

$$\epsilon(n)\mu^2|\omega|<\tfrac{1}{3}K_1^{-1}\mu^2|\omega|$$

and $\quad K\mu<\tfrac{1}{3}K_1^{-1}\mu^2|\omega|.$

Also $\quad 2\delta\mu^2|\omega|<\tfrac{1}{9}K_1^{-1}\mu^2|\omega|$

provided δ is chosen less than $\tfrac{1}{18}K_1^{-1}$. We thus have from (i) and (ii)

$$\mathfrak{R}(x\phi\iota-\tfrac{1}{2}x^2\omega-x^3\omega_1-\dots)$$
$$<-K_1^{-1}\mu^2|\omega|+\tfrac{8}{9}K_1^{-1}\mu^2|\omega|<-K^{-1}\mu^2|\omega|.$$

The result (3) above follows immediately.

† The series

$$\Sigma\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\dots)\,e^{2r\pi x\iota}$$

is uniformly convergent on the contour C_1''.

It will then follow that

$$\sum_{r=2}^{h}\int_{C''}\exp(x\phi\iota-\ldots)\,e^{2r\pi x\iota}\,dx = hO\{\exp[-\tfrac{1}{2}\omega^{-1}\phi^2 - K^{-1}|\omega|^{-1}]\}$$
$$= O\{\exp[-\tfrac{1}{2}\omega^{-1}\phi^2 - K^{-1}|\omega|^{-1}]\} \quad\ldots\ldots\ldots\ldots(\mathrm{K})_3,$$

since $h < K\nu^{\frac{1}{3}}$ [by (2) of § 42] $< K|\omega|^{-K^{-1}}$ [by condition 3°].

We have $\qquad |r\omega^{-1}| \leqslant h|\omega|^{-1} < K\mu$ [by (2) of § 42] $\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$

Let $\qquad y_1 = A\mu + (2r\pi+\phi)\,\iota\omega^{-1}, \quad y_2 = -A\mu + (2r\pi+\phi)\,\iota\omega^{-1},$

where A is a large constant. It is evident that, if ϵ is properly chosen, y_1 and y_2 lie within the region $|x| < \epsilon^{-1}\mu$, i.e. within the region in which $\exp(x\phi\iota - \frac{1}{2}x^2\omega - \ldots)$ is regular. The integration in (1) may therefore be taken along the straight lines $(\mu+\frac{1}{2})+\iota\delta\mu$ to y_1 to y_2 to $-(\mu+\frac{1}{2})+\iota\delta\mu$. On the straight line from $(\mu+\frac{1}{2})+\iota\delta\mu$ to y_1 we have $x = (\mu+\frac{1}{2})+\iota\delta\mu+e^{\iota\beta}y$, where β and y are real and y is positive. By choosing A sufficiently large we can [by (2)] make $|\beta|$ as small as we please.

Now $\mathfrak{R}\omega < K^{-1}|\omega|$, and therefore $|\arg\omega| < \frac{1}{2}\pi - K^{-1}$. We have

$$\mathfrak{R}x^2\omega = \mathfrak{R}\,[(\mu+\tfrac{1}{2}+\iota\delta\mu)^2\,\omega + 2\,(\mu+\tfrac{1}{2}+\iota\delta\mu)\,e^{\iota\beta}y\omega + e^{2\iota\beta}y^2\omega].$$

By choosing first δ and then β sufficiently small, we can make

$$\mathfrak{R}\,[(\mu+\tfrac{1}{2}+\iota\delta\mu)\,e^{\iota\beta}\omega] > 0, \quad \mathfrak{R}e^{2\iota\beta}\omega > 0.$$

We then have $\qquad \mathfrak{R}x^2\omega > \mathfrak{R}\,(\mu+\frac{1}{2}+\iota\delta\mu)^2\,\omega > K^{-1}\mu^2|\omega| \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$

provided δ is chosen sufficiently small.

Again, when $x = \mu+\frac{1}{2}+\iota\delta\mu$, we have $\mathfrak{R}x\iota < 0$, and when $x = y_1$ we have

$$\mathfrak{R}x\iota = \mathfrak{R}\,[-(2r\pi+\phi)\,\omega^{-1}] = \mathfrak{R}\,[-(2r\pi+\theta)\,\omega^{-1}] < 0.$$

It follows easily that $\mathfrak{R}x\iota < 0$ and $|e^{2r\pi\iota x}| < 1$ for all points x on the straight line from $\mu+\frac{1}{2}+\iota\delta\mu$ to y_1. We therefore have, by the help of (3),

$$\left|\int_{\mu+\frac{1}{2}+\iota\delta\mu}^{y_1}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\ldots)\,.\,e^{2r\pi x\iota}\,.\,dx\right| < \int_{\mu+\frac{1}{2}+\iota\delta\mu}^{y_1}\exp[K\mu - K^{-1}\mu^2|\omega| + K\mu^3n^{-1}|\omega|]\,.\,1\,.\,|dx|$$
$$< K\mu\,.\,\exp[K\mu - K^{-1}\mu^2|\omega| + K\mu^3n^{-1}|\omega|]$$

(since the length of the path of integration is less than $K|y_1| + K|\mu+\frac{1}{2}+\iota\delta\mu| < K\mu$)

$$= O\,[\exp\{-K^{-1}\mu^2|\omega|\}].$$

A similar result holds for the integral from y_2 to $-(\mu+\frac{1}{2})+\iota\delta\mu$. We therefore have

$$\int_{C_1''}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\ldots)\,e^{2r\pi x\iota}\,dx = \int_{y_1}^{y_2}\exp(x\phi\iota-\ldots)\,e^{2r\pi x\iota}\,dx + O\,[\exp\{-K^{-1}\mu^2|\omega|\}]\ldots(4).$$

Now, writing $x = y + (2r\pi+\phi)\,\iota\omega^{-1}$, we have

$$\left|\int_{y_1}^{y_2}\exp(x\phi\iota-\ldots)\,e^{2r\pi x\iota}\,dx\right| = \left|\int_{y_1}^{y_2}\exp[x\iota(\phi+2r\pi) - \tfrac{1}{2}x^2\omega + O(x^3n^{-1}\omega)]\,dx\right|$$
$$= \left|\int_{A\mu}^{-A\mu}\exp[-(2r\pi+\phi)^2\,\omega^{-1} - \tfrac{1}{2}\omega y^2 + \tfrac{1}{2}\omega\,\{(2r\pi+\phi)\,\omega^{-1}\}^2 + O(\{y+(2r\pi+\phi)\,\omega^{-1}\iota\}^3\,n^{-1}\omega)]\,dy\right|$$
$$\ldots\ldots\ldots(5).$$

Now it may be shown that*

$$O(\{y+(2r\pi+\phi)\,\iota\omega^{-1}\}^3 n^{-1}\omega) = \epsilon(n)\,\omega y^2 + \epsilon(n)\,(2r\pi+\phi)^2\omega^{-1}.$$

The right-hand side of (5) is therefore less than

$$\begin{aligned}
&|\exp[-\{\tfrac{1}{2}+\epsilon(n)\}(2r\pi+\phi)^2\omega^{-1}]|\,.\int_{-A\mu}^{A\mu}|\exp[-\{\tfrac{1}{2}+\epsilon(n)\}\,\omega y^2]\,dy| \\
&\quad < |\exp[-\{\tfrac{1}{2}+\epsilon(n)\}(2r\pi+\phi)^2\omega^{-1}]|\,.\,K\,|\omega|^{-\frac{1}{2}} \\
&\quad < |\exp(-\tfrac{1}{2}\phi^2\omega^{-1})|\,.\,|\exp[-\tfrac{1}{2}\omega^{-1}2r\pi(2r\pi+2\phi)+\epsilon(n)(2r\pi+\phi)^2\omega^{-1}]|\,.\,K\,|\omega|^{-\frac{1}{2}} \quad\ldots(6).
\end{aligned}$$

Now

$$\begin{aligned}
\mathfrak{R}[-(2r\pi+2\phi)\,\omega^{-1}] &= \mathfrak{R}[-(2r\pi+2\theta)\,\omega^{-1}] \\
&\leqslant \mathfrak{R}[-(2r\pi-2\pi+\epsilon)\,\omega^{-1}] \\
&< -K^{-1}r\,.\,\mathfrak{R}\omega^{-1}, \quad \text{since } r\geqslant 2, \\
&< -K^{-1}r\,|\omega|^{-1}.
\end{aligned}$$

The right-hand side of (6) is therefore less than

$$\begin{aligned}
&|\exp(-\tfrac{1}{2}\phi^2\omega^{-1})|\,.\exp[-K^{-1}r^2|\omega|^{-1}+\epsilon(n)\,r^2|\omega|^{-1}]\,.\,K\,|\omega|^{-\frac{1}{2}} \\
&\quad < |\exp(-\tfrac{1}{2}\phi^2\omega^{-1})|\,.\exp(-K^{-1}r^2|\omega|^{-1})\,K\,|\omega|^{-\frac{1}{2}}, \quad \text{if } n \text{ is large,} \\
&\quad < |\exp(-\tfrac{1}{2}\phi^2\omega^{-1})\,.\exp(-K^{-1}|\omega|^{-1})| \quad\ldots\ldots(7).
\end{aligned}$$

From (4) and (7) we have

$$\begin{aligned}
\int_{C_1''}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\ldots)\,e^{2r\pi x\iota}\,dx &= O\{\exp[-K^{-1}\mu^2|\omega|]\} + O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1}-K^{-1}|\omega|^{-1}]\} \\
&= O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1}-K^{-1}|\omega|^{-1}]\},
\end{aligned}$$

since $\mu^2|\omega| = \nu^{\frac{2}{3}}|\omega|^{-1}$. We thus have the result (1), and therefore, as we have seen above, $(\mathrm{K})_3$.

§ 44. From $(\mathrm{K})_2$ and $(\mathrm{K})_3$

$$\sum_{r=2}^{\infty}\int_{C_1''}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\ldots)\,e^{2r\pi x\iota}\,dx = O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1}-K^{-1}|\omega|^{-1}]\},$$

and, from this equation and $(\mathrm{K})_1$,

$$T_2' = \int_{L_1}\exp(x\phi\iota-\tfrac{1}{2}x^2\omega-\ldots)\,\frac{e^{4\pi x\iota}}{e^{2\pi x\iota}-1}\,dx = O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1}-K^{-1}|\omega|^{-1}]\}.$$

A similar equation holds for the second integral T_2'' in (H), and we obtain the result (K) concerning T_2.

We now return to T_1 in the equation (H). In the integrals in the expression for T_1 given by that equation, the integrands are regular functions of x upon and within the contour L_1+L_2. The paths L_1 and L_2 may therefore be deformed into the part of the real axis between $\mu+\frac{1}{2}$ and $-(\mu+\frac{1}{2})$ described respectively from right to left and from left to right. We therefore obtain from (H)

$$T_1 = g(\phi) + g(2\pi+\phi) + g(-2\pi+\phi),$$

* We have

$$\begin{aligned}
|\{y+(2r\pi+\phi)\,\iota\omega^{-1}\}^3| &< Ky^3+Ky^2\,|(2r\pi+\phi)\,\iota\omega^{-1}| \\
&\quad + Ky\,|(2r\pi+\phi)\,\iota\omega^{-1}|^2 + K\,|(2r\pi+\iota\phi)\,\omega^{-1}|^3 \\
&< Ky^2[A\mu+Kh\,|\omega|^{-1}] + K\,|(2r\pi+\phi)\,\iota\omega^{-1}|^2\,[A\mu+Kh\,|\omega|^{-1}] \\
&< K\mu\,.\,y^2 + K\mu\,|(2r\pi+\phi)^2\omega^{-2}|.
\end{aligned}$$

On multiplying each side of this inequality by $n^{-1}\omega$, we obtain the desired result, for μn^{-1} is of the form $\epsilon(n)$.

where $$g(\phi) = \int_{-(\mu+\frac{1}{2})}^{\mu+\frac{1}{2}} \exp(x\phi\iota - \tfrac{1}{2}x^2\omega - x^3\omega_1 - \dots)\, dx \quad \dots\dots\dots\dots\dots\dots\dots(1).$$

Summing up from (G), (H) and (K), and recalling that

$$\exp[-K^{-1}\mu^2 |\omega|] = O\{\exp(-\tfrac{1}{2}\phi^2\omega^{-1} - K^{-1}|\omega|^{-1})\},$$

we have, when $|\theta| < 2\pi - \epsilon$,

$$\frac{F(z)}{c_n z^n} = g(\phi) + g(2\pi+\phi) + g(-2\pi+\phi) + O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1} - K^{-1}|\omega|^{-1}]\} \quad \dots\dots(L),$$

where $g(\phi)$ is given by (1).

§ 45. The integral for $g(\phi)$ is very similar to the integral $\int_{-\eta}^{\eta} \exp(\gamma_0 + \gamma_1\phi + \dots)\, d\phi$ discussed in § 28. The integrand in $g(\phi)$ contains an oscillating factor $\exp(x\phi\iota)$. By a substitution of the form $x = x_0 + y$ we may obtain an expression for $x\phi\iota - \frac{1}{2}x^2\omega - \dots$, in which the coefficient of y is zero. The approximation for $g(\phi)$ then proceeds on the lines of §§ 28—31 and is in all essential principles identical with the latter investigation. We shall content ourselves by indicating the leading results.

The only condition which we shall impose on ϕ is $|\phi| < K$.

By a sufficient number of successive approximations we may calculate a value x_0 which differs from the root (of least modulus) of

$$\phi\iota - 2\,.\,\tfrac{1}{2}x\omega - 3x^2\omega_1 - 4x^3\omega_2 - \dots = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1),$$

by an expression of the form $O(\nu^{-\lambda_1})$, where λ_1 is an arbitrary constant. We evidently have for the first approximation

$$x_0 = \phi\iota\omega^{-1} + \text{lower terms}.$$

We now substitute $x = x_0 + y$ in

$$x\phi\iota - \tfrac{1}{2}x^2\omega - x^3\omega_1 - \dots,$$

which, when $|x| < K\mu$, becomes of the form

$$\left.\begin{aligned} &\gamma_0 + \gamma_2 y^2 + \dots + \gamma_{\lambda_2} y^{\lambda_2} + O(\nu^{-\lambda}),\\ \text{where}\quad &\gamma_0 = x_0\phi\iota - \tfrac{1}{2}x_0^2\omega - \dots - x_0^{\lambda_3}\omega_{\lambda_3-2},\\ &\gamma_2 = -\tfrac{1}{2}\omega - {}_3C_2 x_0\omega_1 - {}_4C_2 x_0^2\omega_2 - \dots - {}_{\lambda_3+2}C_2 x_0^{\lambda_3}\omega_{\lambda_3},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots(2),$$

where λ_2 and λ_3 depend on λ.

It may next be shown that we have

$$\left.\begin{aligned} g(\phi) &= [1 + O(\nu^{-\lambda})]\exp(\gamma_0)\, I,\\ I &= \int_{-\mu}^{\mu} \exp[\gamma_2 y^2 + \gamma_3 y^3 + \dots + \gamma_{\lambda_2} y^{\lambda_2}]\, dy \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots(M).$$

For the integral I we may approximate as follows. Let

$$\exp(\gamma_3 y^3 + \gamma_4 y^4 + \dots) = 1 + \beta_3 y^3 + \beta_4 y^4 + \dots.$$

Then, when λ_4 is chosen sufficiently large,

$$I = \int_{-\infty}^{\infty} \exp(\gamma_2 y^2)\,[1 + \beta_3 y^3 + \ldots + \beta_{\lambda_4} y^{\lambda_4} + O(\nu^{-\lambda})]\,dy$$

$$= \left(\frac{\pi}{-\gamma_2}\right)^{\frac{1}{2}} \left[1 + \sum_{2r=4}^{\lambda_4} \frac{1\,.\,3\ldots(2r-1)\,\beta_{2r}}{(-2\gamma_2)^r} + O(\nu^{-\lambda})\right] \quad \text{.................(N).}$$

§ 46. As we have now found $g(\phi)$ save for a factor $[1 + O(\nu^{-\lambda})]$, it is easy to find an asymptotic formula of the same type for $F(z)$.

We shall suppose now that ϕ satisfies the conditions

$$\pi - |\theta| > \nu^{\delta-1} + |\omega|^{1-\delta}, \quad |\alpha| < K \quad \text{..............................(1),}$$

where δ is an arbitrarily small positive constant.

We have $\quad \gamma_0 = -\tfrac{1}{2}\phi^2\omega^{-1} + O(n^{-1}\omega^{-2}) = -\tfrac{1}{2}\phi^2\omega^{-1} + O(\nu^{-1}\omega^{-1}),$

$$\gamma_2 = [-\tfrac{1}{2} + \epsilon(n)]\,\omega, \quad \beta_{2r}(\gamma_2)^{-r} = \epsilon(n).$$

Therefore, from (M) and (N),

$$g(\phi) = [1 + \epsilon(n)]\,.\,(\tfrac{1}{2}\omega)^{-1} \exp[-\tfrac{1}{2}\phi^2\omega^{-1} + O(\nu^{-1}\omega^{-1})] \quad \text{.................(2),}$$

with similar results in which ϕ is replaced by $\pm 2\pi + \phi$. Thus we have

$$\left|\frac{g(\pm 2\pi + \phi)}{g(\phi)}\right| < K\,|\exp[\tfrac{1}{2}\omega^{-1}\{\phi^2 - (2\pi \pm \phi)^2\} + O(\nu^{-1}\omega^{-1})]|$$

$$< K\,|\exp[-2\pi\omega^{-1}(\pi \pm \phi) + K_2\nu^{-1}|\omega|^{-1}]|$$

$$< K\,|\exp[-2\pi\omega^{-1}\,.\,(\pi \pm \theta) + K_2\nu^{-1}|\omega|^{-1}]|$$

$$< K\,|\exp[-K_1^{-1}|\omega|^{-1}\,.\,(|\omega|^{1-\delta} + \nu^{\delta-1}) + K_2\nu^{-1}|\omega|^{-1}]|$$

(since $\pi - |\theta| > \nu^{\delta-1} + |\omega|^{1-\delta}$)

$$< K \exp[-\tfrac{1}{2}K_1^{-1}|\omega|^{-1}(|\omega|^{1-\delta} + \nu^{\delta-1})] < K \exp(-K^{-1}|\omega|^{-\delta})$$

[since $K_2\nu^{-1}|\omega|^{-1} < \tfrac{1}{2}K_1^{-1}|\omega|^{-1}(|\omega|^{1-\delta} + \nu^{\delta-1})$]

$$= \chi(\omega) \quad \text{..(3).}$$

Now, from (1), we have

$$O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1} - K^{-1}|\omega|^{-1}]\} = \chi(\omega)\,g(\phi) \quad \text{......................(4).}$$

From (L), and (3), (4) we have

$$F(z) = c_n z^n g(\phi)\,.\,[1 + \chi(\omega)] \quad \text{................................(P).}$$

Summing up from (M), (N), (P), we have the following theorem:

Let z be put in the form $e^{\iota\phi}\exp[\Omega(n)]$, where $\phi = \theta + \iota\alpha\omega$, $|\theta| \leqslant \pi$, $|\alpha| < K$, and let z be confined to the region defined by $\pi - |\theta| > |\omega(n)|^{1-\delta} + [\nu(n)]^{\delta-1}$. Then if λ be any assigned positive constant, we have

$$F(z) = c_n z^n \left(\frac{\pi}{-\gamma_2}\right)^{\frac{1}{2}} \exp(\gamma_0)\,.\left[1 + \sum_{2r=4}^{\lambda_3} \frac{1\,.\,3\ldots(2r-1)}{(-2\gamma_2)^r}\,\beta_{2r} + O(\nu^{-\lambda}) + \chi(\omega)\right],$$

where the β's and γ's are determined as follows.

x_0 is determined by a certain number of successive approximations from the equation

$$\phi\iota - 2\,.\,\tfrac{1}{2}x\omega - 3x^2\omega_1 - \ldots = 0,$$

so that $x_0 = x + O(\nu^{-\lambda_1})$. The γ's are then determined so that

$$(x_0+y)\,\phi\iota - \tfrac{1}{2}\omega\,(x_0+y)^2 - \omega_1\,(x_0+y)^3 - \ldots = \gamma_0 + \gamma_2 y^2 + \ldots + \gamma_{\lambda_2} y^{\lambda_2} + O(\nu^{-\lambda})$$

when $|y| < K\mu$. The β's are then the coefficients in the formal expansion

$$\exp(\gamma_3 y^3 + \gamma_4 y^4 + \ldots) = 1 + \beta_3 y^3 + \beta_4 y^4 + \ldots.$$

The numbers λ_1, λ_2, λ_3 depend on λ. They are to be so chosen (which is always possible) that the above expression for $F(z)$ changes only by a factor of the form $[1 + O(\nu^{-\lambda})]$ when λ_1, λ_2, λ_3 are increased in any manner.

§ 47. As an example, let us find an approximation of the form $[1+\epsilon(z)]\,f(z)$ for the function

$$F(z) = \sum_{s=0}^{\infty} \exp(-\rho s^{1+k})\, z^s,$$

where

$$\mathfrak{R}\rho > 0, \quad \tfrac{1}{3} < k < 1.$$

The conditions 1° to 6° are easily shown to be satisfied.

We have $\qquad \Omega = (1+k)\,\rho n^k, \quad \omega = (1+k)\,k\rho n^{k-1}, \quad \omega_1 = \rho_{1+k} C_3 n^{k-2},$

and we find that

$$x_0 = \phi\iota\omega^{-1} + 3\phi^2\omega_1\omega^{-3} + \epsilon(n), \quad \gamma_0 = -\tfrac{1}{2}\phi^2\omega^{-1} + \phi^3\iota\omega_1\omega^{-3} + \epsilon(n), \quad -\gamma_2 = \tfrac{1}{2}\omega\,[1+\epsilon(n)].$$

The general formula then gives

$$F(z) = [1+\epsilon(n)]\,(2\pi\omega^{-1})^{\frac{1}{2}} \exp\left[k\rho n^{k+1} + n\phi\iota - \tfrac{1}{2}\phi^2\omega^{-1} + \iota\phi^3\omega_1\omega^{-3}\right] \quad \ldots\ldots(1)$$

$$= [1+\epsilon(z)] \left\{\frac{2\pi}{k(1+k)\rho}\right\}^{\frac{1}{2}} \left[\frac{\log z}{\rho(1+k)}\right]^{\frac{1}{2}\frac{1-k}{k}} \exp\left[k\rho \left\{\frac{\log z}{\rho(1+k)}\right\}^{\frac{1+k}{k}}\right] \quad \ldots\ldots(2),$$

the terms in ϕ disappearing when we substitute for n in (1) from the equation

$$(1+k)\,\rho n^k = \Omega = \log z - \iota\phi.$$

The part of the plane from which z is excluded may be defined as follows. Let C_s be the circle with centre at the point $-\exp[\rho(1+k)\,s^k]$ and of radius

$$\rho_s = s^{\delta - h} \exp[\mathfrak{R}\rho\,(1+k)\,s^k],$$

where δ is arbitrarily small, and h is the lesser of k and $1-k$. Then the formula (2) is valid when z is excluded from all the circles C.

The circles overlap (after a certain value of s), and the aggregate of the points included by them forms, roughly speaking, a spiral strip. The ratio of the part of this strip included within the circle $|z| = R$, to the area of the whole circle, tends to zero with $1/R$. These results may be proved on the lines of § 16.

It is obvious that all but a finite number of the zeros of $F(z)$ must lie within the strip. The more precise determination of the position of the nth zero of $F(z)$ is a problem which, in its generality, will occupy us in the next section of the paper.

If we assume that, in the more general case when we have $0 < k < 1$, ϕ disappears from the final result, the formula (2) may be proved still valid. In the general theorem of § 46, we have, as a particular case,

$$F(z) = [1+\epsilon(z)]\,(2\pi\omega^{-1})^{\frac{1}{2}} \exp(\gamma_0)\, c_n z^n.$$

In any case in which ϕ disappears from the final result we may write $\phi = 0$; it is then easy to see that $x_0 = 0$, $\gamma_0 = 0$. Then

$$F(z) = [1+\epsilon(z)]\,(2\pi\omega^{-1})\, c_n \exp[n\Omega(n)],$$

where for n we are to substitute *formally* in terms of z from the equation

$$\Omega(n) = \log z.$$

In the particular case before us this leads immediately to the equation (2).

§ 48. As in Part I, Section I, we have in our general formula for $F(z)$ a remainder-term of the form

$$O(\nu^{-\lambda}) + \chi(\omega) \quad \ldots\ldots\ldots\ldots (1).$$

The $\chi(\omega)$ is of less order than $\exp(-K^{-1}|\omega|^{-\delta})$. If, however, we subject z to the conditions (more stringent than those of § 46) $\pi - |\theta| > \delta, |\alpha| < K$, $\chi(\omega)$ is of less order than $\exp(-K^{-1}|\omega|^{-1})$: it may be shown, in fact, that*

$$\chi(\omega) = \{1 + \epsilon(n)\} . [\exp\{-2\pi(\pi+\phi)\omega^{-1} + O(n^{-1}\omega^{-2})\} + \exp\{-2\pi(\pi-\phi)\omega^{-1} + O(n^{-1}\omega^{-2})\}].$$

As in Part I, we have a different form for the remainder-term according as $\exp(\omega^{-1})$ is of a type of order higher than, comparable with, or lower than that of ν. The critical case corresponding to that considered in § 22 occurs when c_n is of the type of order

$$\exp[-\rho n^2 (\log n)^{-1}].$$

As an example of the critical case let us take the function†

$$F(z) = \sum_{s=1}^{\infty} \exp[-\tfrac{1}{4}\rho x_s^2 (2\log x_s + 1)] . z^s,$$

where $\mathfrak{R}\rho > 0$, and $x_s \log x_s = s$.

We have

$$\log z - \iota\phi = \Omega = \rho x_n,$$

$$\omega = \rho \frac{dx_n}{dn} = \rho(\log x_n + 1)^{-1},$$

and we obtain

$$F(z) = [2\pi\rho^{-1}\log(e\rho^{-1}\log z)]^{\frac{1}{2}} \exp[\rho^{-1}\log(\rho^{-1}\log z) . (\log z)^2 - \tfrac{1}{4}\rho^{-1}(\log z)^2]$$
$$\times \left[\begin{array}{l} 1 + \tfrac{1}{12}\rho(\log z)^{-2}\{\log(e\rho^{-1}\log z)\}^{-3}[4 + 3\log(\rho^{-1}\log z)] + O\{(\log z)^{-4}\} \\ \qquad + \{1+\epsilon(z)\}\{(e\rho^{-1}\log z)^{-2\pi\rho^{-1}(\pi+\phi)} + (e\rho^{-1}\log z)^{-2\pi\rho^{-1}(\pi-\phi)}\} \end{array}\right].$$

The terms containing ϕ are of the form $O\{(\log z)^{-4}\}$ provided $\pi - |\theta| > 2/(\pi\mathfrak{R}\rho^{-1})$. On the other hand, if $\pi - |\theta|$ is less than $[\pi\mathfrak{R}\rho^{-1}]^{-1}$, one of these terms is more important than the term in $(\log z)^{-2}$.

§ 49. In the case of the function

$$F(z) = \sum_{s=1}^{\infty} \exp[-\tfrac{1}{4}\rho x_s^2\{2(\log x_s)^2 - 2\log x_s - 1\}] z^s,$$

where

$$x_s(\log x_s - 1)^2 = s,$$

we obtain

$$F(z) = (2\pi\rho^{-1})^{\frac{1}{2}}[\log(e\rho^{-1}\log z)]^{\frac{1}{2}} \exp[\tfrac{1}{4}\rho^{-1}(\log z)^2\{2(\log\{\rho^{-1}\log z\})^2 - 10\log\{\rho^{-1}\log z\} + 1\}]$$
$$\times [1 + t_1(\log z)^{-2} + t_2(\log z)^{-4} + \ldots],$$

where the t's are algebraic functions of $\log(\rho^{-1}\log z)$ and (as a theoretical possibility) ϕ. We may calculate $t_1, t_2 \ldots$ as far as any numerically assigned suffix. The remainder is then of the order of the succeeding term, and the series is not disturbed by terms of the form $\exp[-2\pi\omega^{-1}(\pi \pm \phi)]$.

SECTION II. THE ZEROS OF $F(z)$.

§ 50. Let us return to the formulae

$$\frac{F(z)}{c_n z^n} = g(\phi) + g(2\pi + \phi) + g(-2\pi + \phi) + O\{\exp[-\tfrac{1}{2}\phi^2\omega^{-1} - K^{-1}|\omega|^{-1}]\} \quad \ldots\ldots (L),$$

$$g(\phi) = \left(\frac{\pi}{-\gamma_2}\right)^{\frac{1}{2}} \exp(\gamma_0)\left[1 + \sum_{2r=4}^{\lambda_4} \frac{1 . 3 \ldots (2r-1)\beta_{2r}}{(-\gamma_2)^r} + O(\nu^{-\lambda})\right] \ldots\ldots\ldots\ldots (1),$$

* The terms within square brackets are respectively furnished by $g(2\pi+\phi)$ and $g(-2\pi+\phi)$. Cf. the work of § 46.

† This function and that in the next article are chosen so that $\Omega(n)$, instead of c_n, may have a simple form.

of which the latter is a direct combination of (M) and (N). It is easily seen that

$$\left(\frac{\pi}{-\gamma_2}\right)^{\frac{1}{2}} = \left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}} [1 + O(\nu^{-1})], \quad \beta_{2r}(-\gamma_2)^{-r} = O(\nu^{-1}).$$

We thus have from (1) $\quad g(\phi) = [1 + O(\nu^{-1})](2\pi\omega^{-1})^{\frac{1}{2}} \exp(\gamma_0)$ ....(Q),

where the processes involved in the calculation of γ_0 need be carried only to such a point that all terms added to γ_0 by further approximation are of the form $O(\nu^{-1})$. The equation (L) is valid when $|\theta| < 2\pi - \epsilon$, $|\alpha\omega| < K$, and (Q) [as also (1)] when $|\phi| < K$.

The formulae (L) and (Q) suggest at once a method of approximating for the nth zero. Since $\gamma_0 = -\frac{1}{2}\omega^{-1}\phi^2 +$ lower terms, it is easily seen from (Q) that

$$g(\pm 2\pi + \phi)/g(\phi) = O\{\exp[-2\pi\omega^{-1}(\pi \pm \phi) + \text{lower terms}]\}.$$

Hence, from (L), in order for $\exp[\Omega(n) + \iota\phi]$ to be a zero of $F(z)$, we must have

$$\phi = \pm\pi + \epsilon(n).$$

Since $\exp(\iota\phi)$ has the period 2π, we need consider only one of these forms, $\phi = -\pi + \epsilon(n)$ say. The terms $g(-2\pi + \phi)$ and $\exp[-\frac{1}{2}\phi^2\omega^{-1} - K^{-1}|\omega|^{-1}]$ in (L) are then easily seen to be of the form $\exp[-K^{-1}|\omega|^{-1}]\, g(\phi) = \epsilon(n)\, g(\phi)$. Now in (Q), γ_0 is a polynomial $Q(\phi)$. We may thus expect that for *some* zero $\exp[\Omega(n) + \iota\phi]$ of $F(z)$, the value of ϕ is approximately one given by

$$\exp[Q(\phi)] + \exp[Q(2\pi + \phi)] = 0,$$

or by $\quad Q(2\pi + \phi) - Q(\phi) = -\pi\iota$ ....(2).

Since $Q(\phi) = -\frac{1}{2}\omega^{-1}\phi^2 + \nu^{-1}\omega^{-1}P(\phi)$* we can obtain by a process of successive substitution, a value ϕ_0 such that the two sides of (2) differ by $O(\nu^{-1})$ when ϕ_0 is substituted for ϕ. It may be expected, and it will appear in the sequel, that $\exp[\Omega(n) + \iota\phi_0]$ is an approximation for the nth zero of $F(z)$. But apart from matters of detail several questions present themselves which involve rather delicate considerations in their resolution.

1°†. Is there, (*a*) actually one, and (*b*) only one, zero corresponding to

$$\exp[\Omega(n) + \iota\phi_0]$$

when n is large?

2°. In the equation (2), $-\pi\iota$ might equally well be replaced by $(2r-1)\pi$. When $r \neq 0$, can we identify the corresponding zero with $\exp[\Omega(n') + \iota\phi_0(n')]$, where $n' \neq n$?

3°. These questions being answered in the affirmative, it follows that *when n is large*, there corresponds exactly one zero to $\exp[\Omega(n) + \iota\phi_0]$, and conversely, and hence that there is a constant integer p (positive, negative, or zero) such that the $(n+p)$th zero

$$b_{n+p} = \exp[\Omega(n) + \iota\phi_0].$$

Can we prove $p = 0$‡?

* $P(\phi)$, $P_1(\phi)$, etc. denote polynomials in which the modulus of each coefficient is less than K.

† These questions, which emphasize points that might possibly be overlooked, do not follow the logical order of the proof; they are, however, intended to provide a clue to that order.

‡ The result $p=0$ is remarkable, for it implies that we can, in general, find the *exact* number of zeros contained within a large circle $|z| = R$.

§ 51. The γ_0 occurring in (Q) being denoted by $Q(\phi)$, we shall give a method of calculating ϕ_0 such that

$$Q(2\pi + \phi_0) - Q(\phi_0) = -\pi\iota + O(\nu^{-1}) \quad \text{(R)}.$$

Since $Q(\phi) = -\tfrac{1}{2}\omega^{-1}\phi^2 + \nu^{-1}\omega^{-1}P(\phi)$, the equation $Q(\phi) - Q(2\pi+\phi) = -\pi\iota$ may be written

$$-2\pi(\pi+\phi)[\omega^{-1} + \nu^{-1}\omega^{-1}P_1(\phi)] + \nu^{-1}\omega^{-1}P_2(\phi) = -\pi\iota,$$

or

$$\pi + \phi = (2\pi)^{-1}[\omega\pi\iota + \nu^{-1}P_2(\phi)] \,.\, [1 + \nu^{-1}P_1(\phi)]^{-1}.$$

When $|\phi| < K$ (so that $|P_1(\phi)| < K$) this last expression is of the form

$$(2\pi)^{-1}[\omega\pi\iota + \nu^{-1}P_2(\phi)]\,[1 + \sum_{r=1}^{\lambda-1}(-)^r \nu^{-r}\{P_1(\phi)\}^r] + O(\nu^{-\lambda}) \quad \text{(1)}.$$

Since $|\omega| > K\nu^{-K}$ we can choose λ so that $O(\nu^{-\lambda}) = O(\omega\nu^{-1})$, and therefore

$$\pi + \phi = u(\phi) + O(\omega\nu^{-1}) \quad \text{(2)},$$

where $u(\phi)$ is the first term in (1).

Let us substitute $-\pi$ for ϕ in $u(\phi)$ and call the result $\pi + \phi_1$, then substitute ϕ_1 for ϕ in $u(\phi)$ and call the result $\pi + \phi_2$, and so on. Then on account of the factor ν^{-1} which multiplies $P_1(\phi)$ and $P_2(\phi)$ in (1), it is easily seen that after a certain number of substitutions we arrive at a value ϕ_r, which we shall call ϕ_0, such that $\pi + \phi_0$ differs from $u(\phi_0)$ by a number of the form $O(\omega\nu^{-1})$, so that (as is easily seen)

$$Q(2\pi + \phi_0) - Q(\phi_0) = -\pi\iota + O(\nu^{-1}),$$

the desired result.

§ 52. It is evident that ϕ_0 is a polynomial in ω^{-1}, ω, and the coefficients of $\nu^{-1}P_1(\phi)$, $\nu^{-1}P_2(\phi)$; therefore a polynomial in ω^{-1}, ω, and the coefficients of $Q(\phi)$; and therefore a polynomial in $\omega^{-1}, \omega, \omega_1, \omega_2, \ldots,\ \ -\pi + \Sigma A\,\omega^{\pm p}\omega_1^{a_1}\omega_2^{a_2}$ say.

It is easily seen that for terms of this expression

$$\omega^{\pm p}\omega_1^{a_1}\ldots = \omega^{\pm p}\{O(n^{-1}\omega)\}^{a_1}\{O(n^{-2}\omega)\}^{a_2}\ldots = O(\omega) \quad \text{(1)}.$$

Now it may be shown that*

$$\omega_r(n+1) - \omega_r(n) = \epsilon(n)[n^{-r}\omega(n)] + \epsilon(n)[n^{-r}\omega(n+1)].$$

From this result and from (1) it follows easily that

$$\{\omega(n+1)\}^{\pm p}\{\omega_1(n+1)\}^{a_1}\{\omega_2(n+1)\}^{a_2}\ldots - \{\omega(n)\}^{\pm p}\{\omega_1(n)\}^{a_1}\ldots = \epsilon(n)\,\omega(n) + \epsilon(n)\,\omega(n+1),$$

* In the expression

$$\sum_{s=1}^{r+2} A_s \log(c_{n+s}/c_n)$$
$$= -\sum_{s=1}^{r+2} sA_s\,.\,\Omega(n) - \sum_{s=1}^{r+2}\tfrac{1}{2}s^2A_s\,.\,\omega(n) - \sum_{s=1}^{r+2}s^3A_s\,.\,\omega_1(n)\ldots$$
$$- \sum_{s=1}^{r+2}s^{r+2}A_s\omega_r(n) + O[n^{-r-1}\omega(n)],$$

we may choose the constants A so that the coefficients of $\Omega(n)$, $\omega(n)$, ... $\omega_{r-1}(n)$ vanish and the coefficient of $\omega_r(n)$ is unity. We thus have

$$\omega_r(n) = \sum_{s=1}^{r+2} A_s\log(c_{n+s}/c_n) + O[n^{-r-1}\omega(n)].$$

Changing n into $n+1$ we have

$$\begin{aligned}\omega_r(n+1) &= \sum_{s=1}^{r+2} A_s\log(c_{n+s+1}/c_{n+1}) + O[n^{-r-1}\omega(n)]\\ &= \sum_{s=1}^{r+2} A_s\log(c_{n+s+1}/c_n) - \sum_{s=1}^{r+2} A_s\log(c_{n+1}/c_n)\\ &\qquad + O[n^{-r-1}\omega(n)] + O[(n+1)^{-r-1}\omega(n+1)]\\ &= \Omega(n)[-\Sigma A_s\{(s+1)-1\}]\\ &\qquad + \omega(n)[-\Sigma\tfrac{1}{2}A_s\{(s+1)^2-1\}] + \ldots\\ &\qquad + \omega_r(n)[-\Sigma A_s\{(s+1)^{r+2}-1\}]\\ &\qquad + O[n^{-r-1}\omega(n)] + O[n^{-r-1}\omega(n+1)]\\ &= \omega_r(n) + O[n^{-r-1}\omega(n)] + O[n^{-r-1}\omega(n+1)],\end{aligned}$$

in virtue of the equations satisfied by the A's. The desired result follows immediately.

and therefore that [denoting ϕ_0 *qua* function of n by $\phi_0(n)$]

$$\phi_0(n+1) - \phi_0(n) = \epsilon(n)\,\omega(n) + \epsilon(n)\,\omega(n+1) \quad \text{.....................(2).}$$

Now it was shown in § 38 that functions $\epsilon_1'(n)$, $\epsilon_2'(n)$ (tending to zero) exist, such that

$$\Omega(n+1) - \{\tfrac{1}{2} + \epsilon_1'(n+1)\}\,\omega(n+1) = \Omega(n) + \{\tfrac{1}{2} + \epsilon_2'(n)\}\,\omega(n).$$

It therefore follows from (2) that functions $\epsilon_1(n)$, $\epsilon_2(n)$ exist, such that

$$\begin{aligned} v(n+1) = \Omega(n+1) + \iota\phi_0(n+1) - \{\tfrac{1}{2} + \epsilon_1(n+1)\}\,\omega(n+1) & \\ = \Omega(n) + \iota\phi_0(n) + \{\tfrac{1}{2} + \epsilon_2(n)\}\,\omega(n). & \end{aligned}$$

§ 53. Let us denote by C_{n+1} the circle $|z| = |\exp[v(n+1)]|$. We shall show that when n is large the circle C_n contains exactly n zeros of $F(z)$ ....(S).

Since, as was seen in the last article, $\phi_0 = -\pi + \epsilon(n)$, the conditions $|\theta| < 2\pi - \epsilon$, $|\alpha\omega| < K$, for the validity of (L), will be satisfied when

$$\phi = \vartheta + \phi_0 + \{\tfrac{1}{2} + \epsilon_1(n)\}\,\iota\omega \quad \text{..................................(1),}$$

where ϑ is real, and $0 \leqslant \vartheta \leqslant 2\pi$. For this value of ϕ the corresponding value of z is $\exp[\iota\vartheta + v(n)]$, so that as ϑ increases from 0 to 2π, the point z describes the circle C_n.

Now since by (Q)

$$g(\phi) = [1 + O(\nu^{-1})]\,(2\pi\omega^{-1})^{\frac{1}{2}} \exp[-\tfrac{1}{2}\omega^{-1}\phi^2 + \nu^{-1}\omega^{-1}P(\phi)],$$

it follows easily that the term $\exp[-\frac{1}{2}\omega^{-1}\phi^2 - K^{-1}|\omega|^{-1}]$ in (L) is of the form

$$O\{\exp[-K^{-1}|\omega|^{-1}]\}\,g(\phi).$$

We therefore have from (L)

$$\begin{aligned} \frac{F(z)}{c_n z^n} &= g(\phi)\left[1 + \frac{g(2\pi+\phi)}{g(\phi)} + \frac{g(-2\pi+\phi)}{g(\phi)} + O\{\exp(-K^{-1}|\omega|^{-1})\}\right] \\ &= [1 + O(\nu^{-1})]\,(2\pi\omega^{-1})^{\frac{1}{2}} \exp\{Q(\phi)\}\,\big[1 + \{1 + O(\nu^{-1})\}\exp\{Q(2\pi+\phi) - Q(\phi)\} \\ &\qquad + \{1 + O(\nu^{-1})\}\exp\{Q(-2\pi+\phi) - Q(\phi)\} + O\{\exp - K^{-1}|\omega|^{-1}\}\big] \text{......(S)}_1. \end{aligned}$$

We shall show that as ϑ increases from 0 to 2π the argument of the right-hand side returns to its original value.

Let us first consider the factor $[1 + O(\nu^{-1})]\,(2\pi\omega^{-1})^{\frac{1}{2}}\exp\{Q(\phi)\}$. As ϑ increases from 0 to 2π the argument of this expression increases by

$$\begin{aligned} O(\nu^{-1}) &+ IQ[2\pi + \phi_0 + \{\tfrac{1}{2} + \epsilon_1(n)\}\,\iota\omega] - IQ[\phi_0 + \tfrac{1}{2}\{1 + \epsilon_1(n)\}\,\iota\omega] \\ &= O(\nu^{-1}) + I\{Q(2\pi+\phi_0) - Q(\phi_0)\} + I\{Q(2\pi+\phi_0+\beta\omega) - Q(2\pi+\phi_0)\} \\ &\qquad - I\{Q(\phi_0+\beta\omega) - Q(\phi_0)\} \quad \text{.................................(2),} \end{aligned}$$

where β is written for $\{\frac{1}{2} + \epsilon_1(n)\}\,\iota$. The second term is of the form $-\pi + \epsilon(n)$ in virtue of (R). The fourth term is of the form

$$\begin{aligned} -I[-\tfrac{1}{2}\omega^{-1}(\phi_0+\beta\omega)^2 + \nu^{-1}\omega^{-1}P(\phi_0+\beta\omega) + \tfrac{1}{2}\omega^{-1}\phi_0^2 - \nu^{-1}\omega^{-1}P(\phi_0)] & \\ = I[\beta\phi_0 - \tfrac{1}{2}\beta^2\omega + \nu^{-1}\omega^{-1}.\,\beta\omega P_3(\phi)] = I[\{\tfrac{1}{2} + \epsilon_1(n)\}\,\phi_0\iota] + \epsilon(n). & \end{aligned}$$

Similarly it may be shown that the third term of (2) is of the form

$$-I[\{\tfrac{1}{2} + \epsilon_1(n)\}\,(2\pi + \phi_0)\,\iota] + \epsilon(n).$$

By addition it is seen that (2) is of the form $\epsilon(n)$..(S)$_2$.

§ 54. Let us now consider the factor

$$h(\vartheta) = 1 + [1 + O(\nu^{-1})] \exp [Q(2\pi + \phi) - Q(\phi)]$$
$$+ [1 + O(\nu^{-1})] \exp [Q(-2\pi + \phi) - Q(\phi)] + O\{\exp(-K^{-1}|\omega|^{-1})\}.$$

We divide the range $\vartheta = 0$ to 2π into 0 to δ, δ to $2\pi - \delta$, and $2\pi - \delta$ to 2π, where δ is a small positive constant. When $\vartheta < \delta$, we have

$$|\exp [Q(-2\pi + \phi)]| = |\exp [Q\{-3\pi + \vartheta + \epsilon(n)\}]| = O\{\exp(-K^{-1}|\omega|^{-1})\} \quad \text{......(1).}$$

Again,

$$Q(2\pi + \phi) - Q(\phi) = Q(2\pi + \phi_0 + \beta\omega + \vartheta) - Q(\phi_0 + \beta\omega + \vartheta)$$
$$= \{Q(2\pi + \phi_0 + \beta\omega) - Q(\phi_0 + \beta\omega)\} + \{Q(2\pi + \phi_0 + \beta\omega + \vartheta) - Q(2\pi + \phi_0 + \beta\omega)\}$$
$$- \{Q(\phi_0 + \beta\omega + \vartheta) - Q(\phi_0 + \beta\omega)\} \quad \text{..............................(2).}$$

The first term in this last expression is seen to be of the form $\epsilon(n)$ by the proof used to establish $(\mathrm{S})_2$. Since $Q(\phi) = -\frac{1}{2}\omega^{-1}\phi^2 + \omega^{-1}\nu^{-1}P(\phi)$, the second term is of the form

$$-\tfrac{1}{2}\omega^{-1}.\,2(2\pi + \phi_0 + \beta\omega)\vartheta - \tfrac{1}{2}\omega^{-1}\vartheta^2 + \vartheta\,O(\omega^{-1}\nu^{-1}),$$

and the third term is of the form

$$-\{-\tfrac{1}{2}\omega^{-1}.\,2(\phi_0 + \beta\omega)\vartheta - \tfrac{1}{2}\omega^{-1}\vartheta^2 + \vartheta\,O(\omega^{-1}\nu^{-1})\}.$$

From these results and from (2),

$$Q(2\pi + \phi) - Q(\phi) = -2\pi\omega^{-1}\vartheta + \vartheta\,O(\omega^{-1}\nu^{-1})$$
$$= -2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)] \quad \text{..............................(3).}$$

By means of (1) and (3) we obtain

$$h(\vartheta) = 1 + \epsilon(n) + [1 + \epsilon(n)] \exp\{-2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\} \quad \text{..................(4).}$$

It is evident that when $\vartheta = 0$, $h(\vartheta) = 2 + \epsilon(n)$, and when $\vartheta = \delta$, $h(\vartheta) = 1 + \epsilon(n)$. When ϑ increases from 0 to δ, the point $h(\vartheta)$ describes a continuous curve in the Argand diagram from $2 + \epsilon(n)$ to $1 + \epsilon(n)$. We have to show that this curve cannot cut the negative real axis.

