CANTOR SETS, BINARY TREES AND LIPSCHITZ CIRCLE HOMEOMORPHISMS

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Abstract. We define the notion of rotations on infinite binary trees, and construct an irrational tree rotation with bounded distortion. This lifts naturally to a Lipschitz circle homeomorphism having the middle-thirds Cantor set as its minimal set. This degree of smoothness is best possible, since it is known that no C^1 circle diffeomorphism can have a linearly self-similar Cantor set as its minimal set.

1. Introduction

Let A_1 and A_2 be disjoint compact subintervals of $[0, 1)$, and let L be the smallest compact interval containing $A_1 \cup A_2$.

Let $S : A_1 \cup A_2 \to L$ be a mapping such that the restrictions $S|_{A_i}$ are affine surjections onto L for each i. Then we define the affine Cantor set K_S by

$$K_S \equiv \{x \in L : S^i(x) \subset A_1 \cup A_2, \text{ for all } i \geq 1\}.$$

We call this a *two-branched affine Cantor set*. If we replace the restriction that S be locally affine with the requirement that $|S'| > 1$, we call K_S a two-branched *hyperbolic Cantor set*. (This is sometimes also called a dynamically defined Cantor set, a self-similar Cantor set, or a "cookie cutter".) A k-branched affine Cantor set or hyperbolic Cantor set is defined similarly for any $k \geq 2$.

A different kind of Cantor set arises as follows. Let f be an orientation preserving homeomorphism of the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$. Poincaré showed that if f has no periodic orbits, then either

(1) every orbit is dense in the circle, and f is topologically conjugate to the irrational rotation $R_\alpha(x) = x + \alpha$, where α is the rotation number of f, or

(2) no orbit is dense, and every orbit accumulates on a unique Cantor set Γ_f. (In this case the homeomorphism is called a *Denjoy counterexample* because of Denjoy's theorem below.) The Cantor set Γ_f is minimal for f, meaning that it is compact, non-empty, f-invariant, and has no compact non-empty f-invariant subsets.

Intuitively, the Cantor sets Γ_f are fundamentally different from the self-similar Cantor sets K_S described above. To make precise the sense in which this is true, we introduce the following terminology.

For $r \geq 0$, denote by $\mathcal{C}(r)$ the class of C^r*-minimal* sets; that is,

$$\mathcal{C}(r) = \{C \subset \mathbf{S}^1 : C \text{ is a minimal Cantor set for some } C^r \text{ diffeomorphism of } \mathbf{S}^1\}.$$

Since any two Cantor sets in $\mathbf{S}^1$ are ambiently homeomorphic, it is easy to see that $\mathcal{C}(0)$ includes every Cantor set. On the other hand, $\mathcal{C}(r)$ is empty for $r \geq 2$ due to:

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Denjoy's Theorem [1]. *If f is a C^1 diffeomorphism of $\mathbf{S}^1$ without periodic points, and if the derivative Df has bounded variation, then f is topologically conjugate to an irrational rotation.*

Herman [3] produced Denjoy counterexamples of class $C^{1+\alpha}$ for all $\alpha < 1$, so $\mathcal{C}(r)$ is non-empty for all $r < 2$. Clearly $\mathcal{C}(r) \subset \mathcal{C}(s)$ if $s < r$. The intuitive idea mentioned above can be stated as follows:

Theorem A [5]. *$\mathcal{C}(1)$ contains no affine Cantor sets. Moreover, if the generating map S is C^2-sufficiently close to affine, $K_S \notin \mathcal{C}(1)$.*

See also McDuff [4], and [6] for other results about $\mathcal{C}(r)$.

In this paper we show that at least some affine Cantor sets do belong to $\mathcal{C}(r)$ for all $r < 1$. In fact we will prove

Theorem 1. *If S is a two-branched affine Cantor set as defined above and $|A_1| = |A_2|$, then there is a bi-Lipschitz homeomorphism f of the circle (Lipschitz with Lipschitz inverse) such that $\Gamma_f = K_S$.*

The rotation number of the f is the golden mean.