Suppose that it does. Then there is a value of ϑ such that

$$[1 + \epsilon(n)] \exp\{-2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\} = -1 + \epsilon(n) - \alpha, \quad (\alpha \geqslant 0).$$

We therefore have $$|\exp\{-2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\}| \geqslant 1 - |\epsilon(n)| \quad \text{..........................(5),}$$

and $$|\arg \exp\{-2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\}| \geqslant \pi - |\epsilon(n)| \quad \text{... ....(6).}$$

From (5) we have $$\mathfrak{R}\{2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\} < |\epsilon(n)|,$$

and from (6), $$|I\{2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\}| > \pi - |\epsilon(n)|,$$

so that when n is large

$$\mathfrak{R}\{2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]\} = \epsilon(n)\,|I 2\pi\omega^{-1}\vartheta\,[1 + \epsilon(n)]|.$$

This inequality is incompatible with $\mathfrak{R}\omega > K^{-1}|\omega|$.

The curve described by $h(\vartheta)$ therefore does not cut the negative real axis. Since $h(\vartheta)$ respectively begins and ends at the points $2+\epsilon(n)$, $1+\epsilon(n)$, it follows that as ϑ increases from 0 to δ, the argument of $h(\vartheta)$ increases by a number of the form $\epsilon(n)$............$(S)_3$.

When $\delta \leqslant \vartheta \leqslant 2\pi-\delta$, it is easily seen that

$$\exp[Q(2\pi+\phi)-Q(\phi)] \quad \text{and} \quad \exp[Q(-2\pi+\phi)-Q(\phi)]$$

are of the form $\epsilon(n)$, so that $h(\vartheta)=1+\epsilon(n)$. As ϑ increases from δ to $2\pi-\delta$, $\arg\{h(\vartheta)\}$ therefore increases by $\epsilon(n)$..$(S)_4$.

Finally, the case $2\pi-\delta \leqslant \vartheta \leqslant 2\pi$ is similar to the case $0 \leqslant \vartheta \leqslant \delta$, the parts played by $Q(2\pi+\phi)$ and $Q(-2\pi+\phi)$ in the latter case becoming interchanged in the former, and $\arg\{h(\vartheta)\}$ increases by $\epsilon(n)$ as ϑ increases from $2\pi-\delta$ to 2π ....$(S)_5$.

From $(S)_2$, $(S)_3$, $(S)_4$, $(S)_5$ it follows that the argument of $F(z)/(c_n z^n)$ increases by $\epsilon(n)$ as z describes the circle C_n. Since $F(z)/(c_n z^n)$ is a uniform function, its argument can increase only by some multiple of 2π, and therefore, when n is large, its argument must return to its original value, when z describes C_n. Then the argument of $F(z)$ increases by $2n\pi\iota$, and it follows that $F(z)$ has exactly n zeros within C_n.

§ 55. Since $F(z)$ has exactly $(n-1)$ zeros within C_{n-1}, it has exactly one zero, the nth in order of increasing moduli, in the annulus A_n bounded by C_{n-1} and C_n. We proceed to find an asymptotic formula for the value of this zero.

Let the value in question be $\exp[\Omega+\iota(\phi_0+\vartheta)]$ (where, of course, ϑ is not now restricted to be real), where we may suppose that

$$-\delta \leqslant \mathfrak{R}\vartheta \leqslant 2\pi-\delta \quad (\delta \text{ a small positive constant}) \quad\text{.........(1)}.$$

It follows from the two forms of the equation of the circle C_n given at the beginning of § 54, that $\vartheta=\theta+\iota\beta\omega$, where θ and β are real, $|\beta| \leqslant \frac{1}{2}+\epsilon(n)$, and hence, from (1), $\theta<2\pi-\delta+\epsilon(n)$. From the last inequality it is easily shown that

$$g(-2\pi+\phi) \quad \text{and} \quad \exp(-\tfrac{1}{2}\omega^{-1}\phi^2-K^{-1}|\omega|^{-1})$$

are of the form $$O\{\exp(-K|\omega|^{-1})\}\,g(\phi), \quad [\phi=\phi_0+\vartheta],$$

and therefore from (L)

$$\begin{aligned}0=\frac{F(z)}{c_n z^n g(\phi)}&=1+O\{\exp(-K^{-1}|\omega|^{-1})\}+\frac{g(2\pi+\phi)}{g(\phi)}\\ &=1+O\{\exp(-K^{-1}|\omega|^{-1})\}+[1+O(\nu^{-1})]\exp[Q(2\pi+\phi_0+\vartheta)-Q(\phi_0+\vartheta)].\end{aligned}$$

We must therefore have

$$Q(2\pi+\phi_0+\vartheta)-Q(\phi_0+\vartheta)=-\pi\iota+2r\pi\iota+O(\nu^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}\text{......(T)},$$

where r is an integer. Since

$$Q(2\pi+\phi_0)-Q(\phi_0)=-\pi\iota+O(\nu^{-1}),$$

we have from (T),

$$\begin{aligned}\{Q(2\pi+\phi_0+\vartheta)-Q(2\pi+\phi_0)\}-\{Q(\phi_0+\vartheta)-Q(\phi_0)\}&\\ =2r\pi\iota+O(\nu^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}&\text{.... (2)}.\end{aligned}$$

The first term on the left-hand side is of the form

$$\tfrac{1}{2}\omega^{-1}[(2\pi+\phi_0+\vartheta)^2-(2\pi+\phi_0)^2]+\nu^{-1}\omega^{-1}[P(2\pi+\phi_0+\vartheta)-P(2\pi+\phi_0)]$$
$$=\omega^{-1}(2\pi+\phi_0)\vartheta+\tfrac{1}{2}\omega^{-1}\vartheta^2+\vartheta\, O(\nu^{-1}\omega^{-1}),$$

and the second term is equal to

$$-[\omega^{-1}\phi_0\vartheta+\tfrac{1}{2}\omega^{-1}\vartheta^2+\vartheta\, O(\nu^{-1}\omega^{-1})].$$

We therefore obtain

$$\vartheta[2\pi\omega^{-1}+O(\omega^{-1}\nu^{-1})]=2r\pi\iota+O(\nu^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\},$$

or

$$\vartheta=r\omega\iota[1+O(\nu^{-1})]+O(\omega\nu^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}.$$

Since $\vartheta=\theta+\iota\beta\omega$, where θ and β are real, we have

$$\beta=r[1+O(\nu^{-1})]+\epsilon(n),$$

and therefore, since $|\beta|\leqslant\frac{1}{2}+\epsilon(n)$, we must have $r=0$, and

$$\vartheta=O(\omega\nu^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}=O(n^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}.$$

Then $$\exp[\Omega(n)+\iota(\phi_0+\vartheta)]=[1+O(n^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}]\exp(\Omega+\iota\phi_0).$$

§ 56. Summing up, we have the following theorem:

Let $F(z)$ be an integral function whose coefficients satisfy the conditions of § 38. *Then we have for the nth zero b_n the formula*

$$b_n=[1+O(n^{-1})+O\{\exp(-K^{-1}|\omega|^{-1})\}]\exp[\Omega(n)+\iota\phi_0],$$

where ϕ_0 is determined by successive substitution so that

$$Q(2\pi+\phi_0)-Q(\phi_0)=-\pi\iota+O(\nu^{-1}),$$

where $Q(\phi)$ is γ_0 in the formula (Q) *of* § 50.

If we describe about the point $\exp[\Omega(s)+\iota\phi_0(s)]$, a circle Γ_s of radius

$$Ks^{-1}+K\exp(-K^{-1}|\omega(s)|^{-1}),$$

it is easily seen, and, indeed, has been proved implicitly in § 55, that after a certain value of s the circles Γ_s are all external to each other. It follows from the theorem that, provided K is sufficiently large, every zero lies within some circle Γ_s, and that after a certain value of s every circle Γ_s contains exactly one zero.

§ 57. The general formula for b_n becomes

$$[1+O(n^{-1})]\exp[\Omega(n)+\iota\phi_0],$$

in all cases for which $\omega(n)=\epsilon(n)\log n$.

As an example let us consider the function

$$F(z)=\sum_{s=1}^{\infty}s^{\sigma}\exp(-\rho s^{1+k})z^s,$$

where $$\Re\rho>0,\quad 1>k\geqslant\tfrac{1}{3}.$$

We may take $$Q(\phi)=-\tfrac{1}{2}\phi^2\omega^{-1}+\phi^3\iota\omega_1\omega^{-3}+a\phi^4,$$

where $a=O(n^{-2}\omega^{-3})$. We then obtain

$$\phi_0=-\pi+\tfrac{1}{2}\iota\omega+\pi^2\iota\omega_1\omega^{-2}.$$

We have $$\Omega=(1+k)\rho n^k-\sigma n^{-1},\quad \omega=(1+k)k\rho n^{k-1}+\sigma n^{-2},\quad \omega_1={}_{1+k}C_3 n^{k-2}-\tfrac{1}{3}\sigma n^{-3},$$

and hence $$\phi_0=-\pi+\tfrac{1}{2}(1+k)k\,.\,\iota\rho n^{k-1}+\tfrac{1}{6}\pi^2\iota(k-1)k^{-1}(1+k)^{-1}\rho^{-1}n^{-k}+O(n^{-1}).$$

The general formula then gives

$$b_n = -[1+O(n^{-1})] \exp [(1+k)\rho n^k - \tfrac{1}{2}(1+k)k\rho n^{k-1} - \tfrac{1}{6}\pi^2(k-1)k^{-1}(1+k)^{-1}\rho^{-1}n^{-k}].$$

It is worth noticing that this formula is independent of σ, from which fact it follows that the nth zero of $zF'(z)$, and therefore the $(n-1)$th zero b'_{n-1} of $F'(z)$, is given by the same formula as b_n. It is easily shown that the distance between b_n and b_{n-1} (or b_{n+1}) is equal to $|b_n| n^{k-1}$, multiplied by a factor finite both ways, while the distance between b_n and b'_{n-1} is equal to $|b_n| n^{-1}$, multiplied by a similar factor. It thus appears that the large zeros of $F'(z)$ lie relatively near to those of $F(z)$, and do not, for example, lie approximately half-way between consecutive zeros of $F(z)$.

§ 58. It was seen in § 55 that the annuli A_n, which together cover the z-plane completely, each contain exactly one zero of $F(z)$ when n is large. This result has an immediate consequence of some interest. If all the coefficients c_n are real, the imaginary zeros of $F(z)$ occur in conjugate pairs. But if A_n contains one imaginary point it evidently contains its conjugate. Consequently, after a certain value of n, the zero contained in A_n must be real. If, therefore, the coefficients of a function $F(z)$, satisfying the conditions of § 38, are real, $F(z)$ can have only a finite number of imaginary zeros

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ON A CHANGE OF ORDER OF INTEGRATION IN AN IMPROPER REPEATED INTEGRAL.

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XIII. *On the Change of Order of Integration in an Improper Repeated Integral.* By W. H. YOUNG, SC.D., F.R.S.

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§ 1. THE object of the present paper is to give a set of rules for determining when the process of reversing the order of integration in a repeated integral, which for simplicity will be supposed to be in two variables, is allowable. It will be found that the account differs both in form and substance from others which have been hitherto presented. As regards form, no use is made, either in the enunciations of theorems or in the applications to examples, of ϵ-machinery. In this subject, even more than in others, the use of it appears to obscure the issue. As regards substance, certain new rules are given, which, though obtained without difficulty, and applied with considerable ease, have apparently not been stated.

De la Vallée Poussin was one of the first to occupy himself with this subject. His conditions—exception being made of one far-reaching theorem due to him, which is capable of remarkable generalisation—involve considerations of *uniform and non-uniform convergence.* The recent trend of research, especially where integration is involved, has been materially to reduce the importance of these concepts as compared with that of being *bounded.* In the present paper the concepts of uniform and non-uniform convergence are not employed, but the rules given include none the less those of de la Vallée Poussin as a particular case.

I have attempted to make the exposition as systematic as possible, and have accordingly begun by stating the known result for bounded integrands, and then have proceeded successively to unbounded integrands and infinite domains.

§ 2. We begin then by considering the integral $\int_0^q dy \int_0^p f(x, y)\, dx$, where p and q are finite and positive and where $f(x, y)$ is a bounded function of (x, y), and not merely of x and y separately. Exception being made of what we may call at most philosophically possible but mathematically non-existent functions, $f(x, y)$ has then necessarily a Lebesgue double integral, and therefore its repeated Lebesgue integrals necessarily exist and are equal. Thus the change of order of Lebesgue integration is always allowable. It will however usually happen that $f(x, y)$ possesses an ordinary Riemann integral with respect to each of the variables separately.

In this case, by a known theorem*, these integrals are functions of the remaining variable in each case, which possess a Riemann integral with respect to that variable.

In other words we have the following theorem:—

THEOREM. *If $f(x, y)$ is a bounded function of (x, y), which is integrable with respect to x, in the ordinary Riemann sense, and integrable with respect to y, in the same sense, then both its repeated integrals exist and are equal.*

In fact a Riemann integral, when it exists, coincides in value with the Lebesgue integral.

NOTE. It is perhaps worth remarking that the truth of the above theorem depends implicitly on the obvious fact that the integral between finite limits of a bounded function is necessarily bounded, so that the second integrations in the repeated integrals are proper integrations.

If the function $f(x, y)$ is merely bounded with respect to each variable separately, the argument would no longer hold. It is easy to give examples which are bounded with respect to each variable separately, and not bounded with respect to the ensemble, e.g.

$$f(x, y) = 0, \text{ on the axes of } x \text{ and of } y, \text{ and elsewhere}$$
$$f(x, y) = x/(x^2 + y^2).$$

This is a bounded function of x for each fixed value of y, and a bounded function of y for each fixed value of x, but is unbounded on every straight line through the origin other than the axes.

§ 3. We next consider the case where, p and q being still finite, $f(x, y)$ is no longer bounded with respect to (x, y). We shall require the following well-known Lemma, which will also be of use subsequently.

LEMMA. *If* $$f_1(x) \leqslant f_2(x) \leqslant \ldots\ldots$$
is a monotone increasing sequence of functions having $f(x)$ as limit, where each $f_n(x)$ is positive but not necessarily bounded, then

$$\underset{n=\infty}{\mathrm{Lt}} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx,$$

whether this latter integral is finite or infinite.

Choose M so that $$\int_a^b f(x, M)\, dx > K,$$

where $f(x, M)$ denotes the function got from $f(x)$ by changing the value of $f(x)$ to M wherever it was greater, and at the other points leaving it unaltered, and K denotes any quantity less than

$$\int_a^b f(x)\, dx.$$

Let $f_n(x, M)$ be formed from $f_n(x)$ in the same manner as $f(x, M)$ from $f(x)$; then evidently

$$f_1(x, M) \leqslant f_2(x, M) \leqslant \ldots\ldots$$

* W. H. Young, "On Parametric Integration," *Monatshefte für Mathematik und Physik*, Jan. 1910, XXI. pp. 125—149.

is also a monotone increasing sequence, and has $f(x, M)$ as limit. Moreover these functions are in their ensemble bounded, each lying between 0 and M, therefore

$$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(x, M)\, dx = \int_a^b f(x, M)\, dx > K.$$

Hence, remembering that $f_n(x)$ is nowhere less than $f_n(x, M)$, we have

$$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(x)\, dx > K.$$

But K is any quantity less than $\int_a^b f(x)\, dx$, therefore

$$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(x)\, dx \geqslant \int_a^b f(x)\, dx.$$

But $f_n(x)$ is nowhere greater than $f(x)$, therefore its integral cannot be the greater, and we must take the sign of equality, that is

$$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx. \qquad \text{Q.E.D.}$$

By means of the above Lemma, we can at once prove the following important theorem and Corollary.

THEOREM. *If $f(x, y)$ is a positive unbounded function of (x, y), change of order of integration between finite limits is always allowable.*

We have, with our usual notation,

$$\int_0^q dy \int_0^p f(x, y)\, dx = \int_0^q dy \operatorname*{Lt}_{M=\infty} \int_0^p f(x, y, M)\, dx,$$

where, the integrand with respect to y being obviously a monotone function of M, we may, by the Lemma, write the right-hand side

$$= \operatorname*{Lt}_{M=\infty} \int_0^q dy \int_0^p f(x, y, M)\, dx.$$

But since $f(x, y, M)$ is bounded, this may be written

$$= \operatorname*{Lt}_{M=\infty} \int_0^p dx \int_0^q f(x, y, M)\, dy,$$

and therefore, by the Lemma,

$$= \int_0^p dx \operatorname*{Lt}_{M=\infty} \int_0^q f(x, y, M)\, dy$$

$$= \int_0^p dx \int_0^q f(x, y)\, dy. \qquad \text{Q.E.D.}$$

COR. *If $f(x, y)$ is not everywhere positive, but continues to have one of its repeated integrals finite when its sign is made everywhere positive, the integration being over a finite rectangle, as before, then the second repeated integral of f is finite and equal to the first.*

In fact, let f_1 be the value of f, wherever positive, and be elsewhere zero, and f_2 the value with the sign changed of f, wherever negative, and elsewhere zero. Thus

$$f = f_1 - f_2.$$

Then the repeated integral in question of both f_1 and f_2 must be finite, and equal therefore to the other repeated integral in each case, by the theorem. Hence, subtracting the two equalities so obtained, we get the required result, provided the subtraction is legitimate, that is, provided the repeated integral of the difference of two functions is equal to the difference of their repeated integrals. To ascertain whether this is allowable, write

$$\int_0^q f_1(x, y)\, dy = g_1(x),$$

with a similar equation for f_2.

The difference of the repeated integrals, taken first with respect to y and then with respect to x, then takes the form

$$\int_0^p g_1(x)\, dx - \int_0^p g_2(x)\, dx.$$

Now $g_1(x)$ and $g_2(x)$ will in general have infinite values for certain, the same, values of x, unless we expressly assume, which is unnecessary, that these integrals converge for every value of x. But even so, if we agree to make the convention that where $g_1(x)$ and $g_2(x)$ are both infinite, and therefore their difference is undefined, the integral of their difference be still regarded as existing, provided, whatever value we give to this difference at the doubtful points we get the same result on integrating, we can still assert the truth of the theorem. For, by hypothesis, the repeated integral of $(f_1 + f_2)$ exists and is finite, and $(f_1 + f_2)$ has at these points the value infinity, so that the values at these points cannot effect the integration.

NOTE. It will be noticed that we cannot any longer assert that if a function possesses an ordinary improper integral with respect to x and also one with respect to y, that it possesses its repeated integrals in the ordinary sense, whether as Riemann proper or improper integrals, even when the function is positive. For the integral with respect to x of a function which is even continuous in the extended sense with respect to (x, y) may be a bounded non-integrable upper semi-continuous function of y*. There is thus no longer in the case of unbounded functions a theorem exactly corresponding to that given in § 2 for bounded functions.

§ 4. From the result of § 3 we obtain at once the following test which is often convenient.

THEOREM. *If $v(x, y)$ is a positive unbounded function of (x, y), possessing finite repeated integrals of equal value, and $u(x, y)$ be any bounded function, then the repeated integrals of uv are finite and equal to one another.*

It is evident that as it suffices to prove the existence of either repeated integral when the integrand is made positive, it will suffice if we prove the above theorem for the case when u is positive. But when u is positive the repeated integral necessarily exists at least in the extended sense, that is, having a finite or infinite value. But the value cannot be infinite, for its value could not be decreased by substituting for the positive function u its upper bound, in which case, however, the repeated integral becomes the product of this upper bound by the repeated integral of v.

* W. H. Young, "On the Inequalities Connecting the Double and Repeated Upper and Lower Integrals of a Function of Two Variables," *Proc. L. M. S.* Ser. 2, Vol. VI. pp. 247—249 (1908).

§ 5. The importance of the theorem and corollary given in § 3, though considerable, might easily be overrated. They do not dispose even of all those cases where the integrand possesses what is known as an absolutely convergent double integral in the ordinary extended Riemann sense, in other words the cases where the integrand, rendered positive by suitably changing its sign, possesses an improper Riemann double integral. As shown in the paper cited in the preceding section, the existence of this double integral does not even carry with it the existence of the corresponding repeated integrals; nor is the converse the case. Indeed the improper double integral even in the Riemann sense of the absolute value of the integrand may exist and yet one or both of the repeated integrals, even in the Lebesgue sense, may be infinite.

We have, however, the following important theorem, which constitutes a material addition to the information contained in the theorem and corollary of § 3.

THEOREM. *If an unbounded function possesses finite absolutely convergent improper Lebesgue double and finite improper Lebesgue repeated integrals, the limits of integration being finite, and the plane area being the corresponding rectangle, then all three integrals are equal in value**.

§ 6. There exists, moreover, a class of theorems, other than those so far given, in which there is no reference to the existence of an absolutely convergent double integral. Thus we have the following theorem:—

THEOREM. *If the single integrals with respect to x and y are, for varying upper limits x and y respectively, each of them bounded functions of (x, y), and if the infinite discontinuities of $f(x, y)$ lie on a finite number of monotone curves, then the repeated integrals are equal.*

This latter theorem may be compared with § 408 of Hobson's book†. We may remark that Hobson's condition (3) necessarily carries with it the first condition of the above theorem, so that our condition is much less stringent than that given by him, which he has quoted from Jordan. Moreover we do not require to pre-assume his condition (1) at all. On the other hand it seems that his condition (2), which he has also taken from Jordan, cannot easily be rendered less stringent.

To prove the theorem in question, we may describe round every point of these curves, whether the point be a discontinuity or not, a rectangle whose sides are e and e' parallel respectively to the axes of x and y. The portion of the fundamental rectangle then left over will certainly be such that f possesses a double Lebesgue integral over it. Moreover we can make e and e' approach zero in such a manner that this double Lebesgue integral has an unique limit, say I.

Let $f_n(x, y)$ be zero at every point of these small rectangles when e and e' have the values e_n and e'_n of the chosen sequence of values of (e, e'), and be equal to $f(x, y)$ elsewhere. Then it is plain that on any definite ordinate, f_n will differ from f only at points of a finite number of intervals, ranged round the sections of the curves by the ordinate. Hence by the Harnack-Lebesgue definition of an improper integral, it follows that

$$\int dx \int_0^b f(x, y)\, dy = \int dx \underset{n=\infty}{\mathrm{Lt}} \int_0^b f_n(x, y)\, dy.$$

* H. Lebesgue, "Intégral, Longueur, Aire," 1902, *Annali di Matematica*, Ser. 3, VII. § 40, p. 280. E. W. Hobson, *Theory of Functions of a Real Variable*, Camb. Univ. Press, p. 582 (1907).

† See also E. W. Hobson, "On Repeated Integrals," (1907), *Proc. L.M.S.* Ser. 2, Vol. v. p. 330.

Now if r be the number of curves $\int f_n(x, y)\, dy$ differs from $\int f(x, y)\, dy$ by at most $2rA$, where A is the upper bound of the absolute value of $\int_0^y f(x, y)\, dy$; for the value of the integral over any interval is certainly not in absolute value greater than $2A$, and there are at most r such intervals on the particular ordinate.

Hence $\int_0^b f_n(x, y)\, dy$ is a bounded function of (n, x).

Hence we obtain, by the Lebesgue Theory,

$$\int dx \int_0^b f(x, y)\, dy = \operatorname*{Lt}_{n=\infty} \int dx \int_0^b f_n(x, y)\, dy.$$

But the right-hand side is the unique limit I of the double integral of f_n or f for the mode of approach of e and e' to zero adopted.

Since the same argument applies to the other repeated integral we have thus proved that the repeated integrals are equal.

COR. *Under the same circumstances, what Hobson calls the restricted Jordan double integral is bound to exist.*

For we have virtually proved that the repeated integrals are each equal to any one of the limits of the double integral obtained by making (e, e') approach $(0, 0)$. Thus all these latter limits must be the same and finite in value.

NOTE. Thus the existence of the restricted Jordan integral, though redundant as an assumption, is none the less necessarily involved. If therefore we find in a particular example that it does not exist, we know at once that it is useless to apply this test. On the other hand if it does not exist, one or other of our conditions (1) and (2) must be violated. In the example given by Hobson on p. 584, it is of course the first of these conditions that is violated. It should be noticed that it is owing to the violation of the same condition (1), and therefore of Hobson's condition (3), that the theorem fails to apply in Hobson's second example, pp. 584, 585; in fact $\frac{y}{x}\sin\frac{1}{x}$ is not a bounded function of (x, y). In this case the integrand $\frac{1}{x}\sin\frac{1}{x}$ is an integrable function of a single variable, so that the existence and equality of the repeated integrals is, of course, obvious *à priori*.

That Hobson's condition (3) involves the fulfilment of our condition (1) as an inevitable consequence may be shown as follows:—

THEOREM. *If $\int_0^y f(x, y)\, dy$ be such that, given any positive e, we can find k_1, independent of (x, y), so that $\left|\int_y^{y+k} f(x, y)\, dy\right| < e$, for all values of k such that $k \leqslant k_1$, the range of variation of x and y being, say, a definite rectangle $(0 \leqslant x \leqslant a;\ 0 \leqslant y \leqslant b)$, then $\int_0^y f(x, y)\, dy$ is a bounded function of (x, y) throughout the rectangle.*

PROOF. When $y = 0$ the integral has the value zero.

Let N be the first integer such that $Nk_1 \geqslant b$. Then, whatever y be chosen, there is a determinate integer $n \leqslant N$, which is the smallest integer such that $nk_1 \geqslant y$.

We shall then have $$\int_0^y = \int_0^{k_1} + \int_{k_1}^{2k_1} + \dots + \int_{(n-1)k_1}^{(n-1)k_1+k},$$
where $k \leqslant k_1$.

Each of the integrals on the right is numerically less than e, by the hypothesis, hence, since the modulus of a sum is not greater than the sum of the moduli,

$$\left| \int_0^y f(x, y)\, dy \right| \leqslant ne \leqslant Ne.$$

This shows that the integral in question is a bounded function of (x, y), since N and e are both fixed quantities.

It is hardly necessary to add that the converse of the above theorem is not true. To assert it would *correspond* in the theory of series to the fallacious assertion that if there were no points of non-uniform convergence with infinite measure the convergence would be uniform.

§ 7. We now consider integrals over what is called *an infinite domain*, or more strictly repeated integrals in which the limits of integration are no longer necessarily finite.

Even if the integrand is bounded, such repeated integrals are not in general expressible as repeated limits of proper repeated integrals. All known tests turn however on the possibility in certain cases of so expressing them. They consist in short and in general of two distinct sets of conditions, viz.

(1) Conditions that they can be so expressed;

(2) Conditions that these repeated limits are equal to one another.

This latter set of conditions breaks up into two parts:—

(2 a) Conditions under which these two repeated limits are repeated limits of the *same* function of two variables, and accordingly

(2 b) Conditions that the two repeated limits of this function are equal.

I propose therefore in the next section to state briefly a fact or two with regard to the circumstances under which the repeated limits of a function of two variables are equal to one another. Before doing so, it may be well to emphasize once more that the conditions with which we are now concerned, however general the form we may give to them, are sufficient, but not necessary, conditions; they fail, for example, as will be seen, to prove the equality of the repeated integrals

$$\int_0^\infty dy \int_0^\infty f(x, y)\, dx, \quad \int_0^\infty dx \int_0^\infty f(x, y)\, dy,$$

when $$f(x, y) = \frac{xy(1 - x^4y^4)}{(1 + x^4y^4)^2}.$$

In this case both the repeated integrals have the value zero, but they cannot be expressed as repeated limits of proper repeated integrals.

§ 8. *Under what circumstances can we assert the equality of the repeated limits*

$$\underset{x=0}{\mathrm{Lt}}\ \underset{y=0}{\mathrm{Lt}}\ F(x, y) = \underset{y=0}{\mathrm{Lt}}\ \underset{x=0}{\mathrm{Lt}}\ F(x, y) \quad \text{.............................(1)?}$$

We can assert the equality if we can prove that there is an unique double limit.

This condition is sufficient, but not necessary. Unfortunately, however, there is no other case in which we can assert it. Any other set of conditions we can give are exactly equivalent except in form to the statement of the equality. *If F be a monotone function of each variable separately, the sense of the monotony being the same for each variable separately, an unique double limit exists and we can accordingly assert the equality.* This is of great importance, the fact being the basis of de la Vallée Poussin's Theorem, alluded to in the preface, and to be given in an extended form shortly. Here, moreover, from the very idea of monotony, we are sure *à priori* that the single limits and repeated limits all exist, if we include infinity as one of the values a function or limit can have. The repeated limits are in this case equal without exception.

This particular case suggests one possible direction in which we may seek to modify the form of the equality whose conditions of existence are in question. By an obvious artifice we can so modify the equality (1) that its truth carries with it, not only implicitly, but as a demonstrable proposition, the existence of the two repeated limits, as well as their equality.

For simplicity we suppose the single limits to exist. Write

$$\underset{y=0}{\mathrm{Lt}}\ F(x, y) = G(x).$$

Then (1) may be written

$$\underset{y=0}{\mathrm{Lt}}\ \underset{x=0}{\mathrm{Lt}}\ [F(x, y) - G(x)] = 0 \quad \text{...................................(2),}$$

provided the repeated limits are finite, in which case only does what follows apply.

It is evident that if (1) holds, so does (2) provided the repeated limits are finite. But conversely if (2) holds, always provided the single limits both exist, we can assert, not only that (1) holds *if* the repeated limits exist, but that the repeated limits *must* exist*.

In fact take such a sequence of values of x that $G(x)$ has an unique limit, say L. Then for this sequence of values of x

$$\underset{y=0}{\mathrm{Lt}}\ \underset{x=0}{\mathrm{Lt}}\ F(x, y) = L \quad \text{..(3).}$$

But $F(x, y)$ has an unique limit with respect to x independent of the particular sequence chosen, therefore the equation (3) is true absolutely. In other words the repeated limit in question exists and is equal to L. But L is any one of the possible limits of $G(x)$, therefore all the limits of $G(x)$ coincide, which proves both the existence and the equality of the two repeated limits. Q.E.D.

So far we have interpreted equation (2) in the obvious manner. If we choose to require its validity only when the set of values to be ascribed to y as it approaches zero is a dis-

* It is the merit of Bromwich to have called attention to the importance of this equation (2) in the theory of repeated integrals. As however the equation is not given explicitly but expressed in ϵ-language in his paper on the subject many readers will probably have missed the intuitiveness of the matter. Cp. T. J. I'A. Bromwich, "The Inversion of a Repeated Infinite Integral," *Proc. L. M. S.* Ser. 2, Vol. I. pp. 176—201 (1903).

continuous one, or even discrete, it is clear that we can still draw the same conclusion from the equation, provided we are sure that $\underset{x=0}{\mathrm{Lt}}\ \underset{y=0}{\mathrm{Lt}}\, F(x, y)$ is known to exist.

§ 9. Though the case in which only one of the integrations has an infinite limit can be implicitly inferred from the more general one, there is perhaps a gain of clearness in discussing it separately.

Consider then *under what circumstances we can assert*

$$\int_0^\infty dy \int_0^p f(x, y)\, dx = \int_0^p dx \int_0^\infty f(x, y)\, dy.$$

Here again it is a particular set of conditions that it is proposed to give. No really general test is known, and in this respect the conditions which follow are no exception.

The first integral is simply $\quad \underset{q=\infty}{\mathrm{Lt}} \int_0^q dy \int_0^p f(x, y)\, dx,$

which, if the change of order of integration when the limits are both finite is allowable, may be written

$$\underset{q=\infty}{\mathrm{Lt}} \int_0^p dx \int_0^q f(x, y)\, dy.$$

We can then assert the equality if

$$(\alpha) \qquad \int_0^p dx \int_0^q f(x, y)\, dy = \int_0^q dy \int_0^p f(x, y)\, dx,$$

the possibility of which we have now fully discussed, both when f is bounded and when it is unbounded and

$$(\beta) \qquad \underset{q=\infty}{\mathrm{Lt}} \int_0^p dx \int_0^q f(x, y)\, dy = \int_0^p dx \int_0^\infty f(x, y)\, dy;$$

or, which is the same thing,

$$\underset{q=\infty}{\mathrm{Lt}} \int_0^p dx \int_q^\infty f(x, y)\, dy = 0.$$

We may, if we please, write (β) in ϵ-language, but we shall in this way gain no further information.

We shall often be able, in particular cases, to prove the truth of (β) directly, but, just as in the Theory of Convergence, it is a great help to have rules which, though they do not apply in all cases, do apply in many, and so enable us almost by inspection to determine the question at issue. We proceed therefore to give a few such rules, whose application is immediate.

These rules are as follows:—

RULE I. *Equation (β) is certainly true if $f(x, y)$ is a positive function.*

RULE II. *Equation (β) certainly holds if we know that either*

$$\underset{q=\infty}{\mathrm{Lt}} \int_0^p dx \int_0^q |f(x, y)|\, dy \text{ or } \int_0^p dx \int_0^\infty |f(x, y)|\, dy \text{ is finite.}$$

RULE III. *Equation (β) certainly holds if we know that $g(x, q)$, where*

$$g(x, q) = \int_0^q f(x, y)\, dy$$

exists and is a bounded function of (x, q) in the infinite rectangle $(0 \leqslant x \leqslant p;\ 0 \leqslant q \leqslant \infty)$.

RULE IV. *Equation* (β) *certainly holds if we know that* $g(x, q)$ *exists and is a monotone function of* q. (This includes Rule I.)

Rules I, II and IV are self-evident in the light of the Lemma stated and proved in § 3.

As regards Rule III, its truth is at once apparent to anyone acquainted with the elements of the Lebesgue Theory.

De la Vallée Poussin has given a condition, included in Rule III, as a particular case. It states, namely, that the equation in question holds if the convergence of $g(x, q)$ to $g(x, \infty)$ is uniform, whereas, as is obvious, our rule only excludes the possibility of the existence of points of infinite non-uniform convergence.

It should also be remarked that certain obvious extensions of the rule may be given. Thus we do not require the existence of $g(x, q)$ for the value $q = \infty$ except at a set of values of x of content p. More conveniently we may say that we do not exclude the possibility of the integral $\int_0^q f(x, y)\,dy$ oscillating finitely at $q = \infty$ for a set of values of x of content zero*.

§ 10. We come now to the general case, where both integrals have infinite limits of integration. *Under what circumstances can we assert that*

$$\int_0^\infty dy \int_0^\infty f(x, y)\,dx = \int_0^\infty dx \int_0^\infty f(x, y)\,dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)?$$

Here $f(x, y)$ may of course be unbounded, this will, however, thanks to the order in which we have treated the subject, present no additional difficulty.

We have
$$\int_0^\infty dy \int_0^\infty f(x, y)\,dx = \operatorname*{Lt}_{q=\infty} \int_0^q dy \operatorname*{Lt}_{p=\infty} \int_0^p f(x, y)\,dx,$$
with a similar statement for the repeated integral on the right-hand side of (4). It is plain, therefore, that, if we can assert that

$$\int_0^q dy \int_0^\infty f(x, y)\,dx = \operatorname*{Lt}_{p=\infty} \int_0^q dy \int_0^p f(x, y)\,dx,$$

and
$$\int_0^p dx \int_0^\infty f(x, y)\,dy = \operatorname*{Lt}_{q=\infty} \int_0^p dx \int_0^q f(x, y)\,dy,$$
or, in other words, that

$$\left.\begin{array}{l} \int_0^q dy \int_p^\infty f(x, y)\,dx \text{ has the unique limit zero as } p \text{ approaches infinity} \\ \text{and } \int_0^p dx \int_q^\infty f(x, y)\,dy \text{ has the unique limit zero as } q \text{ approaches infinity} \end{array}\right\} \ldots\ldots(\mathrm{A}),$$

for which conditions have been given in the last section, then the equality (4) takes the form

$$\operatorname*{Lt}_{q=\infty} \operatorname*{Lt}_{p=\infty} \int_0^q dy \int_0^p f(x, y)\,dx = \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{q=\infty} \int_0^p dx \int_0^q f(x, y)\,dy.$$

Hence, if
$$\int_0^q dy \int_0^p f(x, y)\,dx = \int_0^p dx \int_0^q f(x, y)\,dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{B}),$$

* The reader who desires further information on this point can obtain it by referring to a paper by the author on "Term-by-Term Integration of Oscillating Series," *Proc. L. M. S.* Ser. 2, Vol. VIII. pp. 99—116 (1909).

for which conditions have been given in §§ 2—6, and we denote their common value by $F(p, q)$, (4) takes the form

$$\underset{q=\infty}{\mathrm{Lt}}\ \underset{p=\infty}{\mathrm{Lt}}\ F(p, q) = \underset{p=\infty}{\mathrm{Lt}}\ \underset{q=\infty}{\mathrm{Lt}}\ F(p, q) \quad \text{.............................(5)},$$

and it remains to express this equation in the alternative form of a condition involving the integrals.

As explained in § 8, we can, if we concern ourselves only with finite possible values of the repeated integrals, write the equation (5) in either of the two forms

$$\underset{q=\infty}{\mathrm{Lt}}\ \underset{p=\infty}{\mathrm{Lt}}\ \{F(p, q) - G(p)\} = 0,$$

and

$$\underset{p=\infty}{\mathrm{Lt}}\ \underset{q=\infty}{\mathrm{Lt}}\ \{F(p, q) - H(q)\} = 0.$$

Taking the former form, we have

$$G(p) = \int_0^\infty dy \int_0^p f(x, y)\, dx = \int_0^p dx \int_0^\infty f(x, y)\, dy,$$

where we have tacitly assumed that the second of the conditions (A) holds.

Hence,
$$G(p) - F(p, q) = \int_0^p dx \int_q^\infty f(x, y)\, dy.$$

Thus, making first p proceed to the limit and then q, our condition takes the form

$$\underset{q=\infty}{\mathrm{Lt}} \int_0^\infty dx \int_q^\infty f(x, y)\, dy = 0 \quad \text{................................(C)}.$$

Similarly we get the other form of condition,

$$\underset{p=\infty}{\mathrm{Lt}} \int_0^\infty dy \int_p^\infty f(x, y)\, dx = 0 \quad \text{................................(C')}.$$

Either of these conditions (C) *or* (C′) *is then, on the assumption of the truth of* (A) *and* (B), *necessary and sufficient to insure the possibility of reversing the order of partial integration in the repeated improper integrals of* $f(x, y)$ *between infinite limits.*

Instead of (C) and (C′) we might have attempted to express directly that $F(p, q)$ has an unique double limit. This would be a sufficient, but not necessary, condition. Thus the condition

$$\underset{\substack{p=\infty\\ q=\infty}}{\mathrm{Lt}} \int_0^p dx \int_0^q f(x, y)\, dy \text{ is unique } \quad \text{..............................(C'')}$$

is sufficient with (A) and (B) to secure our purpose. This may, in particular examples, be easier to show than (C) or (C′), or to show directly that

$$\underset{p=\infty}{\mathrm{Lt}}\ \underset{q=\infty}{\mathrm{Lt}} \int_0^p dx \int_0^q f(x, y)\, dy = \underset{q=\infty}{\mathrm{Lt}}\ \underset{p=\infty}{\mathrm{Lt}} \int_0^p dx \int_0^q f(x, y)\, dy \quad \text{..............(C''')}.$$

Apart from the fact, however, that (C″) might not be true, and yet change of order of integration be allowable, the conditions (C) and (C′) will be often easier to apply, for frequently in practice definite integrals become more and not less difficult to evaluate in whole or in part when the limits of integration are infinite. In one case (C‴) is immediately applicable, namely if $f(x, y)$ be a bounded mixed differential coefficient.

§ 11. De la Vallée Poussin gives a different condition, namely that

$$\int_0^\infty dx \int_0^q f(x, y)\, dy$$

should converge uniformly in an unlimited interval $q_0 \leqslant q \leqslant \infty$.

It will be found, however, on examination that this condition is quite unnecessarily stringent and therefore very limited in its application. It is even more stringent than (C″), for it requires $F(p, q)$ to be continuous with respect to (p, q), not only at the point (∞, ∞), but at each point (∞, q), for all values of q, $q_0 \leqslant q \leqslant \infty$. Moreover if we proceed in any given particular example to attempt to apply this condition, we shall have to go through work similar to that required for the test (C), or (C′), with the serious disadvantage that we shall have to prove much more than these tests require.

We may, if we please, express the test (C″) slightly differently in a manner analogous to that by means of which we obtained the tests (C) and (C′). If we do this, it states that

$$\int_0^q dy \int_p^\infty f(x, y)\, dx$$

must have an unique double limit as (p, q) *approaches* (∞, ∞), *in any manner*, with a corresponding alternative condition that comes by interchanging p and q, and x and y as variables of integration. De la Vallée Poussin's condition would require these repeated integrals to possess an unique double limit, not only at (∞, ∞), but also at (p, ∞) and (∞, q) respectively.

§ 12. That uniform convergence, even in the case of a series of discontinuous functions, involves the boundedness of the whole set of functions in their entirety is well known. As, however, I do not know of any formal proof of this statement, I here append it.

THEOREM. *If a series of bounded functions* $f_1(x)$, $f_2(x)$, ... *converges uniformly to the function* $f(x)$, *then*

(1) $f(x)$ *is bounded;*

(2) $f_n(x)$, *regarded as a function of* (x, n), *is bounded.*

PROOF. Since the convergence is uniform, we can find an m, such that for $n \geqslant m$

$$|f(x) - f_n(x)| < e,$$

for all values of x, e being any chosen positive quantity.

Let the upper bound of the absolute value of $f_m(x)$ be B. Then this shows that the upper bound of $f(x)$ is not greater than $B + e$, which proves (1).

Further, for every value of x and every $n > m$,

$$|f_n(x)| < B + 2e,$$

which proves (2).

This theorem shows that, even when all the functions concerned are discontinuous, series of functions which are in their ensemble bounded include uniformly convergent series as particular cases, so that any theorem true of the former is true of the latter.

Thus de la Vallée Poussin's conditions are, as has already been asserted, special cases of those given above.

§ 13. Before proceeding to illustrative examples we now state the remarkable theorem, already alluded to, which may with propriety be called de la Vallée Poussin's Theorem. In the general form we have now given to it, this theorem is as follows:—

THEOREM. (De la Vallée Poussin's*.) *If the integrand is always positive, whether it be bounded or unbounded, and whether the limits of integration are finite or infinite, then change in the order of generalised Lebesgue integration is always allowable. Moreover this theorem is still true, when the integrand is not everywhere positive, provided one of the repeated integrals continues to be finite when the integrand is everywhere made positive, by changing its sign wherever it is negative.*

It is unnecessary to prove this theorem now, its truth will be self-evident to anyone who has followed the preceding sections.

In practice it is convenient to ascertain first if this theorem applies, and, if not, to consider the conditions (A), (B) and (C) in detail.

We now give illustrative examples. We have purposely chosen some of those discussed already either by Bromwich or by Hobson, so that the advantages of the rules expounded above may be properly tested.

Ex. 1†. Let $$f(x, y) = \sin y e^{-yx^2}.$$

This is an essentially bounded function of (x, y), and is integrable with respect to x and is integrable with respect to y. Hence its repeated integrals over a finite rectangle exist and are equal, so that condition (B) is fulfilled.

Again, $$\int_0^q f(x, y)\, dy = \frac{1}{x^4+1}\{1 - e^{-x^2 q}(\cos q + x^2 \sin q)\},$$

$$\int_0^p dx \int_q^\infty f(x, y)\, dy = \int_0^p \frac{1}{1+x^4} e^{-qx^2}(\cos q + x^2 \sin q)\, dx.$$

Now the integrand on the right-hand side is a bounded function of q for all values of x from 0 to p, and all values of q up to and including infinity, and it has the unique limit zero as q approaches infinity. Hence the integral itself has the unique limit zero. Thus one of the conditions (A) is satisfied.

Again, if $g(t) = \int_t^\infty e^{-t^2} dt$, so that $g(t)$ is a bounded monotone decreasing function of t, we have

$$\int_0^p e^{-yx^2} \sin y\, dx = \frac{\sin y}{\sqrt{y}}\{g(0) - g(p\sqrt{y})\},$$

* A special case of this theorem of great use in practice is given in Bromwich's *Infinite Series*. The proof there given, which is essentially de la Vallée Poussin's, would be shortened if in the small print on pp. 457, 458 the method of monotone sequences were adopted.

† This example is discussed in Hobson, p. 593. It should be noticed that there is a slight oversight in the statement of conditions (2) on p. 589, they add nothing to his conditions (1), as we see by changing the order of integration. Cp. Bromwich, *Proc. L. M. S.* Ser. 2, Vol. I. p. 188, or Hobson, p. 588, line 16 for the correct statement. As Hobson uses, however, his conditions (2), his discussion of the example is not quite satisfactory. It should also be noticed that there is an apparent oversight on pp. 590, 591 of his account of de la Vallée Poussin's conditions. The modification he introduces in condition (2) would seem to admit of $\int_1^\infty f(x, y)\, dy$ not converging at certain points, or even if converging, of assuming values which are not bounded. The reasoning on p. 591, line 7 would not then apply.

and is plainly therefore a bounded function of (p, y) for all values of (p, y) such that $(0 \leqslant p \leqslant \infty;\ 0 \leqslant y \leqslant q)$. Hence the second condition (A) holds.

Finally we have

$$\int_0^\infty dx \int_q^\infty f(x, y)\, dy = \int_0^\infty \frac{e^{-qx^2}}{1+x^4}(\cos q + x^2 \sin q)\, dx.$$

Breaking the integral on the right-hand side up into two parts, viz. from 0 to 1 and from 1 to infinity, we see that, as the integrand has zero as unique limit when q approaches infinity, the first partial integral will have the unique limit zero. The second partial integral is plainly numerically less than $\int_1^\infty e^{-qx}\, dx$, that is, numerically less than e^{-q}/q, and therefore has the unique limit zero. Hence condition (C) holds, and therefore, since (A) and (B) hold, we are sure that the repeated integrals over the infinite quadrant exist and are equal.

Ex. 2. Let
$$f(x, y) = \frac{d^2 V}{dx\, dy},$$
where
$$V = xy/(1 + x^2 + y^2).$$

It is here evident, by inspection, that both repeated integrals between the limits 0 and ∞ exist and are equal, both being zero, since their first integrals with respect to x and with respect to y are both zero.

Following Hobson, however, who quotes it from Bromwich, we take it as an example of the use of our conditions. Here condition (B) is, of course, satisfied of itself.

Also
$$\int_0^p dx \int_q^\infty f(x, y)\, dy = \int_0^p \frac{q(1 + q^2 - x^2)}{(1 + q^2 + x^2)^2}\, dx.$$

Here the integrand is a bounded function of (x, q), and has the unique limit zero as q increases. Hence the integral has the unique limit zero, so that the first condition (A) is satisfied, and, by symmetry, so is the other.

Finally $F(p, q)$ is in our case $pq/(1 + p^2 + q^2)$ and we notice that, though it has no unique double limit as p and q approach infinity, its repeated limits are both zero. Here, as will be seen, we have used condition (C‴).

Ex. 3. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$, and let both the lower limits of integration be unity, the upper limits being infinite.

Here, if we integrate twice up to the limits p and q for x and y respectively, we get by inspection

$$F(p, q) = \left(\tan^{-1}\frac{p}{q} - \tan^{-1}\frac{1}{q}\right) - \left(\tan^{-1} p - \frac{\pi}{4}\right).$$

If now we first of all make q infinite and then p, we get $-\frac{\pi}{4}$; but if we first make p infinite and then q, we get $+\frac{\pi}{4}$. Hence (C‴) is violated. Accordingly the repeated integrals under discussion cannot be equal, unless it should appear that conditions (A) do not hold.

Now $$\int_1^p dx \int_q^\infty f(x, y)\, dy = -\int_1^p q\, dx/(x^2+q^2).$$

Here the integrand is bounded and has the limit zero when q becomes infinite.

Hence the limit of the integral is zero. Thus the first condition (A) is satisfied, and, by symmetry, so is the second.

Thus $-\dfrac{\pi}{4}$ and $+\dfrac{\pi}{4}$ are the actual values of the repeated integrals.

This example has also been taken from Hobson to illustrate the use of our conditions.

Ex. 4. Let $$f(x, y) = \{e^{-ay} \sin(xy) \cos x\}/x, \text{ where } a > 0.$$

Here f is evidently a bounded function of (x, y) for all values of (x, y) in the finite rectangle, and change of order of integration when the limits of integration are finite is allowable. Further

$$\int_0^p f(x, y)\, dx = \tfrac{1}{2} e^{-ay} \left(\int_0^p \frac{\sin(y+1)x}{x} dx + \int_0^p \frac{\sin(y-1)x}{x} dx \right).$$

If in the first of the integrals inside the bracket we take $x(y+1)$ and in the second $x(y-1)$ for new variable, we see, what is otherwise well known, that the quantity inside the bracket is bounded, being in fact certainly numerically less than 2π. Hence for all values of p from 0 to infinity, and all values of y from 0 to q, the integral on the left-hand side is certainly a bounded function of (p, y). Thus, by Rule III, one of the conditions (A) is certainly satisfied.

Again, $$\int_0^p dx \int_q^\infty f(x, y)\, dy = \int_0^p \frac{e^{-aq} \cos x}{x(a^2+x^2)} \{a \sin(xq) + x \cos(xq)\}\, dx.$$

Now e^{-aq} may be brought outside the integral, and the integrand will then be such that $\int_0^p$ whether p be finite or infinite is clearly a bounded function of (p, q). Hence the limit of the left-hand side, whether p be finite or not, q being indefinitely increased, is certainly zero. Thus condition (C) and the remaining condition (A) are both fulfilled.

Ex. 5. Let $$f(x, y) = x^{n-1} e^{-x} (e^{-y} - e^{-xy})/y.$$

Here the limits of integration are once more to be 0 to infinity for both variables. If we take the limits of integration for x to be 1 to infinity, instead of 0 to infinity, f is essentially positive, and if we take the limits of integration for x to be 0 to 1, f is essentially negative. Thus it is convenient to break up the repeated integral into the two corresponding parts.

By de la Vallée Poussin's Theorem, therefore, change of order of integration is certainly allowable, provided only we show that one of the repeated integrals has a finite value, both when the limits of x are 0 and 1 and when they are 1 to ∞. Now if we first integrate with respect to x between the limits 0 and ∞, we get a multiple of

$$\int_0^\infty \{e^{-y} - (1+y)^{-n}\} \frac{dy}{y},$$

where the integrand, being bounded, we need only consider the portion of the integra involving large values of y. Hence we easily see that this has a finite value.

Also $$\int_1^\infty dx \int_0^\infty x^{n-1} e^{-x} (e^{-y} - e^{-xy}) \frac{dy}{y} = \int_1^\infty dx \int_0^\infty x^{n-1} e^{-x} e^{-ky} (x-1)\, dy,$$

where k is a function of (x, y) whose magnitude lies between 1 and x and is therefore greater than unity.

Hence this repeated integral is numerically less than

$$\int_1^\infty x^{n-1} e^{-x} (x-1)\, dx \int_0^\infty e^{-y}\, dy,$$

and therefore less than $$\int_1^\infty x^n e^{-x}\, dx.$$

It is accordingly numerically less than $\Gamma(n+1)$, and is therefore finite.

Hence also $\int_0^\infty dy \int_1^\infty f(x, y)\, dx$ has a finite value.

But we have shown that $\int_0^\infty dy \int_0^\infty f(x, y)\, dx$ has a finite value. Hence also

$$\int_0^\infty dy \int_0^1 f(x, y)\, dx \text{ has a finite value.}$$

Thus change of the order of integration was certainly allowable in the original integral between the limits 0 to 1 for x and 0 to ∞ for y, and the limits 1 to ∞ for x and 0 to ∞ for y, and therefore, by addition, also for 0 to ∞ for x and 0 to ∞ for y.

It will also be noticed that we have nowhere assumed more than that n is positive. It need not be restricted to be greate than unity.

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THE STRESSES IN A THICK HOLLOW CYLINDER SUBJECTED TO INTERNAL PRESSURE.

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XIV. *The Stresses in a Thick Hollow Cylinder subjected to Internal Pressure.*

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(Communicated by Professor Hopkinson.)

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CONTENTS.

In the course of an experimental research on the nature of the stress-distribution determining the limit of elasticity in steel, the author has had occasion to use large hydrostatic pressures. It was desired at one time to measure these pressures with gauges formed of twisted, flattened, steel tube*, and in experimenting with such it was found that with thick tubes the pressure needed to induce permanent set could be much raised by a preliminary operation of overstraining, followed by boiling in water to hasten the recovery of elasticity†. These observations suggested the present analysis, in the attempt to see in what manner, and to what extent, there may be automatically produced the condition of initial stress sought in a wire-wound gun.

* After J. J. Guest, *Phil. Mag.* II. 1900.

† See Muir, *Phil. Trans.* 193 A.

The analysis presupposes (1) that the material is initially in the isotropic condition, elastic effects of overstraining during manufacture having been removed by annealing or otherwise; (2) that the elastic limit* at any point in the material is reached when the maximum shear at the point exceeds a certain definite value, which is independent of the actual distribution of the stress—i.e. that the shear theory of breakdown is accurate; and (3) that the yield point coincides with the elastic limit, and the yield is completely plastic—i.e. the stress-strain graph is a straight line right up to the yield point, where it becomes parallel to the axis of strain. It is not asserted that these suppositions, necessary for the mathematical treatment which follows, are accurately true for every or any steel. Let it suffice that they form a close approximation to what may be observed; so that with these hypotheses quantitative results may be obtained which, though not accurate, yet indicate the kind of effects to be anticipated in actual experimental tests.

With such qualifications, the third of these assumptions needs little justification. A stress-strain relation of this description is the property of most, if not all, samples of annealed mild steel and iron†; and the writer has found it eminently so for annealed samples also of a 3 °/ₒ nickel steel, and a tool steel containing as much as 1·2 °/ₒ of carbon. The common belief that an intermediate stage of considerable range intervenes between states of perfect elasticity and perfect plasticity is probably due to two causes. Firstly, the usual test is one in simple tension in the Testing Machine, the specimen being an un-annealed rolled bar; and secondly, insufficient precautions are taken to ensure centrality of pull. Probably the rolling during manufacture, and certainly the inequality of distribution of stress, will produce or increase the phenomenon of apparent gradual yield.

With regard to the second assumption, there is no published evidence for the case of three-dimensional stress with which we are dealing. A large amount of experimental work has been undertaken to demonstrate—as it turns out—the approximate truth of the shear theory for mild steel and iron in various cases of TWO-dimensional stress‡, but only some rather inconclusive experiments by the author have, as far as he is aware, extended the investigation to THREE-dimensional stress. So far as these go, however, they show that the shear theory is at least not far from accurate for the only material examined, mild steel.

It is likely that almost any theory of the elastic limit would suggest a strengthening of the cylinder by overstrain, and each theory would lead to appropriate numerical results. In this paper the implications of the most probable of such theories are developed.

Let:

S = the elastic limit shear for the material;

E, σ = Young's Modulus and Poisson's Ratio, respectively;

a, b = internal and external radii of cylinder, respectively;

r = distance of any point from axis of cylinder;

R_1, R_2, R_3 = radial, hoop, and longitudinal tensions at any point, respectively [fig. 1];

P = internal hydrostatic pressure; ($P = -R_1$ at inner layer).

* Throughout the paper, it is the Limit of Linear Elasticity which is referred to.

† As an example, see fig. 4, p. 387.

‡ For some recent experiments, and a short account of former work, see "The elastic breakdown of materials submitted to compound stress," *Engineering*, Feb. 5 and 12, 1909.

Assume the cylinder is initially unstressed; and that the portion under consideration is far from the ends, so that from symmetry plane sections remain plane.

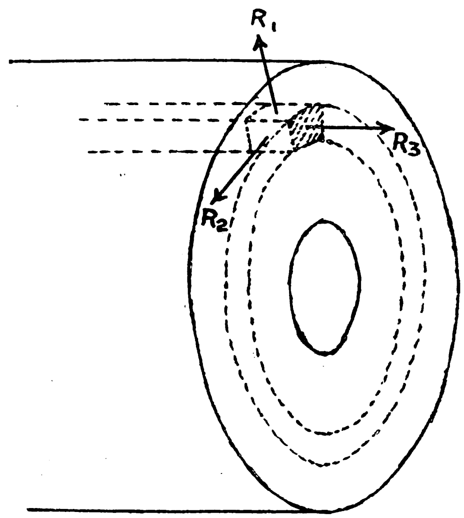

Fig. 1.

1. *When the material is elastic throughout.*

Let u = radial shift of element.

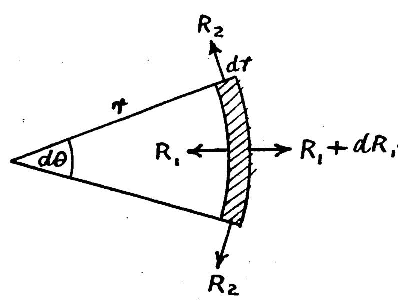

Fig. 2.

For radial equilibrium of element [fig. 2], we have

$$R_1 r\,d\theta - (R_1 + dR_1)(r + dr)\,d\theta + R_2\,dr\,d\theta = 0,$$

$$R_1\,dr + r\,dR_1 = R_2\,dr,$$

$$R_2 - R_1 = r\frac{dR_1}{dr} \quad \text{......................................(i).}$$

$$\text{Radial stretch} = \frac{du}{dr} = \frac{1}{E}[R_1 - \sigma(R_2 + R_3)] \quad \text{............................(ii).}$$

$$\text{Hoop stretch} = \frac{u}{r} = \frac{1}{E}[R_2 - \sigma(R_3 + R_1)] \quad \text{............................(iii).}$$

$$\text{Longitudinal stretch} = \text{const.} = \frac{1}{E}[R_3 - \sigma(R_1 + R_2)] \quad \text{.................(iv).}$$

Eliminating u from (ii) and (iii),

$$\frac{1}{E}[R_1 - \sigma(R_2 + R_3)] = \frac{du}{dr} = \frac{1}{E}[R_2 - \sigma(R_3 + R_1)] + \frac{r}{E}\left[\frac{dR_2}{dr} - \sigma\frac{dR_3}{dr} - \sigma\frac{dR_1}{dr}\right],$$

$$R_1(1+\sigma) - R_2(1+\sigma) - r\frac{dR_2}{dr} + \sigma r\frac{dR_1}{dr} = -\sigma r\frac{dR_3}{dr}.$$

But from (iv)

$$\frac{dR_3}{dr} = \sigma\frac{dR_1}{dr} + \sigma\frac{dR_2}{dr},$$

$$\therefore\ (R_1 - R_2)(1+\sigma) - r\frac{dR_2}{dr} + \sigma r\frac{dR_1}{dr} = -\sigma^2 r\frac{dR_1}{dr} - \sigma^2 r\frac{dR_2}{dr},$$

$$(R_1 - R_2)(1+\sigma) + \sigma r\frac{dR_1}{dr}(1+\sigma) - r\frac{dR_2}{dr}(1-\sigma^2) = 0,$$

$$R_1 - R_2 + \sigma r\frac{dR_1}{dr} + (\sigma - 1)r\frac{dR_2}{dr} = 0.$$

Whence from (i)

$$r\frac{dR_1}{dr}(\sigma - 1) + r\frac{dR_2}{dr}(\sigma - 1) = 0,$$

$$R_1 + R_2 = \text{const.} = 2A \text{ (say)} \qquad \text{(v)}.$$

Whence from (iv)

$$R_3 = \text{const.}^*$$

If the cylinder has open ends, and there is no external pull,

$$R_3 = 0.$$

If the cylinder has closed ends, and there is an additional axial pull L,

$$R_3 = \frac{1}{\pi(b^2 - a^2)}(L + \pi a^2 P) \qquad \text{(vi)}.$$

From (i) and (iv)

$$2A - 2R_1 = r\frac{dR_1}{dr},$$

$$\frac{dR_1}{2(A - R_1)} = \frac{dr}{r},$$

$$-\tfrac{1}{2}\log(A - R_1) = \log r + \text{const.},$$

$$\log(r\sqrt{A - R_1}) = \text{const.},$$

$$r^2(A - R_1) = \text{const.} = -B \text{ (say)},$$

$$R_1 = A + \frac{B}{r^2} \qquad \text{(vii)},$$

$$R_2 = A - \frac{B}{r^2}, \text{ from (v)} \qquad \text{(viii)}.$$

* It is usual to take $R_3 = 0$; or, if the cylinder is supposed to have closed ends, to assume R_3 is uniform over a section. This assumption is unwarranted. The above proof that $R_1 + R_2$ is constant, whence it follows that R_3 is constant, over a section, is due to Mr C. E. Inglis of King's College.

To find A and B:

when $r=b$, $R_1=0$, when $r=a$, $R_1=-P$,

$$\therefore\ A+\frac{B}{b^2}=0, \qquad \therefore\ A+\frac{B}{a^2}=-P,$$

$$\therefore\ B=-P\frac{a^2b^2}{b^2-a^2},$$

and
$$A=P\frac{a^2}{b^2-a^2}.$$

Putting these values of A and B in (vii) and (viii),

$$R_1=P\frac{a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) \quad \text{..................................(ix)},$$

$$R_2=P\frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) \quad \text{.......................................(x)}.$$

These are the ordinary formulae of the text-books.

Assuming that L is not so large as to make $R_3>R_2$, the greatest value of P consistent with elasticity of the inner layer is given by

$$2S=R_2-R_1 \text{ at } r=a$$

$$=P\frac{a^2}{b^2-a^2}\cdot\frac{2b^2}{a^2},$$

i.e.
$$P=S\frac{b^2-a^2}{b^2} \quad \text{...(xi)}.$$

2. *When part of the cylinder is plastic, part elastic.*

When P exceeds the above value (xi), the inner part of the cylinder becomes plastic; and as P increases, the radius of the plastic region increases from a to b. Now it is assumed that the material becomes plastic when the maximum shear at any point becomes equal to S. Hence throughout the plastic region there is a uniform maximum shear S at each point*, and we have

$$R_2-R_1=2S,$$

or
$$R_3-R_1=2S,$$

according as
$$R_3 \lesseqgtr R_2, \text{ respectively.}$$

Let us assume here that $R_3<R_2$, so that we have

$$R_2-R_1=2S \quad \text{.......................................(xii)}.$$

We shall find how far this is true in the next section, (3).

To find R_1 and R_2 for the plastic portion, we have in addition to (xii) the former equation (i) for the equilibrium of an element; viz.

$$R_2-R_1=r\frac{dR_1}{dr}.$$

* This has been demonstrated for any material in the plastic state by Tresca (see Todhunter and Pearson, *History of Elasticity*, vol. II. art. 247). It is implied in our hypothesis that maximum shear determines the elastic limit, and that the elastic limit coincides with the plastic yield point.

From these,
$$R_1 = 2S \log r + \text{const.},$$
$$\therefore\ R_1 = -P + 2S \log \frac{r}{a} \quad \text{(xiii)},$$
and
$$R_2 = -P + 2S\left(1 + \log \frac{r}{a}\right) \quad \text{(xiv)}.$$

3. *Longitudinal tension as component of plastic shear.*

Having found R_1 for the plastic region, we will now investigate the limitations of the assumption (xii) that
$$R_2 - R_1 = 2S.$$

To evaluate R_3, some datum of plastic yield is necessary. Hence the present investigation is not rigorous, but must itself rest on some more acceptable or more elementary assumption. [See footnote, p. 383.]

We shall see that if at any point in a section R_3 is a component of the plastic shear, then R_3 is a component at every point in the section; i.e.
$$R_3 - R_1 = 2S \text{ throughout.}$$
For suppose at some point A, R_3 is a component of the plastic shear. Of the corresponding plastic slide the component principal strains at A are plastic radial and longitudinal stretches. But since plane sections remain plane, the latter must be accompanied by equal longitudinal stretch throughout; i.e. plastic longitudinal stretch at A must be accompanied by equal longitudinal stretch at points, such as B, where the material is still elastic. Now the stretch of a fibre at B is
$$\frac{1}{E}\left[R_3 - \sigma(R_1 + R_2)\right];$$
and here we may apply the results (ix) and (x), if we take the symbols a, b, P therein to refer to the cylindrical elastic portion of the material. Thus
$$R_2 = P_\rho \frac{\rho^2}{b^2 - \rho^2}\left(1 + \frac{b^2}{r^2}\right),$$
and
$$R_1 + R_2 = 2P_\rho \frac{\rho^2}{b^2 - \rho^2} = R_2 \frac{2}{1 + \dfrac{b^2}{r^2}},$$

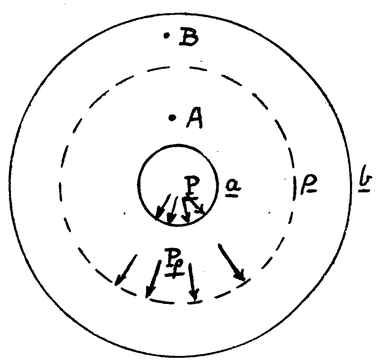

Fig. 3.

where P_ρ is the pressure of the plastic on the elastic region, and ρ is the radius of the common surface. Plastic yield at A is accompanied by increase of the hoop tension R_2 at B*; and hence our supposed plastic longitudinal stretch at A, while requiring at B an (equal) increase of

$$\frac{1}{E}\left[R_3 - \sigma(R_1 + R_2)\right],$$

is also accompanied by increase of $(R_1 + R_2)$. Therefore R_3 must increase.

But the total longitudinal force across a section is fixed (being $L + \pi a^2 P$); hence the longitudinal tension at our point A must be relieved. Hence if the longitudinal tension is a component of the plastic shear anywhere, it is so everywhere; and conversely, if at any place it is not a component of the plastic shear, then nowhere is it a component.

At any point, the condition that the longitudinal tension is not a component of the plastic shear is

$$R_3 < 2S + R_1,$$

i.e.

$$R_3 < -P + 2S\left(1 + \log\frac{r}{a}\right) \text{ [from (xiii)]}.$$

With this condition, we have

$$\text{Total pull across section} = \int_a^b R_3 \,.\, 2\pi r\,dr$$

$$< 2\pi\int_a^b (-P + 2S)\,r\,dr + 2\pi\int_a^b 2S\left(\log\frac{r}{a}\right) r\,dr$$

$$< \pi(-P + 2S)(b^2 - a^2) + 4\pi S a^2\left[\frac{r^2}{4a^2}\left(2\log\frac{r}{a} - 1\right)\right]_a^b$$

$$< \pi(-P + 2S)(b^2 - a^2) + \pi S\left[b^2\left(2\log\frac{b}{a} - 1\right) + a^2\right]$$

$$< \pi S\left[b^2\left(2\log\frac{b}{a} + 1\right) - a^2\right] - \pi P(b^2 - a^2).$$

Thus for any longitudinal force less than

$$\pi S\left[b^2\left(2\log\frac{b}{a} + 1\right) - a^2\right] - \pi P(b^2 - a^2),$$

whether due to internal pressure alone or with additional axial pull, the longitudinal tension R_3 does not enter into the plastic shear at any point; and for the plastic portion we may write

$$R_2 - R_1 = 2S.$$

The condition that such axial force is not produced by the internal pressure on the closed ends alone is

$$\pi P a^2 < \pi S\left[b^2\left(2\log\frac{b}{a} + 1\right) - a^2\right] - \pi P(b^2 - a^2),$$

i.e.

$$S\left[b^2\left(2\log\frac{b}{a} + 1\right) - a^2\right] - Pb^2 \text{ is positive.}$$

* This is an assumption, but one that is likely to be readily accepted. For a note on the assumptions which may be made regarding plastic behaviour, see section 15 at the end of this paper.

The greatest value of P occurs when the cylinder is all plastic, and is found from (xiii) by putting $r = b$, $R_1 = 0$. This gives

$$P = 2S \log \frac{b}{a} \quad \text{...(xv).}$$

So the above condition becomes that the coefficient of S in

$$(b^2 - a^2) S$$

is positive; and this is obviously always the case.

Hence whatever the dimensions of the cylinder, the internal pressure never produces a longitudinal tension large enough to be a component of the plastic shear. We shall continue to stipulate that any additional axial pull applied to our cylinder is insufficient to make $R_3 > R_2$.

4. *Cylinder plastic throughout.*

When the cylinder is plastic throughout, we have seen

$$\text{(xv)} \qquad P = 2S \log \frac{b}{a}.$$

Substituting for P in equations (xiii) and (xiv), we have

$$R_1 = 2S \log \frac{r}{b} \quad \text{...(xvi),}$$

$$R_2 = 2S \left(1 + \log \frac{r}{b}\right) \text{....................................(xvii).}$$

5. *Internal pressure removed and reapplied. Ideal case of unimpaired elasticity.*

The overstrain which the material has suffered will, actually, impair its elasticity. But let us approach the actual conditions by first considering the imaginary case in which, when the internal pressure is removed, the material returns to its old elastic state between limits of shear $+S$ and $-S$.

Assuming then that the material remains elastic, to find the stresses when the internal pressure is removed, we must subtract from the values for R_1 and R_2 in (xvi) and (xvii) the values for R_1 and R_2 in (ix) and (x) respectively, putting P in the last equal to $2S \log b/a$ [see (xv)]. Thus we get

$$R_1 = 2S \left[\log \frac{r}{b} - \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \log \frac{b}{a}\right] \quad \text{........................(xviii),}$$

$$R_2 = 2S \left[1 + \log \frac{r}{b} - \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \log \frac{b}{a}\right] \quad \text{.....................(xix).}$$

The maximum shear at each point is

$$\frac{R_2 - R_1}{2} = S\left(1 - \frac{2a^2}{b^2 - a^2} \cdot \frac{b^2}{r^2} \cdot \log \frac{b}{a}\right).$$

This is greatest where $r = b$, but can never exceed $+S$. It is least where $r = a$, and is there

$$S\left(1 - \frac{2b^2}{b^2 - a^2}\log\frac{b}{a}\right) \quad \text{(xx).}$$

Here the coefficient of S is negative for all values of b/a; it is zero when $b = a$.

If the expression (xx) for the reversed shear at the inner layer is not less (algebraically) than the elastic limit value $-S$, the stress-strain change during removal of the internal pressure is perfectly elastic; so that reapplication of the pressure merely restores the state of stress existing before the removal, without involving any further overstrain. Subsequent repetition of the process entails repetition of a perfectly elastic stress-strain cycle.

But if expression (xx) is less than $-S$, removal of the pressure involves a reversed overstrain at the inner layer; and on every subsequent application and removal, overstrain will occur.

6. *Influence of dimensions of cylinder.*

The condition that no overstrain shall occur on removal of the pressure is

$$S\left(1 - \frac{2b^2}{b^2 - a^2}\log\frac{b}{a}\right) \not< -S \quad \text{[see (xx)].}$$

The limiting ratio b/a is given by

$$\frac{b^2}{b^2 - a^2}\log\frac{b}{a} = 1,$$

$$\log\left(\frac{b}{a}\right)^2 = \frac{2(b^2 - a^2)}{b^2} = 2\left[1 - \frac{1}{\left(\frac{b}{a}\right)^2}\right],$$

$$\left(\frac{b}{a}\right)^2 = 5{\cdot}005 \text{ approx.,}$$

$$\frac{b}{a} = 2{\cdot}24.$$

Hence in a cylinder subjected to repeated applications and removals of that internal pressure which at first just renders it all plastic, overstrain will not continue to occur if $b/a < 2{\cdot}2$; whereas if $b/a > 2{\cdot}2$, overstrain will occur on each application and removal of the pressure, and must ultimately lead to rupture. Such a cylinder may be safely subjected to a fraction of that pressure needed to overstress the whole material. In the next section we proceed to determine the greatest safe pressure for any cylinder whose $b/a > 2{\cdot}2$.

7. *Cylinders for which* $\frac{b}{a} > 2{\cdot}2$.

In section 2 we found expressions for the stresses in the plastic portion due to internal pressure P; viz.

$$\text{(xiii)} \qquad R_1 = -P + 2S\log\frac{r}{a},$$

$$\text{(xiv)} \qquad R_2 = -P + 2S\left(1 + \log\frac{r}{a}\right).$$

When the pressure is removed, the material is again elastic, and to find the stresses we must subtract from the above the values of R_1 and R_2 given in section 1; viz.

$$\text{(ix)} \qquad R_1 = P\frac{a^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right),$$

$$\text{(x)} \qquad R_2 = P\frac{a^2}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right).$$

Assuming the material remains elastic, we thus get, as the stresses in the erstwhile plastic region, the expressions

$$R_1 = 2S\log\frac{r}{a} - P\frac{b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \qquad \text{(xxi)},$$

$$R_2 \doteq 2S\left(1+\log\frac{r}{a}\right) - P\frac{b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \qquad \text{(xxii)}.$$

The greatest shear due to these stresses is

$$\frac{R_2-R_1}{2} = S - P\frac{a^2b^2}{b^2-a^2}\cdot\frac{1}{r^2}.$$

As before (section 5), the pressure P may be applied and removed repeatedly with impunity only on the condition that this shear does not (arithmetically) exceed S; i.e.

$$S - P\frac{a^2b^2}{b^2-a^2}\cdot\frac{1}{r^2} \nless -S,$$

$$P\frac{a^2b^2}{b^2-a^2}\cdot\frac{1}{r^2} \ngtr 2S.$$

The L.H.S. is greatest where $r = a$, so the condition becomes

$$P\frac{b^2}{b^2-a^2} \ngtr 2S.$$

Thus the greatest permissible pressure is

$$P = 2S\frac{b^2-a^2}{b^2} \qquad \text{(xxiii)}.$$

This is just twice the pressure producing limit of elasticity of the inner layer of the fresh cylinder.

8. *Actual case: elasticity impaired by overstrain.*

In the last three sections we have supposed that the overstrain produced by the first application of the internal pressure did not impair the elasticity of the material for stresses of reduced value. Such is not actually the case.

A ductile material, after being overstressed, is approximately elastic for stress between zero and the old yield point; but it is not elastic for stress in the reverse direction. But although not elastic for the reversed stress, the material is not plastic. It will support any stress numerically less than the old yield stress, but with greater accompanying strain than that given by the straight line law. This is illustrated in figure 4, which is the stress-strain graph for a torsion test of an annealed thin tube of mild steel (1″ diam.,

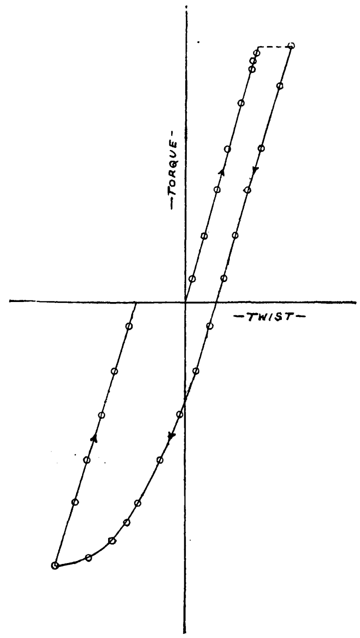

Fig. 4.

and ·022″ thick). Consequently equations (xviii) and (xix) represent a state of stress which does not actually obtain, but to which the actual state more or less closely approximates according as the stress-strain graph is more or less perfectly rectilinear.

Now elasticity impaired by overstrain is a condition from which recovery takes place. It is well known that prolonged rest at atmospheric temperature induces such recovery, and it has been shown by Prof. Muir* that a few minutes' immersion in boiling water hastens the recovery immensely. Muir's experiments included tests under reversed stress—overstrain in tension, followed by subjection to a temperature of 100° C., and then a test in compression—and here also recovery occurred. In a highly interesting paper published recently†, it is shown that the partial-plasticity introduced by overstrain gradually disappears as the material is subjected to cyclically varying stress, if the range of the stress is not greater than the "safe range" for resistance to fatigue. "In all cases cyclical permanent set" (i.e. the maximum width of the hysteresis loop) "was produced by the initial permanent extension" (i.e. non-cyclic change of length of specimen), "the amount of which decreased as the test proceeded and in some cases disappeared."

The kind of stress-strain cycle which is to be expected in our thick cylinder is indicated in figures 5 and 6. These refer to a cylinder of the critical thickness, $b/a = 2{\cdot}2$; they show the stress at the inside and outside layers during the first two cycles of internal

* *loc. cit.* p. 377.

† L. Bairstow, "The elastic limits of iron and steel under cyclical variations of stress," *Phil. Trans.* 210 A.

pressure; figure 5 refers to the ideal case of unimpaired elasticity, and figure 6 to the actual case of impaired but recovering elasticity.

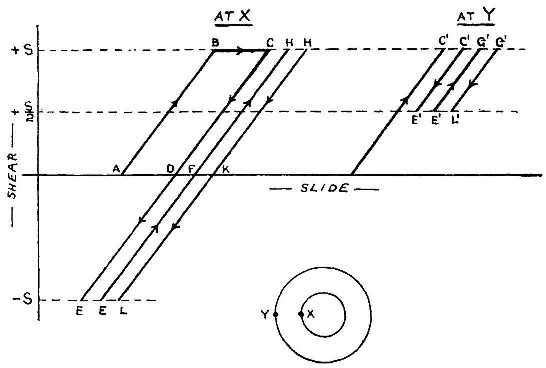

Fig. 5.

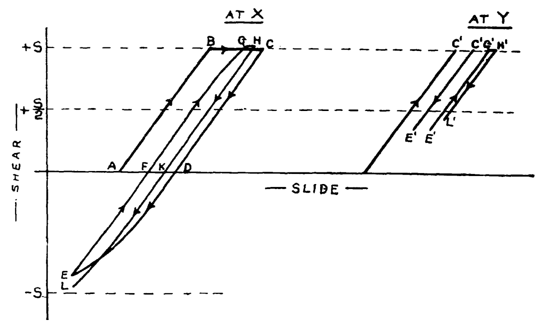

Fig. 6.

Explanation of figure 6.

At X.

AB, *BC*. Pressure first applied, till all plastic.

CD, *DE*. Pressure removed. *CD* straight; *DE* curving over to greater slide and smaller shear than in figure 5.

EF, *FG*, *GH*. Pressure reapplied. *EF* straight; *FG* curving over owing to impaired elasticity; *GH*, yield due to insufficient stress at *E*.

HK, *KL*. Pressure removed. *HK* straight; *KL* curving over, but less so than *DE*, so that *L* is nearer $-S$ than is *E*.

At Y.

$C'E'$. Comes below $S/2$ owing to curvature of *DE*.

$G'H'$. Yield due to bending of *FGH*.

$H'L'$. Comes below $S/2$, owing to curvature of *KL*, but less than does E'.

The overstrain produced by the repeated application and removal of the pressure is greatest at the inner layer, and figure 6 illustrates the kind of recovery which may be expected to occur there. The rapidity with which the cylinder regains its elasticity will be greatly enhanced by a brief subjection to 100° C. after each of the first few removals of pressure.

The gradual closing in of the hysteresis loop indicated in figure 6 can only be completed if the range of stress which would thereby be produced does not exceed the elastic range for repetition of stress. According to Bauschinger's hypothesis of the "natural elastic limits," this range would be the primitive elastic range $2S$ of our annealed material; and this is implied in the case of figure 6. Bairstow's experiments* show, so far as they go, that the limiting range for no hysteresis loop is approximately equal to the limiting range for resistance to fatigue. And experiments by the author (as yet unpublished) are corroborative in showing that the limiting range for resistance to fatigue, both in simple tension and in simple shear, of annealed mild steel rods, is approximately equal to the range between the primitive elastic limits.

* *loc. cit.* p. 387.

If, however, the safe range is less than $2S$, the recovery indicated in figure 6 can never be completed. In such a case the applied pressure must be reduced during applications subsequent to the first, from the all-plastic value to a value which shall not produce a range of stress at the inner layer exceeding the safe range for the material. Equations (xxi) and (xxii) enable the necessary calculations to be made. Bairstow's experiments render it probable that if only this reduced pressure were applied in the first instance, and the material were never made plastic throughout, adjustment would take place, and the same all-elastic distribution of stress would ultimately be established. It should be noted that, in the ideal case of figure 5, if b/a had been less than 2·2, the lower limits E, L of the stress at the inner layer would be above $-S$, so that the range from this lower limit to the upper limit $+S$ would itself be less than $2S$, even with the full all-plastic pressure.

In consequence of the discrepancy between the actual conditions and the ideal conditions of figure 5, we can with certainty only regard the critical thickness $b/a = 2{\cdot}2$ as referring to an upper limit to the possible strengthening effect of the overstrain. How closely that limit can be approached is a question which calls for experimental investigation.

9. *Collected Results.*

At elastic limit of inner layer of fresh cylinder:

$$\text{(xi)} \qquad P = S\,\frac{b^2 - a^2}{b^2} \quad \ldots\ldots\ldots\ldots(\alpha).$$

When cylinder is all plastic:

$$\text{(xv)} \qquad P = S \,.\, 2 \log \frac{b}{a} \quad \ldots\ldots\ldots\ldots(\beta).$$

Greatest safe repeated pressure:

$$\text{(xv)} \qquad \text{if } \frac{b}{a} < 2{\cdot}2, \;\; P = S \,.\, 2 \log \frac{b}{a},$$

$$\text{(xxiii)} \qquad \text{if } \frac{b}{a} > 2{\cdot}2, \;\; P = S \,.\, 2\,\frac{b - a^2}{b^2} \quad \ldots\ldots\ldots\ldots(\gamma).$$

These results are shown graphically in figure 7.

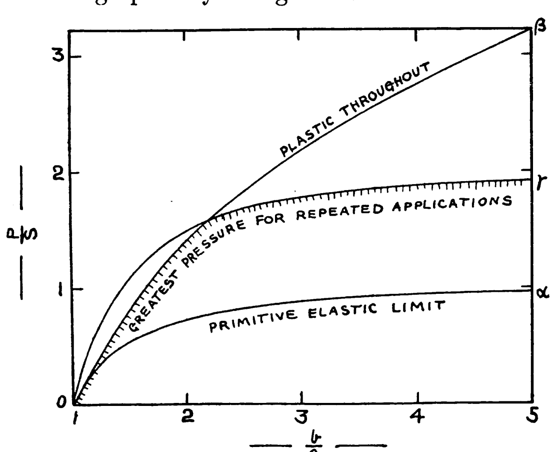

Fig. 7.

The curves show how much the strength of any cylinder may be increased above the primitive-elastic-limit strength. Thus the ratio

$$\frac{\text{safe repeated pressure}}{\text{primitive-elastic-limit pressure}}$$

increases from unity for thin cylinders, to 2 for $\frac{b}{a} = 2{\cdot}2$; after which it is constant at 2 for any greater thickness. Since the primitive-elastic-limit pressure increases only very slowly for thicknesses greater than $\frac{b}{a} = 3$, reaching the asymptotic value $\frac{P}{S} = 1$ when the cylinder is infinitely thick, very little can be gained by increasing the thickness above $\frac{b}{a} = 3$. The highest value of the greatest safe pressure for repeated applications that can be obtained, however thick the cylinder, is given by $\frac{P}{S} = 2$, thus making this pressure equal to the elastic limit tension of the material as found in the Testing Machine. If we take 50 tons/sq. in. as the highest elastic limit tension practicably obtainable in any steel that could be used for the purpose, then no hollow cylinder can be made to withstand the repeated application of an internal pressure greater than 50 tons/sq. in.

It should be noted that the ratio 2 : 1, as the limit of increase of strength of a cylinder by the treatment examined in this paper, applies to every method of increasing the strength by producing an initial stress; e.g. by the wire-winding of a gun. If this is not obvious it may be seen thus.

Suppose $\pm K$ are the elastic limit stresses for the material (whether maximum shear, tension, stretch, or any other function of the state of stress); and let P denote the greatest safe internal pressure. The application of P produces stress

$$P \,.\, f(r) \quad \text{(say)},$$

greatest at (say) $r = r_1$. Then if the cylinder is initially unstressed, P is determined by

$$P \,.\, f(r_1) = K;$$

i.e.
$$P = K/f(r_1).$$

Now we must not produce in the cylinder an initial stress-distribution in which the stress at any point is (algebraically) less than $-K$. If the initial stress at $r = r_1$ is made equal to $-K$, the resultant stress there is

$$P \,.\, f(r_1) - K;$$

and this reaches its maximum permissible value K when

$$P \,.\, f(r_1) = 2K;$$

i.e.
$$P = 2 \,.\, K/f(r_1).$$

Hence the strength can not be more than doubled.

10. *An experimental test.*

As illustration, in some degree, of the strengthening effect we have been considering, the following experiment may be cited.

The specimen was a freshly annealed, weldless, mild steel tube, of $\frac{3}{8}''$ bore. It was turned down outside for a length of about 15″ to diameter $\frac{5}{8}''$, and strains were measured over the middle of this portion.

Earlier tests under axial pull in the Testing Machine showed an elastic limit in simple tension of 30,300 lbs./sq. in. (13·6 tons/sq. in.), and this was closely followed by complete (plastic) yield.

The tube was subjected to internal oil-pressure, and readings of the accompanying diametral stretch across the middle section were taken with the "latometer," an instrument devised for the purpose. Longitudinal stretch, as measured by an Ewing's Extensometer, was much too small to be used for indication of yield under the internal pressure.

As previously found (sections 1 and 4), the inner layer of the tube begins to yield with an internal pressure

$$\text{(xi)} \qquad P = S\frac{b^2 - a^2}{b^2};$$

and when all is plastic

$$\text{(xv)} \qquad P = 2S \log \frac{b}{a};$$

where S is the yield point shear, which in this case, as given by the simple tension test, is $\frac{13 \cdot 6}{2} = 6 \cdot 8$ tons/sq. in. These expressions give:

primitive elastic limit when $P = 4 \cdot 35$ tons/sq. in.;
all plastic when $P = 7 \cdot 0$ tons/sq. in.

The first test showed departure from elasticity at a pressure of 4·5 tons/sq. in. The pressure was increased up to 6·6 tons/sq. in., and was then reduced to zero. The cycle of pressure was then repeated a number of times, between the same limits, 0 and 6·6 tons/sq. in. The curves plotted between internal pressure (ordinates) and latometer reading (abscissae) are shown in figure 8. They satisfactorily exhibit the great increase of strength after the first application, and the gradual establishment of a perfectly elastic condition.

11. *Suggested application.*

In the foregoing pages we have examined a process whereby a thick tube may be strengthened to resist internal pressure. The added strength is due to the formation, in the material of the tube, of an initial stress-distribution, such that the resultant stress when the pressure is applied is more uniform throughout the material than if the initial stress did not exist. Since about 1850 cannon have been strengthened by the formation of a similar initial distribution of stress: but methods of producing the stress have been to chill the core, in the old cast cannon; and to shrink on tube over tube, or to wind on layers of wire, in the modern construction. It is suggested that the method treated in this paper might be employed in place of, or in combination with, the existing methods. At least as much strengthening effect could be obtained, with the obvious advantage of greatly decreased complexity of construction.

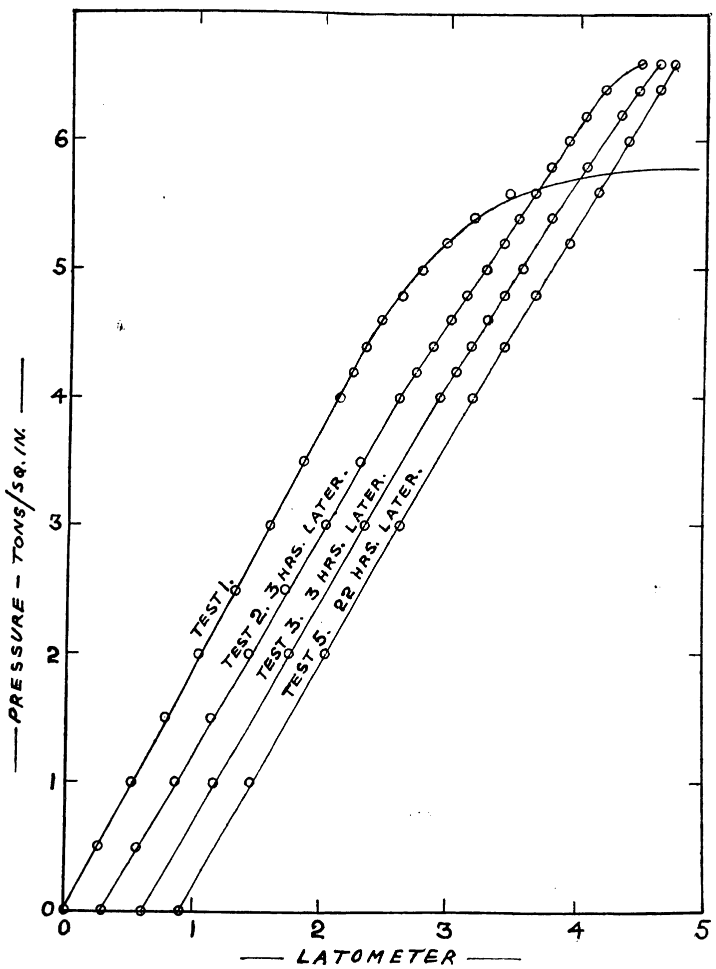

Fig. 8.

In modern large cannon, the greatest ratio b/a along the barrel is some 3 or 4; and the tubes are commonly of steel with an elastic limit of some 30 tons/sq. in. Hence the internal pressure, $P = 2S\,\frac{b^2 - a^2}{b^2}$ [(xxiii)], required to prepare the gun by the proposed method, would be not more than some 27 tons/sq. in. Fluid pressures of such magnitude are not too great to be manipulated with tolerable ease*.

* Messrs Schäffer and Budenberg catalogue hydraulic gauges reading up to 34 tons/sq. in.

It may be desirable to make the rifled inside surface of the gun of a harder steel than the main body; so that it might be necessary to use a liner incapable of taking the overstrain involved in the process contemplated. It would seem that this need be no obstacle, for modern wire guns are fitted with a renewable liner. If in the case of the wire gun a liner can be removed and another substituted, a liner could be fitted to a thick tube separately prepared by the overstraining process.

Since this process of producing the desired initial stress in a thick cylinder consists merely of applying and reapplying the appropriate internal pressure, it may be asked: Why go to any trouble at all? Will not the normal explosion of the charge in the gun do all that is required? Presumably experience has shown that it will not; and probably the reason lies in the different effects of a gradual and a sudden application of the pressure producing the overstrain.

That a gun-barrel is not of uniform thickness need cause no great difficulty. The pressure could be applied between a pair of pistons, movable along the length of the gun. Thus the tube could be treated piece by piece along the length, each piece receiving a pressure appropriate to the thickness there.

It may be that these remarks relating to the manufacture of cannon have little practical value, for the author is unacquainted with the technical processes involved. But the analysis should, at any rate, be of use in the design of hydraulic cylinders for high pressures.

APPENDIX.

We have already carried the investigation of the stresses as far as is necessary for the purpose of examining the strengthening effect, according to the shear theory of elastic breakdown, of overstraining the cylinder. It may be of interest, however, to pursue the problem further. We have seen, subject to a certain assumption, under what circumstances the longitudinal tension is or is not a component of the shear producing yield, but we have not, in general, found expressions for this tension. Nor have we found the radial and hoop tensions in the elastic part of the cylinder after some portion has become plastic. And further, we have not found any relation subsisting between the internal pressure and the depth of the plastic region. These points shall now receive attention.

12. *Stresses in the elastic region.*

Let $\rho =$ radius of plastic region.

For the elastic portion, $r=\rho$ to $r=b$, we have the former equations:

(vii) $$R_1 = A + \frac{B}{r^2} \qquad \text{(xxiv)},$$

(viii) $$R_2 = A - \frac{B}{r^2} \qquad \text{(xxv)},$$

(iv) Longitudinal stretch = constant,

or $$C = \frac{1}{E}[R_3 - 2\sigma A] \qquad \text{(xxvi)}.$$

A an B are constants across a section. They have not the values they had when the whole cylinder was elastic, for the boundary conditions have changed. To find A and B:

when $r=\rho$, $R_2 - R_1 = 2S$, when $r=b$, $R_1 = 0$,

$$\therefore\ 2S = -\frac{2B}{\rho^2}, \qquad \therefore\ 0 = A + \frac{B}{b^2},$$

$$B = -S\rho^2. \qquad A = \frac{S}{b^2}\rho^2.$$

Substituting these values of A and B in equations (xxiv), (xxv), and (xxvi), we have

$$R_1 = S\rho^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) \quad\text{......(xxvii)},$$

$$R_2 = S\rho^2\left(\frac{1}{b^2} + \frac{1}{r^2}\right) \quad\text{......(xxviii)},$$

$$R_3 = EC + \frac{2\sigma S}{b^2}\rho^2 \quad\text{......(xxix)}.$$

Thus R_1, R_2 are known as functions of P when the relation between ρ and P is known. This is found in the next section.

For R_3 we require also to know C in terms of ρ or P. An equation between ρ and C is given in section 14.

13. *Depth of plastic region.*

At the plastic-elastic surface, the values of R_1 on each side must be equal. Thus, equating the expressions for R_1 as given in the last section and in section 2 [(xiii)], and putting $r=\rho$, we have

$$S\rho^2\left(\frac{1}{b^2} - \frac{1}{\rho^2}\right) = -P + 2S\log\frac{\rho}{a},$$

$$P = S\left(1 + 2\log\frac{\rho}{a} - \frac{\rho^2}{b^2}\right) \quad\text{......(xxx)}.$$

To obtain ρ in terms of P, this equation must be solved empirically for any particular numerical values of a and b. Equations (xxvii) and (xxviii) then express R_1 and R_2 in terms of the internal pressure.

14. *Plastic region; longitudinal tension.*

We have already found for the plastic region

$$\text{(xiii)} \qquad R_1 = -P + 2S\log\frac{r}{a},$$

$$\text{(xiv)} \qquad R_2 = -P + 2S\left(1 + \log\frac{r}{a}\right).$$

To find the longitudinal tension R_3, we have further:

(a) and (b), the two plastic conditions of constant volume, and zero slide-velocity where shear is zero*; and

(c), the surface integral of longitudinal tension across a section equals the total axial pull.