In particular, the usual middle-thirds Cantor set (scaled down to fit inside the fundamental domain $[0, 1)$ of $\mathbf{R}/\mathbf{Z}$) is the minimal set for some bi-Lipschitz circle homeomorphism, but not for any C^1 circle diffeomorphism.

When S is defined as in Theorem 1 with $|A_1| = |A_2|$, we call K_S a *linear* Cantor set to distinguish this special case from the more general affine case.

The method of proof of Theorem 1 is the following. A hyperbolic Cantor set K has a natural tree structure, discussed below. Such a tree has a natural circular ordering, and the resulting order topology. The rotation number of a tree homeomorphism can be defined in the usual way. We construct a tree homeomorphism with rotation number equal to the golden mean. This can be lifted in a natural way to a circle homeomorphism f with $\Gamma_f = K$. The tree homeomorphism can be constructed in such a way that it has bounded distortion in the sense that it changes the "depth" of any node in the tree by a bounded amount. In the case when K is a linear Cantor set, bounded distortion on the tree level lifts to a Lipschitz condition on f.

Remark. It might have been tempting to think that the reason for Theorem A is an unbounded distortion forced by a conflict between the scaling of gaps of an affine Cantor set and the order property of orbits for an irrational rotation. Theorem 1 shows that this is not the case; instead the difficulty rests more delicately on the continuity of the derivative of f.

2. Trees

Intuitively, a tree is simply a graph containing no closed loops; it can be specified by giving a list of nodes, together with a description of which nodes are connected by edges. In this paper we will be considering infinite binary trees where every node (except one isolated node) is the parent of two other nodes (its children), and the only edges are those connecting parent and child.

Since our tree rotations will always take nodes to nodes, we will henceforth ignore the edges of the tree, and simply consider the nodes, along with the relation of parenthood, as follows. (The whole discussion is determined by the structure of Cantors sets on the circle.)

First, we define the complete circular binary tree T. T is a countable set of elements called *nodes*, with a certain labelling. There are two special nodes, r and $r*$, called the root node and the isolated node, respectively. The remaining nodes are in one-to-one correspondence with the set of nonempty finite words from the alphabet $\{0, 1\}$. The node corresponding to $i_1 \dots i_k$, $k \geq 1$, is denoted $r(i_1 \dots i_k)$.

The node $r(i_1 \dots i_k)$ is called the parent of each of the two nodes $r(i_1 \dots i_k 0)$ and $r(i_1 \dots i_k 1)$; they are its children. r is the parent of $r(0)$ and $r(1)$. The isolated node n has no children. Descendant and ancestor are defined in the obvious way. To visualize T as a graph, connect every non-isolated node with each of its children by an edge. (See figure 1.)

[figure 1]

If u is the node $r(i_1 \dots i_k)$, we will use the notation $u(j_1 \dots j_l)$ to denote the node $r(i_1 \dots i_k j_1 \dots j_l)$.

The *level* of a node $u \in T$ is defined by $\ell(u) = 0$ if $u = r$ or $r*$; $\ell(u) = k$ if $u = r(i_1 \dots i_k)$ for some choice of $i_1 \dots i_k$.

There is also a natural (left-right) linear order structure we can place on T. This simplest way to specify this is to define an injection $\mu : T \to (0, 1]$. We let $\mu(r) = 1/2$, $\mu(r*) = 1$, and

$$\mu(r(i_1, \dots, i_k)) = 2^{-k-1} + \sum_{j=1}^{k} i_j 2^{-j}.$$

The range of μ is the set of dyadic rationals in $(0, 1]$. We pull back the natural circular ordering on $(0, 1]$ to T via μ, and denote it by $\prec$. That is, if $u, v, w \in T$, then $u \prec v \prec w$ iff $\mu(u) < \mu(v) < \mu(w) \bmod 1$.

This ordering induces the usual order topology on T, for which the open intervals form a basis. In this topology, T is homeomorphic to a countable dense subset of the circle.