These three principles may be shown to lead to the following expression for the longitudinal tension in the plastic region:

$$R_3 = -P + S\left(1 + 2\log\frac{r}{a}\right) + \frac{2r^2}{\rho^4 - a^4}\left[(\rho^2 - b^2)\left(EC + \frac{2\sigma S\rho^2}{b^2}\right) - (\rho^2 - a^2)\left(-P + S + 2S\log\frac{\rho}{a}\right) + Pa^2\right] \quad \text{......(xxxi).}$$

Thus for the plastic, as for the elastic [(xxix)], region, the longitudinal tension is known when we know C, the uniform longitudinal stretch across a section. A differential equation between C and ρ may be found, and is given below. Since no solution is offered, the steps are omitted by which it, and the above equation (xxxi), are obtained.

$$(\rho^2 - b^2)\left(EC + \frac{2\sigma S\rho^2}{b^2}\right) - (\rho^2 - a^2)\left(-P + S + 2S\log\frac{\rho}{a}\right) + Pa^2 = \frac{\frac{3}{4}(\rho^4 - a^4)\dfrac{dC}{d\rho}}{2\dfrac{S}{E}(1+\sigma)\rho^3\left(\dfrac{1-2\sigma}{b^2} + \dfrac{1}{\rho^2}\right) + \dfrac{1-2\sigma}{2}\rho^2\dfrac{dC}{d\rho}} \quad \text{..................(xxxii).}$$

15. *Note on the assumptions regarding plastic behaviour.*

In section 3, to show that internal pressure alone could never render the cylinder plastic in the longitudinal direction, we made the assumption that an increment of plastic yield at the inner region increases the hoop tensions in the still-elastic part. In section 14, following Saint-Venant, we assumed that plastic yield of an element makes no change in the volume of the element, and thus found an expression (xxxi) for the longitudinal tension in the plastic region. The result obtained in section 3 for the general case when the cylinder is plastic in part, may be derived independently for the particular case when it is plastic throughout, from the (subsequent) equations (xvi) and those of section 14. The argument is on these lines: Assuming there is no plastic longitudinal stretch—for we know this is so for some range of b/a, since in an indefinitely thin cylinder $R_3 = \frac{1}{2}R_2$—find R_3 in terms of P and b/a. Examine this expression [(xxxi)] to find for what values of b/a R_3 is everywhere less than R_2. We can do this because the term in the expression involving the undetermined constant C vanishes when the cylinder is plastic throughout. We thus find, as the condition that

$$R_3 - R_1 < 2S$$

* In *Comptes Rendus*, 1872, by the use of these principles, Saint-Venant found the stresses in a thick cylinder subjected to internal and external hydrostatic pressures sufficient to render it plastic throughout, the ends of the cylinder being between rigid fixed planes perpendicular to the axis. "This problem can be solved by introducing the velocities...of the points of the material. The principles which determine these velocities for a plastic material are: 1st, that there is no change in the volume of the element; and 2nd, that on each elementary area in the material the direction for which the shear is zero must be that for which the slide-velocity is zero. The latter principle involves the ratios of the half-differences of the tractions to the corresponding stretch-velocities being equal two and two." Todhunter and Pearson, *History of Elasticity*, vol. II. art. 261. See the next section for some remarks on the constant-volume assumption.

throughout the material, that

$$\frac{2b^2}{b^4 - a^4}\left[a^2\left(1 + 2\log\frac{b}{a}\right) - b^2\right] < 1;$$

which relation always obtains, since $b/a > 1$.

So far, then, our assumption of section 3, and Saint-Venant's constant-volume assumption, yield concordant results. But on a more searching comparison, it is seen that the two results are not the same. Having found in each case that with no added axial pull, there is never plastic longitudinal stretch, let us see how much added axial pull may be applied according to the respective assumptions. This pull, as found from section 3, is

$$\pi S(b^2 - a^2);$$

while from section 14 it is

$$\pi S\left[(b^2 - a^2) + \left(\frac{b^4 - a^4}{2b^2} - 2a^2\log\frac{b}{a}\right)\right].$$

The two expressions obviously do not agree.

Saint-Venant's assumption of constant volume during plastic strain seems to be applied without any sense of danger. As far as I have been able to see, it is recorded without comment in the *History of Elasticity** in the plain remark: "The material in the plastic state is treated as incompressible." The assumption is quite a common one, but the most specific reference to any experimental or other substantiation which I have noticed refers to the case of a specimen in the Testing Machine, and no figures are given†. If we imagine a material, otherwise maintained in the plastic condition, subjected also to a fluid pressure, we cannot but suppose that this pressure will produce a shrinkage of volume. So that although it may be established that in the case of simple tension, plastic deformation is accompanied by no change of volume, yet it would seem that further justification is wanted before we may confidently apply the principle to the present, or to Saint-Venant's, complicated problem.

A third assumption has been put forward‡, enabling us to find R_3 in the all-plastic condition. It is that the longitudinal stretch C is the same function of the tensions R_1, R_2, R_3 when the material is plastic under the shear $(R_2 - R_1)/2$, as when it is elastic; i.e. always

$$C = \frac{1}{E}[R_3 - \sigma(R_1 + R_2)].$$

This hypothesis yields the condition for no plastic longitudinal stretch

$$\frac{b^2}{b^2 - a^2}\log\frac{b}{a} < \frac{1-\sigma}{1-2\sigma};$$

i.e. $b/a < 4{\cdot}1$ if $\sigma = 1/4$,

or $< 7{\cdot}1$ if $\sigma = 1/3$.

Thus these three hypotheses all give different results. Fortunately for our investigation, they agree that in a cylinder, whose b/a does not exceed something between 4 and 7, and which is rendered plastic throughout by an internal pressure alone, the longitudinal tension is nowhere a component of the maximum shear.

* Todhunter and Pearson, *History of Elasticity*, vol. II. art. 246.

† "During the ductile elongation, the area of cross-section decreases in practically the same proportion that the length increases, or in other words, the volume of the material remains practically unchanged." A. Morley, *Strength of Materials*.

‡ By Prof. B. Hopkinson, to whom I am much indebted for valuable criticism and advice.

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ON THE DIFFERENTIATION OF FUNCTIONS DEFINED BY INTEGRALS.

BY

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XV. *On the Differentiation of Functions Defined by Integrals.*

By W. H. Young, Sc.D., F.R.S.

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§ 1. In a paper presented to the Society last year, and published recently in its *Transactions**, I discussed the problem of determining when the process of reversing the order of integration in a repeated integral is allowable, in other words the problem of integration under the sign of integration. In the present paper I propose to give a set of rules for determining when differentiation under the sign of integration is allowable, or more generally when it is allowable to make use of the usual formulae for the differential coefficient of a function defined by an integral, when the limits, as well as the integrand, involve a parameter. This matter has been treated, it is hardly necessary to say, with considerable care in the more recent English text-books; it is hoped, however, that the account here given will be found more complete, and more up to date, than any at present in existence. Several of the theorems to be found below are, it is believed, new in substance or in form, or are the extensions of known results. I have once more avoided the use of the ϵ-machinery and I have always stated my conditions without reference to the uniformity, or non-uniformity, of the convergence of the integrals, when these are improper, and accordingly defined as the limits of proper integrals. In this way greater generality has been secured, and in certain cases increased facility of application. For the sake of clearness I have, here and there, given not only the enunciations but also the proofs of certain results recently obtained by myself and published elsewhere. I have, of course, wherever desirable, made use of the results of the companion paper, above referred to, on "Change of Order of Integration." Our present problem cannot indeed, with our present methods of research, be successfully attacked, when the integrals are improper, without in certain cases making use of the theory of change of order of integration. This is due to the circumstance that, in applying the Theorem of the Mean to a function of two variables, which is being differentiated with respect to one of them, the θ which occurs is a function of the remaining variable; this limits to some extent the use we are able to make of this theorem. The present account will be found, however, to

* W. H. Young, "On Change of Order of Integration in an Improper Repeated Integral," *Camb. Phil. Trans.* 1909, vol. xxi. pp. 361—376.

make very much less use of the method of repeated integration than previous accounts. The number of results which we are able to get by the method of sequences alone is large and increasing, thanks to the facility with which we are now becoming able to treat the integration of such sequences. By employing this method, we are able to obtain results beyond the reach of the method of repeated integration, and to obtain other results somewhat more simply.

Our problem in its most general form is, as already remarked, that of finding the differential coefficient of an integral with respect to a parameter when the limits of integration, as well as the integrand, are functions of that parameter. We begin accordingly by shewing how to reduce this problem to that of differentiation under the sign of integration, properly so called. The theorems given in this connection shew that all turns on the existence, and in certain cases on the continuity, of the function constituted by the partial differential coefficient of the integral with respect to the parameter in the integrand. After these preliminaries our main problem then becomes that of determining when

$$\int_a^b f_y(x, y)\, dx \qquad \text{(I)},$$

supposed to exist either over the whole interval (a, b), or when taken over all but a set of points of content zero, and supposing the limits of integration to be fixed, is the differential coefficient with respect to y of $\int_a^b f(x, y)\, dx$.

The nature of the integral (I) depends on the boundedness or unboundedness of the integrand $f_y(x, y)$, and on the finitude or non-finitude of the limits of integration. Evidently it is sufficient to assume that the inferior limit of integration is always finite, for if not, the integral can always be expressed as the sum of two others in each of which the inferior limit is finite. The method of sequences enables us, however, (i) to obtain a large number of general results applicable in all cases, (ii) to reduce the case in which the superior limit is infinite to that in which it is finite. I am therefore naturally led, in discussing this main problem, to give, in the first instance, an account of the two distinct methods of sequences and repeated integration, and then to develope the general results in question. When this has been done, the discussion of the separate cases necessarily loses a certain amount of its interest, owing to the comparative paucity in some of the cases of the results which are left. The case of finite limits of integration and unbounded integrand requires, on the other hand, a number of investigations not included in the general theorems. For the convenience of the reader, I have accordingly grouped all such remaining theorems under the respective headings, Bounded Integrand and Finite Limits, Bounded Integrand and Infinite Limits, Unbounded Integrand and Infinite Limits.

It may perhaps not be amiss to remark that the integrals considered are almost always Lebesgue integrals, so that all the theorems obtained hold for absolutely convergent improper integrals, properly so-called, as particular cases; they do not, however, hold for Harnack integrals, at least in the cases in which the integrand is unbounded. I have, however, given one or two theorems for Harnack integrals.

I have not thought it necessary on this occasion to give illustrative examples. On the other hand, besides treating the general theory rather fully, I have considered various special cases which may present themselves. I have, moreover, shewn how the theory may be employed to obtain sufficient conditions for the equality of $\frac{d^2u}{dxdy}$ and $\frac{d^2u}{dydx}$. As examples of sets of conditions obtained in this way which do not involve in their statement the concept of integration, we have the following:

If throughout a fundamental rectangle $\frac{du}{dx}$, $\frac{du}{dy}$ and $\frac{d}{dy}\left(\frac{du}{dx}\right)$ exist and are bounded, and if $\frac{d}{dy}\left(\frac{du}{dx}\right)$ has the property of being a differential coefficient with respect to x (e.g. if it is continuous with respect to x), then $\frac{d}{dx}\left(\frac{du}{dy}\right)$ exists and is equal to $\frac{d}{dy}\left(\frac{du}{dx}\right)$ everywhere.

If $\frac{du}{dx}$ and $\frac{du}{dy}$ exist, while the former is a continuous function of y and the latter a continuous function of x, and if $\frac{d}{dy}\left(\frac{du}{dx}\right)$ exists and is, except in the neighbourhood of a countable set of points, a bounded function of the ensemble (x, y), then the necessary and sufficient condition that $\frac{d}{dx}\left(\frac{du}{dy}\right)$ should exist, and be equal to it, is that $\frac{d}{dy}\left(\frac{du}{dx}\right)$ should have the property of being a differential coefficient with respect to x.

Theorems including these as special cases, and others involving in their statement the concept of integration, are given.

§§ 2, 3. *Preliminary; on the Continuity of a Function defined by an Integral.*

§ 2. It will be convenient in this and the following article to make some remarks with regard to the nature of the function of the variables (y, a, b) defined by an expression of the form $\int_a^b f(x, y)\,dx$. Such a function is, by the ordinary theory of Harnack, Lebesgue and Harnack-Lebesgue integrals, wherever it exists, a continuous function of a, a continuous function of b, and, since it is the sum of two functions each containing a and b separately, a continuous function of the ensemble (a, b). That it is a continuous function of y in the case when the integrand $f(x, y)$ is a continuous function of the ensemble (x, y) has been long known; comparatively recently it has been recognised, thanks to the work of Lebesgue, that this is also the case if the integrand is only a bounded function of the ensemble (x, y), provided that it is a continuous function of y alone; or more generally, it is continuous with respect to y at $y = y_0$, if the integrand is bounded and is continuous with respect to y at $y = y_0$, except at most for a set of values of x of zero content. If the integrand is unbounded, we can no longer assert the continuity of the integral with respect to y, it may, however, happen that, either for all values of the limits of integration a and b, or for particular values, the integral is a continuous function of y, when the integrand is unbounded, but has a suitable special form. Various rules have been given enabling us to recognise, under certain circumstances, such continuity. Thus the integral is continuous if the integrand is a monotone continuous function of y, whether, or no, it is continuous with respect to (x, y), and whether, or no, the limits of integration are finite. Again, if we know that the integral of the positive unbounded function $v(x, y)$, taken between finite limits, is a continuous function of y, we can assert that the same is true when the integrand is replaced by the product of this function and any other function $u(x, y)$ which is bounded and has its discontinuities with respect to (x, y) forming for each fixed value of y a set of zero content*. Again, if the integrals of $g(x, y)$ and of $h(x, y)$ are continuous functions of y, the same is true of the integral of a function $f(x, y)$, which lies everywhere between $g(x, y)$ and $h(x, y)$, provided all three functions are continuous with respect to y, except for a set of values of x of zero content†.

§ 3. We have next to consider the continuity of the integral regarded as a function of the ensemble (y, a, b). In the theorem about to be given we suppose that the continuity with respect to y is known beforehand, at the point $y = y_0$ considered, whatever be the values of the limits of integration, but we do not assume in cases (i) and (ii) the continuity for other values of y. In case (iii), we assume that this continuity still continues to hold in a suitable neighbourhood of $y = y_0$.

THEOREM. *If* $\int_a^b f(x, y)\, dx$ *is continuous with respect to* y, *it is continuous with respect to the ensemble* (y, a, b) *at the point considered, provided a certain rectangle* $(a, y;\ b, y+c)$ *can be assigned, or provided two-dimensional neighbourhoods of the points* (a, y) *and* (b, y) *can be assigned, in which one of the following conditions holds :*

(i) $f(x, y)$ *is a bounded function of* (x, y);

(ii) $f(x, y)$ *is a monotone function of* (x, y);

(iii) $f(x, y)$ *does not change sign, and is continuous with respect to* y, *not only at the point* y *considered, but also in the neighbourhood.*

For

$$\int_{a+h}^{b+H} f(x, y+k)\, dx = \int_a^b f(x, y+k)\, dx + \int_b^{b+H} f(x, y+k)\, dx - \int_a^{a+h} f(x, y+k)\, dx.$$

Since $\int_a^b f(x, y)\, dx$ is continuous with respect to y, the first integral on the right has the unique limit $\int_a^b f(x, y)\, dx$, when the ensemble (k, h, H) approaches $(0, 0, 0)$ in any manner whatever. It will be seen that in each of the cases (i), (ii) and (iii), the two remaining integrals on the right have each the unique limit zero.

For in case (i), H and k being suitably restricted, so that the point $(x, y+k)$ lies in the rectangle $(a, y;\ b, y+c)$, or in the two-dimensional neighbourhood of (b, y) which can be assigned, the second integral on the right is less than HU, where U is any quantity greater than the upper bound of the numerical values of $f(x, y)$ in that rectangle, or neighbourhood. Since U is fixed, this shews that the integral in question has zero as unique limit, when H decreases down to zero as limit, whatever k may be, less than c, or than a certain value determined by the

* For these theorems see W. H. Young, "On Parametric Integration," *Monatshefte für Mathematik und Physik*, 1910.

† This follows from results given in a paper entitled "On Semi-integrals and Oscillating Successions of Functions," 1910, presented by the author to the London Mathematical Society.

neighbourhood. Thus however the ensemble (k, h, H) approaches $(0, 0, 0)$, the second integral has, in case (i), the unique limit zero.

In case (ii) $\int_b^{b+H} f(x, y+k)\,dx$ is a monotone function of k, since the integrand is a monotone function of k, if k and H are sufficiently small. This integral is also by hypothesis continuous with respect to k at $k=0$, and it is a continuous function of H, therefore by a known theorem, easily proved*, it is continuous with respect to the ensemble (H, k), and has therefore the unique limit zero, since when k is fixed and H approaches zero, this is the case.

In case (iii) $\int_b^{b+H} f(x, y+k)\,dx$ is a monotone function of H, and, since it is also continuous with respect to H, and, by hypothesis, with respect to k, it is, as before, continuous with respect to the ensemble (H, k), and has, therefore, the unique limit zero.

Thus, in each of the three cases, $\int_b^{b+H} f(x, y+k)\,dx$ has the unique limit zero, and, similarly, $\int_a^{a+h} f(x, y+k)\,dx$ has the unique limit zero. Hence the left-hand side of the identity at the beginning of the present proof has $\int_a^b f(x, y)\,dx$ as unique limit, when the ensemble (k, h, H) approaches $(0, 0, 0)$ in any manner, that is to say $\int_a^b f(x, y)\,dx$ is a continuous function of the ensemble (y, a, b), which proves the theorem.

§§ 4, 5. *On the Differentiability of the Integral* $\int_a^b f(x, y)\,dx$ *with respect to each of its Limits of Integration, and on its Possession of a Differential, when regarded as a Function of* (y, a, b).

§ 4. As already remarked, it follows from the very definition of an integral, that it is a continuous function of the ensemble of its limits of integration; it by no means necessarily, however, possesses a differential coefficient† with respect to either of them; still less does it necessarily possess a differential with respect to the ensemble (y, a, b). In this and the next article I propose to give theorems shewing us under what circumstances the differential coefficients with respect to the limits of integration a and b exist, and under what circumstances the integral has a differential with respect to the ensemble (y, a, b), when it is known to possess a differential coefficient with respect to y. It should be remarked that, when we are considering properties relative to one limit, there is nothing to prevent the other limit being infinite.

There are two main circumstances under which we can assert the differentiability of $\int_a^b f(x, y)\,dx$ with regard to its upper limit of integration b.

THEOREM. (i) *If $f(x, y)$ is for the value of y considered continuous with respect to x at $x=b$, then, $F(y, a, b)$ denoting the integral $\int_a^b f(x, y)\,dx$, we can assert that $\frac{\delta F}{\delta b}$ exists and is equal to $f(b, y)$;*

* W. H. Young, "A Note on Monotone Functions," *Quart. Journ.* 1909, p. 84, § 7; "On Uniform Oscillation," *Camb. Phil. Trans.* 1909, vol. XXI. p. 255, § 13.

† For brevity we shall not usually specify whether the differential coefficient we are considering is a right-handed one, or a left-handed one, or the differential coefficient *par excellence*. It will always be obvious from the context which of these is intended, whenever anything is to be gained by their separate consideration.

(ii) *if a neighbourhood of the point $x=b$ can be assigned, such that, for the value of y considered, $f(x, y)$ is throughout that neighbourhood a finite and summable differential coefficient with respect to x, then the same is true.*

If neither of these conditions holds, then we can only assert that the derivates of F with respect to b lie between the upper and lower limits $\phi(b, y)$ and $\psi(b, y)$ of $f(x, y)$ at the point (b, y).

To prove (i) let U denote the upper bound of $f(x, y)$ in the interval

$$b \leqslant x \leqslant b+H,$$

y being fixed. Then

$$\{F(y, a, b+H) - F(y, a, b)\}/H = \int_b^{b+H} f(x, y)\, dx/H \leqslant \int_b^{b+H} U\, dx/H \leqslant U.$$

Letting H decrease indefinitely, U has $f(b, y)$ as unique limit, if $f(x, y)$ is continuous with respect to x at (b, y), and otherwise has $\phi(b, y)$ as unique limit. Hence, in the former case all the derivates of $F \leqslant f(b, y)$, and, in the latter case, $\leqslant \phi(b, y)$.

Similarly, taking the lower instead of the upper bound, all the derivates of $F \geqslant f(b, y)$ in the former case, and, in the latter case, $\geqslant \psi(b, y)$.

Combining these results the first and last statements of the theorem follow.

To prove the theorem in case (ii) we remark that if

$$f(x, y) = \frac{\partial}{\partial x} U(x, y),$$

since $f(x, y)$ is finite and summable, we have, by Lebesgue's Theorem,

$$F(y, a, b) = \int_a^b f(x, y)\, dx = U(b, y) - U(a, y),$$

whence differentiating

$$\frac{\partial F}{\partial b} = f(b, y).$$

§ 5. We have now to examine the existence of a differential. We use the following set of sufficient conditions*:

If $F(x, y, z)$ has partial differential coefficients F_x, F_y and F_z, then it also possesses a differential with respect to the ensemble (x, y, z) provided F_x is continuous with respect to (x, y, z) and F_y is continuous with respect to (y, z) at the point considered respectively.

According to the *rôles* which we ascribe to the variables y, a, b, we deduce from these conditions the following alternative theorems.

THEOREM. *If $f(x, y)$ is continuous with respect to (x, y) at the points (a, y), (b, y), and if $F(y, a, b)$, that is $\int_a^b f(x, y)\, dx$, possesses a differential coefficient F_y with respect to y at the point y considered, then F has a differential with respect to the ensemble (y, a, b).*

* W. H. Young, "The Fundamental Theorems of the Differential Calculus," *Cambridge Tracts in Mathematics and Mathematical Physics*, 1910, No. 11, p. 31.

For F_y then exists, and, by the Theorem of § 4, F_b exists, and, being equal to $f(b, y)$, is by hypothesis continuous with respect to (b, y). Similarly F_a exists, and is continuous with respect to (a, y), and independent of b, so that F_a is continuous with respect to (y, a, b). Hence, by the conditions above stated, F has a differential with respect to (y, a, b).

THEOREM. *If $f(x, y)$ is, in neighbourhoods of the points $x=a$, $x=b$, for the value of y considered, a finite and summable differential coefficient with respect to x, and is continuous with respect to x at the point (b, y); and if $F(y, a, b)$ has a differential coefficient F_y with respect to y which is continuous with respect to (y, a, b) at the point (y, a, b), then F possesses a differential with respect to the ensemble (y, a, b).*

For F_a exists, and F_b exists and is continuous with respect to (a, b), by § 4, while by hypothesis F_y exists and is continuous with respect to (y, a, b). Hence, by the conditions stated above, F has a differential with respect to (y, a, b).

COR. *Under the circumstances mentioned in this or the preceding theorem, if y, a and b are all differentiable functions of a single variable t,*

$$\frac{\delta F}{\delta t} = \dot{y} F_y - \dot{a} f(a, y) + \dot{b} f(b, y),$$

the dots denoting differentiation with respect to t.

§§ 6, 7. *Explanatory Remarks.*

§ 6. It will be noticed that in § 4 we have adopted as one of our assumptions the condition that the integrand should be a differential coefficient with respect to one of the variables. It will be found that in the sequel similar conditions are a common occurrence. In the present state of our knowledge we cannot give any but very special sets* of sufficient conditions that a function should have the property of being a differential coefficient, so that the introduction of a condition of this form is not often of direct use in practice. Its importance in theory is, however, not affected by these considerations, and it has on other grounds seemed to me desirable, that, when we are concerned with a neighbourhood, it is the fact of a function being a differential coefficient, and not its continuity, that we usually require to assume. Evidently a continuous function is a differential coefficient, so that the usual statement of the corresponding portion of a set of conditions follows as a corollary.

We shall in what follows also occasionally assume as one of our conditions that a function with which we are concerned is an indefinite integral (Lebesgue integral). In this way once more our statements will gain in generality. Necessary and sufficient conditions have been obtained by Lebesgue and Vitali that a given function should be an integral, though these conditions are not always easily applied. Moreover a very useful sufficient condition is known, namely that one of the derivates of the function should be summable, and finite except at most at a reducible† set of points. One important property

* For theorems giving cases in which the product of a differential coefficient by a continuous function is a differential coefficient, see a forthcoming paper by the author entitled "Note on the Property of Being a Differential Coefficient," *Proc. L. M. S.*

† [Note added May 15th, 1911. *Countable* may here, and throughout the present paper, be substituted for *reducible*: see my "Note on the Fundamental Theorem of Integration," Nov. 4, 1910, *Proc. Camb. Phil. Soc.* XVI. pp. 35—38.]

of a function which is an integral we shall employ, namely that which asserts that it is the integral of its derivate. It is perhaps well to add that it may sometimes be convenient to employ the fact that if $g(y)$ is a finite and summable differential coefficient, it is necessarily the differential coefficient of its integral, as a means of determining whether $g(y)$ is, or is not, a differential coefficient. In particular this method is applicable when $g(y)$ is a bounded function*.

In certain cases, moreover, it may be sufficient to take as our condition that the function in question should be a finite derivate, which at the point considered is a differential coefficient. The reader acquainted with the theory of derivates will have no difficulty in generalising in a suitable manner the conditions given below, in which, for the sake of brevity, the case of a differential coefficient is alone usually contemplated.

§ 7. The reader should also note that a greater generality will be secured for our results if we do not always require from the expressions defining functions which occur in our equations, that these functions should be one-valued. Thus we may regard the repeated integrals of a function of two variables as existing when they have unique values, while the simple integrals have not; and, by the term *derivate* we may understand, where it is advantageous to do so, the many-valued function which becomes one-valued only where a differential coefficient exists. More generally, we may regard it as denoting any of the various one-valued, or many-valued functions obtainable by selecting at each point certain of the values between the upper and lower derivates at that point. This will be specially advantageous when the function in question is one-valued, except for a set of values of the variable of zero content. There is sometimes an advantage in specially calling attention to this possibility in the enunciation of theorems, as it removes any doubt from the mind of the reader as to requirements at certain exceptional points of the interval considered, e.g. at the proper infinite end-point, if it is an infinite interval.

It may perhaps be well to add that we shall always use the notation $\int_a^b u(x)\,dx$ in the case in which $u(x)$ has no meaning at certain points of the interval (a, b), forming at most a set of content zero, as denoting the integral of $u(x)$ taken over those points at which it does exist, or, which is the same thing, as the integral over the whole interval of the function which agrees with $u(x)$ where the latter exists and is elsewhere zero.

§§ 8—18. *On the Methods employed in the Sequel, and certain Rules of a general Nature.*

§ 8. Two quite distinct methods may be employed with advantage, the method of sequences and that of repeated integration. Considering first the former method, which though it suggests itself at once has not been sufficiently employed in existing accounts of the subject, it is evident we have to investigate the conditions under which we are at liberty to integrate term-by-term the succession, or sequence,

$$\frac{f(x, y+k)-f(x, y)}{k} \quad \text{.......................................} \quad (1),$$

over the finite, or infinite, closed interval (a, b) of values of x.

* "Function" is throughout used for measurable function; a bounded measurable function is always summable.

Whenever

(i) term-by-term integration of this succession is possible,

(ii) the upper and lower functions of the succession differ only at a set of content zero of values of x,

we can evidently assert that, at the point y considered, $\int_a^b f(x, y)\,dx$ has a differential coefficient with respect to y, and that this differential coefficient is $\int_a^b f_y\,dx$, where the latter integral is to be understood to have the usual meaning in the case when f_y does not exist, that is (1) has not an unique limit, at a set of exceptional values of x of content zero.

Thus we might at once dismiss the subject from this point of view as included in the general theory. It is desirable, however, to select those rules for term-by-term integration of a succession, or sequence, which are here most easily applied; and apart from this the facts that (a) the upper and lower bounds of the succession, or sequence, are the same as those of the limiting functions, or function, and (b) in the case of a sequence, each of the functions of the sequence can be expressed, by the Theorem of the Mean, in the form of the limiting function for a different value of y, enable us in certain cases to give a particularly simple form to the rules selected.

It will be noted that in the theorems obtained by this method we are able to prove both that $\int_a^b f_y\,dx$ is a differential coefficient and that it is the differential coefficient of $\int_a^b f(x, y)\,dx$. In the second method it will be found that we have to include the fact that $\int_a^b f_y\,dx$ is a differential coefficient among our assumptions before we can prove it is the differential coefficient in question.

§ 9. It has been usual until recently to lay great stress in the integration of sequences on uniform convergence. We may however entirely avoid all reference to this theory, and it is, on the whole, most convenient to do so, as our results gain in this way more generality and greater flexibility. The main theorem in the theory of integration of sequences, which enables us to dispense with such considerations, is the following:

Theorem. *A sequence $f_1(x)$, $f_2(x)$, ... is certainly integrable term-by-term if two other sequences can be found which are integrable term-by-term, say*

$$g_1(x),\ g_2(x),\ \dots \text{ and } h_1(x),\ h_2(x),\ \dots$$

such that, for all values of n,

$$g_n(x) \leqslant f_n(x) \leqslant h_n(x).$$

Cor. *If a summable function of (x) can be found, such that for all values of n and x*

$$|f_n(x)| \leqslant U(x),$$

then the sequence of which the general term is $f_n(x)$ is integrable term-by-term.

For a proof of the theorem the reader is referred to a paper on "Semi-integrals and Oscillating Successions of Functions," by the author.

It is particularly to be noticed that the range of integration is not necessarily finite; the theorem still holds when it is infinite.

The special case of the corollary when $U(x)$ is a constant, the functions of the sequence are continuous, and the range of integration is finite, was first proved by Osgood, while Lebesgue's celebrated extension consisted in dropping the condition of continuity.

§ 10. It follows at once from the preceding article that, if $f_y(x, y)$ is a bounded function of the ensemble (x, y) in the rectangle determined by the range of integration with respect to x and a suitable neighbourhood with respect to y, our integral $\int_a^b f_y\,dx$ is a differential coefficient, namely that of $\int_a^b f(x, y)\,dx$, in the case in which the limits of integration are finite. We have however the following striking theorem, applying to all the four cases mentioned in the introduction.

THEOREM. *If a function $U(x)$ can be found, such that the modulus of the bounded, or unbounded, function $f_y(x, y)$ is $\leqslant U(x)$, where $U(x)$ is itself summable in the finite or infinite interval (a, b), and the inequality holds in the whole finite or infinite rectangle*

$$a \leqslant x \leqslant b, \quad y_0 \leqslant y \leqslant y_0 + k,$$

then

$$\int_a^b f_y(x, y)\,dx = \frac{d}{dy}\int_a^b f(x, y)\,dx.$$

In the sequence $\{f(x, y+k) - f(x, y)\}/k$ every term for every value of x, y and k considered, is numerically $\leqslant U(x)$, by the Theorem of the Mean, or the well-known theorem of the bounds of the derivates. Hence term-by-term integration with respect to x is allowable, by a known theorem*, which proves the theorem.

§ 11. A theorem of a general character which follows at once by the method of sequences, equally whether the limits of integration are finite, or infinite, and the integrand is bounded or not, is the following:

THEOREM†. *If $f(x, y)$ is less than $h(x, y)$, and is greater than $g(x, y)$, for all values of x and y concerned, and in each case the difference is a monotone increasing function of y, then, provided differentiation under the sign of integration is allowable when the integrand is either g or h, it is allowable when the integrand is f.*

We have, since

$$f(x, y) - g(x, y) \text{ and } h(x, y) - f(x, y)$$

are monotone increasing functions of y,

$$\frac{g(x, y+k) - g(x, y)}{k} \leqslant \frac{f(x, y+k) - f(x, y)}{k} \leqslant \frac{h(x, y+k) - h(x, y)}{k}.$$

* "Semi-integrals and Oscillating Successions of Functions," § 32. † *Ibid.* § 36.

Now

$$\mathop{\mathrm{Lt}}_{k=0}\int\{g(x, y+k)-g(x, y)\}\,dx/k=\frac{d}{dy}\int g(x, y)\,dx=\int\frac{dg}{dy}\,dx=\int\mathop{\mathrm{Lt}}_{k=0}\{g(x, y+k)-g(x, y)\}\,dx/k,$$

with a similar set of equalities involving h in place of g. Hence, by the theorem of § 9,

$$\mathop{\mathrm{Lt}}_{k=0}\int\{f(x, y+k)-f(x, y)\}\,dx/k=\int\mathop{\mathrm{Lt}}_{k=0}\{f(x, y+k)-f(x, y)\}\,dx/k,$$

that is

$$\frac{d}{dy}\int f(x, y)\,dx=\int\frac{df}{dy}\,dx,$$

which proves the theorem.

§ 12. THEOREM. A special case of the theorem of § 10 is the following: *if $f_y(x, y)$ is a bounded function of (x, y), and $h(x)$ is a summable function of x in the finite or infinite interval (a, b),*

$$\int_a^b f_y(x, y)\,h(x)\,dx=\frac{d}{dy}\int_a^b f(x, y)\,h(x)\,dx.$$

In fact the sequence obtained by multiplying $\{f(x, y+k)-f(x, y)\}/k$ by $h(x)$ has the property that every one of its terms is numerically less than $A\,|h(x)|$, where A is any quantity greater than the upper bound of the modulus of f_y; the sequence is accordingly integrable term-by-term.

§ 13. The result just obtained is, as we have seen, an immediate consequence of a theorem giving a sufficient condition for the integrability of a sequence. From another special case in which the sequence is integrable*, we have the following theorem, which holds equally, like the one just mentioned, whether the integrand $f_y(x, y)$ is bounded or unbounded, and whether the limits of integration are finite or infinite.

THEOREM. *If $f_y(x, y)$ is a monotone function of y, then*

$$\int_a^b f_y(x, y)\,dx=\frac{d}{dy}\int_a^b f(x, y)\,dx.$$

For, by the Theorem of the Mean,

$$\int_a^b\{f(x, y_0+k)-f(x, y_0)\}\,dx/k=\int_a^b f_y(x, y_0+\theta k)\,dx,$$

where θ denotes a function of x, whose value lies between 0 and 1.

But, by hypothesis, f_y is a monotone function of y, therefore for each value of x, $f_y(x, y_0+\theta k)$ lies between $f_y(x, y_0)$ and $f_y(x, y_0+k)$, so that the integral of the former function between fixed limits lies between the integrals of the two latter functions between the same limits. Hence, by the preceding equality,

$$\int_a^b\{f(x, y_0+k)-f(x, y_0)\}\,dx/k \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

lies between

$$\int_a^b f_y(x, y_0)\,dx \text{ and } \int_a^b f_y(x, y_0+k)\,dx.$$

* W. H. Young, "On Parametric Integration," *Monatshefte f. Math. u. Physik*, 1910, p. 136, § 13.

Now f_y, being a right-hand differential coefficient, is one of the limits of its values on the right, and is therefore the unique limit, since, by hypothesis, f_y is monotone. Thus f_y is continuous with respect to y, and therefore the same is true of its parametric integral*. In other words, letting k diminish towards zero in any manner, $\int_a^b f_y(x, y_0 + k)\,dx$ has the unique limit $\int_a^b f_y(x, y_0)\,dx$. The same must therefore be true of the integral (A), which lies between $\int_a^b f_y(x, y_0 + k)\,dx$ and its limit. Thus $\int_a^b f(x, y)\,dx$ has a differential coefficient at $y = y_0$, and this differential coefficient is $\int_a^b f_y(x, y_0)\,dx$, which proves the theorem.

§ 14. In the application of the method of sequences the following theorem will be found to be of importance:

THEOREM. *If $f_n(x)$ be the general term of a sequence of functions whose differential coefficients are finite except at a reducible set of points and form an integrable succession which converges, except at a set of content zero, to an unique limiting function, which is itself a summable differential coefficient, and finite except at a reducible set of points, then*

$$\frac{d}{dx}\{\underset{n=\infty}{\mathrm{Lt}}\, f_n(x)\} = \underset{n=\infty}{\mathrm{Lt}}\, \frac{d}{dx} f_n(x).$$

For, denoting the limiting function of the sequence of functions $f_n(x)$ by $f(x)$, and by $g_n(x)$ the differential coefficient of $f_n(x)$, and by $g(x)$ the unique limiting function referred to in the statement of the theorem, we have,

$$\underset{n=\infty}{\mathrm{Lt}}\, \frac{1}{h}\{f_n(x+h) - f_n(x)\} = \underset{n=\infty}{\mathrm{Lt}}\, \frac{1}{h}\int_x^{x+h} g_n(x)\,dx = \frac{1}{h}\int_x^{x+h} \underset{n=\infty}{\mathrm{Lt}}\, g_n(x)\,dx,$$

these two steps being allowable, since the g-succession is integrable.

Hence
$$\{f(x+h) - f(x)\}/h = \int_x^{x+h} g(x)\,dx/h.$$

But since $g(x)$ is a summable and finite differential coefficient except at a reducible set of points, it is the differential coefficient of its integral. Hence, proceeding to the limit zero with h, we have

$$f'(x) = g(x),$$

which proves the theorem.

§ 15. We are now able to prove the following general theorem, which enables us to pass from the case in which the limits of integration are finite, to that in which one of them is infinite.

THEOREM. *If the equation*

$$\int_a^b f_y\,dx = \frac{d}{dy}\int_a^b f(x, y)\,dx$$

holds for all finite values of b, however great, or for a suitable sequence of values of b, having infinity as limit, then it holds also when we put $b = \infty$, provided only that

* *Loc. cit.* § 3.

(i) *the functions of y obtained by giving to b in the expression* $\int_a^b f_y \, dx$ *the suitable set of values in question, form an integrable sequence;*

(ii) $\int_a^\infty f_y \, dx$ *exists*, and is a summable differential coefficient throughout a neighbourhood of the point y considered.*

We have, in fact, $$\int_a^\infty f_y \, dx = \underset{b=\infty}{\mathrm{Lt}} \int_a^b f_y \, dx = \underset{b=\infty}{\mathrm{Lt}} \frac{d}{dy} \int_a^b f(x, y) \, dx,$$

by the hypothesis. Therefore, by the theorem of the preceding article, in virtue of the provisos (i) and (ii),

$$\int_a^\infty f_y \, dx = \frac{d}{dy} \underset{b=\infty}{\mathrm{Lt}} \int_a^b f(x, y) \, dx = \frac{d}{dy} \int_a^\infty f(x, y) \, dx.$$

We may replace the proviso (i) by any of the special provisos which secure the integrability of the sequence in question. E.g.

Cor. 1. *We may replace* (i) *by the proviso that* $\int_a^x f_y \, dx$ *is a bounded function of the ensemble* (x, y) *in the infinite rectangle* $a \leqslant x \leqslant \infty$, $y_0 \leqslant y \leqslant y_0 + k$.

Cor. 2. *We may replace* (i) *by the proviso that*

$$\left| \int_a^x f_y \, dx \right| < g(y),$$

for all values of x in the infinite interval $a \leqslant x \leqslant \infty$, *where* $g(y)$ *is summable.*

Note. No assumption is made in the above as to the nature of the integrals with respect to x, they may be equally well Harnack, Lebesgue, or Harnack-Lebesgue integrals.

§ 16. Passing to the second method, that of repeated integration, we have the following theorem, on which the method depends.

Theorem. *If*

(i) $f(x, y)$ *is a y-integral, except for a set of values of x of zero content,*

(ii) $\int_a^b f_y \, dx$ *exists, and is a summable differential coefficient, when regarded as a function of y, then the necessary and sufficient condition that throughout a certain range of values of y,*

$$y_0 \leqslant y \leqslant y_0 + k,$$

we may have $$\int_a^b f_y \, dx = \frac{d}{dy} \int_a^b f(x, y) \, dx,$$

is that f_y *should have its repeated integrals taken over the whole finite or infinite rectangle*

$$a \leqslant x \leqslant b, \quad y_0 \leqslant y \leqslant y_0 + k$$

equal in value.

For, except at the exceptional set of values of x, we have, by a theorem of Lebesgue's,

$$f(x, y) = \int f_y(x, y) \, dy,$$

the integral being taken over the set of points at which f_y exists.

* And is therefore finite, this being understood to be involved in the existence of an integral.

Hence, denoting $\int_a^b f(x, y)\, dx$ by $F(y)$, we have

$$\{F(y+k) - F(y)\}/k = \int_a^b dx \int_y^{y+k} f_y\, dy/k.$$

Now if $g(y)$ is the function of which $\int_a^b f_y(x, y)\, dx$ is, by hypothesis, the finite and summable differential coefficient, we may write

$$\int_y^{y+k} dy \int_a^b f_y\, dx/k = \int_y^{y+k} g'(y)\, dy/k = \{g(y+k) - g(y)\}/k,$$

since, by Lebesgue's Theorem, $g(y)$ is the integral of its differential coefficient. Hence the theorem follows.

Hence, if f_y has its repeated integrals equal,

$$\{F(y+k) - F(y)\}/k = \{g(y+k) - g(y)\}/k,$$

whence, letting k diminish indefinitely,

$$F'(y) = g'(y),$$

which proves the sufficiency of the condition.

If, on the other hand,
$$\int_a^b f_y\, dx = \frac{d}{dy}\int f(x, y)\, dx,$$

that is
$$g'(y) = F'(y) \qquad \text{...(1)},$$

we have
$$\int_y^{y+k} g'(y)\, dy = \int_y^{y+k} F'(y)\, dy \qquad \text{.................................(2)}.$$

Moreover $F'(y)$, being equal to $g'(y)$, i.e. to $\int_a^b f_y\, dx$, is a finite and summable differential coefficient, hence (2) gives us

$$F(y+k) - F(y) = \int_y^{y+k} g'(y)\, dy = \int_y^{y+k} dy \int_a^b f_y\, dx,$$

that is
$$\int_a^b \{f(x, y+k) - f(x, y)\}\, dx = \int_y^{y+k} dy \int_a^b f_y\, dx,$$

or
$$\int_a^b dx \int_y^{y+k} f_y\, dy = \int_y^{y+k} dy \int_a^b f_y\, dx,$$

which proves the necessity of the condition.

§ 17. From the preceding article we may deduce at once the following general theorem:

THEOREM. *If in the finite or infinite rectangle*

$$a \leqslant x \leqslant b, \quad y_0 \leqslant y \leqslant y_0 + k$$

$f(x, y)$ is a monotone function of y, and is, except for a set of values of x of zero content, a y-integral, then the necessary and sufficient condition that, for the range of values of y given,

$$\int_a^b f_y\, dx = \frac{d}{dy}\int_a^b f(x, y)\, dx$$

is that the integral on the left should be, for this range of values of y, a differential coefficient with respect to y.

In fact, since f_y has the same sign throughout the whole finite or infinite rectangle and one at least of its repeated integrals, viz. $\int_a^b dx \int_{y_0}^{y_0+k} f_y\, dy$, is finite, it follows by de la Vallée Poussin's Theorem in its generalised form* that the repeated integrals of f_y over the rectangle in question both exist and are equal. Hence the result follows.

NOTE. In the above theorem we do not assume that $f(x, y)$ is a bounded function of the ensemble (x, y).

§ 18. In the two preceding articles we have considered differentiation throughout an interval; it is evident that similar theorems may be stated when we are considering differentiation at a point only. It is obvious, however, that the necessary conditions must have reference none the less in part at least to a finite range of values of y, as we integrate over a rectangle. In § 16 we may plainly replace condition (ii) in this case by the following:

$\int_a^b f_y\, dx$ *is a finite and summable derivate, which, at the point* $y = y_0$, *at which differentiation is required, is a differential coefficient.*

Still more generally we may replace it by the condition *that* $\int_a^b f_y\, dx$ *should at the point* $y = y_0$ *be the differential coefficient of its y-integral, of which a particular condition is that it should be a function of y, which at the point* $y = y_0$ *is continuous.*

Corresponding modifications hold of course with respect to the theorem of the last article.

§§ 19, 20. *On Differentiability under the Sign of Integration when the Limits of Integration are finite, and the Integrand has bounded Derivates.*

§ 19. In this case the theory is of the simplest possible character; not only do we not need to have recourse to the theory of repeated integration, but the method of sequences itself takes its easiest form. The most important rule is the following:

If $f_y(x, y)$ *is for a closed neighbourhood of the value of y considered, and for all values of x in the finite interval under consideration* (a, b), *a bounded function of* (x, y), *then*

(i) $\int_a^b f_y\, dx$ *is a differential coefficient at the point and in its neighbourhood, and*

(ii) *it is the differential coefficient of* $\int_a^b f(x, y)\, dx$.

This result is, of course, well-known. It follows at once from the following theorem and corollary, which are given here for completeness in a somewhat more general form than is usual.

THEOREM. *If for the value of y considered* f_y *exists for all values of x in the interval* (a, b), *except at most for a set of values of x of content zero, and if for that value of y the incrementary ratio* $\{f(x, y+k) - f(x, y)\}/k$ *is bounded as x varies in* (a, b) *and k approaches zero continuously, then*

(i) $\int_a^b f(x, y)\, dx$ *has at the point y considered a differential coefficient, and*

(ii) *this differential coefficient is* $\int_a^b f_y\, dx$, *the integration being over the set of points at which* f_y *exists.*

* "On change of Order of Integration in a Repeated Improper Integral," p. 373, § 13.

For the incrementary ratio in question is the general term of an oscillating succession of functions of x, bounded both above and below, and therefore semi-integrable both above and below, so that the limits of the integrals of the succession lie between those of the upper and lower functions. But the upper and lower functions are the upper and lower derivates with respect to y of $f(x, y)$, and have therefore, as they agree except at a set of values of x of content zero, the same integral, viz. $\int_a^b f_y dx$.

Also $$\int_a^b \{f(x, y+k) - f(x, y)\}\, dx/k = \{F(y+k, a, b) - F(y, a, b)\}/k,$$

so that the limits of the integrals are the derivates of F with respect to y. Hence these derivates all coincide and their common value is $\int_a^b f_y dx$, which proves the theorem.

COR. *If we can determine a neighbourhood $(y, y+c)$ of the point y considered, such that one, and therefore any, of the derivates of f with respect to y is a bounded function of (x, y) throughout the rectangle $(a, y;\ b, y+c)$, then the incrementary ratio involved in the enunciation of the theorem is bounded, and therefore, the remaining conditions being fulfilled, the result still holds.*

§ 20. The preceding theorem, which tells us that differentiation is always allowable when the integrand $f(x, y)$ has bounded derivates with respect to y, and the limits of integration are finite, leads at once to a sufficient condition for the equality of $\frac{d^2u}{dxdy}$ and $\frac{d^2u}{dydx}$. We have, in fact, the following theorem.

THEOREM. *If throughout a certain area $\frac{du}{dx}$ exists and is a bounded function of x (or, more generally, is summable with respect to x, and finite, except at most at a reducible set of points for each fixed value of y), and if it has a differential coefficient $\frac{d}{dy}\left(\frac{du}{dx}\right)$ with respect to y which is a bounded function of (x, y) and is, for each fixed value of y, a differential coefficient, then provided $u(x, y)$ has for some fixed value of x a differential coefficient with respect to y, $\frac{du}{dy}$ exists, and has throughout the area a differential coefficient with respect to x, viz. $\frac{d}{dx}\left(\frac{du}{dy}\right)$, which is everywhere equal to $\frac{d}{dy}\left(\frac{du}{dx}\right)$.*

Put $$\frac{du}{dx} = f(x, y),$$

so that f_y exists, and is a bounded function of (x, y).

Then $$\int_a^x f_y dx = \int_a^x \operatorname*{Lt}_{k=0} \{f(x, y+k) - f(x, y)\}\, dx/k \text{..........................(1)}.$$

Now the succession of functions of x

$$\{f(x, y+k) - f(x, y)\}/k \text{ ....(2)}$$

got by keeping y fixed and letting k diminish down to zero, is a sequence, since f_y exists, and is a bounded sequence, since the constituent function may, by the Theorem of the Mean, be expressed in the form

$$f_y(x, y+\theta k),$$

where θ is an unknown function of (x, y) which lies between zero and unity, so that the point $(x, y+\theta k)$ lies within the area in which f_y is a bounded function of (x, y).

Hence the sequence (2) may be integrated term-by-term, so that, by (1),

$$\int_a^x f_y\,dx = \underset{k=0}{\mathrm{Lt}} \int_a^x \{f(x, y+k) - f(x, y)\}/k = \frac{d}{dy}\int_a^x f(x, y)\,dx$$

$$= \frac{d}{dy}\{u(x, y) - u(a, y)\} \quad \ldots\ldots\ldots\ldots (3),$$

since $f(x, y) = u_x$ satisfies sufficient conditions for its integral to be the primitive function $u(x, y)$.

Thus, if a is the value of x hypothecated, for which $u(x, y)$ has a differential coefficient with respect to y, (3) shews that $\frac{du}{dy}$ exists for all values of the ensemble (x, y) in the area considered, and, to a function of y près, is equal to $\int_a^x f_y\,dx$.

That is $$\frac{du}{dy} + \text{a function of } y = \int_a^x \frac{d}{dy}\left(\frac{du}{dx}\right)dx \quad \ldots\ldots\ldots\ldots (4).$$

Now, by hypothesis, the integrand on the right is bounded, and a differential coefficient, when regarded as a function of x, consequently it is the differential coefficient with respect to x of the right-hand side of (4), and therefore of the left-hand side. Hence $\frac{d}{dx}\left(\frac{du}{dy}\right)$ exists, and we have

$$\frac{d}{dx}\left(\frac{du}{dy}\right) = \frac{d}{dy}\left(\frac{du}{dx}\right),$$

which completes the proof of the theorem.

COR. 1. *If* $\frac{du}{dx}$, $\frac{du}{dy}$, *and* $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *exist, and are bounded functions of* (x, y), *then the necessary and sufficient condition that* $\frac{d}{dx}\left(\frac{du}{dy}\right)$ *should exist and be equal to* $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *is that* $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *should be a differential coefficient with respect to* x.

COR. 2. *Under the same circumstances a sufficient condition is that* $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *should be a continuous function of* x.

§§ 21, 22. *On Differentiability under the sign of Integration when the Integrand has bounded derivates and one of the Limits of Integration is infinite.*

§ 21. As we have just seen, the differentiation is always allowable when the limits of integration are finite; the further conditions of § 15 become therefore sufficient in our present case. If we employ the method of sequences, therefore, we have merely to repeat the contents of that article verbatim. Thus we have the theorem:

THEOREM. *If*

(i) *in every finite portion of the rectangle* $a \leqslant x \leqslant \infty$, $y_0 \leqslant y \leqslant y_0 + k$, *a derivate* f_y *of* $f(x, y)$ *with respect to* y *is bounded**,

(ii) $\int_a^b f_y dx$ *gives as* b *varies a set of functions of* y *forming an integrable sequence,*

(iii) $\int_a^\infty f_y dx$ *exists and is a summable differential coefficient for the range of values of* y, $y_0 \leqslant y \leqslant y_0 + k$, *then*

$$\int_a^\infty f_y dx = \frac{d}{dy} \int_a^\infty f(x, y)\, dx.$$

It is unnecessary to repeat the corollaries of § 15.

§ 22. It is now natural to examine whether additional information can be obtained by means of the second method. Returning to § 16, we remark at once that, as the derivates of $f(x, y)$ with respect to y are bounded, $f(x, y)$ is necessarily a y-integral for every value of x, except the value $x = \infty$. Thus the first condition of § 16 fulfils itself. That article gives us at once therefore the following theorem:

THEOREM. *If*

(i) *in every finite portion of the rectangle* $a \leqslant x \leqslant \infty$, $y_0 \leqslant y \leqslant y_0 + k$, *a derivate* f_y *of* $f(x, y)$ *with respect to* y *is bounded,*

(ii) $\int_a^\infty f_y\, dx$ *exists* (*and is finite*) *and is a summable differential coefficient with respect to* y, *then the necessary and sufficient condition that*

$$\int_a^\infty f_y\, dx = \frac{d}{dy} \int_a^\infty f(x, y)\, dx$$

is that the repeated integrals of f_y *should be equal, viz.*

$$\int_a^\infty dx \int_y^{y+k} f_y\, dy = \int_y^{y+k} dy \int_a^\infty f_y\, dx.$$

COR. 1. *It is a necessary and sufficient condition that*

$$\operatorname*{Lt}_{b=\infty} \int_y^{y+k} dy \int_a^b f_y\, dx = \int_y^{y+k} dy \int_a^\infty f_y\, dx.$$

For, since f_y is bounded, change of order of integration between finite limits is allowable; hence the result follows.

But the equation of corollary 1 is nothing more nor less than an assertion that the functions $\int_a^b f_y\, dx$, as b increases indefinitely, form when regarded as functions of y an integrable sequence, or, more accurately, a succession of functions which is semi-integrable both above and below and, except at a set of points of content zero, converges to an unique limiting function.

We are thus led to see that *condition* (ii) *of the preceding article constitutes when slightly modified, not only a sufficient, but a necessary condition, if the conditions* (i) *and* (iii) *are supposed satisfied.*

* Thus f_y is the differential coefficient of $f(x, y)$, except at a set of points of plane content zero, whose section on every ordinate is of linear content zero. Moreover, one of the derivates of $f(x, y)$ being bounded, so are the others.

§§ 23—33. *On Differentiability under the sign of Integration when both the Limits of Integration are finite, and the derivates of the Integrand are unbounded.*

§ 23. We shall assume that f_y exists for the value, or values, of y considered, except for a set of values of x of content zero, whenever such an assumption is not included or virtually included in those explicitly given.

It may happen in a special case that f_y is bounded except at, say, the superior limit of integration. We then have, by definition,

$$\int_a^b f_y\, dx = \operatorname*{Lt}_{e=0} \int_a^{b-e} f_y\, dx,$$

where the integral on the right is, in the most general case, an ordinary Lebesgue integral. This equation is equally true whether the integration on the left-hand side is Harnack integration, or Lebesgue integration, or Harnack-Lebesgue integration. It is evident that the discussion of the circumstances under which the above integral on the left is the differential coefficient of $\int_a^b f(x, y)\, dx$ is then virtually identical with that required in § 15. This case requires, therefore, no further consideration, always provided, be it remarked, that b is independent of y, so that the integrand f_y becomes infinite for the sole value b of x.

§ 24. We have still to consider the case where b is a function of y, say $b(y)$, so that all the infinities of f_y lie on the curve $x = b(y)$. Then it may, and usually will happen that $b(y)$ is a monotone function of y. In this case we shall be able to prove the following theorem.

THEOREM. *If throughout the area $a \leqslant x \leqslant b(y)$, $y_0 \leqslant y \leqslant y_0 + k$,*

(i) *$f(x, y)$ is a bounded function of (x, y), which vanishes when $x = b(y)$,*

(ii) *$b(y)$ is a monotone function of y,*

(iii) *$f(x, y)$ has everywhere a differential coefficient f_y with respect to y,*

(iv) *f_y is a bounded function of (x, y) except in the neighbourhood of $x = b(y)$, along which curve $f(x, y)$ is still continuous with respect to y,*

(v) *$\int_a^x f_y\, dx$ is a bounded function of (x, y),*

then the necessary and sufficient condition that

$$\int_a^b f_y\, dx = \frac{d}{dy}\int_a^{b(y)} f(x, y)\, dx$$

is that the first member of this equation should be a differential coefficient.

Write $g(x, y) = f(x, y)$ when $x \leqslant b(y)$, and $g(x, y) = 0$ when $b(y) \leqslant x \leqslant B$, where B is a constant greater than the upper bound of $b(y)$.

Then $g(x, y)$ is a continuous function of x, and the infinite discontinuities of g_y lie on the single monotone curve $x = b(y)$.

Further $\int_a^x g_y\, dx = \int_a^x f_y\, dx$, and is therefore a bounded function of the ensemble (x, y) for all values of (x, y) in the rectangle $a \leqslant x \leqslant B$, $y_0 \leqslant y \leqslant y_0 + k$.

Also, fixing x, and remembering that, since $b(y)$ is a monotone function of y, the equation

$$x = b(y)$$

has only one solution, say y_x, we have

$$f(x, y) - f(x, y_0) = \int_{y_0}^{y} f_y(x, y)\, dy$$

for all values of the superior limit y less than y_x, and therefore, since $f(x, y)$ is continuous with respect to y up to and including $y = y_x$, this equation still holds when $y = y_x$. Hence also, for all values of (x, y) in the rectangle considered, $\int_{y_0}^{y} g_y\, dy$, since it is equal to the same integral when the integrand is changed from g to f, is a bounded function of (x, y). By § 6 of the paper on "Change of Order of Integration," we have therefore

$$\int_{y_0}^{y_0+k} dy \int_a^B g_y(x, y)\, dx = \int_a^B dx \int_{y_0}^{y_0+k} g_y(x, y)\, dy.$$

Now evidently $g(x, y)$ is the integral of its differential coefficient $g_y(x, y)$.

Hence, after a few obvious reductions the above equation becomes

$$\int_{y_0}^{y_0+k} dy \int_a^B g_y\, dx = \int_a^{b(y)} \{g(x, y_0+k) - g(x, y_0)\}\, dx$$

$$= \int_a^{b(y_0+k)} f(x, y_0+k)\, dx - \int_a^{b(y_0)} f(x, y_0)\, dx.$$

Now, by hypothesis (vi), $\int_a^{b(y)} f_y\, dx$ is a differential coefficient, say $U'(y)$; and it is, by hypothesis (v), bounded. Hence its integral is $U(y)$, to a constant *près*.

Thus, writing
$$F(y) = \int_a^{b(y)} f(x, y)\, dx,$$

we have
$$\frac{U(y_0+k) - U(y_0)}{k} = \frac{F(y_0+k) - F(y_0)}{k}.$$

Letting k approach zero, we see from this equation that $F(y)$ has a differential coefficient, and that it is the same as that of U, namely $\int_a^{b(y)} f_y\, dx$, which proves the theorem.

COR. *If $f(x, y)$ does not vanish when $x = b(y)$, then $f(x, y) - f\{b(y), y\}$ does, and applying the above theorem to this function of (x, y), which we can do if $f\{b(y), y\}$ has a differential coefficient, or if $f(x, y) - f\{b(y), y\}$ has a differential coefficient which is bounded except in the neighbourhood of $x = b(y)$, we get*

$$\frac{d}{dy} \int_a^{b(y)} f(x, y)\, dx = \int_a^{b(y)} f_y\, dx + b'(y) f\{b(y), y\}.$$

§ 25. In the preceding article we employed the method of repeated integration, partly with the object of illustrating the use of the theorem in § 6 of the paper on "Change of Order of Integration." In the present article we propose to use the method of sequences to obtain a different, and in some respects more general form, of the conditions for the applicability of the usual rule. We first prove the following general theorem:

THEOREM. *If* $F(y) = \int_a^{b(y)} f(x, y)\,dx$, *then, assuming* $b'(y)$ *to exist,*

$$F'(y) = b'(y) f\{b(y), y\} + \int_a^{b(y)} f_y\,dx \quad\ldots\ldots(1),$$

provided

(i) $f(x, y)$ *is a continuous function of the ensemble* (x, y),

(ii) *the right-hand side of* (1) *is a summable differential coefficient with respect to* y, *and is infinite at most at a reducible set of points,*

(iii) *the functions obtained by diminishing* e *indefinitely in the expression*

$$b'(y) f\{b(y) - e, y\} + \int_a^{b(y)-e} f_y\,dx$$

form an integrable sequence.

For, denoting $\int_a^{b(y)-e} f(x, y)\,dx$ by $F_e(y)$,

$$\{F_e(y+k) - F_e(y)\}/k = \int_a^{b(y+k)-e} f(x, y+k)\,dx/k - \int_a^{b(y)-e} f(x, y)\,dx/k$$

$$= \int_{b(y)-e}^{b(y+k)-e} f(x, y+k)\,dx/k + \int_a^{b(y)-e} \{f(x, y+k) - f(x, y)\}\,dx/k \quad\ldots\ldots(2).$$

Here the first of the two integrals on the right may be written

$$\int_0^{\{b(y+k)-b(y)\}/k} f\{b(y) + kt - e, y + k\}\,dt,$$

which has, as k approaches zero, the unique limit $b'(y) f\{b(y) - e, y\}$, since $f(x, y)$ is, by hypothesis, continuous with respect to the ensemble (x, y), and therefore

$$\int_0^z f\{b(y) - e, y + k\}\,dt$$

is continuous with respect to the ensemble (k, z), by case (i) of the theorem of § 3 of the present paper.

We next proceed to consider the second of the two integrals on the right of (2), and to find its limit as k diminishes. We first remark that we can suppose k during its progress to zero to be so small that the integrand is a bounded function of (x, k) for the whole rectangle of points (x, k) determined by this range of values of k, and the range of values of x between the superior and inferior limits of integration, y, as it should be remarked, having the particular value we are considering. In fact, either by the Theorem of the Mean, or the Theorem of the Bounds of Derivates, the integrand $\{f(x, y+k) - f(x, y)\}/k$ lies between the upper and lower bounds of $f_y(x, y+k)$ for the same rectangle of points. But $f_y(x, y+k)$ is a bounded function of (x, k) for every rectangle which does not include inside it or on its boundary any point of the curve

$$x = b(Y).$$

But since $b(y)$ is continuous, we can evidently choose the range of values of k to be so small that

$$|b(y+k)|$$

is greater than

$$|b(y)| - e$$

by at least a sufficiently small fixed quantity, the same for all such values of k. This secures that the point $\{b(y+k), y+k\}$ lies for all values of k outside the rectangle in question, which proves the statement made above, since $f_y(x, y+k)$ is only unbounded in the neighbourhood of $x = b(y)$.

Hence, by Lebesgue's theorem, referred to repeatedly, we may obtain the limit of the integral under consideration as k diminishes, by performing the limiting operation inside the integral, and so obtain finally, instead of (2), the equation

$$F_e'(y) = b'(y) f\{b(y) - e, y\} + \int_a^{b(y)-e} f_y \, dx \quad \text{.........................(3).}$$

It should be particularly noticed that the equation (3) just obtained holds for all values of e and all values of y, although the range of k's used in proving it depended, of course, on the particular values of y and e chosen.

The theorem as enunciated now follows immediately, by hypotheses (ii) and (iii), applying § 14, and bearing in mind that since $f(x, y)$ is a continuous function of x the limit of $f\{b(y) - e, y\}$ is $f\{b(y), y\}$.

§ 26. It is clear that, by giving to condition (iii) various special forms, with which in the earlier part of the paper we became acquainted, we can obtain from the preceding general theorem a number of convenient special theorems. The most important of these is perhaps the following:

THEOREM. *If* $F(y) = \int_a^{b(y)} f(x, y)\, dx$, *then*

$$F'(y) = b'(y) f\{b(y), y\} + \int_a^{b(y)} f_y \, dx \quad \text{............................(1),}$$

provided

(i) $f(x, y)$ *is a continuous function of the ensemble* (x, y) *for the whole range of integration, and a closed neighbourhood of* y;

(ii) $\int_a^x f_y\, dx$ *is a bounded function of the ensemble* (x, y) *for the whole range of integration, and a closed neighbourhood of* y;

(iii) $\int_a^{b(y)} f_y\, dx$ *is a differential coefficient with respect to* y;

(iv) $b'(y)$ *is a bounded function of* y, *or is finite, and bounded either above or below, in some closed neighbourhood of the point* y *considered;*

or,

(iv′) $f\{b(y), y\}$ *possesses a bounded differential coefficient;*

or, more generally,

(iv″) $f\{b(y), y\}$ *is an integral, and possesses a finite differential coefficient, which is bounded either above or below;*

or,

(iv‴) $b'(y)$ *is finite and summable and* $f\{b(y), y\}$ *has bounded variation;*

or,

(iv⁗) $b(y)$ *is a function of bounded variation, and* $f\{b(y), y\}$ *possesses a finite and summable differential coefficient.*

For in each of the cases (iv)...(iv'''') the first term of the right-hand side of (1) is a differential coefficient*, and the functions $b'(y) f\{b(y)-e, y\}$ form an integrable sequence, since they are numerically less than the summable function obtained by multiplying the modulus of $b'(y)$ by the upper bound of the modulus of $f(x, y)$.

COR. *The equation* (1) *holds if we replace the conditions* (ii) *and* (iii) *by the single condition that* $\int_a^x f_y\, dx$ *is a continuous function of the ensemble* (x, y)†.

§ 27. By slightly modifying the mode in which we have applied the method of sequences, we may obtain the following additional theorem on the differentiation of the integral $\int_a^b f(x, y)\, dx$, in a case which presents itself frequently in practice.

THEOREM. *If* $F(y) = \int_a^{b(y)} f(x, y)\, dx$, *then*

$$F'(y) = b'(y) f\{b(y), y\} + \int_a^{b(y)} f_y\, dx,$$

provided

(i) $f(x, y)$ *is a continuous function of the ensemble* (x, y) *for the whole range of integration, and a closed neighbourhood of the point* y *considered;*

(ii) f_y *is a bounded function of the ensemble* (x, y) *except in the neighbourhood of the curve* $x = b(y)$;

(iii) $f_y(x, y+k)$ *is a monotone function of* k *in a certain neighbourhood of the point* $x = b(y)$, $k = 0$;

(iv) $b'(y)$ *exists at the point* y *considered, and is finite.*

We obtain as before

$$\{F(y+k) - F(y)\}/k = \int_{b(y)}^{b(y+k)} f(x, y+k)\, dx/k + \int_a^{b(y)} \{f(x, y+k) - f(x, y)\}\, dx/k,$$

where the first integral on the right is seen, as before, to have, as k approaches zero, the unique limit $b'(y) f\{b(y), y\}$.

Considering then the second integral on the right we remark that the integrand is a monotone function of k, by a lemma proved in my paper on "Parametric Integration," § 11, so that, by the theorem used more than once in the present paper, we may introduce the limit under the sign of integration, as k approaches zero. This gives the required result.

§ 28. So far we have confined our attention to the case when the unboundedness of the integrand f_y is due to infinite discontinuities at one of the limits of integration. We now pass to the more general case, when these discontinuities may be situated anywhere. The method of sequences furnishes at once two results of considerable theoretical interest.

* W. H. Young, "On the Property of being a Differential Coefficient," 1910, *Proc. L. M. S.*

† This corollary will be seen to bear a close resemblance to a theorem on p. 439 of Bromwich's *Theory of Infinite Series*. If in the statement of that theorem we insert Bromwich's tacit assumptions, and suitably interpret his requirement that $\int_a^{b(y)} f_y\, dx$ should be "uniformly convergent," it will be seen that his statement becomes a special case of our result. The proof given by Bromwich, being couched in ϵ-language and somewhat condensed, is a little difficult to follow.

THEOREM. *If*

(i) *the points of the line* $y = y_0$ *in the neighbourhood of which* f_y *is an unbounded function of* (x, y) *be only countable in number;*

(ii) *the differential coefficient with respect to* y *of* $\int_a^x f(x, y)\,dx$ *be a continuous function of* x *on the line* $y = y_0$, *then*

$$\int_a^b f_y\,dx = \frac{d}{dy}\int_a^b f(x, y)\,dx,$$

provided the former integral exists.

Consider in fact the sequence of functions of x,

$$\{f(x, y_0 + k) - f(x, y_0)\}/k,$$

whose limiting function is $f_y(x, y_0)$.

A point at which the peak-function of this sequence is infinite, say $x = c$, $y = y_0$, must be a limiting point of a set of points $(x_n, y_0 + k_n)$ such that the limit

$$\operatorname*{Lt}_{n=\infty} \{f(x_n, y_0 + k_n) - f(x_n, y_0)\}/k_n,$$

that is, the limit
$$\operatorname*{Lt}_{n=\infty} f_y\{x_n, y_0 + \theta k_n\},$$

has an infinite value. Thus the point $x = c$, $y = y_0$ would be a point in the neighbourhood of which f_y is unbounded. Hence there is only a countable number of points at which the peak-function is infinite.

Again, $\dfrac{d}{dy}\displaystyle\int_a^x f(x, y)\,dx$ is the limit

$$\operatorname*{Lt}_{k=0} \int_a^x \{f(x, y + k) - f(x, y)\}\,dx/k,$$

that is, it is the limit of the integrals of our sequence. Hence by a theorem in the integration of sequences, term-by-term integration is here allowable. Thus the required result at once follows.

NOTE 1. The theorem quoted in the integration of sequences holds equally for successions which converge except at a set of content zero. Thus we can avoid assuming that f_y exists for every value of x, provided the exceptional values of x form a set of content zero.

NOTE 2. We do not require to assume that $\int_a^x f(x, y)\,dx$ has a differential coefficient, for, in place of (ii) we can substitute the condition that the upper and lower derivates of $\int_a^x f(x, y)\,dx$ with respect to y, must be both continuous functions of x along the line $y = y_0$. This follows from a theorem in Oscillating Successions, it is, however, then a result of the theorem that these upper and lower derivates coincide.

§ 29. In precisely the same way, except that we make appeal to another theorem* in the theory of the integration of sequences or successions, we prove the following theorem:

* The theorem here used will be found stated and proved in its most general form in the paper "On Semi-integrals and Oscillating Successions of Functions" already cited. For the special theorem see Vitali, "Sull' Integrazione per Serie," 1907, *Rend. di Palermo*, Tomo XXIII. pp. 137—155.

THEOREM. *If*

(i) *the points on the line* $y = y_0$ *in the neighbourhood of which* f_y *is an unbounded function of the ensemble* (x, y) *be of content zero;*

(ii) *the differential coefficient of* $\int_a^x f(x, y)\,dx$ *with respect to* y *be, for the value* y_0 *of* y, *an* x*-integral, then* $\int_a^b f_y\,dx$ *exists and we have*

$$\int_a^b f_y\,dx = \frac{d}{dy}\int_a^b f(x, y)\,dx.$$

The notes made at the end of the preceding article apply equally here.

§ 30. Each of the theorems in §§ 28 and 29 may be employed to obtain conditions for the equality of $\dfrac{d^2u}{dx\,dy}$ and $\dfrac{d^2u}{dy\,dx}$, in a manner analogous to that used in § 19. We first prove the theorem which results in this way from § 28.

THEOREM. *If* $\dfrac{d}{dy}\left(\dfrac{du}{dx}\right)$ *exists, and is, except in the neighbourhood of a countable set of points, a bounded function of the ensemble* (x, y), *then the necessary and sufficient conditions that* $\dfrac{d}{dx}\left(\dfrac{du}{dy}\right)$ *exists and is equal to* $\dfrac{d}{dy}\left(\dfrac{du}{dx}\right)$, *are*

(i) $\dfrac{du}{dx}$ *is a summable function of* x,

(ii) $\dfrac{du}{dy}$ *exists and is a continuous function of* x,

(iii) $\dfrac{d}{dy}\left(\dfrac{du}{dx}\right)$ *is, when regarded as a function of* x, *a differential coefficient.*

These conditions are evidently necessary. It remains to prove that they are sufficient.

Since $\dfrac{d}{dy}\left(\dfrac{du}{dx}\right)$ exists, $\dfrac{du}{dx}$ is a continuous function of y.

Put $\dfrac{du}{dx} = f(x, y)$. Then the succession of functions $\{f(x, y+k) - f(x, y)\}/k$ is a sequence, since its limiting function is $f_y(x, y)$, and, since the generating function may, by the Theorem of the Mean, be expressed in the form $f_y(x, y+\theta k)$, is a bounded sequence, except in the neighbourhood of a countable set of points. Also

$$\operatorname*{Lt}_{k=0}\int_a^x \left\{\frac{f(x, y+k) - f(x, y)}{k}\right\} dx = \frac{d}{dy}\int_a^x f(x, y)\,dx = \frac{d}{dy}\{u(x, y) - u(a, y)\},$$

since, by hypothesis, $f(x, y) = u_x$ satisfies conditions which make its integral the primitive function $u(x, y)$. Thus the limit of the integrals of the sequence is, by hypothesis (ii), continuous with respect to x.

Moreover the sequence is, as k diminishes to zero, always finite, and is, if we give k any discrete set of values, unbounded only in the neighbourhood of a countable number of points (x, y) on the line $k = 0$, forming necessarily a closed set of points. Hence, by the theory of term-by-term integration of series, the integral of the limit exists, at least as a Harnack-Lebesgue integral*, and is equal to the limit of the integrals. Thus, we may write

$$\frac{d}{dy}\{u(x, y) - u(a, y)\} = \int_a^x f_y\, dx = \int_a^x \frac{d}{dy}\left(\frac{du}{dx}\right) dx \quad \text{.....................(1)},$$

where the integral on the extreme right may be a Harnack, or Harnack-Lebesgue integral.

If we take the points x and $x+h$ in a black interval of the closed countable set on $y = \text{const.}$, $\int_x^{x+h} f_y\, dx$ is a Lebesgue integral, and, consequently, as f_y is by hypothesis a differential coefficient with respect to x, and is bounded between x and $x+h$, it follows that, at a point x in such a black interval,

$$\int_a^x f_y\, dx$$

has f_y as differential coefficient with respect to x. Hence, except at the countable closed set of points in question, the differential coefficient of this integral exists and is equal to a function which is known to be a differential coefficient everywhere. Call the primitive function of which f_y is the differential coefficient $g(x)$. Then

$$g(x) - \int_a^x f_y\, dx$$

has, except at the countable closed set in question, a differential coefficient which is zero. Hence, since the upper and lower bounds of a derivate are unaltered by the omission of any countable set of points†, the differential coefficient at the remaining points exists, and is zero. This proves that $\int_a^x f_y\, dx$ has everywhere a differential coefficient, and that this differential coefficient is f_y; that is, the right-hand side, and therefore also the left-hand side, of equation (1) has f_y for differential coefficient with respect to x.

Hence, we may differentiate the equation (1), and we get

$$\frac{d}{dx}\left(\frac{du}{dy}\right) = \frac{d}{dy}\left(\frac{du}{dx}\right),$$

which proves the theorem.

§ 31. We next apply the theorem of § 29.

THEOREM. *If*

(i) $\frac{dy}{dx}$ *is a summable function of* x,

(ii) $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *exists (and is finite) throughout a certain rectangle, and the points on the line* $y = y_0$ *in the neighbourhood of which it is an unbounded function of* (x, y) *be of content zero,*

(iii) $\frac{du}{dy}$ *exists, and is a Lebesgue integral, when regarded as a function of* x*, then the*

* The proof of this by Cantor induction is straightforward.

† See "The Fundamental Theorems of the Differential Calculus," by W. H. Young, *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 11, p. 65, note η.

necessary and sufficient condition that $\frac{d}{dx}\left(\frac{du}{dy}\right)$ *should exist, and be equal to* $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *at every point of the line* $y=y_0$, *is that*

(iv) $\frac{d}{dy}\left(\frac{du}{dx}\right)$ *is, when regarded as a function of* x, *a differential coefficient.*

Clearly we need only prove the sufficiency of the condition. To do this we remark that the argument already used in § 30 again applies, except that we are now sure, by Vitali's Theorem*, that $\int_a^x \frac{d}{dy}\left(\frac{du}{dx}\right) dx$ exists as a Lebesgue integral, and consequently has $\frac{d}{dy}\left(\frac{du}{dx}\right)$ for differential coefficient since $\frac{d}{dy}\left(\frac{du}{dx}\right)$ is a finite differential coefficient.

Now the equation (1) of § 30 holds for the value $y=y_0$, and, as we have just seen, the right-hand side can be differentiated with respect to x. Hence the differential coefficient of the left-hand side exists and is equal to that of the right-hand side. Thus

$$\frac{d}{dx}\left(\frac{du}{dy}\right)=\frac{d}{dy}\left(\frac{du}{dx}\right).$$

§ 32. The Method of Sequences will give us one more theorem of a rather special nature, concerning the differentiation of a Harnack integral; it has some resemblance to a theorem already proved in § 12 on the differentiation of a Lebesgue integral; there differentiation under the sign of integration was possible, whether the limits of integration were finite or infinite; here they have to be finite. In that theorem, moreover, f_y was any bounded function, whereas here it must also be monotone with respect to x. These differences illustrate the loss of simplicity when we have to pass from the theory of Lebesgue integrals to that of Harnack integrals.

THEOREM. *If* f_y *is a bounded function of the ensemble* (x, y), *and* $h(x)$ *possesses in the finite interval* (a, b) *a Harnack integral, then provided only that* $f_y(x, y)$ *is a monotone function of* x, *we have*

$$\int_a^b h(x) f_y\, dx=\frac{d}{dy}\int_a^b h(x) f(x, y)\, dx.$$

Since f_y is, say, a monotone increasing function of x, it follows that

$$f_y(x+h, y) \geqslant f_y(x, y),$$

for all values of x and y. Therefore

$$\frac{d}{dy}\{f(x+h, y)-f(x, y)\} \geqslant 0.$$

Therefore $\{f(x+h, y)-f(x, y)\}$ is a monotone increasing function of y. Hence

$$f(x+h, y+k)-f(x, y+k) \geqslant f(x+h, y)-f(x, y),$$

whence $$f(x+h, y+k)-f(x+h, y) \geqslant f(x, y+k)-f(x, y).$$

* *loc. cit.*

Hence the sequence of which the general term is

$$\{f(x, y+k)-f(x, y)\}/k$$

is a bounded sequence of monotone functions of x. By a theorem in the Harnack integration of series* it follows that the sequence whose general term is

$$h(x)\{f(x, y+k)-f(x, y)\}/k$$

is integrable term-by-term, which proves the theorem.

COR. *Under the same circumstances, if, as b increases without limit,* $\int_a^b h(x)\,dx$ *is a bounded function of b, we have*

$$\int_a^\infty h(x) f_y\,dx = \frac{d}{dy}\int_a^\infty h(x) f(x, y)\,dx,$$

provided further that the left-hand side not only exists, but is a differential coefficient with respect to y.

In fact, it follows from the Second Theorem of the Mean for Harnack integrals, that, under the circumstances stated in the theorem, $\int_a^b h(x) f_y\,dx$ traces out, as b increases, a bounded succession, and therefore in our case a bounded sequence, whose limiting function is a differential coefficient with respect to y. The corollary then follows by § 15, Cor. 1.

NOTE. If f_y is a continuous function of y, then, by the theorem in the Harnack Integration of Series already quoted, $\int_a^b h(x) f_y\,dx$ is a continuous function of y, since

$$h(x) f_y(x, y+k)$$

traces out, as k approaches zero, a sequence, which, in accordance with the theorem in question, is integrable. In this case it is evident that $\int_a^\infty h(x) f_y\,dx$ is, if $\int_a^\infty h(x)\,dx$ exists, a continuous function of y, and therefore a differential coefficient.

§ 33. We conclude this section by calling attention to one or two results which follow immediately by the method of Repeated Integration, using the general theorem of § 16. We have to specify special sets of conditions which will secure the equality of the repeated integrals of f_y, when the limits of integration are all finite, but f_y is unbounded. We have then the following theorem:

THEOREM. *If*

(I) *f_y possesses a Lebesgue double integral over the finite rectangle*

$$a \leqslant x \leqslant b, \quad y_0 \leqslant y \leqslant y_0 + k;$$

or,

(II) *if $f(x, y)$ and $\int_a^x f_y\,dx$ are each bounded functions of (x, y), and if the infinite discontinuities of f_y lie on a finite number of monotone curves;*

* See a forthcoming paper on this subject by the author in the November number of the *Messenger of Mathematics*.

or,

(III) *if f_y is finite, except at most at a reducible set of points, and if the total variation in the interval $y_0 \leqslant y \leqslant y_0 + k$ of $f(x, y)$, regarded as a function of y, has a Lebesgue integral with respect to x in the interval (a, b); then provided the conditions* (i) *and* (ii) *of* § 16 *hold, we have*

$$\int_a^b f_y\, dx = \frac{d}{dy}\int_a^b f(x, y)\, dx.$$

That (I) and (II) are sufficient is an immediate consequence of theorems given in the paper on "Change of Order of Integration." To prove that (III) also suffices, note that the total variation of $f(x, y)$ is, by a theorem of Lebesgue's, $\int |f_y|\, dy$, and therefore this latter integral has a Lebesgue integral with respect to x in the interval (a, b). Thus one of the repeated integrals of $|f_y|$ exists and is finite. Hence, applying de la Vallée Poussin's theorem in its extended form, given in the paper just referred to, both the repeated integrals of f_y exist, and are finite, and equal.

§ 34. *On Differentiability under the sign of Integration when one of the Limits of Integration is infinite, and f_y is unbounded.*

§ 34. We have only a few remarks to add on the final case, when f_y is unbounded, and one of the limits of integration is infinite. We have already in our general theorems, and in § 15, exploited to the full the use that we are able to make of the method of sequences. It remains only, as in the preceding article, except that in the present case one of the limits of integration is infinite, to give one or two special sets of conditions which may with convenience replace the general one given in § 16, that the repeated integrals of f_y should be equal. We have then the following theorem:

THEOREM. *If*

(i) $\int_{y_0}^{y_0+k} dy \int_a^\infty |f_y|\, dx$ *is finite;*

(ii) $\int_a^\infty dx \int_{y_0}^{y_0+k} |f_y|\, dy$ *is finite;*

provided also that change of order of integration between finite limits is known to be allowable, and the conditions (i) *and* (ii) *of* § 16 *also hold, then*

$$\int_a^\infty f_y\, dx = \frac{d}{dy}\int_a^\infty f(x, y)\, dx.$$

For other sets of special conditions, applicable either in this article or in the preceding, reference may be made to the paper so often already quoted.

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FOURIER'S DOUBLE INTEGRAL AND THE THEORY OF DIVERGENT INTEGRALS.

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XVI. *Fourier's Double Integral and the Theory of Divergent Integrals.*

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I. *Introduction.*

1. Fourier's "double-integral theorem" is expressed by the equation

$$\int_0^\infty dx \int_{-\infty}^\infty f(\lambda) \cos x (\lambda - \xi)\, d\lambda = \tfrac{1}{2}\pi \{f(\xi + 0) + f(\xi - 0)\} \text{ ....(1).}$$

Here ξ is a constant which may, without loss of generality, be taken to be zero, as appears at once if we make the substitution $\lambda - \xi = \mu$, and then write λ for μ and $f(\lambda)$ for $f(\mu + \xi)$.

It has long been known that the following conditions (which I shall call the "classical" conditions) are sufficient for the validity of the equation (1):

(i) $f(\lambda)$ *is integrable and absolutely integrable* in any finite interval;*

(ii) $f(\xi + 0)$, $f(\xi - 0)$ *exist;*

(iii) $f(\lambda)$ *is monotonic on each side of* $\lambda = \xi$ (*more generally, of limited total fluctuation in some interval including* $\lambda = \xi$);

(iv) *the integral*

$$\int_{-\infty}^\infty |f(\lambda)|\, d\lambda$$

is convergent.

For condition (iii) we may substitute

(iii *a*) *the integrals*

$$\int_\xi^{\xi+\delta} \left| \frac{f(\lambda) - f(\xi + 0)}{\lambda - \xi} \right| d\lambda, \quad \int_{\xi-\delta}^\xi \left| \frac{f(\xi - 0) - f(\lambda)}{\xi - \lambda} \right| d\lambda$$

are convergent†.

2. These results are deduced from a consideration of Dirichlet's integral

$$\int_{-\infty}^\infty f(\lambda) \frac{\sin w\lambda}{\lambda}\, d\lambda \text{ ....(2),}$$

* If a function is *limited* in an interval, its integrability throughout that interval involves its absolute integrability, whereas the converse is not true. Thus the function $f(\lambda)$ which is equal to 1 or -1, according as λ is rational or irrational, is absolutely integrable but not integrable in any finite interval. On the other hand many functions are integrable, in intervals in which they are *unlimited*, without being absolutely integrable.

† See Hobson, *Theory of functions of a real variable*, pp. 758 *et seq*. Prof. Hobson states all these results for the more general case in which the integrals concerned exist only as Lebesgue integrals. In this paper I do not consider this generalisation, which has no particular relevance to the theorems that I have in view.

which is known to have the limit

$$\tfrac{1}{2}\pi\{f(+0)+f(-0)\},$$

as $w \to \infty$, if $f(\lambda)$ is subject to conditions similar to those laid down in § 1. In a paper published in 1908* Prof. Hobson has shown that this last result is still correct if the condition (iv) is replaced by the less stringent condition that

$$\int^{\infty}\left|\frac{f(\lambda)}{\lambda}\right|d\lambda,\quad \int_{-\infty}\left|\frac{f(\lambda)}{\lambda}\right|d\lambda$$

are convergent.

It is, however, not permissible to replace condition (iv) by this condition, in enunciating conditions sufficient for the validity of Fourier's result. Fourier's theorem is in fact deduced from Dirichlet's by the transformation

$$\begin{aligned}\int_0^{\infty}dx\int_{-\infty}^{\infty}f(\lambda)\cos\lambda x\,d\lambda &= \lim_{w\to\infty}\int_0^{w}dx\int_{-\infty}^{\infty}f(\lambda)\cos\lambda x\,d\lambda\\ &= \lim_{w\to\infty}\int_{-\infty}^{\infty}f(\lambda)\,d\lambda\int_0^{w}\cos\lambda x\,dx\\ &= \lim_{w\to\infty}\int_{-\infty}^{\infty}f(\lambda)\frac{\sin w\lambda}{\lambda}d\lambda,\end{aligned}$$

and Prof. Hobson's condition is not sufficient to justify the inversion of the order of integration, or even to ensure the convergence of

$$\int_{-\infty}^{\infty}f(\lambda)\cos\lambda x\,d\lambda.$$

This is easily shown by an example.

Let $$a_1 < b_1 < a_2 < b_2 < a_3 < \ldots\ldots$$

be a sequence of numbers increasing above all limit, and let

$$f(\lambda)=1\,(a_n<\lambda<b_n),\quad f(\lambda)=0\,(b_n<\lambda<a_{n+1}).$$

Then $$\int^{\infty}\frac{f(\lambda)}{\lambda}d\lambda$$

is convergent if $$\sum^{\infty}\log\left(\frac{b_n}{a_n}\right)$$

is convergent; and $$\int^{\infty}f(\lambda)\cos x\lambda\,d\lambda$$

is certainly not convergent unless

$$\sum^{\infty}\{\sin(b_n x)-\sin(a_n x)\}$$

is so; and it is easy to choose a_n and b_n so that the first of these series is convergent but not the second. Thus if

$$a_n=n^2,\quad b_n=n^2+1$$

the series are $$\sum^{\infty}\log\left(1+\frac{1}{n^2}\right),\quad 2\sin\tfrac{1}{2}x\sum^{\infty}\cos(n^2+\tfrac{1}{2})\,x,$$

and the latter series is in general oscillatory.

* *Proc. Lond. Math. Soc.* vol. VI. p. 372.

3. Quite recently there has appeared a paper by Pringsheim* in which he generalises the conditions to be imposed upon $f(\lambda)$, in so far as they relate to the behaviour of $f(\lambda)$ for large values of λ, considerably further than had been effected by any previous writer. In the first place he shows that condition (iv) may be replaced by

(iv a) *$f(\lambda)$ is monotonic for $\lambda > \Lambda$ or $\lambda < -\Lambda$, and tends to zero as $\lambda \to \infty$ or $\lambda \to -\infty$; or, more generally, $f(\lambda)$ is of limited total fluctuation in the infinite intervals* (Λ, ∞), $(-\infty, -\Lambda)$.

In the second place he introduces a function $F(\lambda)$ defined by

$$F(\lambda) = \sum_0^\infty c_\nu \cos(q_\nu \lambda + r_\nu),$$

where *either* $c_\nu = 0$ for $\nu > n$ (so that the series is finite), *or* $\Sigma |c_\nu|$ is convergent, $q_\nu \to \infty$, and $F(\lambda)$ possesses a derivative integrable and absolutely integrable in any finite interval. Writing $f(\lambda) F(\lambda)$ in the place of $f(\lambda)$ he shows that

$$\int_0^\infty dx \int_{-\infty}^\infty f(\lambda) F(\lambda) \cos(\lambda - \xi) x \, d\lambda = \tfrac{1}{2}\pi \{f(\xi + 0) + f(\xi - 0)\} F(\xi) \quad \text{.........(3)}$$

if $f(\lambda)$ is still subject to the conditions (i), (ii), (iii) or (iii a), and (iv) or (iv a), provided that when we chose the condition (iv a), and $F(\lambda)$ is not a mere constant, we impose the additional condition

(v) *the integrals* $$\int^\infty \left|\frac{f(\lambda)}{\lambda}\right| d\lambda, \quad \int_{-\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$$
are convergent†.

Further, he shows that the formula (3) remains valid if we replace its right-hand side by

$$\tfrac{1}{2}\pi \lim_{\epsilon \to 0} \{f(\xi + \epsilon) + f(\xi - \epsilon)\} F(\xi + \epsilon)$$

and suppose only that this limit exists and that $f(\lambda) F(\lambda)$, and not necessarily $f(\lambda)$ itself, is subject to the conditions (i) and (iii).

Finally he shows that, when $F(\lambda)$ reduces to a finite sum, we can dispense with condition (v), provided we regard Fourier's integral as a *principal value* (in respect to the integration with regard to x).

It should be observed that these conditions of Pringsheim's are not valuable merely on account of their greater theoretical generality. There are perfectly simple and obvious forms of $f(\lambda)$, such as

$$f(\lambda) = \frac{\sin \lambda}{\lambda},$$

with which no theorems previously established were sufficient to deal.

4. Generalisations of the ordinary form of Fourier's integral theorem, of a different type, have been considered by Sommerfeld‡ and by the present writer.

The essence of these generalisations is that they show that, if we agree to regard Fourier's integral as *summable* only (by one method or another) and not necessarily convergent, we can

* *Math. Annalen*, Bd. 68, S. 367.

† If $f(\lambda)$ is ultimately monotonic, the signs of the absolute value may be omitted; but they must be retained if all that we know is that $f(\lambda)$ is of limited total fluctuation.

‡ Sommerfeld, "Die willkürlichen Funktionen in der Math. Physik," *Inaugural-Dissertation*, Königsberg, 1901; Hardy, "Further researches in the theory of divergent series and integrals," *Camb. Phil. Trans.* vol. XXI. p. 39.

dispense entirely with all conditions which have regard to the behaviour of $f(\lambda)$ near $\lambda=\xi$, except the conditions (i) and (ii). That is to say we can omit the conditions (iii) or (iii a), or the conditions relating to the function $f(\lambda)F(\lambda)$ substituted for them by Pringsheim. Similarly, we shall find, we can omit the condition, used by Pringsheim, that $F(\lambda)$ possesses an integrable and absolutely integrable derivative.

In the paper in the *Cambridge Philosophical Transactions* referred to above I considered three definitions of the generalised or summable integral, viz. those expressed by the equations

$$G\int_0^\infty f(x)\,dx=\lim_{\delta\to 0}\int_0^\infty e^{-\delta x}f(x)\,dx \quad\text{.............................(4)},$$

$$G\int_0^\infty f(x)\,dx=\lim_{\delta\to 0}\int_0^\infty e^{-(\delta x)^2}f(x)\,dx \quad\text{.............................(5)},$$

$$G\int_0^\infty f(x)\,dx=\lim_{x\to\infty}\frac{1}{x}\int_0^x dt\int_0^t f(u)\,du \quad\text{.............................(6)}.$$

But I considered (as Sommerfeld had done) only the case in which the interval of integration with respect to λ is finite. The result was to establish the formula

$$G\int_0^\infty dx\int_\beta^\gamma f(\lambda)\cos(\lambda-\xi)\,x\,d\lambda=\tfrac{1}{2}\pi\,\{f(\xi+0)+f(\xi-0)\}$$

(where $\beta<\xi<\gamma$) under the conditions that

(i) $f(\lambda)$ *is integrable and absolutely integrable in* (β,γ),

(ii) $f(\xi+0)$ *and* $f(\xi-0)$ *exist.*

In the present paper I propose to complete and generalise these results by

(a) adopting a more general form of the definition of the summable integral,

(b) supposing the interval of integration with respect to λ infinite,

(c) supposing $f(\lambda)$ to have one of the more general forms considered by Pringsheim.

II. *Some Properties of the Summable Integral.*

5. The integral

$$\int_a^\infty f(x)\,dx \quad\text{...(1)}$$

will be said to be *summable* ($C1$) to sum s if

$$\frac{1}{x}\int_a^x dt\int_a^t f(u)\,du\to s \quad\text{.......................................(2)}$$

as $x\to\infty$. We may suppose $a>0$. An alternative and equivalent definition is given by the formula

$$\int_a^x\left(1-\frac{t}{x}\right)f(t)\,dt\to s \quad\text{.......................................(3)}.$$

Similarly the integral is said to be *summable* (Cr) if

$$\frac{r!}{x^r}\left(\int_a^x dt\right)^{r+1}f(t)\,dt\to s$$

or

$$\int_a^x\left(1-\frac{t}{x}\right)^r f(t)\,dt\to s.$$

More general types of definition, modelled on the above, and analogous to those given by Riesz* for summable series, are easily framed; and r need not necessarily be integral. But in what follows we shall only have occasion to use the simplest form—viz. that expressed by (2) or (3).

We shall in the main be concerned with a different type of definition, a generalisation of that expressed by (4) and (5) of § 4.

Let $\phi(x)$ be a function of x subject to the following conditions:

(i) $\phi(x)$ has at most a finite number of maxima and minima†;

(ii) $\phi''(x)$ is continuous and ultimately positive;

(iii) the integral $\int^{\infty} \phi(x)\,dx$ is convergent;

(iv) $\phi(0) \neq 0$.

We note in passing that these properties imply a number of others. Thus $\phi'(x)$ is ultimately negative and increases steadily to zero as $x \to \infty$, and $\phi(x)$ is ultimately positive and decreases steadily to zero. Also

$$\int^{\infty} \phi'(x)\,dx$$

is convergent, and so, as $\phi'(x)$ is ultimately monotonic, we must have

$$x\phi'(x) \to 0.$$

Finally $$\int_a^x t\phi''(t)\,dt = x\phi'(x) - a\phi'(a) - \phi(x) + \phi(a) \to \phi(a) - a\phi'(a),$$

so that $$\int^{\infty} x\phi''(x)\,dx$$

is convergent.

Then we shall say that (1) is *summable* (ϕ), to sum s, if

$$\frac{1}{\phi(0)}\int_a^{\infty} \phi(\delta x) f(x)\,dx \to s$$

as $\delta \to 0$ by positive values: this condition implies the convergence of the integral on the left-hand side for all positive values of δ.

6. The following properties of summable integrals will be used in the sequel.

(i) *If the integral* (1) *is convergent and equal to* s, *it is summable* $(C1)$ *to sum* s‡.

This result is very easily extended to the more general definitions: but it is not necessary for our present purpose to enter into this.

(ii) *If the integral* (1) *is convergent, and has the value* s, *it is summable* (ϕ), *and has the sum* s.

For the integral $$\int_a^{\infty} \phi(\delta x) f(x)\,dx$$

is uniformly convergent for $0 \leqq \delta \leqq \delta_0$§.

* *Comptes Rendus*, 5 July, 1909.

† This is not a consequence of condition (ii). Consider, for example, the function $\phi(x) = x^5 \sin(1/x)$.

‡ Hardy, *Quarterly Journal*, vol. xxxv. p. 54; C. N. Moore, *Trans. Amer. Math. Soc.* vol. VIII. p. 312.

§ Bromwich, *Infinite Series*, pp. 434–5.

(iii) *If* (*a*) *the integral* (1) *is summable* (C1),

(*b*) $\phi(\delta x)\int_a^x f(t)\,dt \to 0$ *for all positive values of* δ,

then the integral (1) *is summable* (ϕ) *to sum* s*.

Let $$f_1(x)=\int_a^x f(t)\,dt,\quad f_2(x)=\int_a^x f_1(t)\,dt,$$

so that $$\frac{1}{x}f_2(x)\to s.$$

Then $$\int_a^X \phi(\delta x)f(x)\,dx=\phi(\delta X)f_1(X)-\delta\phi'(\delta X)f_2(X)+\delta^2\int_a^X \phi''(\delta x)f_2(x)\,dx.$$

Now $\int^\infty \phi''(\delta x)f_2(x)\,dx$ is convergent (absolutely), by comparison with $\int^\infty x\phi''(\delta x)\,dx$; and

$$\delta\phi'(\delta X)f_2(X)=\left\{\frac{1}{X}f_2(X)\right\}\{\delta X\,\phi'(\delta X)\}\to 0.$$

Using condition (*b*) we see that

$$\int_a^\infty \phi(\delta x)f(x)\,dx=\delta^2\int_a^\infty \phi''(\delta x)f_2(x)\,dx;$$

the convergence of each integral having been established incidentally. The integral on the right-hand side is equal to

$$\delta^2\int_a^\infty (s+\epsilon_x)x\phi''(\delta x)\,dx=s\{\phi(\delta a)-\delta a\phi'(\delta a)\}+\delta^2\int_a^\infty x\phi''(\delta x)\,\epsilon_x dx,$$

where $\epsilon_x\to 0$ as $x\to\infty$: and it is easily shown (by a type of proof so familiar that it is hardly worth repeating) that the limit of the right-hand side, when $\delta\to 0$, is $s\phi(0)$.

(iv) *If* (*a*) *the integral* (1) *is summable* (ϕ) *to sum* s,

(*b*) $xf(x)\to 0$ *as* $x\to\infty$,

then (1) *is convergent, and has the value* s†.

In the first place $$\phi(0)-\phi(\delta x)=-\delta x\phi'(\xi),$$
where $0<\xi<\delta x$; and so $$|\phi(0)-\phi(\delta x)|<K\delta x.$$

Hence $$\left|\int_a^X \{\phi(0)-\phi(\delta x)\}f(x)\,dx\right|<K\delta\int_a^X x\,|f(x)|\,dx.$$

Again, if $F(X)$ denotes the upper limit of $x|f(x)|$ for $x\geqq X$, we have

$$\left|\int_X^\infty \phi(\delta x)f(x)\,dx\right|\leqq\frac{F(X)}{X}\int_X^\infty \phi(\delta x)\,dx=F(X)\,\Phi(\delta X)/\delta X,$$

where $$\Phi(x)=\int_x^\infty \phi(t)\,dt.$$

* Cf. Moore, *loc. cit.*; Bromwich, *Math. Annalen*, Bd. 65, S. 367.

† Cf. Tauber, *Monatshefte für Math.* Bd. 8, S. 273; Landau, *ibid.* Bd. 18, S. 8; Bromwich, *Infinite Series*, p. 251. It has been shown by Mr Littlewood, with the aid of more elaborate analysis, that the condition (*b*) may be replaced by the less stringent condition that $|xf(x)|<K$, as in theorem (vi) below. See *Proc. Lond. Math. Soc.* vol. IX. p. 434: the theorem there proved is the analogous theorem for series, but the same method is applicable to integrals.

Suppose that $X = 1/\delta$. Then

$$\left| \phi(0) \int_a^X f(x)\, dx - \int_a^\infty \phi(\delta x) f(x)\, dx \right|$$
$$< \frac{K}{X} \int_a^X x\, |f(x)|\, dx + F(X)\, \Phi(1) \to 0 ;$$

and so

$$\int_a^X f(x)\, dx \to s$$

as $X \to \infty$; which proves the theorem.

(v) *The necessary and sufficient condition that the integral* (1), *when summable* ($C\,1$), *should also be convergent, is that*

$$\frac{1}{x} \int_a^x t f(t)\, dt \to 0$$

as $x \to \infty$.

This follows at once from the equations

$$\frac{f_2(x)}{x} = \frac{1}{x} \int_a^x f_1(t)\, dt = f_1(x) - \frac{1}{x} \int_a^x t f(t)\, dt.$$

(vi) *If* (i) *the integral* (1) *is summable* ($C\,1$) *to sum* s,

(ii) $|x f(x)| < K$,

then the integral (1) *is convergent, and has the value* s.

Let

$$g(x) = x f(x), \quad G(x) = \int_a^x g(t)\, dt.$$

Then

$$f_1(x) = \int_a^x f(t)\, dt = \int_a^x \frac{1}{t} G'(t)\, dt$$
$$= \frac{G(x)}{x} - \frac{G(a)}{a} + \int_a^x G(t) \frac{dt}{t^2},$$

and so

$$\frac{1}{x} \int_a^x f_1(t)\, dt = f_1(x) - \frac{G(x)}{x}$$
$$= -\frac{G(a)}{a} + \int_a^x G(t) \frac{dt}{t^2}.$$

Hence, as the left-hand side tends to a limit as $x \to \infty$, the integral

$$\int_a^\infty G(t) \frac{dt}{t^2}$$

is convergent. I shall now prove that this cannot be the case unless

$$\frac{1}{x} G(x) \to 0 :$$

from which it will follow, by (v), that the integral (1) is convergent.

If this last relation does not hold, it must be possible to find a positive number K such that

$$G(x) > K_1 x,$$

or $G(x) < -K_1 x$, for values of x surpassing all limit. Let us adopt the first hypothesis: we may clearly suppose $K_1 < K$. And let X be a value of x for which the inequality above written is satisfied. Let

$$X_1 = \left(1 - \frac{K_1}{2K}\right) X.$$

Then, for $X_1 \leqq x \leqq X$, we have

$$\begin{aligned} |G(x) - G(X)| &= \left| \int_x^X t f(t)\, dt \right| \\ &< K(X - x) \\ &\leqq K(X - X_1); \end{aligned}$$

and so

$$\begin{aligned} G(x) &\geqq G(X) - |G(x) - G(X)| \\ &> \tfrac{1}{2} K_1 X. \end{aligned}$$

Thus

$$\begin{aligned} \int_{X_1}^X G(t) \frac{dt}{t^2} &> \tfrac{1}{2} K_1 X \int_{X_1}^X \frac{dt}{t^2} \\ &= \tfrac{1}{2} K_1 \left(\frac{X}{X_1} - 1\right) \\ &= \tfrac{1}{2} K_1^2/(2K - K_1) \\ &= K_2, \end{aligned}$$

say. And this inequality is plainly inconsistent with the convergence of the integral $\int^\infty G(t) \dfrac{dt}{t^2}$. Thus the theorem follows*.

III. *Theorems relating to the inversion of the order of integration in a repeated infinite integral.*

7. This section will be devoted to the formulation of a variety of sets of sufficient conditions for the truth of the equations

$$\int_0^\infty \phi(x)\, dx \int_0^\infty f(\lambda) \frac{\cos}{\sin} \lambda x\, d\lambda = \int_0^\infty f(\lambda)\, d\lambda \int_0^\infty \phi(x) \frac{\cos}{\sin} \lambda x da \quad \text{...........(1)}.$$

We shall suppose throughout that $\phi(x)$ and $f(\lambda)$ arc subject to the following condition:

Condition of integrability. *Each of the functions ϕ, f is integrable and absolutely integrable throughout any finite integral*—or, as we shall say, **regularly integrable** throughout any finite interval.

8. It will be convenient to begin by stating certain lemmas. These lemmas are proved by Pringsheim†, but, as the proofs are very short, it seems worth while to repeat them here.

Lemma A. *The integrals*

$$\int_\Lambda^\infty f(\lambda) \frac{\sin \lambda\xi}{\lambda}\, d\lambda, \quad \int_\Lambda^\infty f(\lambda) \frac{\cos \lambda\xi}{\lambda}\, d\lambda \qquad (\Lambda > 0)$$

* For similar theorems relating to *series*, see *Proc. L. M. S.* vol. VIII. pp. 301 *et seq.*

† *Math. Annalen, loc. cit.*

will tend to zero, as $\xi \to \infty$, *if* $\psi(\lambda) = f(\lambda)/\lambda$ *satisfies any one of the following conditions*—

(*a*) $\int_\Lambda^\infty |\psi(\lambda)| d\lambda$ *is convergent;*

(*b*) $\psi(\lambda)$ *tends steadily to zero as* $\lambda \to \infty$;

(*c*) $\psi(\lambda)$ *is of limited total fluctuation in the interval* (Λ, ∞);

—*and, in the case of conditions* (*b*) *and* (*c*), *it is sufficient to suppose them satisfied for values of* λ *greater than some definite value* Λ'.

In the first place, it is well known* that

$$\int_\Lambda^{\Lambda'} \psi(\lambda) \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi d\lambda \to 0$$

as $\xi \to \infty$, for any finite value of Λ'; and all that we have to do is to consider whether we may replace Λ' by ∞.

Now
$$\int_\Lambda^\infty \psi(\lambda) \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi d\lambda = \int_\Lambda^{\Lambda'} + \int_{\Lambda'}^\infty,$$

if only the last integral is convergent. And it is sufficient to show that this is so and that we can make

$$\left| \int_{\Lambda'}^\infty \right| < \epsilon$$

by choice of Λ', independently of ξ. For then, Λ' being fixed, we can choose ξ_0 so that

$$\left| \int_\Lambda^{\Lambda'} \right| < \epsilon$$

for $\xi \geqq \xi_0$; and the truth of the lemma will follow.

(*a*) If this condition is satisfied,

$$\left| \int_{\Lambda'}^\infty \psi(\lambda) \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi d\lambda \right| \leqq \int_{\Lambda'}^\infty |\psi(\lambda)| d\lambda,$$

and the result follows at once.

(*b*) We have (Λ' being large enough)

$$\int_{\Lambda'}^\infty = \lim_{\Lambda'' \to \infty} \int_{\Lambda'}^{\Lambda''} = \lim \psi(\Lambda') \int_{\Lambda'}^{\Lambda'''} \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi d\lambda,$$

where $\Lambda' < \Lambda''' < \Lambda''$: that the limit exists and the integral is convergent follows at once from Dirichlet's test†.

Also
$$\left| \int_{\Lambda'}^\infty \right| \leqq \frac{2}{\xi} |\psi(\Lambda')|,$$

and the result of the lemma follows immediately.

(*c*) In this case we may write

$$\psi(\lambda) = \psi_1(\lambda) - \psi_2(\lambda),$$

where ψ_1 and ψ_2 are subject to (*b*).

* See, *e.g.*, Hobson, *Theory of functions of a real variable*, pp. 672 *et seq.*

† Bromwich, *Infinite Series*, p. 430.

Lemma B. *If $f(\lambda)$ satisfies any one of the conditions imposed upon $f(\lambda)/\lambda$ in Lemma A, then*

$$\int_{\Lambda}^{\infty} f(\lambda) \frac{\sin \lambda\xi}{\lambda} d\lambda \to 0$$

as $\xi \to 0$.

This will certainly be the case if the integral is uniformly convergent in an interval including $\xi = 0$. If condition (a) is satisfied this is obviously the case. If condition (b) is satisfied we have

$$\int_{\Lambda'}^{\infty} f(\lambda) \frac{\sin \lambda\xi}{\lambda} d\lambda = f(\Lambda') \int_{\Lambda'}^{\Lambda''} \frac{\sin \lambda\xi}{\lambda} d\lambda,$$

which is numerically less than $\pi \,|f(\Lambda')|$

for all values of ξ; and the result follows.

Lemma C. *If*

$$\int^{\infty} \frac{|f(\lambda)|}{\lambda} d\lambda$$

is convergent, then

$$\int_{\Lambda}^{\infty} f(\lambda) \frac{\sin \lambda\xi}{\lambda} d\lambda \to 0, \quad \int_{\Lambda}^{\infty} f(\lambda) \frac{\cos \lambda\xi}{\lambda} d\lambda \to \int_{\Lambda}^{\infty} \frac{f(\lambda)}{\lambda} d\lambda,$$

as $\xi \to 0$.

For each integral is plainly uniformly convergent in an interval including $\xi = 0$. In so far as the sine-integral is concerned, this lemma includes case (a) of Lemma B.

9. THEOREM I. *If the integrals*

$$\int^{\infty} |\phi(x)|\, dx, \quad \int^{\infty} |f(\lambda)|\, d\lambda$$

are convergent, then

$$\int_0^{\infty} \phi(x)\, dx \int_0^{\infty} f(\lambda) \frac{\cos}{\sin} \lambda x d\lambda = \int_0^{\infty} f(\lambda)\, d\lambda \int_0^{\infty} \phi(x) \frac{\cos}{\sin} \lambda x dx.$$

This may be deduced at once from the existence of the double integral

$$\int_0^{\infty} \int_0^{\infty} |\phi(x)|\, |f(\lambda)| \left| \frac{\cos}{\sin} \lambda x \right| dx d\lambda.$$

In view of our subsequent results, however, it is more convenient to proceed as follows.

(a) In virtue of the "condition of integrability," we have

$$\int_{x_0}^{X} \int_{\lambda_0}^{\Lambda} = \int_{\lambda_0}^{\Lambda} \int_{x_0}^{X}$$

for any finite values of the limits, zero included.

(b) The integral $\int_0^{\infty} \phi(x) \frac{\cos}{\sin} \lambda x\, dx$

is uniformly convergent for all values of λ. Hence

$$\int_0^{\infty} \int_0^{\Lambda} = \int_0^{\Lambda} \int_0^{\infty},$$

however great be Λ; and it only remains to show that

$$\int_0^\infty \int_\Lambda^\infty$$

is convergent, and tends to zero as $\Lambda \to \infty$: for then

$$\int_0^\infty \phi(x)\,dx \int_0^\infty f(\lambda) \frac{\cos}{\sin} \lambda x\,d\lambda = \lim_{\Lambda\to\infty} \int_0^\infty \int_0^\Lambda = \lim_{\Lambda\to\infty} \int_0^\Lambda \int_0^\infty$$

$$= \int_0^\infty f(\lambda)\,d\lambda \int_0^\infty \phi(x) \frac{\cos}{\sin} \lambda x\,dx,$$

by the definition of the latter repeated integral.

Now
$$\int_\Lambda^\infty f(\lambda) \frac{\cos}{\sin} \lambda x\,d\lambda$$

is uniformly convergent, and so continuous, throughout any finite interval of values of x; and so

$$\phi(x) \int_\Lambda^\infty f(\lambda) \frac{\cos}{\sin} \lambda x\,d\lambda$$

is regularly integrable throughout any such interval. Also

$$\left| \phi(x) \int_\Lambda^\infty \right| < |\phi(x)| \int_0^\infty |f(\lambda)|\,d\lambda;$$

and so
$$\int_0^\infty \phi(x)\,dx \int_\Lambda^\infty f(\lambda) \frac{\cos}{\sin} \lambda x\,d\lambda$$

is convergent. It is moreover plainly less in absolute value than

$$\int_0^\infty |\phi(x)|\,dx \int_\Lambda^\infty |f(\lambda)|\,d\lambda$$

and so tends to zero as $\Lambda \to \infty$. Thus the theorem is established.

10. Theorem II. *The inversion of the order of integration, in the case of the* **cosine** *integral, is also legitimate under the following conditions:*

(i)
$$\int_0^\infty |\phi(x)|\,dx$$
is convergent;

(ii) $\phi(x)$ *tends steadily to a limit* $\phi(+0)$ *as* $x \to 0$;

(iii) $f(\lambda)$ *tends steadily to zero as* $\lambda \to \infty$, *or, more generally, is of limited total fluctuation in an interval* (Λ, ∞).

(*a*) Precisely as in the proof of Theorem I, we show that

$$\int_0^\Lambda \int_0^\infty = \int_0^\infty \int_0^\Lambda.$$

All that remains, therefore, is to shew that

$$\int_0^\infty \int_\Lambda^\infty$$

is convergent, and tends to zero as $\Lambda \to \infty$.

(b) The problem of establishing the truth of the last statement may be divided into three, viz. that of establishing the same property in the cases of the integrals

$$(1) \qquad \int_0^{x_0} \int_\Lambda^\infty,$$

$$(2) \qquad \int_{x_0}^{X} \int_\Lambda^\infty,$$

$$(3) \qquad \int_X^\infty \int_\Lambda^\infty.$$

And, in the first place, it is obviously possessed by the integral (2), inasmuch as

$$\int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda$$

is uniformly convergent throughout any interval of the type $0 < x_0 \leqq x \leqq X$.

Again, it follows at once from the second theorem of the mean that

$$\left| \int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda \right| < 2 \,|\, f(\Lambda) \,|\, / X \qquad (x \geqq X);$$

and from this, and condition (i), it follows that the property in question is possessed by the integral (3). There remains (1), the treatment of which is slightly more difficult.

We observe first that if $\bar{x} > 0$ then

$$\int_{\bar{x}}^{x_0} dx \int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda = \int_\Lambda^\infty f(\lambda) \frac{\sin \lambda x_0 - \sin \lambda \bar{x}}{\lambda} d\lambda.$$

Making $\bar{x} \to 0$, and using Lemma B, we see that

$$\int_0^{x_0} dx \int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda$$

is convergent and may be calculated by inverting the order of integration.

Now $\phi(x)$ is monotonic near $x = 0$, and we may suppose x_0 so small that it is monotonic throughout $(0, x_0)$. It then follows from the second theorem of the mean that

$$\int_0^{x_0} \phi(x) dx \int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda$$

is convergent and equal to

$$\phi(+0) \int_0^{x_1} dx \int_\Lambda^\infty f(\lambda) \cos \lambda x \, d\lambda + \phi(x_0) \int_{x_1}^{x_0} dx \int_\Lambda^\infty f(\lambda) \cos \lambda x d\lambda$$

$$= \phi(+0) \int_\Lambda^\infty f(\lambda) \frac{\sin \lambda x_1}{\lambda} d\lambda + \phi(x_0) \int_\Lambda^\infty f(\lambda) \frac{\sin \lambda x_0 - \sin \lambda x_1}{\lambda} d\lambda,$$

where $0 < x_1 < x_0$. And as (Lemma B) the integral

$$\int_\Lambda^\infty f(\lambda) \frac{\sin \lambda \xi}{\lambda} d\lambda$$

is uniformly convergent in an interval of values of ξ including $\xi = 0$, it follows that

$$\int_0^{x_0} \phi(x)\,dx \int_\Lambda^\infty f(\lambda)\cos\lambda x\,d\lambda \to 0$$

as $\Lambda \to \infty$. Thus the proof of Theorem II is completed*.

11. Theorem III. *The same result holds of the* **sine** *integral if the additional condition is satisfied that*

$$\int^\infty \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$$

is convergent.

The proof is the same as that of Theorem II except that at the end we are left with an expression of the type

$$\phi(+0)\int_\Lambda^\infty f(\lambda)\frac{1-\cos\lambda x_1}{\lambda}\,d\lambda + \phi(x_0)\int_\Lambda^\infty f(\lambda)\frac{\cos\lambda x_1 - \cos\lambda x_0}{\lambda}\,d\lambda,$$

and that we use Lemma C in the place of Lemma B.

12. We have now to consider the question of the inversion of the order of integration in the more general integral

$$\int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda \cos(\lambda-\xi)\,x\,d\lambda \qquad (a \geqq 0).$$

Putting $\lambda - \xi = \mu$, and writing $F(\mu)$ for $f(\mu+\xi)$, we obtain

$$\cos a\xi \int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty F(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, a\mu \cos\mu x\,d\mu \mp \sin a\xi \int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty F(\mu) \begin{matrix}\sin\\ \cos\end{matrix}\, a\mu\cos\mu x\,d\mu,$$

so that the question is reduced to the same question for the original integral, with $\xi = 0$. It may then be expressed in the form

$$\tfrac{1}{2}\int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\,(x+a)\,\lambda\,d\lambda \pm \tfrac{1}{2}\int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\,(x-a)\,\lambda\,d\lambda.$$

We now put $x + a = y$ or $x - a = y$, and, using our previous results, we obtain the theorem which follows.

Theorem IV. *The equation*

$$\int_0^\infty \phi(x)\,dx \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda\cos(\lambda-\xi)\,x\,d\lambda = \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda\,d\lambda \int_0^\infty \phi(x)\cos(\lambda-\xi)\,x\,dx$$

is true if

(i) *f and ϕ satisfy the condition of integrability,*

(ii) $$\int_0^\infty |\phi(x)|\,dx$$

is convergent, and (iii) **either**

* I have established a more general theorem dealing with the case in which neither of the integrals

$$\int^\infty |\phi(x)|\,dx, \quad \int^\infty |f(\lambda)|\,d\lambda$$

is convergent. The theorem here proved is sufficiently general for the purposes of this paper.

(iii a) $$\int_{-\infty}^{\infty} |f(\lambda)|\, d\lambda$$
is convergent, **or**

(iii b) *the following three conditions are satisfied—*

(iii $b\alpha$) $\phi(x)$ *tends steadily to a limit* $\phi(a)$ *as* $x \to a$,

(iii $b\beta$) $f(\lambda)$ *tends steadily to zero as* $\lambda \to \infty$ *or* $-\infty$,

(iii $b\gamma$) *the integrals* $$\int^{\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda, \quad \int_{-\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$$

are convergent—where however it is to be observed that (iii $b\beta$) *may be replaced by the more general condition that* $f(\lambda)$ *is of limited total fluctuation in the intervals* $(-\infty, -\Lambda)$, (Λ, ∞), *and that* (iii $b\gamma$) *may be dispensed with when* $a = 0$.

IV. *The limit of the integral* $\int_0^{\infty} \phi(\delta x)\, dx \int_{-\infty}^{\infty} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix} a\lambda \cos(\lambda - \xi) x\, d\lambda$.

13. If $\phi(x)$ satisfies the first three conditions of § 5, viz. that

(i) $\phi(x)$ has at most a finite number of maxima and minima,

(ii) $\phi''(x)$ exists, and is ultimately positive,

(iii) the integral $$\int_0^{\infty} \phi(x)\, dx$$

is convergent, it is clear that $\phi(\delta x)$, where δ is any positive number, satisfies the conditions of the preceding theorems.

Hence, if $f(\lambda)$ satisfies one or other of the sets of conditions stated in Theorem IV, we have the right to invert the order of integration in the integral written at the head of this paragraph.

Put $$\lambda - \xi = \mu, \quad f(\mu + \xi) \begin{matrix}\cos\\ \sin\end{matrix} a(\mu + \xi) = \psi(\mu).$$
Then the integral reduces to

$$\int_0^{\infty} \phi(\delta x)\, dx \int_{-\infty}^{\infty} \psi(\mu) \cos \mu x\, d\mu = \int_{-\infty}^{\infty} \psi(\mu)\, d\mu \int_0^{\infty} \phi(\delta x) \cos \mu x\, dx.$$

We may write this in the form

$$\frac{1}{\delta} \int_{-\infty}^{\infty} \psi(\mu)\, \Delta\left(\frac{\mu}{\delta}\right) d\mu,$$

where $$\Delta(\mu) = \int_0^{\infty} \phi(x) \cos \mu x dx.$$

We shall have occasion to use the following lemma.

Lemma. *The integral* $$\int_{-\infty}^{\infty} |\Delta(\mu)|\, d\mu$$
is convergent.

In fact, integrating twice by parts, we obtain

$$\Delta(\mu) = -\frac{1}{\mu} \int_0^{\infty} \phi'(x) \sin \mu x dx = \frac{1}{\mu^2} \int_0^{\infty} \phi''(x)(1 - \cos \mu x)\, dx.$$

From this equation, and the fact that $\Delta(\mu)$ is plainly a continuous function of μ, the truth of the lemma follows at once.

It should be observed that the convergence of the integral

$$\int_{-\infty}^{\infty} \Delta(\mu)\, d\mu = \int_{-\infty}^{\infty} d\mu \int_0^{\infty} \phi(x) \cos \mu x dx$$

is an immediate consequence of the ordinary forms of Fourier's integral-theorem: the value of the integral is in fact $\pi\phi(0)$. In the particular case (which in point of fact includes all the most interesting cases) in which $\phi'(x)$ is monotonic not merely ultimately but for all positive values of x, it is clear that

$$\Delta(\mu) = -\frac{1}{\mu}\int_0^{\infty} \phi'(x) \sin \mu x dx > 0;$$

and then the convergence of $\int_{-\infty}^{\infty} \Delta(\mu)\, d\mu$ of course implies its absolute convergence.

We observe also that $\qquad |\Delta(\mu)| < K/\mu^2.$

14. Theorem V. *If*

(i) $\phi(x)$ *is subject to the first three conditions of* § 5,

(ii) $f(\lambda)$ *satisfies the condition of integrability*,

(iii) $f(\xi+0)$ *and* $f(\xi-0)$ *exist*,

then will
$$\int_0^{\infty} \phi(\delta x)\, dx \int_{\Lambda_1}^{\Lambda_2} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda \cos \lambda(x-\xi)\, d\lambda,$$
where $a \geqq 0$, $\Lambda_1 < \xi < \Lambda_2$, *tend, as* $\delta \to 0$, *to the limit*

$$\tfrac{1}{2}\pi\phi(0) \begin{matrix}\cos\\ \sin\end{matrix}\, a\xi \{f(\xi+0) + f(\xi-0)\}.$$

Further, if condition (iii) *is replaced by the condition, sometimes more general, that* $\psi(\xi+0)$ *and* $\psi(\xi-0)$ *exist, where*

$$\psi(\lambda) = f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda,$$

the limit will still have the value

$$\tfrac{1}{2}\pi\phi(0)\{\psi(\xi+0) + \psi(\xi-0)\}.$$

Finally, if the inequalities $\Lambda_1 < \xi < \Lambda_2$ *are not satisfied, the result remains true if the final formula is modified in the manner usual in the theory of Fourier's series*.*

Putting
$$\lambda - \xi = \mu, \quad f(\mu+\xi) \begin{matrix}\cos\\ \sin\end{matrix}\, a(\mu+\xi) = \psi(\mu),$$
$$\Lambda_1 - \xi = M_1, \quad \Lambda_2 - \xi = M_2,$$

we reduce the integral to the form

$$\int_0^{\infty} \phi(\delta x)\, dx \int_{M_1}^{M_2} \psi(\mu) \cos \mu x d\mu,$$

which may be transformed into
$$\frac{1}{\delta}\int_{M_1}^{M_2} \psi(\mu)\, \Delta\left(\frac{\mu}{\delta}\right) d\mu.$$

* The result is zero if ξ falls outside (Λ_1, Λ_2), etc.

The problem therefore reduces itself to proving that

$$\frac{1}{\delta}\int_0^{\Lambda} \psi(\lambda)\,\Delta\left(\frac{\lambda}{\delta}\right) d\lambda \to \tfrac{1}{2}\pi\phi(0)\,\psi(+0)$$

as $\delta \to 0$. Since

$$\frac{1}{\delta}\int_0^{\Lambda} \Delta\left(\frac{\lambda}{\delta}\right) d\lambda = \int_0^{\Lambda/\delta} \Delta(\mu)\,d\mu \to \tfrac{1}{2}\pi\phi(0),$$

it is sufficient to prove that

$$\frac{1}{\delta}\int_0^{\Lambda} \chi(\lambda)\,\Delta\left(\frac{\lambda}{\delta}\right) d\lambda \to 0,$$

where $\chi(\lambda) = \psi(\lambda) - \psi(+0)$. I shall prove first that the integral tends to zero when the lower limit 0 is replaced by any positive number λ_0. That this is so follows in fact at once from the inequality

$$\frac{1}{\delta}\left|\int_{\lambda_0}^{\Lambda} \chi(\lambda)\,\Delta\left(\frac{\lambda}{\delta}\right) d\lambda\right| < K\delta\int_{\lambda_0}^{\Lambda} \frac{|\chi(\lambda)|}{\lambda^2}\,d\lambda,$$

which is itself an immediate consequence of the remark at the end of the last paragraph.

We can suppose λ_0 so chosen that $|\chi(\lambda)| < K$ for $0 < \lambda \leqq \lambda_0$. Then

$$\frac{1}{\delta}\int_0^{\lambda_0} \chi(\lambda)\,\Delta\left(\frac{\lambda}{\delta}\right) d\lambda = \int_0^{\lambda_0/\delta} \chi(\delta\mu)\,\Delta(\mu)\,d\mu = \left(\int_0^M + \int_M^{\lambda_0/\delta}\right) \chi(\delta\mu)\,\Delta(\mu)\,d\mu.$$

The second integral is less than

$$K\int_M^{\lambda_0/\delta} |\Delta(\mu)|\,d\mu,$$

which may be made less than ϵ by choice of M, independently of δ. And when M is fixed we can choose δ_0 so that $|\chi(\delta\mu)| < \epsilon$ for $0 < \delta \leqq \delta_0$, $0 < \mu \leqq M$. Then

$$\left|\int_0^M \chi(\delta\mu)\,\Delta(\mu)\,d\mu\right| < \epsilon\int_0^M |\Delta(\mu)|\,d\mu,$$

and the proof of the theorem is completed.

It is interesting to consider an example in which the result is not true. Let

$$\phi(x) = 1 \quad (0 \leqq x \leqq X), \quad = 0 \quad (x > X).$$

Then

$$\int_0^{\infty} \phi(\delta x)\,dx \int_0^{\Lambda} f(\lambda)\cos\lambda x\,d\lambda = \int_0^{X/\delta} dx \int_0^{\Lambda} f(\lambda)\cos\lambda x\,d\lambda$$
$$= \int_0^{\Lambda} f(\lambda)\,\frac{\sin(\lambda X/\delta)}{\lambda}\,d\lambda,$$

and it is known that the continuity of $f(\lambda)$ is not a sufficient condition to ensure that this integral shall tend to a limit as $\delta \to 0$.

15. THEOREM VI. *The result of Theorem V remains true for* $\Lambda_1 = -\infty$, $\Lambda_2 = \infty$ *provided* **either**

(α) $\int_{-\infty}^{\infty} |f(\lambda)|\,d\lambda$ *is convergent,*

or (β) *the following conditions are satisfied,*

(β 1) $f(\lambda)$ *tends steadily to zero as* $\lambda \to -\infty$ *or* ∞ (*or, more generally, is of limited total fluctuation in the intervals* $(-\infty, \Lambda_1)$, $(\Lambda_2, -\infty)$),

(β 2) *the integrals* $\displaystyle\int_{-\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda, \quad \int^{\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$

are convergent.

If $a=0$ *the condition* (β 2) *may be dispensed with.*

We have only to show that Λ_1 and Λ_2 may be chosen so that

$$\int_0^\infty \phi(\delta x)\, dx \int_{-\infty}^{\Lambda_1} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda \cos(\lambda-\xi)\, x\, d\lambda \to 0,$$

$$\int_0^\infty \phi(\delta x)\, dx \int_{\Lambda_2}^{\infty} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda \cos(\lambda-\xi)\, x\, d\lambda \to 0,$$

as $\delta \to 0$. Take the second relation, for example. Putting $\lambda - \xi = \mu$ we reduce it to

$$\int_0^\infty \phi(\delta x)\, dx \int_{M_2}^\infty \psi(\mu) \cos \mu x\, d\mu \to 0,$$

or, as the conditions of Theorem IV are satisfied, to

$$\frac{1}{\delta}\int_{M_2}^\infty \psi(\mu)\, \Delta\left(\frac{\mu}{\delta}\right) d\mu \to 0.$$

We can suppose M_2 chosen so that

$$|\psi(\mu)| < K \qquad (\mu \geqq M_2).$$

Then
$$\frac{1}{\delta}\int_{M_2}^\infty \psi(\mu)\, \Delta\left(\frac{\mu}{\delta}\right) d\mu = \int_{M_2/\delta}^\infty \psi(\delta\lambda)\, \Delta(\lambda)\, d\lambda,$$

which is in absolute value less than

$$K\int_{M_2/\delta}^\infty |\Delta(\lambda)|\, d\lambda$$

and so tends to zero with δ.

16. We can now state

Theorem VII. *If*

(i) $\phi(x)$ *is subject to the conditions of* § 5,

(ii) $f(\lambda)$ *is subject to the conditions* (α) *or* (β),

(iii) $f(\xi+0)$, $f(\xi-0)$ *exist, or more generally,* $\psi(\xi+0)$, $\psi(\xi-0)$ *exist, where*

$$\psi(\lambda) = f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda\,;$$

then will the integral $\displaystyle\int_0^\infty dx \int_{-\infty}^\infty f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix}\, a\lambda \cos(\lambda-\xi)\, x\, d\lambda$

be summable (ϕ), *and its sum will be*

$$\tfrac{1}{2}\pi \begin{matrix}\cos\\ \sin\end{matrix}\, a\xi \{f(\xi+0) + f(\xi-0)\},$$

or
$$\tfrac{1}{2}\pi \{\psi(\xi+0) + \psi(\xi-0)\}.$$

V. *Summability by Cesàro's method.*

17. It will be convenient at this stage to prove

THEOREM VIII. *If the conditions of Theorem VII are satisfied* (*except* (i)) *then the same result holds in respect of summability* (C 1).

We have to prove that

$$\frac{1}{X}\int_0^X dx \int_0^x dt \int_{-\infty}^{\infty} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix} a\lambda \cos(\lambda - \xi)\, t\, d\lambda \to \tfrac{1}{2}\pi \begin{matrix}\cos\\ \sin\end{matrix}(a\xi)\{f(\xi+0) + f(\xi-0)\}$$

as $X \to \infty$. It is clear that the generality of the investigation is in no way affected by supposing $\xi = 0$.

The triple repeated integral is then the same as

$$\frac{1}{X}\int_0^X (X-x)\, dx \int_{-\infty}^{\infty} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix} a\lambda \cos\lambda x\, d\lambda = \int_{-\infty}^{\infty} f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix} a\lambda d\lambda \frac{1}{X}\int_0^X (X-x)\cos\lambda x dx$$

$$= \int_{-\infty}^{\infty} \psi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda,$$

where $\psi(\lambda) = f(\lambda) \begin{matrix}\cos\\ \sin\end{matrix} a\lambda$ and $w = \tfrac{1}{2}X$. We have therefore to determine the limit of this integral as $w \to \infty$. I shall prove first that, as $w \to \infty$,

$$\int_0^{\Lambda} \psi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda \to \tfrac{1}{2}\pi\psi(+0).$$

Let $$\psi(\lambda) = \psi(+0) + \chi(\lambda);$$

then $\chi(\lambda) \to 0$ as $\lambda \to 0$. And it is clear that what we have to prove is that

$$\int_0^{\Lambda} \chi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda \to 0.$$

Choose λ_0 so that $$|\chi(\lambda)| < \epsilon \qquad (0 < \lambda \leqq \lambda_0);$$

then $$\left|\int_0^{\lambda_0} \chi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda\right| < \epsilon \int_0^{\infty} \left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda = \tfrac{1}{2}\pi\epsilon.$$

Also $$\left|\int_{\lambda_0}^{\Lambda} \chi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda\right| < \frac{1}{w\lambda_0^2}\int_0^{\Lambda} |\chi(\lambda)|\, d\lambda;$$

and from these two inequalities our conclusion follows at once.

In order to complete the proof of the theorem we have only to prove that

$$\int_{\Lambda}^{\infty} \psi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\, d\lambda \to 0.$$

We can suppose Λ so chosen that

$$|\psi(\lambda)| < K \qquad (\lambda \geqq \Lambda),$$

and then $$\left|\int_{\Lambda}^{\infty} \psi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\,d\lambda\right| < K\int_{\Lambda}^{\infty}\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\,d\lambda = K\int_{w\Lambda}^{\infty}\left(\frac{\sin u}{u}\right)^2 du,$$

which plainly tends to zero as $w \to \infty$. Thus

$$\int_0^{\infty} \psi(\lambda)\left(\frac{\sin w\lambda}{w\lambda}\right)^2 w\,d\lambda \to \tfrac{1}{2}\pi\psi(+0);$$

and the truth of the theorem follows immediately*.

VI. *The introduction of Cauchy's Principal Value.*

18. The condition relating to the convergence of the integrals

$$\int_{-\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda, \quad \int^{\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$$

was introduced in Theorem III. In fact, if the condition is *not* satisfied, it is not generally true that

$$\int_0 \phi(x)\,dx \int_{\Lambda}^{\infty} f(\lambda)\sin\lambda x\,d\lambda$$

is convergent: although this is true of the corresponding integral involving $\cos\lambda x$.

Suppose, for example, that

$$f(\lambda) = \frac{1}{\log\lambda}$$

so that the condition is not satisfied. Then it is not hard to show that, as $x \to 0$,

$$\int_{\Lambda}^{\infty} \frac{\cos\lambda x}{\log\lambda}\,d\lambda \sim \tfrac{1}{2}\pi\frac{1}{x\{\log(1/x)\}^2},$$

$$\int_{\Lambda}^{\infty} \frac{\sin\lambda x}{\log\lambda}\,d\lambda \sim \frac{1}{x\{\log(1/x)\}},$$

the first, but not the second, function, being integrable down to $x = 0$.

If the condition is not satisfied, however, we can impose an additional condition on $\phi(x)$, in the neighbourhood of $x = 0$, so that†

$$P\int_{-x_0}^{\infty} \phi(x)\,dx \int_{\Lambda}^{\infty} f(\lambda)\sin\lambda x\,d\lambda$$

$(x_0 > 0)$ shall still be convergent and equal to

$$\int_{\Lambda}^{\infty} f(\lambda)\,d\lambda \int_{-x_0}^{\infty} \phi(x)\sin\lambda x\,dx,$$

as I shall now proceed to prove.

* It is not difficult to prove that, if $f(\lambda)$ is of limited total fluctuation in an interval including $\lambda = \xi$, then

$$\left|\int_{-\infty}^{\infty} \psi(\lambda)\cos(\lambda - \xi)x\,d\lambda\right| < K/x$$

for large values of x. It then follows from Theorem (v) of § 6 that Fourier's double-integral is convergent in the ordinary sense, so that his integral-theorem becomes a corollary of Theorem VIII. Similarly we can use Mr Littlewood's extension of Theorem (iv) of § 6 to exhibit Fourier's integral-theorem as a corollary of Theorem VII. Analogous remarks apply to Dirichlet's form of Fourier's series theorem: cf. Littlewood, *loc. cit.*, and Hardy, *Proc. Lond. Math. Soc.* vol. VIII. p. 308.

† For a detailed exposition of the theory of Cauchy's "Principal Values" I may refer to four papers in the *Proc. Lond. Math. Soc.* (*Old Series*, vol. XXXIV. p. 16 and p. 55 and vol. XXXV. p. 81, and *New Series*, vol. VII. p. 181).

19. Since $$\int_{x_0}^{\infty}\int_{\Lambda}^{\infty} = \int_{\Lambda}^{\infty}\int_{x_0}^{\infty},$$

it is clear that what we have to prove is that

$$P\int_{-x_0}^{x_0}\int_{\Lambda}^{\infty} = \int_{\Lambda}^{\infty}\int_{-x_0}^{x_0}.$$

The principal value will be convergent, and it will be legitimate to calculate its value by a change in the order of integration, if and only if the same is true of the ordinary integral

$$\int_0^{x_0} x\chi(x)\,dx\int_{\Lambda}^{\infty} f(\lambda)\sin\lambda x\,d\lambda$$

where $$x\chi(x) = \phi(x) - \phi(-x).$$

We suppose $\chi(x)$ continuous and monotonic near $x=0$.

Now $$\int_0^{x_0}\int_{\Lambda}^{\Lambda'}$$

is convergent, and may be calculated by inversion, for any finite value of Λ'. Hence what we have to prove is that

$$\int_0^{x_0}\int_{\Lambda'}^{\infty}$$

is convergent and tends to zero as $\Lambda'\to\infty$.

Now, if $f(\lambda)$ is monotonic, $\left|\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda\right| < \dfrac{2}{x}|f(\Lambda')|$

and so $$\int_0^{x_0} x\,dx\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda$$

is convergent. Hence, by the second mean-value theorem,

$$\int_0^{x_0} x\chi(x)\,dx\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda$$

is convergent and equal to

$$\chi(0)\int_0^{x_1} x\,dx\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda + \chi(x_0)\int_{x_1}^{x_0} x\,dx\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda,$$

where $0<x_1<x_0$. These repeated integrals are convergent, and may be calculated by inversion. The only point in this statement that requires proof is that the first of them may be calculated by inversion. We have, however small $\bar{x}$,

$$\int_{\bar{x}}^{x_1} x\,dx\int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda = \int_{\Lambda'}^{\infty}\int_{\bar{x}}^{x_1} = \int_{\Lambda'}^{\infty}\frac{f(\lambda)}{\lambda^2}\{\theta(x_1)-\theta(\bar{x})\}\,d\lambda,$$

where $$\theta(x) = \sin\lambda x - \lambda x\cos\lambda x.$$

We have therefore to show that

$$\bar{x}\int_{\Lambda'}^{\infty}\frac{f(\lambda)}{\lambda}\cos\lambda\bar{x}\,d\lambda - \int_{\Lambda'}^{\infty}\frac{f(\lambda)}{\lambda^2}\sin\lambda\bar{x}\,d\lambda$$

tends to zero as $\bar{x} \to 0$. This is obvious in the case of the second integral. In the case of the first we have

$$\bar{x}\int_{\Lambda'}^{\infty} = \bar{x}\int_{\Lambda'}^{\Lambda''} + \bar{x}\int_{\Lambda''}^{\infty} .$$

The second integral is numerically less than

$$\bar{x} \cdot \frac{|f(\Lambda'')|}{\Lambda''} \cdot \frac{2}{\bar{x}},$$

which may be made less than ϵ by choice of Λ'', independently of $\bar{x}$; and the first term may then be made less than ϵ by choice of $\bar{x}$.

Hence
$$\int_0^{x_0} x\,\chi(x)\,dx \int_{\Lambda'}^{\infty} f(\lambda)\sin\lambda x\,d\lambda = \chi(0)\int_{\Lambda'}^{\infty} \frac{f(\lambda)}{\lambda^2}\,\theta(x_1)\,d\lambda$$
$$+\chi(x_0)\int_{\Lambda'}^{\infty} \frac{f(\lambda)}{\lambda^2}\{\theta(x_0) - \theta(x_1)\}\,d\lambda,$$

and, by what is practically a repetition of the argument which immediately precedes, we can prove that this tends to zero as $\Lambda' \to \infty$.

20. Theorem IX. *If the condition that the integrals*

$$\int_{-\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda, \quad \int^{\infty} \left|\frac{f(\lambda)}{\lambda}\right| d\lambda$$

be dropped, in Theorem III, the result of that theorem will in general not be correct; but if we impose upon $\phi(x)$ the additional condition that

$$\chi(x) = \frac{1}{x}\{\phi(x) - \phi(-x)\}$$

is continuous and monotonic near $x = 0$, it will still be true that

$$P\int_{-x_0}^{\infty} \phi(x)\,dx \int_{-\infty}^{\infty} f(\lambda)\sin\lambda x\,d\lambda = \int_{-\infty}^{\infty} f(\lambda)\,d\lambda \int_{-x_0}^{\infty} \phi(x)\sin\lambda x\,dx,$$

where $x_0 > 0$, and P is the sign of Cauchy's Principal Value.

Theorem X. *In Theorem IV we may drop condition* (iii $b\gamma$), *if we introduce the additional condition that*

$$\chi(x) = \frac{\phi(a+x) - \phi(a-x)}{x}$$

is continuous and monotonic near $x = 0$, and insert the sign of the principal value before the outer integration with regard to x.

21. We have now to consider how far these modifications affect the work of § IV. In the first place, the effect on the proof of Theorem V is obviously *nil*. Secondly, the missing condition affects the proof of Theorem VI only in so far as it is required to justify a certain inversion of integrations—its effect on which we have already discussed. Hence

Theorem XI. *Theorem VI remains valid if modified as Theorem IV was modified above.*

VII. *The introduction of a function defined by an infinite series.*

22. Let
$$F(\lambda) = \sum_0^\infty c_\nu \cos(q_\nu \lambda + r_\nu),$$
where (q_ν) is an ascending sequence of positive numbers, tending to ∞ with ν, and
$$\Sigma\, |c_\nu|$$
is convergent. I shall now prove that if we assume that $\phi(x)$ and $f(\lambda)$ satisfy the conditions of Theorem IV, and in addition that $\phi(x)$ is monotonic, or has at most a finite number of maxima and minima, then we may invert the order of integration in the integral
$$\int_0^\infty \phi(x)\, dx \int_{-\infty}^\infty f(\lambda)\, F(\lambda + y) \cos \lambda x \, d\lambda$$
(for any value of y).

This will be proved if we can justify the following series of inversions:
$$\int_0^\infty \phi(x)\, dx \int_{-\infty}^\infty f(\lambda) \cos \lambda x \, d\lambda \sum_0^\infty c_\nu \cos\{q_\nu(\lambda + y) + r_\nu\}$$
$$= \int_0^\infty \sum_0^\infty \int_{-\infty}^\infty \quad \text{..........(A)}$$
$$= \sum_0^\infty \int_0^\infty \int_{-\infty}^\infty \quad \text{..........(B)}$$
$$= \sum_0^\infty \int_{-\infty}^\infty \int_0^\infty \quad \text{..........(C)}$$
$$= \int_{-\infty}^\infty \sum_0^\infty \int_0^\infty \quad \text{..........(D)}$$
$$= \int_{-\infty}^\infty \int_0^\infty \sum_0^\infty \quad \text{..........(E)}$$

(A) The equation
$$\int_{-\infty}^\infty f(\lambda) \cos \lambda x \, d\lambda \sum_0^\infty c_\nu \cos\{q_\nu(\lambda + y) + r_\nu\} = \sum_0^\infty \int_{-\infty}^\infty$$
will be true if

(i) the series is uniformly convergent throughout any finite interval of values of λ (as is certainly the case),

(ii)
$$\sum_0^\infty \int_\Lambda^\infty$$
is convergent and tends to zero as $\Lambda \to \infty$,

(iii)
$$\sum_0^\infty \int_{-\infty}^{-\Lambda}$$
is convergent and tends to zero as $\Lambda \to \infty$. It is clearly enough to justify the assertion (ii). Its truth is practically obvious if $f(\lambda)$ satisfies condition (iii a) of Theorem IV: we shall therefore suppose that $f(\lambda)$ is subject to the set of conditions (iii b).

We can choose ν_0 so that $q_{\nu_0} > x$, and it is enough to consider the series

$$\sum_{\nu_0}^{\infty} \int_{\Lambda}^{\infty}.$$

But $\int_{\Lambda}^{\infty}$ is numerically less than the product of $|c_\nu|$ and

$$\frac{1}{2}\left|\int_{\Lambda}^{\infty} f(\lambda) \cos\{(q_\nu + x)\lambda + q_\nu y + r_\nu\}\, d\lambda\right| + \frac{1}{2}\left|\int_{\Lambda}^{\infty} f(\lambda) \cos\{(q_\nu - x)\lambda + q_\nu y + r_\nu\}\, d\lambda\right|$$

and so less than
$$|f(\Lambda)|\left(\frac{1}{q_\nu + x} + \frac{1}{q_\nu - x}\right).$$

It follows that our series is convergent and numerically less than

$$|f(\Lambda)| \sum_{\nu_0}^{\infty} |c_\nu| \left(\frac{1}{q_\nu + x} + \frac{1}{q_\nu - x}\right);$$

and so tends to zero as $\Lambda \to \infty$.

(B) The equation
$$\int_0^{\infty} \sum_0^{\infty} \int_{-\infty}^{\infty} = \sum_0^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty}$$
will certainly be true if

(i)
$$\int_0^{X} \sum_0^{\infty} \int_{-\infty}^{\infty} = \sum_0^{\infty} \int_0^{X} \int_{-\infty}^{\infty}$$
for any finite X, and

(ii)
$$\sum_0^{\infty} \int_X^{\infty} \int_{-\infty}^{\infty}$$
is convergent and tends to zero as $X \to \infty$. As above I confine myself to the case in which $f(\lambda)$ satisfies the conditions (iii b), leaving the easier case in which it satisfies (iii a) to the reader.

It is easily proved, by a slight modification of the argument used under (A) above, that

$$\sum_0^{\infty} \int_{-\infty}^{\infty}$$

is uniformly convergent for $0 \leqq x \leqq X$. In the first place, this is obviously true of

$$\sum_0^{\infty} \int_{\Lambda_1}^{\Lambda_2}$$

for any finite values of Λ_1 and Λ_2; for the series may be compared with

$$\sum_0^{\infty} |c_\nu| \int_{\Lambda_1}^{\Lambda_2} |f(\lambda)|\, d\lambda.$$

We have therefore only to justify the assertion for

$$\sum_0^{\infty} \int_{\Lambda}^{\infty}.$$

Choose ν_0 so that $q_{\nu_0} > X$. Then we need only consider

$$\sum_{\nu_0}^{\infty} \int_{\Lambda}^{\infty}$$

and the result follows at once from the analysis given under (A). It should be observed that each term of the series may become infinite for one special value of x, viz. $x = q_\nu$. But as this only happens, in the interval $(0, X)$, to a finite number of terms, no difficulty is caused thereby.

Next, as regards (ii), we observe first that

$$\sum_0^\infty \int_X^\infty \int_{\Lambda_1}^{\Lambda_2}$$

is convergent and tends to zero as $X \to \infty$, for any finite values of Λ_1 and Λ_2. For the series may be compared with

$$\sum_0^\infty |c_\nu| \left| \int_X^\infty \right| \phi(x) | \, dx \int_{\Lambda_1}^{\Lambda_2} | f(\lambda) | \, d\lambda,$$

which has certainly this property. We need therefore only establish the property for a series of the type

$$\sum_0^\infty \int_X^\infty \int_\Lambda^\infty .$$

Now $$\int_X^\infty \int_\Lambda^\infty = \int_\Lambda^\infty \int_X^\infty ,$$

and $$\left| \int_X^\infty \phi(x) \cos \lambda x \, dx \right| < \frac{2 \, | \phi(X) |}{\lambda} *,$$

and so $$\left| \int_\Lambda^\infty \int_X^\infty \right| < 2 \, | \phi(X) | \int_\Lambda^\infty \frac{| f(\lambda) |}{\lambda} d\lambda .$$

Hence $$\left| \sum_0^\infty \int_X^\infty \int_\Lambda^\infty \right| < 2 \, | \phi(X) | \sum_0^\infty | c_\nu | \int_\Lambda^\infty \frac{| f(\lambda) |}{\lambda} d\lambda$$

and the truth of (ii) is established.

(C) The legitimacy of this inversion of the order of integration has already been established.

(D) In order to prove that

$$\sum_0^\infty \int_{-\infty}^\infty \int_0^\infty = \int_{-\infty}^\infty \sum_0^\infty \int_0^\infty$$

we must prove that (i) $$\sum_0^\infty \int_{\Lambda_1}^{\Lambda_2} \int_0^\infty = \int_{\Lambda_1}^{\Lambda_2} \sum_0^\infty \int_0^\infty$$

for any finite values of Λ_1 and Λ_2, and

(ii) $$\sum_0^\infty \int_\Lambda^\infty \int_0^\infty$$

is convergent and tends to zero as $\Lambda \to \infty$.

The truth of (i) follows at once from the fact that the general term of the series

$$\sum_0^\infty \int_0^\infty$$

is numerically less than $$| c_\nu | \int_0^\infty | \phi(x) | \, dx.$$

* Provided $\phi(x)$ is monotonic for $x \geqq X$, a condition certainly satisfied if X is large enough.

The truth of (ii) follows at once from the inequalities

$$\left|\int_0^\infty \phi(x)\cos\lambda x\,dx\right| < \frac{K}{\lambda},$$

$$\left|\int_\Lambda^\infty f(\lambda)\cos\{q_\nu(\lambda+y)+r_\nu\}\,d\lambda\int_0^\infty \phi(x)\cos\lambda x\,dx\right| < K\int_\Lambda^\infty \frac{|f(\lambda)|}{\lambda}\,d\lambda.$$

(E) Finally, in this inversion there is nothing to justify, the two limit operations applying to entirely distinct factors of the subject of integration and summation.

Hence we deduce

THEOREM XII. *If* $$F(\lambda) = \sum_0^\infty c_\nu \cos(q_\nu\lambda + r_\nu),$$

where (q_ν) *is an ascending sequence of positive numbers whose limit is infinity, and* Σc_ν *is an absolutely convergent series: if further* $\phi(x)$, *besides satisfying the conditions of Theorem IV, has at most a finite number of maxima and minima; and* $f(\lambda)$ *is subject to the conditions of Theorem IV (including* (iii $b\gamma$)); *then will*

$$\int_0^\infty \phi(x)\,dx\int_{-\infty}^\infty f(\lambda)\,F(\lambda+y)\cos\lambda x\,d\lambda = \int_{-\infty}^\infty f(\lambda)\,F(\lambda+y)\,dy\int_0^\infty \phi(x)\cos\lambda x\,dx.$$

THEOREM XIII. *We have, under similar conditions,*

$$\int_0^\infty \phi(x)\,dx\int_{-\infty}^\infty f(\lambda)\,F(\lambda)\cos(\lambda-\xi)\,x\,d\lambda = \int_{-\infty}^\infty f(\lambda)\,F(\lambda)\,d\lambda\int_0^\infty \phi(x)\cos(\lambda-\xi)\,x\,dx.$$

23. THEOREM XIV. *Let* $F(\lambda)$ *be defined as in Theorem XII. Let* $\phi(x)$ *and* $f(\lambda)$ *be subject to the conditions of Theorem VII. Then will the integral*

$$\int_0^\infty dx\int_{-\infty}^\infty f(\lambda)\,F(\lambda)\cos(\lambda-\xi)\,x\,d\lambda$$

be summable (ϕ), *and its sum will be*

$$\tfrac{1}{2}\pi\,F(\xi)\,\{f(\xi+0)+f(\xi-0)\},$$

or, more generally, $$\tfrac{1}{2}\pi\,\{\psi(\xi+0)+\psi(\xi-0)\},$$

where $$\psi(\lambda) = f(\lambda)\,F(\lambda).$$

In fact no substantial modification is required in the argument of Section IV in consequence of this more general hypothesis as to the structure of the function $\psi(\lambda)$.

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CYCLIC PATHS FOR RAYS REFLECTED AT AN ELLIPTICAL BOUNDARY.

BY

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XVII. *Cyclic Paths for Rays reflected at an Elliptical Boundary.*

By R. Hargreaves.

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In an enclosure, of which the boundary is not circular or spherical, the shape of the boundary and the geometrical law of reflexion together determine any paths which may be complete circuits of finite length. Such paths it is proposed to consider for a plane space bounded by an ellipse.

The problem was suggested by recent Thermodynamical methods of dealing with the radiation in a closed space; the particular point with respect to which it may have significance is the resonance of the space. This aspect of the question is not considered here; we are concerned exclusively with the geometry, which presents two features of interest. The first is the existence of a group of cycles specially characteristic of the ellipse, as distinct from the more obvious group obtained by deformation of regular polygons in a circle. The second is the treatment of construction by means of a theorem of correspondence which may be applied to either group. This method of construction requires only the use of a parallel ruler, and in actual drawing is at once easier and susceptible of greater accuracy than the direct application of the law of reflexion. The theorem also makes it possible to present the analytical conditions in a symmetrical way, and gives an important property of two corresponding paths, viz. that if one is a circuit the other has the same property, and the two circuits have an equal length of path.

§ 1. Circuits may be divided into two groups according as they are intra-focal or extra-focal. In the former each section of path from boundary to boundary crosses the major axis between the foci, in the latter each section crosses (or would if continued cross) the major axis outside the foci. A ray traversing the major axis has a cyclic path, but no other ray through a focus is part of a cycle: it will be found either by drawing or calculation that any such ray pursued in either sense ultimately approaches the line of the major axis.

In the search for cycles it is important to attend to conditions of symmetry. If when a certain point or line is reached a symmetrical continuation of the path is assured, then for the purpose of determining a cycle such point or line may be regarded as terminal. For *extra-focal* circuits (*a*) the ends of either axis, and (*b*) lines parallel to either axis, have this character. When (*a*) or when (*b*) is used for each terminal the figure has $4n$ sides, in the former case $n-1$ points in the quadrant need determination, in the latter n points. When (a) is used for one terminal and (b) for the other, different cases arise according as we use the same axis for each, or one axis in (a) the other in (b). When the same axis is used we have figures of $4n+2$ sides, in which n points in the quadrant need determination, these points being differently placed according as the major or minor axis is used. If the point terminal is an end of the major axis, the line terminal a parallel to the minor axis or *vice-versâ*, we have figures of $2n+1$ sides, n points of the quadrant being differently placed in the two cases.

All these figures may be regarded as deformations of regular polygons in a circle, if we include amongst the polygons such variants as the pentagon whose sides subtend angles of 144° at the centre or the octagon with 135°. Each type in the circle has two representatives in the ellipse.

For *intra-focal* cycles it is evident that two of the above terminals, an end of the major axis and a parallel to the same, are inadmissible. They are replaced by a point at which a ray is normal, and a central chord; the different types are then furnished by combinations of the four terminals

(a) a point at which a ray is normal,

(b) an end of the minor axis,

(c) a central chord,

(d) a chord parallel to the minor axis.

The three types into which a normal chord enters are open paths, the whole circuit comprising a journey from end to end and back. With n for the number of points needing determination,

an open path	(ab)	has	$2n$	lines,	circuit	$4n$	lines,	V or W pattern,
„ „ „	(ac)	„	$2n-1$	„	„	$4n-2$	„	N pattern,
„ „ „	(ad)	„	$2n-1$	„	„	$4n-2$	„	open lattice pattern;
a closed path	(cd)	has	$4n$	sides,	closed lattice pattern,			
„ „ „	(bd)	„	$4n+2$	„	„ „ pattern,			
„ „ „	(bc)	„	$4n+2$	„	star pattern.			

A distinguishing feature of intra-focal cycles is that they originate at some definite lower limit of eccentricity as configurations of lines indefinitely close to the minor axis, and continue to exist for all higher values of eccentricity. With increase of eccentricity points move away from the ends of the minor axis, but remain in their respective quadrants with an unchanged order of succession in a quadrant. The initial stage, in virtue of the smallness of all angles concerned, admits of a general treatment for any number of variables, by which the limit of eccentricity and the character of the configuration are determined.

§ 2. For the purpose of this treatment the use of eccentric angle as commonly defined is not convenient; it distinguishes four quadrants while we need only a distinction between two, as every ray crosses the major axis. On the auxiliary circle we use a deviation θ

from the minor axis which for points above the major axis is positive in the first and negative in the second quadrant, and for points below the major axis is positive in the third and negative in the fourth quadrant. With θ_1 always in the first quadrant, it is known whether the point to which any subsequent θ refers is above or below the major axis. For the small angles belonging to the initial stage the x coordinates of θ_1 and θ_2 are $a\theta_1$ and $-a\theta_2$, the y coordinates $+b$ and $-b$. The inclination to the minor axis of the chord joining $\theta_1\theta_2$ is $a(\theta_1+\theta_2)/2b$ or $(\theta_1+\theta_2)/2\sqrt{1-e^2}$, that of the normal at θ_2 is $\theta_2\sqrt{1-e^2}$, and the angle of incidence is $(\theta_1+\theta_2)/2\sqrt{1-e^2}-\theta_2\sqrt{1-e^2}$. The angle of reflexion is

$$\theta_2\sqrt{1-e^2}-(\theta_2+\theta_3)/2\sqrt{1-e^2},$$

and the law of reflexion makes $\theta_1+\theta_3+2\theta_2=4(1-e^2)\theta_2$ or $\theta_1+\theta_3=2(1-2e^2)\theta_2$. We may therefore write the typical sequence-equation in the form

$$\theta_{m+1}+\theta_{m-1}=2(1-2e^2)\theta_m \quad\text{.......}(1).$$

This is to be supplemented by relations connecting two angles only at each terminal.

(a) For a normal at θ_1 as terminal, the angle of incidence being zero,

$$(\theta_1+\theta_2)/2\sqrt{1-e^2}=\theta_1\sqrt{1-e^2} \quad\text{or}\quad \theta_2=(1-2e^2)\theta_1 \quad\text{.......}(2a).$$

(b) For a chord joining θ_n to an end of the minor axis as terminal, by writing $\theta_{n+1}=0$ in (1)

$$\theta_{n-1}=2(1-2e^2)\theta_n \quad\text{.......}(2b).$$

(c) For a central chord through θ_n as terminal, by writing $\theta_{n+1}=\theta_n$ in (1)

$$\theta_{n-1}=(1-4e^2)\theta_n;\ \text{with}\ \theta_2=(1-4e^2)\theta_1 \quad\text{.......}(2c)$$

as alternative form if the central chord passes through θ_1.

(d) For a line through θ_n parallel to the minor axis as terminal, by writing $\theta_{n+1}=-\theta_n$ in (1)

$$\theta_{n-1}=(3-4e^2)\theta_n;\ \text{with}\ \theta_2=(3-4e^2)\theta_1 \quad\text{.......}(2d)$$

as alternative form if the parallel passes through θ_1.

§ 3. *Open Paths.* If we use (a) as starting point for all open paths, the ratios of θ's are given by quantities $u_m=\theta_m:\theta_1$ having the properties

$$u_1=1,\quad u_2=1-2e^2,\quad u_{m+1}+u_{m-1}=2(1-2e^2)u_m \quad\text{.......}(3),$$

so that u_{m+1} is a polynomial of degree m in e^2.

For the type (ab) the condition at the other terminal is

$$u_{n+1}=0 \quad\text{.......}(4).$$

This condition gives n values of e^2, the substitution of which in u_m yields n groups of values for the initial ratios of angles defining the configurations introduced at the several limiting values of eccentricity. If we write $e=\sin\epsilon$ or $b=a\cos\epsilon$, u_m is defined by

$$\left.\begin{array}{l} u_1=1,\quad u_2=\cos 2\epsilon,\quad u_{m+1}+u_{m-1}=2u_m\cos 2\epsilon \\ u_{m+1}=\cos 2m\epsilon. \end{array}\right\} \quad\text{.......}(5).$$

of which the solution is (the second line above).

The condition for circuit (ab) is

$$\cos 2n\epsilon=0,\quad\text{or}\quad \epsilon=\frac{\pi}{4n}\{1,\ 3,\ 5,\ \ldots(2n-1)\} \quad\text{.......}(6).$$

For the least root $u_{m+1} = \cos\frac{m\pi}{2n}$, for the greatest $u_{m+1} = (-1)^m \cos\frac{m\pi}{2n}$, and in both cases the last chord of the quarter-cycle proceeds to an end of the minor axis; in the configuration introduced at the least root every previous chord, in that introduced at the greatest root no previous chord, crosses the minor axis; for the first the figure passes from 1st and 3rd to 2nd and 4th quadrants in the next quarter-cycle, while in the second it passes from 1st and 4th to 2nd and 3rd.

The position is less simple for intermediate roots. With $n=1$, $\epsilon = \frac{\pi}{4}$ or $2e^2 = 1$ is the limit of entry for configuration V (fig. 1); with $n=2$, $\epsilon = \frac{\pi}{8}$ or $\frac{3\pi}{8}$, i.e. $4e^2 = 2 \mp \sqrt{2}$, there are two forms, that admitted at the higher root is a direct W (fig. 9), that at the lower root a form with crossed legs (fig. 4).

For the type (ac) the condition $u_{n+1} - u_n = 0$ or $u_{n-1} = (1-4e^2)\, u_n$ is in trigonometrical form $\sin\epsilon \sin(2n-1)\,\epsilon = 0$. The root $\epsilon = 0$ corresponds to the minor axis itself, and is not the condition of entry of a form which afterwards deviates from the minor axis. Other roots are

$$\epsilon = \frac{\pi}{2n-1}\{1, 2, \dots (n-1)\} \quad \dots\dots\dots(7),$$

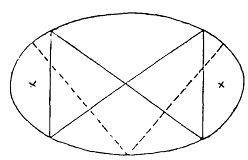

Fig. 1.

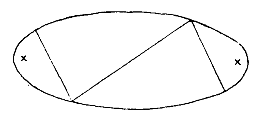

Fig. 2.

and the general description resembles that for (ab). The N form (fig. 2) has $n=2$, and condition $4e^2 > 3$.

For the type (ad) the condition

$$u_{n+1} + u_n = 0 \text{ or } u_{n-1} = (3-4e^2)\, u_n \text{ is } \cos\epsilon \cos(2n-1)\,\epsilon = 0.$$

The root $\epsilon = \frac{\pi}{2}$, meaning an eccentricity $=1$, is not a condition of entry for the ellipse problem, and other roots are

$$\epsilon = \frac{\pi}{2(2n-1)}\{1, 3, \dots (2n-3)\} \quad \dots\dots\dots(8).$$

For the simplest case $n=2$, the condition of entry is $e = \frac{1}{2}$ (fig. 3). This pattern, the open lattice, has symmetry with respect to the major axis, the case (ab) symmetry with respect to the minor axis, and (ac) has skew symmetry.

§ 4. *Closed Paths.* For (dc) and (db) we can use a parallel to the minor axis as starting position, and the ratios of θ's to θ_1 are then given by functions v having the properties

$$\left.\begin{array}{l} v_1 = 1,\ \ v_2 = 3-4e^2,\ \ v_{m+1} + v_{m-1} = 2(1-2e^2)\, v_m = 2v_m \cos 2\epsilon \\ \text{of which } v_{m+1} = \sin(2m+1)\,\epsilon/\sin\epsilon \text{ is the solution.} \end{array}\right\} \quad \dots\dots\dots(9).$$

For the type (dc) the condition at the other terminal is $v_{n+1} - v_n = 0$ or $\cos 2n\epsilon = 0$, and

$$\epsilon = \frac{\pi}{4n}\{1, 3, 5, \dots (2n-1)\} \quad \dots\dots(10)$$

as in (6) for (ab); but here the ratios of angles are different. See figs. 1, 4, 9.

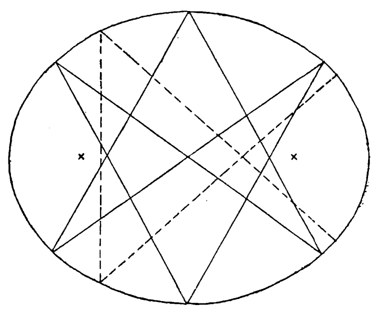

Fig. 3.

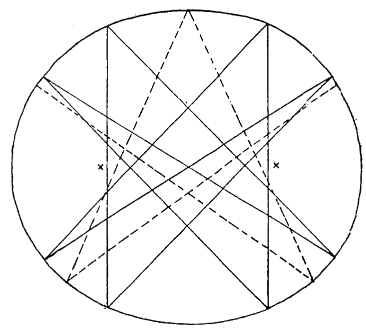

Fig. 4.

For the type (db) the condition at the other terminal is $v_{n+1} = 0$, and

$$\epsilon = \frac{\pi}{2n+1}\{1, 2, \dots n\} \quad \dots\dots(11),$$

which is similar to (7). The closed lattice patterns for the form admitted at the greatest root have the parallels to the minor axis at the wings, for that admitted at the least root the parallels lie nearest the minor axis. See figs. 5 and 6.

For the type (cb) if we take θ_1 to refer to one end of a central chord, the ratios of θ's to θ_1 are given by functions w with the properties

$$w_1 = 1,\ w_2 = 1 - 4e^2,\ w_{m+1} + w_{m-1} = 2(1 - 2e^2)\, w_m, \text{ making } w_{m+1} = \cos(2m+1)\,\epsilon/\cos\epsilon \quad \dots(12).$$

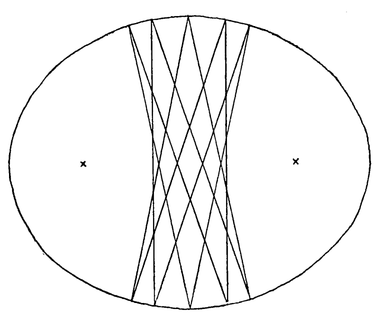

Fig. 5.

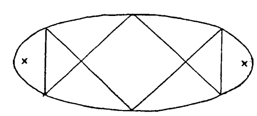

Fig. 6.

The condition at the other terminal is $w_{m+1} = 0$, and

$$\epsilon = \frac{\pi}{2(2n+1)}\{1, 3, \dots (2n-1)\} \quad \dots\dots(13)$$

similar to (8). The simplest case $n = 1$ (fig. 3) gives a star-shaped figure for which the lower limit of eccentricity is $\frac{1}{2}$.

It will be noted that the same limits of eccentricity apply to (ab) and (cd), to (ac) and (bd), and to (ad) and (bc).

If for polygons in a circle we seek new forms by taking a multiple of the angle at the centre, some cases are resolved into figures of an earlier type, e.g. for twelve sides only the multiple 5 gives an independent figure, others give two hexagons, ... The same kind of short-circuiting occurs in intra-focal types, e.g. for $n=3$ in (ab), where e has the values $\sin 15°$, $\sin 45°$, $\sin 75°$. The value $e=\sin 45°$ makes $u_2=0$, $u_3=-1$, $u_4=0$, and the initial configuration is resolved into three V's. The open circuit of 6 lines has two forms admitted at $e=\sin 15°$ and $e=\sin 75°$; the value $e=\sin 45°$ which has already occurred for the type V leads to no new form.

§ 5. We proceed to state and prove the theorem of correspondence which will be used in the actual drawing of figures, and in the statement of analytical conditions.

Mark the points of a configuration 1, 2, 3, ... *and from the centre draw lines parallel to the chords joining* 1 *to* 2, 2 *to* 3, ... *in order, to meet the auxiliary circle. The projections of these points of the circle on the ellipse, say* (12), (23), ..., *form a second configuration derived without ambiguity from the first. Apply this operation to the second configuration with one element of difference, viz. that the cycle is traversed in the reverse direction. If now the first configuration is reproduced, the point* 2 *in the first corresponding to the chord from* (23) *to* (12) *in the second, and so on, there is a reciprocal correspondence. Each configuration will then represent the course of a ray; if one is cyclic so is the other, and the lengths of the two circuits are equal*.*

To see the way in which the law of reflexion is involved in the reciprocal correspondence, let ϕ' correspond to the chord joining ϕ_1 to ϕ_2, ϕ'' to the chord from ϕ_2 to ϕ_3, and reciprocally the point ϕ_2 to the chord joining ϕ'' to ϕ', each variable being an eccentric angle. The condition that the radius to ϕ' on the circle may be parallel to the chord $\phi_1\phi_2$ of the ellipse is $\tan \frac{1}{2}(\phi_1+\phi_2)\tan\phi' = -\sqrt{1-e^2}$, a condition which also makes the tangent at ϕ' on the ellipse parallel to the radius to $\frac{1}{2}(\phi_1+\phi_2)$ on the circle. The reciprocal correspondence therefore makes the chords $\phi_1\phi_2$, $\phi_2\phi_3$, and the tangent at ϕ_2, all on the ellipse, parallel respectively to radii to the points ϕ', ϕ'' and $\frac{1}{2}(\phi'+\phi'')$ on the circle; the chords are therefore equally inclined to the tangent, and the law of reflexion is implied by the mutual correspondence.

Or again if the first configuration is a path for reflected rays, and ϕ', ϕ'' are drawn as above, it follows that ϕ_2 corresponds to the chord $\phi''\phi'$, i.e. the correspondence is reciprocal and the second configuration also a path for reflected rays.

Further the theorem may be used as a method of drawing the course of a ray with parallel ruler, and then we develope the two figures *pari passu*. Let the chord from ϕ_1 to ϕ_2, given coordinates, be a line of the ray whose further course we wish to follow. Find ϕ' as before, and from ϕ' on the ellipse draw chord $\phi'\phi''$ parallel to the line from ϕ_2 on the circle to its centre, which gives point ϕ''. Then draw $\phi_2\phi_3$ parallel to the line from centre to ϕ'' on the circle; the new chord is the path after reflexion at ϕ_2. The angle

* All the diagrams are examples of this theorem. In fig. 7 the auxiliary circle is drawn, and also the radii necessary to show the parallelisms occurring in one quarter-cycle. The positions of foci are marked by small crosses.

ϕ_3 in turn determines the direction of chord $\phi''\phi'''$, and so on; the construction giving the path required, and a companion path.

In these statements the completion of a cycle is immaterial.

The total length of a cyclic path* admits of a different partition in the form

$$\Sigma l = \Sigma 2p \cos i,$$

where l is a chord, p a perpendicular from the centre on a tangent at which reflexion takes place, and i an angle of incidence. We shall show that a chord l_{12} of one configuration is equal to an element $2p' \cos i'$ of the alternative formula for length of the reciprocal configuration. If p' is the perpendicular on the tangent at ϕ', then $p' = b/\sqrt{1 - e^2 \cos^2 \phi'}$. Also by the parallelism $\tan \phi' = \dfrac{y_2 - y_1}{x_2 - x_1}$, or $\cos^2 \phi' = \dfrac{(x_2 - x_1)^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; therefore

$$1 - e^2 \cos^2 \phi' = \{(1 - e^2)(x_2 - x_1)^2 + (y_2 - y_1)^2\}/l_{12}^2 = b^2 \{(\cos \phi_2 - \cos \phi_1)^2 + (\sin \phi_2 - \sin \phi_1)^2\}/l_{12}^2$$
$$= 2b^2 \{1 - \cos(\phi_1 - \phi_2)\}/l_{12}^2,$$

and $2p' \sqrt{\sin^2 \frac{1}{2}(\phi_1 - \phi_2)} = l_{12}$, the length of the chord joining ϕ_1, ϕ_2 on the ellipse.

But if i' is the angle of incidence at ϕ', $\pi - 2i'$ is the angle between the two chords meeting at ϕ', which if the correspondence is reciprocal is the difference of ϕ_1 and ϕ_2; and consequently $l_{12} = 2p' \cos i'$ as stated above.

The sum $\Sigma 2p \cos i$ always represents a length traversed by the ray, but that length is measured from the foot of perpendicular from centre on one ray-line to another such point. For a cyclic path it is the equivalent of Σl, and it is only in connexion with this point that the question of cycle enters into the argument.

§ 6. To find in what way the types of cycle analysed above enter into the scheme of correspondence, we must attend to the correspondence of terminal positions, which is of two kinds:

(α) a parallel to either axis, and an end of the same axis,

(β) a central chord, and a point at which the ellipse is met normally.

The first is evident, and with respect to the second we observe that a normal chord is traversed in opposite directions, and the points corresponding to these directions have eccentric angles differing by two right angles, or the chord joining them passes through the centre. For intra-focal cycles then we have the following cases of correspondence.

(i) (*ab*) and (*cd*). To obtain the closed figure the open figure must be followed from end to end and back. If the closed figure (*cd*) is traversed in a reverse direction the open figure must be turned upside down. This applies to Figs. 1, 4, 9.

(ii) (*ad*) and (*bc*). If the closed figure (*bc*) is traversed in a reverse direction, the open figure must be reversed from left to right. This applies to Figs. 3, 10.

For the cases (*ac*) and (*bd*) a reference to their meaning and to (α), (β) shews that the terminals admit of self-correspondence in each, not of mutual correspondence; and in fact (iii) (*ac*) has correspondence with its own image in the major axis regarded as a

* Cf. A Kinematical theorem in Radiation, *Proc. Camb. Phil. Soc.* Vol. XVI. part 4, p. 333.

mirror (Figs. 2, 8), while (iv) (*bd*) (Fig. 6) has complete self-correspondence. In each case the correspondence theorem is valid as a method of construction, and the terms of Σl correspond to those of $\Sigma 2p \cos i$ in its own configuration but taken from opposite ends of the quarter-cycle. No correspondence is established between (*ac*) and (*bd*) either as regards construction or cyclic length.

Extra-focal cycles have correspondence as follows.

(v) For an odd number of sides there is correspondence between a case which has an end of the major axis and a parallel to the minor, and the case in which major and minor are interchanged. For pentagon and other figures presenting two or more species, those correspond which give like figures in the circle. The correspondence holds for the circle, but the two figures then differ only by rotation through a right angle.

(vi) For figures of $4n$ sides there is correspondence between those which have four points at the ends of the axes, and those which have four lines parallel to the axes, like species being implied as in (v).

(vii) Of the remaining extra-focal cases, those with $4n+2$ sides, one type has points at the ends of the major axis and parallels to the same; in the other minor takes the place of major. Each type has self-correspondence, but no correspondence between them is established.

§ 7. It is an appreciable advantage of the method of correspondence and use of the parallel ruler that no subsidiary *lines* need be drawn in the figure; subsidiary *points* on the auxiliary circle must of course be marked. If we invoke the aid of analysis to form an eliminant and obtain a numerical value of *one* terminal variable, a correct drawing may be secured at the first attempt. Either the terminal found or that of the reciprocal figure presents a chord given in position, so that we have a basis for the construction of figures *pari passu* as on p. 458. Also in one or the other figure the last line drawn in the quarter-cycle will be a chord which should pass through an end of an axis or through the centre. If we rely entirely on graphical work we may expect a deviation from the true position of this last line, and then aim at securing a deviation in the contrary sense in a second attempt, when if the figure is possible an adjustment of starting point between the positions used should furnish a good approximation. If self-correspondence takes the place of correspondence of two figures, the construction proceeds from the two terminals and when the figure is completed one parallelism remains to be used as a verification of accuracy, or criterion of success or failure in the choice of initial position.

The analysis consists in writing the tangent of each eccentric angle (or its complement) equal to the tangent of an angle of inclination of a chord to one or the other axis. I have taken inclination to the minor axis, and written the formulae with such signs as to make all angles θ positive angles less than a right angle. The points to which the angles refer will be clear from formulae and figure taken together. The eliminant is often applicable to other cases than that for which it is originally obtained. The examples include cases up to cycles of 10 lines, intra-focal and extra-focal.

Intra-focal cycles. Ex. (1). (ab) and (cd) with $n=1$, V and lattice of 4 sides (fig. 1). The correspondence is given by

$$\tan\theta' = \frac{\sin\theta}{\sqrt{1-e^2}\,(1+\cos\theta)},\quad \tan\theta = \frac{\tan\theta'}{\sqrt{1-e^2}};\ \text{whence } 1-e^2 = e^2\cos\theta = e^2\cos^2\theta'.$$

The condition is $2e^2>1$, and an easy construction is to make the central chord equal to the distance between the foci.

Ex. (2). (ab) and (cd) with $n=2$, W and lattice of 8 sides. The formulae with signs suited to the type admitted at the greater limit $e=\sin 67\frac{1}{2}°$ are

$$\tan\theta_1 = \frac{\tan\theta_1'}{\sqrt{1-e^2}},\quad \tan\theta_2 = \frac{\sin\theta_2' - \sin\theta_1'}{\sqrt{1-e^2}\,(\cos\theta_1' + \cos\theta_2')},$$

$$\tan\theta_1' = \frac{\sin\theta_1 - \sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_1+\cos\theta_2)},\quad \tan\theta_2' = \frac{\sin\theta_2}{\sqrt{1-e^2}\,(1+\cos\theta_2)};$$

leading to $(1-e^2)^4 - 4e^2(1-e^2)^2\cos\theta_1 - 2e^4(1-e^2)^2\cos^2\theta_1 - 4e^4(1-e^2)\cos^3\theta_1 + e^8\cos^4\theta_1 = 0$, or to $(1-e^2)^4 - 4e^2(1-e^2)(1+e^2)^2\cos^2\theta_1' + 2e^4(1-e^2)(3+5e^2)\cos^4\theta_1' - 4e^6(1-e^2)\cos^6\theta_1' + e^8\cos^8\theta_1' = 0$, either eliminant valid also for the type admitted at the lower limit $e=\sin 22\frac{1}{2}°$. For $b/a = 7/8$ the solution is $\cos\theta_1 = {\cdot}542407$, which applies to the type admitted at $e=\sin 22\frac{1}{2}°$, fig. 4. The other type is shown in fig. 9 for which $b/a=1/3$ and $\cos\theta_1 = {\cdot}73468$.

Ex. (3). (ac) with $n=2$, N form (fig. 2). The self-correspondence is expressed by

$$\tan\theta_1 = \frac{\tan\theta_2}{\sqrt{1-e^2}},\quad \tan\theta_2 = \frac{\sin\theta_1 - \sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_1+\cos\theta_2)},$$

leading to $$e^2\sin^2\theta_2 = 1 - 2\sqrt{1-e^2},\quad e^6\sin^2\theta_1 = 3e^2 - 2 - 2(1-e^2)^{\frac{3}{2}}.$$

The condition of existence is $4e^2>3$, and a construction is got by making the central chord $=2(a-b)$.

Ex. (4). (ac) with $n=3$, open figure of 5 lines. The self-correspondence with signs for the form admitted at $e=\sin 36°$ is

$$\tan\theta_1 = \frac{\tan\theta_3}{\sqrt{1-e^2}},\quad \tan\theta_2 = \frac{\sin\theta_3 - \sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_3+\cos\theta_2)},\quad \tan\theta_3 = \frac{\sin\theta_1+\sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_1+\cos\theta_2)},$$

leading to

$$(1+\sqrt{1-e^2})(1-e^4\sin^2\theta_1)(2e^2-1-e^4\sin^2\theta_1) + 2e^2\sqrt{1-e^2}\,\{1-e^2(2-e^2)\sin^2\theta_1\}\sqrt{1-e^2\sin^2\theta_1} = 0.$$

With $z=\sqrt{1-e^2\sin^2\theta_1}$, if $b/a=4/5$ the equation is $256+128z-328z^3-81z^4=0$, and $z={\cdot}97823$, $\sin\theta_1={\cdot}34603$, the value used in fig. 8. The eliminant for the form admitted at $e=\sin 72°$ is

$$(1-\sqrt{1-e^2})(1-e^4\sin^2\theta_1)(2e^2-1-e^4\sin^2\theta_1) - 2e^2\sqrt{1-e^2}\,\{1-e^2(2-e^2)\sin^2\theta_1\}\sqrt{1-e^2\sin^2\theta_1} = 0.$$

For $$b/a=1/4,\quad 1-10z+170z^3-225z^4=0,\quad z={\cdot}671717,\quad \sin\theta_1={\cdot}765103.$$

Ex. (5). (bd) with $n=1$, lattice with 6 sides (fig. 6). Self-correspondence gives

$$\tan\theta = \frac{\sin\theta}{\sqrt{1-e^2}\,(1+\cos\theta)},\quad \text{or}\quad e^2\cos\theta = \sqrt{1-e^2}+1-e^2.$$

The condition of existence is $4e^2 > 3$, and a simple construction is got by making a chord through an end of the minor axis $= a$.

Ex. (6). (*bd*) with $n = 2$, lattice with 10 sides (fig. 5). The self-correspondence with signs as for the form admitted at $e = \sin 36°$ is

$$\tan\theta_1 = \frac{\sin\theta_1 + \sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_1 + \cos\theta_2)}, \quad \tan\theta_2 = \frac{\sin\theta_1}{\sqrt{1-e^2}\,(1+\cos\theta_1)},$$

leading to

$$(1-e^2)\sqrt{1-e^2} + \{1 - e^2 - (1+e^2)\sqrt{1-e^2}\}\cos\theta_1$$
$$+ \{2(1-e^2) - (2-e^2)\sqrt{1-e^2}\}\cos^2\theta_1 - e^2(1-\sqrt{1-e^2})\cos^3\theta_1 = 0.$$

Fig. 5 has $b/a = 4/5$ which makes $\cos\theta_1 = \cdot 944$. The eliminant for the form admitted at $e = \sin 72°$ is

$$(1-e^2)\sqrt{1-e^2} - \{1 - e^2 + (1+e^2)\sqrt{1-e^2}\}\cos\theta_1 - \{2(1-e^2) + (2-e^2)\sqrt{1-e^2}\}\cos^2\theta_1$$
$$+ e^2(1+\sqrt{1-e^2})\cos^3\theta_1 = 0.$$

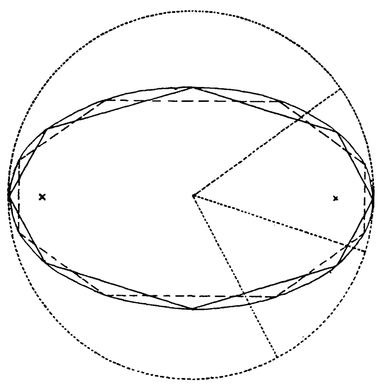

Fig. 7.

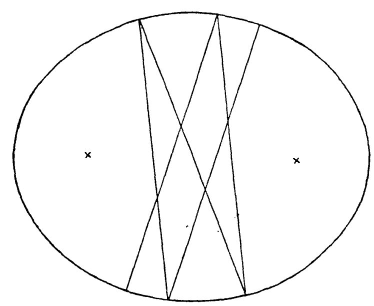

Fig. 8.

Ex. (7). (*ad*) and (*bc*), open lattice of 3 lines and star with 6 sides (fig. 3). The correspondence is

$$\tan\theta_1 = \frac{\tan\theta}{\sqrt{1-e^2}}, \quad \tan\theta_2 = \frac{\sin\theta}{\sqrt{1-e^2}\,(1+\cos\theta)}, \quad \tan\theta = \frac{\sin\theta_1 + \sin\theta_2}{\sqrt{1-e^2}\,(\cos\theta_1 + \cos\theta_2)};$$

whence $\sin^2\theta + 2\sin\theta = (1-e^2)/e^2$ or $\sin\theta = \dfrac{1}{e} - 1$, requiring $e > \frac{1}{2}$.

For $n = 3$ in (*ad*) and (*bc*) the solution in fig. 10 for $b/a = 1/2$ was obtained by graphical work only.

Extra-focal cycles. Ex. (8). Triangle (fig. 11). The correspondence is

$$\tan\theta' = \frac{1+\sin\theta}{\sqrt{1-e^2}\cos\theta}, \quad \tan\theta = \frac{\sin\theta'}{\sqrt{1-e^2}\,(1+\cos\theta')},$$

whence $e^2\sin^2\theta + 2(1-e^2)\sin\theta - 1 = 0$, and $e^2\sin\theta = \sqrt{1-e^2+e^4} - 1 + e^2$. Note result

$$\frac{\cos\theta}{b} = \frac{\sin\theta'}{a}.$$

Ex. (9). Pentagon. The correspondence with signs as for uncrossed pentagon is

$$\tan\theta_2' = \frac{1-\sin\theta_1}{\sqrt{1-e^2}\cos\theta_1}, \quad \tan\theta_1' = \frac{\sin\theta_2+\sin\theta_1}{\sqrt{1-e^2}(\cos\theta_1-\cos\theta_2)},$$

$$\tan\theta_1 = \frac{\sin\theta_1'-\sin\theta_2'}{\sqrt{1-e^2}(\cos\theta_1'+\cos\theta_2')}, \quad \tan\theta_2 = \frac{\sin\theta_1'}{\sqrt{1-e^2}(1-\cos\theta_1')};$$

leading to

$$1-2(1+e^2)\sin\theta_1-(4-9e^2)\sin^2\theta_1+8e^2(1-e^2)\sin^3\theta_1-e^4(9-4e^2)\sin^4\theta_1$$
$$+2e^4(1+e^2)\sin^5\theta_1-e^6\sin^6\theta_1=0,$$

which for $e=0$ is satisfied by $\sin\theta_1=\sin 18°$ or $-\sin 54°$, the latter root referring to the pentagon with 144° at the centre. The value $\sin\theta_1 = \cdot 3578$ for $b/a=4/5$ is used in fig. 12.

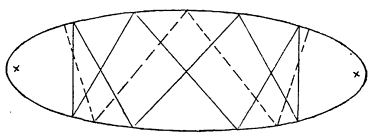

Fig. 9.

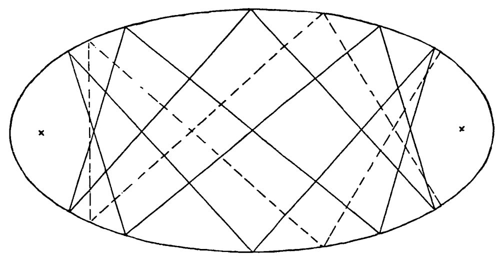

Fig. 10.

Ex. (10). Hexagon. The self-correspondence for the form with parallels to the minor axis is $\tan\theta = \dfrac{\sin\theta}{\sqrt{1-e^2}(1-\cos\theta)}$, or $(1+\sqrt{1-e^2})\cos\theta=\sqrt{1-e^2}$; that for the form with parallels to the major axis is $\tan\theta' = \dfrac{1-\sin\theta'}{\sqrt{1-e^2}\cos\theta'}$, or $(1+\sqrt{1-e^2})\sin\theta'=1$, with $\dfrac{\cos\theta}{b}=\dfrac{\sin\theta'}{a}$ as in Ex. (8).

Simple constructions are got by making the chords not parallel to an axis equal to a or b in the respective forms.

Ex. (11). Octagon. The correspondences are

$$\tan\theta = \frac{\sin\theta_1-\sin\theta_2}{\sqrt{1-e^2}(\cos\theta_2-\cos\theta_1)}, \quad \tan\theta_1 = \frac{\sin\theta}{\sqrt{1-e^2}(1-\cos\theta)}, \quad \tan\theta_2 = \frac{1-\sin\theta}{\sqrt{1-e^2}\cos\theta};$$

from which $(1-e^2)\sin\theta=\cos\theta(1-e^2\sin\theta)$, leading to

$$1-2e^2\sin\theta-2(1-e^2)\sin^2\theta+2e^2\sin^3\theta-e^4\sin^4\theta=0.$$

The value $\sin\theta=\cdot 80344$ for $b/a=3/5$ is used in fig. 7. The crossed form of octagon has $(1-e^2)\sin\theta=\cos\theta(1+e^2\sin\theta)$, giving the previous eliminant with $-\sin\theta$ for $\sin\theta$. In this form we pass from A to a point in the 2nd quadrant, and thence to the lower end of the minor axis. The value $\sin\theta=-\cdot 90038$ for $b/a=4/5$ is used in fig. 13.

Ex. (12). Decagon. The self-correspondence for the case with parallels to the minor axis is

$$\tan\theta_2 = \frac{\sin\theta_1}{\sqrt{1-e^2}\,(1-\cos\theta_1)},\quad \tan\theta_1 = \frac{\sin\theta_2 - \sin\theta_1}{\sqrt{1-e^2}\,(\cos\theta_1-\cos\theta_2)};$$

leading to

$$(1-e^2)\sqrt{1-e^2} + \{1-e^2+(1+e^2)\sqrt{1-e^2}\}\cos\theta_1 - \{2(1-e^2)+(2-e^2)\sqrt{1-e^2}\}\cos^2\theta_1$$
$$- e^2(1+\sqrt{1-e^2})\cos^3\theta_1 = 0,$$

which for $e=0$ has roots $\cos\theta_1 = \cos 36^\circ$ or $\cos 108^\circ$. If for the case with parallels to the major axis we use eccentric angles,

$$\tan\phi_2 = \frac{\sqrt{1-e^2}\sin\phi_1}{1-\cos\phi_1},\quad \tan\phi_1 = \frac{\sqrt{1-e^2}(\sin\phi_2-\sin\phi_1)}{\cos\phi_1-\cos\phi_2};$$

with eliminant

$$1+\{1-2e^2+\sqrt{1-e^2}\}\cos\phi_1 - \{2-e^2+2\sqrt{1-e^2}\}\cos^2\phi_1 + e^2(1+\sqrt{1-e^2})\cos^3\phi_1 = 0,$$

derivable from the previous by using ϕ for θ and a/b for b/a. The value used in fig. 14, a deformation of the decagon with angle 108° at the centre, is $\cos\phi_1 = -\cdot 40204$ for $b/a = 4/5$.

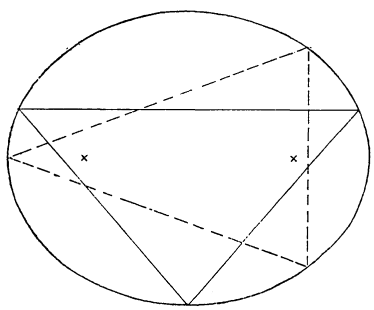

Fig. 11.

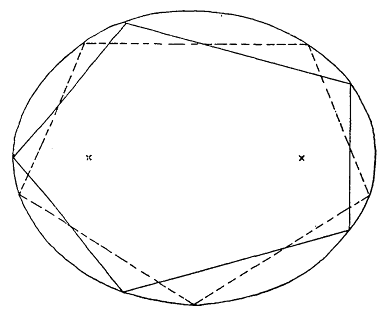

Fig. 12.

§ 8. The conditions of existence for intra-focal types formed a part of the treatment of the initial stage; it may be useful in connexion with two or three examples to point out how these conditions enter into and affect the eliminant in its general form. In Ex. (2) $\cos\theta_1$ is given by

$$(1-e^2)^4 - 4e^2(1-e^2)^2\cos\theta_1 - 2e^4(1-e^2)^2\cos^2\theta_1 - 4e^4(1-e^2)\cos^3\theta_1 + e^8\cos^4\theta_1 = 0.$$

We are concerned only with values of $\cos\theta_1$ between 0 and $+1$, at which limits the left-hand member has the values $(1-e^2)^4$ and $1-8e^2+8e^4$ respectively. The equation of condition for the type is $u_3 = 0$, and $u_3 = 1-8e^2+8e^4$, so that a connexion is at once apparent. It is hardly necessary to give a detailed argument, but the position is that for $e < \sin 22\frac{1}{2}^\circ$ (the roots of the equation in e are $\sin 22\frac{1}{2}^\circ$ and $\sin 67\frac{1}{2}^\circ$) there is no suitable value of $\cos\theta_1$, for $\sin 67\frac{1}{2}^\circ > e > \sin 22\frac{1}{2}^\circ$ there is one value, and for $e > \sin 67\frac{1}{2}^\circ$ there are two values. Each form persists for values of e greater than that at which it is admitted.

With this may be contrasted the extra-focal case of Ex. (11), where the left-hand member of the equation in $\sin\theta$ has for $\sin\theta = -1,\ 0,\ +1$ the values $-(1-e^2)^2,\ 1,\ -(1-e^2)^2$ in which the signs are not dependent on the numerical value of e.

There is one case in which extra-focal and intra-focal cycles are or may be given by a single equation of condition, viz. (bd) and the extra-focal of $4n+2$ sides with two lines parallel to the minor axis. The first eliminant in Ex. (12) is

$$(1-e^2)\sqrt{1-e^2}+\{1-e^2+(1+e^2)\sqrt{1-e^2}\}\cos\theta_1 - \{2(1-e^2)+(2-e^2)\sqrt{1-e^2}\}\cos^2\theta_1 - e^2(1+\sqrt{1-e^2})\cos^3\theta_1 = 0,$$

which differs only in the sign of $\cos\theta_1$ from the second eliminant in Ex. (6) corresponding to the root admitted at $e=\sin 72^\circ$. For $\cos\theta_1 = 1,\ 0,\ -1,\ -\infty$ the left-hand member has the values -1, $+(1-e^2)\sqrt{1-e^2}$, $4e^2-3-2\sqrt{1-e^2}$, $+\infty$; the third of these values is negative for values of $e<\sin 72^\circ$ (the higher root of $16e^4-20e^2+5=0$), but is positive for $e>\sin 72^\circ$. There is always one root between 0 and $+1$, and another root between 0 and -1, these roots corresponding to the extra-focal cycles. If $e<\sin 72^\circ$ the third root falls between -1 and $-\infty$, but if $e>\sin 72^\circ$ it falls between 0 and -1 and is then applicable to the intra-focal case. The two negative roots are clearly separated, for a chord from B to a point θ passes through a focus if $1-e^2+(1+e^2)\cos\theta=0$, which real value makes the left-hand member of the equation negative. The first eliminant in Ex. (6) corresponding to the form admitted at $e=\sin 36^\circ$ is quite different in character, it has no roots for less values of e and has no connexion with extra-focal forms.

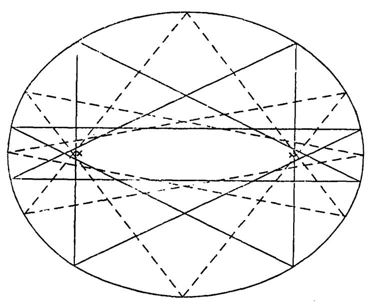

Fig. 13.

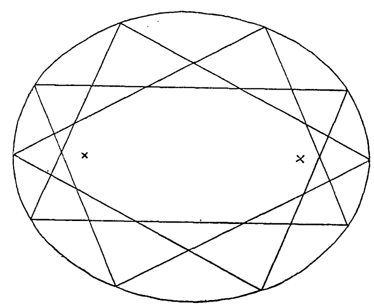

Fig. 14.

§ 9. The lengths of the cycles may be stated explicitly when the equation of condition is not of degree higher than the second. The axes themselves are cycles of lengths $4a$ and $4b$; with L for length of cycle other cases are as follows.

Intra-focal. (ab) and (cd), V and lattice of 4 sides, $L=4a/e$, condition $2e^2>1$.

(ad) and (bc), open lattice and star pattern of 6 sides, $L=4a\sqrt{2-e}\left(\sqrt{e}+\dfrac{1}{\sqrt{e}}\right)$, condition $e>\tfrac{1}{2}$.

(ac) and (bd), N and lattice of 6 sides, $L=4(a^2-ab+b^2)/(a-b)$, condition $4e^2>3$.

Extra-focal. Rhombus and rectangle, $L=4\sqrt{a^2+b^2}$.

Triangles with sides parallel to respective axes have

$$L = 6a\sqrt{1+e^2+2\sqrt{1-e^2+e^4}}/(1+e^2+\sqrt{1-e^2+e^4})$$
$$= 6a^2\sqrt{2a^2-b^2+2\sqrt{a^4-a^2b^2+b^4}}/(2a^2-b^2+\sqrt{a^4-a^2b^2+b^4})$$
$$= 2\sqrt{3}\,(a^2+b^2+\sqrt{a^4-a^2b^2+b^4})/\sqrt{a^2+b^2+2\sqrt{a^4-a^2b^2+b^4}};$$

in reality symmetrical with regard to a and b, though in the earlier form it appears unsymmetrical.

Hexagons whether they have sides parallel to the major or to the minor axis have

$$L = 4(a^2 + ab + b^2)/(a + b).$$

The reader will recall that in two out of the three intra-focal pairs, and in two out of the three extra-focal pairs, equality in the length of circuit for the members of a pair was established as part of the correspondence theorem. In the third pair of each kind there is self-correspondence for each member of the pair, but no connexion was established between them. Though not proved, it seems probable that the equality which we find to exist in the simplest cases of these types is a general property. In the extra-focal case of $4n + 2$ sides it depends on the cyclic path being a symmetrical function of a and b, which is probably the case for all extra-focal cycles. For the intra-focal types (ac) and (bd) I have some evidence in graphical work that the equality exists for cases of higher order.

With respect to each intra-focal type it will be observed that with increase of the number of variables the limiting values of eccentricity are extended in both directions, towards 0 and towards 1. For an ellipse of small eccentricity, apart from the minor axis itself, only intra-focal circuits for which n is great are possible. Thus in a process of gradual deformation of a circle to an ellipse the intra-focal circuits first admissible have a great length of path, while extra-focal circuits are slightly altered from the shapes and lengths of polygons in a circle.

§ 10. In conclusion it may be shown that the theorem of correspondence applies also to the ellipsoid. If (lmn) is a point on a unit sphere, (al, bm, cn) may be taken for coordinates of a point on an ellipsoid. The correspondence of the point $(x'y'z')$ with the chord joining $(x_1y_1z_1)$ to $(x_2y_2z_2)$ is then expressed by

$$\frac{x_2 - x_1}{r_{12}} = l', \quad \frac{y_2 - y_1}{r_{12}} = m', \quad \frac{z_2 - z_1}{r_{12}} = n'.$$

Thus $$\frac{1}{p'^2} = \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4} = \frac{l'^2}{a^2} + \frac{m'^2}{b^2} + \frac{n'^2}{c^2} = \frac{1}{r_{12}^2}\left\{\frac{(x_2 - x_1)^2}{a^2} + \frac{(y_2 - y_1)^2}{b^2} + \frac{(z_2 - z_1)^2}{c^2}\right\},$$

i.e. $r_{12}^2 = 2p'^2(1 - l_1l_2 - m_1m_2 - n_1n_2)$, or with a reciprocal correspondence $r_{12} = 2p' \cos i'$ as for the ellipse.

The original relation then gives $2p'l' \cos i' = a(l_2 - l_1)$, and two others. But if $(\lambda'\mu'\nu')$ are direction-cosines of the normal at $(x'y'z')$,

$$a^2\lambda' = p'x' \quad \text{or} \quad a\lambda' = p'l', \quad b\mu' = p'm', \quad c\nu' = p'n',$$

and so $$2\lambda' \cos i' = l_2 - l_1, \quad 2\mu' \cos i' = m_2 - m_1, \quad 2\nu' \cos i' = n_2 - n_1,$$

i.e. the normal in question is parallel to the chord from $(l_1m_1n_1)$ to $(l_2m_2n_2)$ on the sphere.

Thus with a reciprocal correspondence the chords from $(x_1y_1z_1)$ to $(x_2y_2z_2)$ and from $(x_2y_2z_2)$ to $(x_3y_3z_3)$ and the normal at $(x_2y_2z_2)$ are parallel respectively to radii to $(l'm'n')$ and $(l''m''n'')$ on the sphere and the chord joining them. This involves the law of reflexion.

No attempt has been made to deal with the problem of cyclic paths in an ellipsoid.

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THE PROBLEM OF 'DERANGEMENT' IN THE THEORY OF PERMUTATIONS

BY

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XVIII. *The Problem of 'Derangement' in the Theory of Permutations.*

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Section I.

1. In a paper entitled 'A Certain Class of Generating Functions in the Theory of Numbers,' *Phil. Trans. Roy. Soc.*, 1894, A. pp. 111—160, I have given a solution of the general problem of 'derangement' in the form of a symmetric function generating function. It was therein established that the number of permutations of the assemblage of letters

$$x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

which are such that exactly m of the letters are in the places they originally occupied, is equal to the coefficient of the term

$$a^m x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n},$$

in the development of the product

$$(ax_1 + x_2 + \dots + x_n)^{\xi_1} (x_1 + ax_2 + x_3 + \dots + x_n)^{\xi_2} \dots (x_1 + x_2 + \dots + x_{n-1} + ax_n)^{\xi_n}.$$

This is the redundant generating function.

It was also shewn that the same number is the coefficient of the term $a^m x_1^{\xi_1} x_2^{\xi_2} \dots x_n^{\xi_n}$ or of $a^m (\xi_1 \xi_2 \dots \xi_n)$, in the notation of symmetric functions, in the development of the algebraic fraction

$$\frac{1}{1 - a\Sigma x_1 + (a-1)(a+1)\,\Sigma x_1 x_2 - (a-1)^2 (a+2)\,\Sigma x_1 x_2 x_3 + \dots + (-)^n (a-1)^{n-1} (a+n-1)\, x_1 x_2 \dots x_n}.$$

This does not involve the numbers $\xi_1, \xi_2, \dots \xi_n$ and is the condensed generating function. If the number in question be denoted by

$$\{m;\ \xi_1 \xi_2 \dots \xi_n\}$$

the condensed generating function is clearly

$$\Sigma \{m;\ \xi_1 \xi_2 \dots \xi_n\} (\xi_1 \xi_2 \dots \xi_n)\, a^m.$$

I propose, in this short paper, to submit the generating function to examination so as to determine the properties of the number

$$\{m;\ \xi_1 \xi_2 \dots \xi_n\}.$$

2. I write $(x-x_1)(x-x_2)\dots(x-x_n)=x^n-p_1x^{n-1}+p_2x^{n-2}-\dots,$
so that the generating function is

$$\frac{1}{1-ap_1+(a-1)(a+1)p_2-(a-1)^2(a+2)p_3+\dots+(-)^n(a-1)^{n-1}(a+n-1)p_n}.$$

When, in the generating functions, we put $a=0$ the numbers generated are

$$\{0;\ \xi_1\xi_2\dots\xi_n\};$$

and the functions become

$$(x_2+x_3+\dots+x_n)^{\xi_1}(x_1+x_3+\dots+x_n)^{\xi_2}\dots(x_1+x_2+\dots+x_{n-1})^{\xi_n},$$

and
$$\frac{1}{1-p_2-2p_3-\dots-(n-1)p_n}.$$

If $\xi_1=\xi_2=\dots=\xi_n=1$ some properties of the numbers are obtained in a very simple manner from the redundant function

$$(x_2+x_3+\dots+x_n)(x_1+x_3+\dots+x_n)\dots(x_1+x_2+\dots+x_{n-1}),$$

for this may be written

$$(p_1-x_1)(p_1-x_2)\dots(p_1-x_n),$$

or
$$p_1^{n-2}p_2-p_1^{n-3}p_3+\dots+(-)^np_n,$$

and observing that

$$p_1^{n-s}p_s=(1)^{n-s}(1^s)=\dots+\frac{n!}{s!}p_n+\dots,$$

when $(1)^{n-s}(1^s)$ is multiplied out so as to be expressed as a linear function of monomial symmetric functions, the coefficient of $x_1x_2\dots x_n$ or of p_n is seen to be

$$\frac{n!}{2!}-\frac{n!}{3!}+\dots+(-)^n\frac{n!}{n!},$$

which may be written

$$n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots+(-)^n\frac{1}{n!}\right\},$$

the well known value of the number

$$\{0;\ 1^n\}$$

which is met with in the 'Problème des Rencontres.'

3. Similarly it is easy to find an expression for

$$\{m;\ 1^n\};$$

for, retaining a in the redundant generating function and putting

$$1-a=b,$$

it becomes
$$(p_1-bx_1)(p_1-bx_2)\dots(p_1-bx_n);$$

or
$$p_1^n-bp_1^n+b^2p_1^{n-2}p_2-\dots+(-)^nb^np_n.$$

Developing this expression, so as to obtain the coefficient of p_n, we find

$$n!\left\{1-b+\frac{b^2}{2!}-\dots+(-)^n\frac{b^n}{n!}\right\};$$

and expressing this in terms of a we readily find

$$\{0;\ 1^n\} + \binom{n}{1} a \{0;\ 1^{n-1}\} + \binom{n}{2} a^2 \{0;\ 1^{n-2}\} + \ldots + \binom{n}{n} a^n;$$

and thence $$\{m;\ 1^n\} = \binom{n}{m} \{0;\ 1^{n-m}\}.$$

It is readily seen *à priori* that this result is correct; for we can select the m letters, which are to remain undisplaced, in $\binom{n}{m}$ ways; and, for each of these selections, the remaining letters can be permuted so as to be all displaced in $\{0;\ 1^{n-m}\}$ ways.

The whole of the permutations can be found by summing $\{m;\ 1^n\}$ from $m=0$ to $m=n$. Hence the well known formula

$$\{0;\ 1^n\} + \binom{n}{1} \{0;\ 1^{n-1}\} + \binom{n}{2} \{0;\ 1^{n-2}\} + \ldots + \binom{n}{n} = n!$$

The reader is reminded that the early values of $\{0;\ 1^n\}$ are

1, 0, 1, 2, 9, 44, 265, 1854, 14833, ...

From the above results it is easy to shew that

$$\{0;\ 1^n\} = (n-1)\left[\{0;\ 1^{n-1}\} + \{0;\ 1^{n-2}\}\right],$$

$$\{0;\ 1^n\} = n\{0;\ 1^{n-1}\} + (-)^n,$$

both of them well known relations.

4. We will now consider the condensed generating function

$$\frac{1}{1 - p_2 - 2p_3 - 3p_4 - \ldots - (n-1)p_n},$$

wherein $\{0;\ \xi_1\xi_2\ldots\xi_n\}$ is the coefficient of the symmetric function $(\xi_1\xi_2\ldots\xi_n)$.

We may regard $\xi_1, \xi_2, \ldots \xi_n$ as numbers in descending order of magnitude and n as indefinitely great. The numbers $\xi_1, \xi_2, \ldots \xi_n$ may be any integers, zero not excluded, but there are certain symmetric functions that, from *à priori* considerations, must be absent. For clearly

$$\{0;\ \xi_1\xi_2\ldots\xi_n\} = 0,$$

if $\xi_1 > \xi_2 + \xi_3 + \ldots + \xi_n$. For example such functions as (21), (421), ... do not present themselves in the development.

In the first instance we will restrict ourselves to the numbers $\{0;\ 1^s\}$ which we will write P_s for convenience.

Write $$\frac{1}{1 - p_2 - 2p_3 - 3p_4 - \ldots} = 1 + P_1p_1 + P_2p_2 + \ldots + P_np_n + \text{other terms};$$

or $$1 = (1 - p_2 - 2p_3 - 3p_4 - \ldots)(1 + P_1p_1 + P_2p_2 + \ldots + P_np_n) + \text{other terms}.$$

On the right-hand side to find the coefficient of (1^n) or p_n the relevant terms are

$$P_np_n - P_{n-2}p_2p_{n-2} - 2P_{n-3}p_3p_{n-3} - \ldots - (n-3)P_2p_{n-2}p_2 - (n-2)P_1p_{n-1}p_1 - (n-1)p_n,$$

and since, if $n > 0$, the coefficient of p_n must be zero we have

$$P_n = \binom{n}{2} P_{n-2} + 2\binom{n}{3} P_{n-3} + 3\binom{n}{4} P_{n-4} + \ldots + (n-1)\binom{n}{n},$$

or

$$\{0;\ 1^n\} = \binom{n}{2}\{0;\ 1^{n-2}\} + 2\binom{n}{3}\{0;\ 1^{n-3}\} + \ldots + (n-1)\binom{n}{n},$$

a new relation, and the verification for $n=6$ is

$$265 = 1.15.9 + 2.20.2 + 3.15.1 + 4.6.0 + 5.1.1.$$

The law that has been established can be exhibited by putting

$$(n-s-1)\,P_s = Q_{s,n};$$

so that

$$Q_{n,n} = -P_n \text{ and } Q_{n-1,n} = 0;$$

then

$$0 = Q_{n,n} + \binom{n}{1} Q_{n-1,n} + \binom{n}{2} Q_{n-2,n} + \ldots + \binom{n}{n} Q_{0,n},$$

and now writing symbolically

$$Q_{s,n} = Q^s_{0,n},$$

$$(Q_{0,n} + 1)^n = 0.$$

5. Next consider the expansion of $(1 - p_2 - 2p_3 - 3p_4 - \ldots)^{-1}$ in ascending powers of $(p_2 + 2p_3 + 3p_4 + \ldots)$; we have

$$(p_2 + 2p_3 + 3p_4 + \ldots)^s = \Sigma \frac{s!}{s_2!\,s_3!\,s_4!\ldots} 1^{s_2}.\,2^{s_2}.\,3^{s_3}\ldots p_2^{s_2} p_3^{s_3} p_4^{s_4}\ldots,$$

where

$$s_2 + s_3 + s_4 + \ldots = s.$$

The coefficient herein of p_n or (1^n) is readily obtained because the coefficient of (1^n) in the development of the product

$$(1^2)^{s_2}(1^3)^{s_3}(1^4)^{s_4}\ldots$$

is, by a well known theorem of symmetry, equal to the coefficient of symmetric function $(2^{s_2}3^{s_3}4^{s_4}\ldots)$ in the expansion of (1^n); this, by the multinomial theorem, is

$$\frac{n!}{(2!)^{s_2}(3!)^{s_3}(4!)^{s_4}\ldots}.$$

Hence the portion of the right-hand side that we require is

$$\Sigma \frac{s!\,n!}{s_2!\,s_3!\,s_4!\ldots(2.0!)^{s_2}(3.1!)^{s_3}(4.2!)^{s_4}\ldots}(1^n),$$

the summation being in respect of all values of s_2, s_3, s_4, ... such that

$$2s_2 + 3s_3 + 4s_4 + \ldots = n,$$

$$s_2 + s_3 + s_4 + \ldots = s.$$

Thence the coefficient of (1^n) in the expansion of $(1 - p_2 - 2p_3 - 3p_4 - \ldots)^{-1}$ is

$$\{0;\ 1^n\} = \sum_s \Sigma \frac{s!\,n!}{s_2!\,s_3!\,s_4!\ldots(2.0!)^{s_2}(3.1!)^{s_3}(4.2!)^{s_4}\ldots},$$

where we have the additional summation in regard to all integer values of s.

This solution depends upon the non-unitary partitions of n; for $n=8$ the calculation is

Partition	gives	Number
2^4	,,	2520
$3^2 2$	,,	6720
42^2	,,	3780
4^2	,,	630
53	,,	896
62	,,	280
8	,,	7
	Total	14833

the value of $\{0; 1^8\}$.

6. The above method is only appropriate for obtaining results over a limited range of the expanded function. We require theorems of a more general character and the symmetric function operators are competent to produce them. The operators available are

$$d_s = \frac{d}{dp_s} + p_1 \frac{d}{dp_{s+1}} + p_2 \frac{d}{dp_{s+2}} + \dots,$$

$$D_s = \frac{1}{s!}(d_1^s).$$

Writing $p_2 + 2p_3 + 3p_4 + \dots = B$ it will be found that for the particular operand $(1-B)^{-1}$ these operators are connected by special relations and that every such relation is of significance in the theory of the generating function. A special object of the following investigation is the discovery of operators which have the effect of leaving the special operand unaltered.

Since $$d_s B = (s-1) + s(p_1 + p_2 + p_3 + \dots) + B, \text{ when } s > 0,$$
we find $$(d_s - d_t) B = (s-t)(1 + p_1 + p_2 + p_3 + \dots), \text{ when } t > 0;$$
whence if u, v be integers also, greater than zero,

$$(u-v)(d_s - d_t) B = (s-t)(d_u - d_v) B;$$

shewing that, for the particular operand B,

$$(u-v)(d_s - d_t) \text{ and } (s-t)(d_u - d_v)$$

are equivalent operators. Moreover since these operators are linear the equivalence obtains for any power of B and for the operand $(1-B)^{-1}$, u, v, s, t being any positive integers, zero excluded. This result has been reached by the elimination of $p_1 + p_2 + p_3 + \dots$ and B; if we only eliminate $p_1 + p_2 + p_3 + \dots$ we find

$$(td_s - sd_t) B = (s-t)(1-B);$$

leading to the important result

$$\frac{td_s - sd_t}{s-t} \frac{1}{1-B} = \frac{1}{1-B};$$

shewing that the operation

$$\frac{td_s - sd_t}{s-t} \text{ or } \frac{1}{s-t}\begin{vmatrix} d_s & d_t \\ s & t \end{vmatrix}$$

leaves the operand $(1-B)^{-1}$ unaltered.

7. The two results that have now been established lead to a large number of relations between the operators which may be applied forthwith to the study of the properties of the coefficients which arise in the development of the generating function $(1-B)^{-1}$. The difficulty lies in selecting the relations so as to best exhibit those properties. Generally, in applying the operator to the expanded form of the generating function, we first express the operators d_s in terms of the operators D_s in order to take advantage of the facility with which the latter operators are performed upon symmetric functions which are denoted by partitions.

It was shewn by Hammond* that the linear operators d_s have the expressions

$$d_1 = D_1,$$
$$d_2 = D_1^2 - 2D_2,$$
$$d_3 = D_1^3 - 3D_2D_1 + 3D_3,$$
$$\dots\dots\dots\dots\dots\dots$$

the law being the same as that which expresses the sums of powers of quantities in terms of their elementary symmetric functions. Moreover Hammond also shewed (*loc. cit.*) that the operators D_s have the effect

$$D_s(\lambda\mu\nu\,\dots) = 0,$$

if s is not included among the integers $\lambda, \mu, \nu, \dots$

$$D_\lambda(\lambda^l\mu\nu\,\dots) = (\lambda^{l-1}\mu\nu\,\dots),$$
$$D_\lambda{}^l D_\mu{}^m D_\nu{}^n \dots (\lambda^l\mu^m\nu^n) = 1.$$

We will in the first place consider the equivalence

$$(u-v)(d_s - d_t) \equiv (s-t)(d_u - d_v),$$

in the simple particular case obtained by putting $(u, v, s, t) = (3, 2, 2, 1)$, viz.

$$d_3 \equiv 2d_2 - d_1.$$

Transforming to the operators D_s, we see that the operation

$$(D_1^3 - 3D_2D_1 + 3D_3) - 2(D_1^2 - 2D_2) + D_1$$

reduces the function $(1-B)^{-1}$ to zero; writing the function in the form

$$\Sigma\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\dots\}.(1^{\pi_1}2^{\pi_2}3^{\pi_3}\dots),$$

and, after operating, equating the coefficient of $(1^{\pi_1}2^{\pi_2}3^{\pi_3}\dots)$ to zero, we find

$$\{0;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\dots\} - 3\{0;\ 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\dots\} + 3\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\dots\}$$
$$- 2[\{0;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\dots\} - 2\{0;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\dots\}] + \{0;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\dots\} = 0.$$

We thus obtain a linear relation between certain groups of numbers which are found throughout the whole extent of the expanded generating function; for the numbers $\pi_1, \pi_2, \pi_3, \dots$ are entirely at our disposal. The way in which the specification of the numbers is connected with the formula which expresses sums of powers in terms of elementary symmetric functions will be noted. In mathematical shorthand we may denote the above relation by

$$(3)\{\,\} - 2(2)\{\,\} + (1)\{\,\} = 0.$$

* *Proc. Lond. Math. Soc.*

As a simple example put $\pi_1 = 1$, $\pi_2 = \pi_3 = \ldots = 0$; then

$$\{0;\ 1^4\} - 3\{0;\ 21^2\} + 3\{0;\ 31\} - 2\{0;\ 1^3\} + 4\{0;\ 21\} + \{0;\ 1^2\} = 0,$$

and this relation, since $\{0;\ 31\} = \{0;\ 21\} = 0$, yields

$$3\{0;\ 21^2\} = \{0;\ 1^4\} - 2\{0;\ 1^3\} + \{0;\ 1^2\} = 9 - 4 + 1 = 6;$$

so that $\{0;\ 21^2\} = 2$, which is obviously correct.

8. In general from the relation

$$(u-v)(d_s - d_t) = (s-t)(d_u - d_v),$$

we proceed to the relation

$$(u-v)[(s)\{\ \} - (t)\{\ \}] = (s-t)[(u)\{\ \} - (v)\{\ \}],$$

a valuable property of the numbers under examination.

9. Next we see that the result

$$\frac{td_s - sd_t}{s-t}\frac{1}{1-B} = \frac{1}{1-B}$$

gives rise to the equivalence $td_s - sd_t \equiv s - t$. It will be found that there are several ways of dealing with this.

We will first consider the particular case

$$td_1 - d_t = -(t-1);$$

putting $t = 2$, we deduce

$$2!\,D_2 = D_1^2 - 2D_1 - 1;$$

putting $t = 3$ and reducing by means of the relation just found there results

$$3!\,D_3 = D_1^3 - 6D_1^2 + 3D_1 + 4,$$

and, thence similarly

$$4!\,D_4 = D_1^4 - 12D_1^3 + 30D_1^2 + 4D_1 - 15,$$

$$5!\,D_5 = D_1^5 - 20D_1^4 + 110D_1^3 - 140D_1^2 - 95D_1 + 56,$$

&c.

and it is clear that we can express D_s in terms of D_1.

To calculate these relations we remark that the algebraic equivalent of the relation $td_1 - d_t = -(t-1)$ is

$$tp_1 - (t) = -(t-1),$$

and since p_s corresponds to D_s we have to express p_s in terms of p_1 being given that

$$tp_1 - s_t = -(t-1),$$

where (t) has been replaced by s_t; thus since

$$2!\,p_2 = s_1^2 - s_2,$$

$$3!\,p_3 = s_1^3 - 3s_2s_1 + 2s_3,$$

$$4!\,p_4 = s_1^4 - 6s_2s_1^2 + 3s_2^2 + 8s_3s_1 - 6s_4,$$

&c.

we find
$$2\,!\,p_2 = p_1^2 - (2p_1+1),$$
$$3\,!\,p_3 = p_1^3 - 3(2p_1+1)\,p_1 + 2(3p_1+2),$$
$$4\,!\,p_4 = p_1^4 - 6(2p_1+1)\,p_1^2 + 3(2p_1+1)^2 + 8(3p_1+2)\,p_1 - 6(4p_1+3),$$
&c.

and now we have merely to write D_s for p_s to arrive at the relations under consideration. We can of course write down the general formula for D_s expressed as a function of the elements
$$D_1,\ 2D_1+1,\ 3D_1+2,\ 4D_1+3,\ \ldots.$$

10. Applying these relations to the expanded generating function we first obtain
$$2\,!\,\{0;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\} = \{0;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\} - 2\{0;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\} - \{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}.$$

A particular case, putting $\{0;\ 1^s\} = P_s$, is
$$2\,!\,\{0;\ 21^s\} = P_{s+2} - 2P_{s+1} - P_s;$$
a convenient formula for $\{0;\ 21^s\}$ which may be given another form by utilizing known properties of the numbers P_s, viz.:—
$$2\{0;\ 21^s\} = (s^2+s-1)\,P_s + (-)^{s-1}(s-1).$$

Thence we obtain values of $\{0;\ 21^s\}$

for
$$s = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ \ldots$$
$$\{0;\ 21^s\} = 0,\ 0,\ 2,\ 12,\ 84,\ 640,\ 5430,\ 50988,\ 526568,\ \ldots.$$

We next obtain
$$3\,!\,\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\} = \{0;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\} - 6\{0;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\}$$
$$+ 3\{0;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\} + 4\{0;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\};$$

and thence
$$6\{0;\ 31^s\} = P_{s+3} - 6P_{s+2} + 3P_{s+1} + 4P_s.$$

We derive values of $\{0;\ 31^s\}$, viz.:—for
$$s = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$
$$\{0;\ 31^s\} = 0,\ 0,\ 0,\ 6,\ 72,\ 780,\ \ldots.$$

Similarly
$$24\{0;\ 41^s\} = P_{s+4} - 12P_{s+3} + 30P_{s+2} + 4P_{s+1} - 15P_s,$$
$$120\{0;\ 51^s\} = P_{s+5} - 20P_{s+4} + 110P_{s+3} - 140P_{s+2} - 95P_{s+1} + 56P_s;$$
and generally, in the relation which expresses D_t in terms of powers of D_1, we are at liberty to substitute
$$\{0;\ t1^s\} \text{ for } D_t \text{ and } P_{s+\kappa} \text{ for } D_1^{\kappa}.$$

Also we may, more generally, substitute
$$\{0;\ 1^{\pi_1}2^{\pi_2}\ldots t^{\pi_t+1}\ldots\} \text{ for } D_t \text{ and } \{0;\ 1^{\pi_1+\kappa}2^{\pi_2}3^{\pi_3}\ldots\} \text{ for } D_1^{\kappa}.$$

The reader may also proceed to the relation
$$24\{0;\ 41^s\} = \left\{\frac{(s+4)!}{s!} - 12\frac{(s+3)!}{s!} + 30\frac{(s+2)!}{s!} + 4(s+1) - 15\right\}P_s + (-)^{s+1}(s-1)(s^2-3s-1);$$
and will find no difficulty in reaching the general formula.

Again, from the relation

$$2D_2 = D_1^2 - 2D_1 - 1,$$

is derived the formula of reduction

$$\{0;\ 2^r 1^s\} = \tfrac{1}{2}\{0;\ 2^{r-1}1^{s+2}\} - \{0;\ 2^{r-1}1^{s+1}\} - \tfrac{1}{2}\{0;\ 2^{r-1}1^s\}:$$

and a multitude of similar results are obtainable.

Section II.

11. The generating function

$$\frac{1}{1 - ap_1 + (a-1)(a+1)p_2 - (a-1)^2(a+2)p_3 + \ldots},$$

which when expanded is

$$\Sigma\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}\, a^m (1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots),$$

may be similarly dealt with. For write it $(1-C)^{-1}$ where

$$C = ap_1 - (a-1)(a+1)p_2 + (a-1)^2(a+2)p_3 - \ldots;$$

then $$d_s C = (-)^{s+1}(a-1)^s(1-C) + (-)^{s+1} s (a-1)^{s-1} E,$$

where $$E = 1 - (a-1)p_1 + (a-1)^2 p_2 - \ldots.$$

From this relation we obtain

$$\{d_s + (-)^{s-t+1}(a-1)^{s-t} d_t\}\, C = (-)^{s-1}(a-1)^{s-1}(s-t)E;$$

and, herein putting $1 - a = b$,

$$(b^{1-s}d_s - b^{1-t}d_t)\, C = (s-t)E;$$

shewing that, for an operand C,

$$(u-v)(b^{1-s}d_s - b^{1-t}d_t) \equiv (s-t)(b^{1-u}d_u - b^{1-v}d_v)$$

are equivalent operations. Since the operations are linear the equivalence persists when the operand is $(1-C)^{-1}$.

If, from the original expression of $d_s C$, we eliminate E we find

$$(tb^t d_s - sb^s d_t)\, C = (s-t)\, b^{s+t}(1-C);$$

whence $$\frac{tb^{-s}d_s - sb^{-t}d_t}{s-t} \cdot \frac{1}{1-C} = \frac{1}{1-C};$$

establishing that the operator

$$\frac{tb^{-s}d_s - sb^{-t}d_t}{s-t}$$

leaves the generating function unchanged. In other words the operator

$$tb^{-s}d_s - sb^{-t}d_t - s + t$$

causes the function to vanish.

In regard to the two operators

$$(u-v)(b^{1-s}d_s - b^{1-t}d_t) - (s-t)(b^{1-u}d_u - b^{1-v}d_v),$$

$$tb^{-s}d_s - sb^{-t}d_t - s + t,$$

which cause the generating function to vanish, it is to be remarked that, regarding b as being of weight unity and d_s as of weight s, the first operator is of weight unity and the second of weight zero. We are thus able to proceed to these operators from those obtained for the case $b=1$, viz.:—

$$(u-v)(d_s-d_t)-(s-t)(d_u-d_v),$$
$$td_s-sd_t-s+t,$$

by introducing, in each term, such a power of b as will make every term of the *same* weight.

12. From the first operator taking the simple case

$$d_3-2bd_2+b^2d_1=0,$$

we apply it to the expanded form

$$\Sigma\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}\,a^m(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)$$

of the generating function. We thus obtain

$$\begin{aligned}&\sum_{m,\pi}\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\}\,a^m\{(1^{\pi_1-3}2^{\pi_2}3^{\pi_3}\ldots)-3(1^{\pi_1-1}2^{\pi_2-1}3^{\pi_3}\ldots)+3(1^{\pi_1}2^{\pi_2}3^{\pi_3-1}\ldots)\}\\&-2\sum_{m,\pi}\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}(a^m-a^{m+1})\{(1^{\pi_1-2}2^{\pi_2}3^{\pi_3}\ldots)-2(1^{\pi_1}2^{\pi_2-1}3^{\pi_3}\ldots)\}\\&+\Sigma\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}(a^m-2a^{m+1}+a^{m+2})(1^{\pi_1-1}2^{\pi_2}3^{\pi_3}\ldots)=0.\end{aligned}$$

Herein selecting the coefficient of $(1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots)$ we have

$$\begin{aligned}&\sum_m[\{m;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots\}-3\{m;\ 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots\}+3\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\}]\,a^m\\&-2\sum_m[\{m;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\}-2\{m;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\}](a^m-a^{m+1})\\&+\sum_m\{m;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\}(a^m-2a^{m+1}+a^{m+2})=0,\end{aligned}$$

and, herein selecting the coefficient of a^m, we find that

$$\begin{aligned}&\{m;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots\}-3\{m;\ 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots\}+3\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\}\\&-2[\{m;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\}-2\{m;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\}]\\&+2[\{m-1;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\}-2\{m-1;\ 1^{\pi_1}2^{\pi_2-1}3^{\pi_3}\ldots\}]\\&+\{m;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\}-2\{m-1;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\}+\{m-2;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\}=0,\end{aligned}$$

a relation connecting ten of the coefficients.

In applying this formula it must be noticed that

$$\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\},$$

denoting as it does the number of permutations in which exactly m of the letters are not displaced, must be zero,

(i) when m is negative,

(ii) when $m>\Sigma s\pi_s$, i.e. greater than the number of letters in the permutation.

Also it is manifest that

$$\{\Sigma s\pi_s;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}=\{\Sigma s\pi_s-1;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\},$$

since if all but one of the letters are undisplaced then all must be so.

Bearing these facts in mind there is no difficulty in verifying the formula in some cases.

13. Generally we may obtain the relation between the coefficients corresponding to the relation

$$(u-v)(b^{1-s}d_s - b^{1-t}d_t) - (s-t)(b^{1-u}d_u - b^{1-v}d_v) = 0,$$

for writing this

$$(u-v)\,d_s - (u-v)\,b^{s-t}\,d_t - (s-t)\,b^{s-u}\,d_u + (s-t)\,b^{s-v}d_v = 0,$$

where s, t, u, v are in descending order of magnitude, and denoting the expressions

$$\{m;\ 1^{\pi_1+1}2^{\pi_2}3^{\pi_3}\ldots\},$$
$$\{m;\ 1^{\pi_1+2}2^{\pi_2}3^{\pi_3}\ldots\} - 2\{m;\ 1^{\pi_1}2^{\pi_2+1}3^{\pi_3}\ldots\},$$
$$\{m;\ 1^{\pi_1+3}2^{\pi_2}3^{\pi_3}\ldots\} - 3\{m;\ 1^{\pi_1+1}2^{\pi_2+1}3^{\pi_3}\ldots\} + 3\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3+1}\ldots\},$$

by $(1)\{\ \}_m$, $(2)\{\ \}_m$, $(3)\{\ \}_m$, ... respectively, we find

$$\begin{aligned}
&(u-v)(s)\{\ \}_m \\
&-(u-v)\left[(t)\{\ \}_m - \binom{s-t}{1}(t)\{\ \}_{m-1} + \binom{s-t}{2}(t)\{\ \}_{m-2} - \ldots\right] \\
&-(s-t)\left[(u)\{\ \}_m - \binom{s-u}{1}(u)\{\ \}_{m-1} + \binom{s-u}{2}(u)\{\ \}_{m-2} - \ldots\right] \\
&+(s-t)\left[(v)\{\ \}_m - \binom{s-v}{1}(v)\{\ \}_{m-1} + \binom{s-v}{2}(v)\{\ \}_{m-2} - \ldots\right] = 0,
\end{aligned}$$

a relation which, if N_s denotes the number of partitions of s and s, t, u, v are all *different*, involves

$$(s+1)(N_s + N_t + N_u + N_v) - (sN_s + tN_t + uN_u + vN_v) \text{ coefficients.}$$

14. Passing now to the relation

$$tb^{-s}d_s - sb^{-t}d_t - s + t = 0,$$

and putting $s=1$, $t=2$, we find

$$2!\,D_2 = D_1^2 - 2bD_1 - b^2;$$

and without difficulty we reach the further results

$$3!\,D_3 = D_1^3 - 6bD_1^2 + 3b^2D_1 + 4b^3,$$
$$4!\,D_4 = D_1^4 - 12bD_1^3 + 30b^2D_1^2 + 4b^3D_1 - 15b^4,$$
$$5!\,D_5 = D_1^5 - 20bD_1^4 + 110b^2D_1^3 - 140b^3D_1^2 - 95b^4D_1 + 56b^5,$$
&c.,

which can be written down, from those given by the case $b=1$, by simply introducing the proper power of b in each term.

Application of the first of these

$$2D_2 = D_1^2 - 2bD_1 - b^2,$$

yields the relation

$$\begin{aligned}
2\{m;\ 1^{\pi_1}2^{\pi_2+1}\ldots\} = \{m;\ 1^{\pi_1+2}2^{\pi_2}\ldots\} &- 2\{m;\ 1^{\pi_1+1}2^{\pi_2}\ldots\} - \{m;\ 1^{\pi_1}2^{\pi_2}\ldots\} \\
&+ 2\{m-1;\ 1^{\pi_1+1}2^{\pi_2}\ldots\} + 2\{m-1;\ 1^{\pi_1}2^{\pi_2}\ldots\} \\
&- \{m-2;\ 1^{\pi_1}2^{\pi_2}\ldots\},
\end{aligned}$$

of which a particular case is

$$2\{m;\ 21^s\} = \{m;\ 1^{s+2}\} - 2\{m;\ 1^{s+1}\} - \{m;\ 1^s\} + 2\{m-1;\ 1^{s+1}\} + 2\{m-1;\ 1^s\} - \{m-2;\ 1^s\};$$

and since

$$\{m;\ 1^s\} = \binom{s}{m}\{0;\ 1^{s-m}\},$$

we find

$$2\{m;\ 21^s\} = \left\{\binom{s+2}{m} + 2\binom{s+1}{m-1} - \binom{s}{m-2}\right\}\{0;\ 1^{s-m+2}\}$$
$$-2\left\{\binom{s+1}{m} - \binom{s}{m-1}\right\}\{0;\ 1^{s-m+1}\} - \binom{s}{m}\{0;\ 1^{s-m}\}.$$

The number $\{m;\ 1^{\pi_1}2^{\pi_2}3^{\pi_3}\ldots\}$ is ultimately expressible as a linear function of the numbers $\{0;\ 1^s\}$.

15. It is worth while remarking that the numbers $\{0;\ 1^s\}$ may be studied by means of the elementary notions of the Theory of Substitutions.

Every substitution which displaces the whole of the letters may be represented by a product of circular substitutions of order not less than 2. Such a substitution, displacing the whole of the letters, may be termed a non-unitary substitution and there is a one-to-one correspondence between the arrangement in which every letter is displaced and the non-unitary substitutions. We have therefore merely to enumerate the non-unitary substitutions. A certain number of such substitutions correspond to a particular non-unitary partition

$$(2^{\pi_2}3^{\pi_3}\ldots) \text{ of the number } n.$$

If we distribute the n letters in any manner into $\pi_2+\pi_3+\ldots$ parcels so that π_s parcels each contain s letters, where s has the values 2, 3, ..., we obtain a definite circular substitution corresponding to any assigned order of the letters in the parcels. Now we observe that a parcel which contains s letters may have the letters permuted in $(s-1)!$ different ways so as to give $(s-1)!$ different circular substitutions because $(s-1)!$ is the number of permutations of s different letters which are arranged in circular order; so that if N be the number of ways of distributing n different letters into $\pi_2+\pi_3+\ldots$ parcels so that $\pi_2, \pi_3, \ldots$ parcels contain 2, 3, ... letters respectively, the number of substitutions thence derivable is

$$N(1!)^{\pi_2}(2!)^{\pi_3}(3!)^{\pi_4}\ldots.$$

We can find N because it is known to be the coefficient of (1^n) in the development which arises when the product

$$\frac{p_2^{\pi_2}p_3^{\pi_3}p_4^{\pi_4}\ldots}{\pi_2!\,\pi_3!\,\pi_4!\ldots}$$

is multiplied out. The coefficient is

$$\frac{n!}{(2!)^{\pi_2}(3!)^{\pi_3}(4!)^{\pi_4}\ldots\pi_2!\,\pi_3!\,\pi_4!\ldots},$$

and

$$N(1!)^{\pi_2}(2!)^{\pi_2}(3!)^{\pi_3}\ldots = \frac{n!}{2^{\pi_2}.\,3^{\pi_3}.\,4^{\pi_4}\ldots\pi_2!\,\pi_3!\,\pi_4!\ldots}.$$

Hence the total number of non-unitary substitutions or of permutations which displace every letter is

$$\{0;\ 1^n\} = \sum_\pi \frac{n!}{2^{\pi_2}.3^{\pi_3}.4^{\pi_4}\ldots\pi_2!\,\pi_3!\,\pi_4!\ldots},$$

the summation being for every non-unitary partition

$$(2^{\pi_2}\,3^{\pi_3}\,4^{\pi_4}\ldots),$$

of the number n.

16. The interest of the above solution lies in the comparison with the result previously reached in Section I. Art. 5, viz. :—

$$\{0;\ 1^n\} = \sum_\pi \left\{\frac{n!}{2^{\pi_2}.3^{\pi_3}.4^{\pi_4}\ldots.\ \pi_2!\,\pi_3!\,\pi_4!\ldots}\cdot\frac{(\pi_2+\pi_3+\pi_4+\ldots)!}{(2!)^{\pi_4}(3!)^{\pi_5}\ldots}\right\}.$$

This old expression gives for $n=6$

$$120.\frac{1}{24}+90.\frac{2}{2}+40\frac{2}{1}+15.6=265.$$

The new expression gives

$$\frac{6!}{6}+\frac{6!}{2.4}+\frac{6!}{9.2}+\frac{6!}{8.6}=265,$$

the four terms corresponding in each case to the partitions

(6), (42), (3^2), (2^3) respectively.

The identity which presents itself, of the form

$$\sum_\pi A_{\pi_1,\pi_2,\pi_3,\ldots} = \sum A_{\pi_1,\pi_2,\pi_3,\ldots}\,B_{\pi_1,\pi_2,\pi_3,\ldots},$$

is remarkable.

Section III.

Laplace's Problem.

17. In the *Théorie des Probabilités* Laplace discusses and solves a problem of a somewhat similar kind. He supposes an urn to contain nr tickets which are in n sets each set involving r tickets. The tickets in a first set are each numbered one, in a second set two and so on till those in an nth set are each numbered n. He supposes the tickets to be well mixed and then n tickets to be drawn in succession. If the mth ticket that is drawn happens to be numbered m he calls this a coincidence and he inquires into the probability of there being *at least* s coincidences. Observing that s cannot be superior to n the method of this paper leads quickly to the solution.

We have to determine the number of permutations of

$$x_1^r x_2^r \ldots x_n^r,$$

which are such that an x_m occurs in the mth place from the left on *at least* s occasions.

Consider a redundant generating function

$$(ax_1 + x_2 + \ldots + x_n)(x_1 + ax_2 + \ldots + x_n) \ldots (x_1 + x_2 + \ldots + ax_n)(x_1 + x_2 + \ldots + x_n)^{rn-m};$$

it is clear that the coefficient C_s of

$$a^s x_1^r x_2^r \ldots x_n^r$$

in this product denotes the number of permutations in which an x_m occurs in the mth place *exactly* s times.

Hence the number of permutations we seek is the sum of the coefficients of the terms

$$a^s x_1^r x_2^r \ldots x_n^r,\ a^{s+1} x_1^r x_2^r \ldots x_n^r,\ \ldots\ a^n x_1^r x_2^r \ldots x_n^r,$$

in the product. Denoting the product by u we have to find the coefficient of a^s in

$$\left(1 + \frac{1}{a} + \frac{1}{a^2} + \ldots \frac{1}{a^{n-s}}\right) u,$$

that is in $\dfrac{u}{1 - \dfrac{1}{a}}$ or in $\dfrac{au}{a-1}$, since a does not occur to a higher power than n.

Now, writing $\qquad p_1 = \Sigma x_1,\ p_2 = \Sigma x_1 x_2,\ \ldots,$

$$\begin{aligned} n &= \{p_1 + (a-1)x_1\}\{p_1 + (a-1)x_2\} \ldots \{p_1 + (a-1)x_n\} p_1^{rn-n}, \\ &= \{p_1^n + (a-1)p_1^n + (a-1)^2 p_1^{n-2} p_2 + \ldots + (a-1)^n p_n\} p_1^{rn-n}, \\ &= ap_1^{rn} + (a-1)^2 p_1^{rn-2} p_2 + (a-1)^3 p_1^{rn-3} p_3 + \ldots + (a-1)^n p_1^{rn-n} p_n \end{aligned}$$

and $\quad \dfrac{au}{a-1} = \dfrac{a^2 p_1^{rn}}{a-1} + a(a-1) p_1^{rn-2} p_2 + a(a-1)^2 p_1^{rn-3} p_3 + \ldots + a(a-1)^{n-1} p_1^{rn-n} p_n.$

Herein the coefficient of a^s is

$$p_1^{rn-s} p_s - \binom{s}{1} p_1^{rn-s-1} p_{s+1} + \binom{s+1}{2} p_1^{rn-s-2} p_{s+2} - \ldots + (-)^{n-s} \binom{n-1}{n-s} p_1^{rn-n} p_n.$$

The coefficient herein of $x_1^r x_2^r \ldots x_n^r$ is obtained as the result of operating upon it with D_r^n.

The reader will have no difficulty in proving that

$$D_r^n p_1^{nr-m} p_m = \binom{n}{m} \frac{(nr-m)!}{(r!)^{n-m} \{(r-1)!\}^m}.$$

Thence the number we require is

$$\begin{aligned} &\binom{n}{s} \frac{(nr-s)!}{(r!)^{n-s}\{(r-1)!\}^s} - \binom{s}{1}\binom{n}{s+1} \frac{(nr-s-1)!}{(r!)^{n-s-1}\{(r-1)!\}^{s+1}} \\ &\qquad + \binom{s+1}{2}\binom{n}{s+2} \frac{(nr-s-2)!}{(r!)^{n-s-2}\{(r-1)!\}^{s+2}} - \ldots + (-)^{n-s}\binom{n-1}{n-s}\binom{n}{n} \frac{(nr-n)!}{\{(r-1)!\}^n}; \end{aligned}$$

and dividing this by the whole number of permutations, viz.:—

$$\frac{(nr)!}{(r!)^n},$$

we find a result which is readily identified with that of Laplace.

18. If we put $s=1$ we obtain the whole number of permutations which exhibit coincidences in the first n places; this number is therefore

$$\frac{(nr)!}{(r!)^n} - \binom{n}{2}\frac{(nr-2)!}{(r!)^{n-2}\{(r-1)!\}^2} + \binom{n}{3}\frac{(nr-3)!}{(r!)^{n-3}\{(r-1)!\}^3} - \ldots + (-)^{n-1}\frac{(nr-n)!}{\{(r-1)!\}^n}.$$

If we now subtract this number from

$$\frac{(nr)!}{(r!)^n},$$

we must obtain the whole number of permutations which do *not* exhibit coincidences in the first n places. This number is

$$\binom{n}{2}\frac{(nr-2)!}{(r!)^{n-2}\{(r-1)!\}^2} - \binom{n}{3}\frac{(nr-3)!}{(r!)^{n-3}\{(r-1)!\}^3} + \ldots + (-)^n\frac{(nr-n)!}{\{(r-1)!\}^n}.$$

If herein we put $r=1$, the first n places are in fact the whole of the places so that the expression becomes the value of the number we have denoted by $\{0;\ 1^n\}$. We thus find again

$$\{0;\ 1^n\} = n!\left\{\frac{1}{2!} - \frac{1}{3!} + \ldots + (-)^n\frac{1}{n!}\right\},$$

a verification.

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