Effectively now there are two orderings on T: the “left-right” circular ordering, and the “vertical” partial order induced by the relation of parenthood. We will write $u \lessdot v$ if v is a descendant of u.

By a subtree we mean a subset of T with the induced orderings.

A subtree S of T is called an *ideal (with respect to $\lessdot$)* if S contains r and $r*$, and if moreover S contains u whenever S contains v and $u \lessdot v$.

Let g be an order-preserving bijection of T (relative to the circular order $\prec$). Then g is a homeomorphism, and the push-forward $\mu(g)$ defined by $\mu(g)(x) = \mu(g(\mu^{-1}(x)))$ is a homeomorphism of the dyadic rationals in the circle. Therefore $\mu(g)$ extends to an orientation-preserving homeomorphism of the circle, and as such it has a rotation number. This will be our definition of the rotation number of g on T. For this reason, we call any such g a *tree rotation*.

There is a natural one-to-one correspondence τ between T and the collection $\mathcal{I}$ of connected components (intervals) of $S^1 \setminus K_S$. To describe this, we rotate K_S so that its left endpoint is at 0. Let $A_1 = [0, a]$, $A_2 = [b, c]$, where $0 < a < b < c < 1$. Let $\phi_0 : L \to A_1$ and $\phi_1 : L \to A_2$ be the two branches of the inverse of S. Then

$\tau : T \to \mathcal{I}$ is defined by

$$\begin{aligned}\tau(r*) &= (c, 1)\\ \tau(r) &= (a, b)\\ \tau(r(i_1 \dots i_k)) &= \phi_{i_1} \circ \dots \circ \phi_{i_k}[(a, b)].\end{aligned}$$

It is easy to check that τ is order-preserving.

Next, we say that the node u *immediately precedes* the node v in a finite ideal S of T if there is no node w of S such that $u \prec w \prec v$.

We need these lemmas for later use.

Lemma 1. *If u immediately precedes v in a finite ideal S of T then $\ell(u) = \ell(v)$ implies $\ell(u) = \ell(v) = 0$, and hence $u = r$, $v = r*$.*

Proof. For contradiction, assume $k := \ell(u) = \ell(v) \neq 0$. Then for some choice of indices, $u = r(i_1 \dots i_k)$ and $v = r(j_1 \dots j_k)$. Let

$$m = \min\{n : i_n \neq j_n\} \geq 1,$$

and let

$$w = r(i_1 \dots i_{m-1})$$

if $m > 1$ or $w = r$ if $m = 1$. Then $u = w(i_m \dots i_k)$ and $v = w(j_m \dots j_k)$. Since S is an ideal and $u \in S$, this means $w \in S$. Also, w lies between u and v. This contradicts the choice of u and v in S.

Lemma 2. *Suppose S is a finite ideal of T, and suppose u immediately precedes v in S. Then there is a unique node $w := S(u, v)$ of $T \setminus S$ such that*

(i) $S \cup \{w\}$ is an ideal in T,

(ii) w is between u and v (i.e. u immediately precedes w and w immediately preceds v in $S \cup \{w\}$), and

(iii) $\ell(w) = \max\{\ell(u), \ell(v)\} + 1$.

Furthermore any other node of T between u and v has level greater than $\ell(w)$.

Proof. If $u = r$ and $v = r*$, let $w = u(1)$. Otherwise, by Lemma 1, $\ell(u) \neq \ell(v)$. Let $w = u(1)$ if $\ell(u) > \ell(v)$, otherwise let $w = v(0)$. Clearly w satisfies (i),(ii), and (iii). We leave uniqueness and the remaining claim as an excercise.

Lemma 3. *Suppose u, v, u', v' are nodes of T satisfying $\ell(u) < \ell(v)$, $\ell(u') > \ell(v')$, and $\max\{|\ell(u) - \ell(u')|, |\ell(v) - \ell(v')|\} \leq 2$.*

If x, y are nodes with $\ell(x) = \ell(v) + 1$ and $\ell(y) = \ell(u') + 1$, then $|\ell(x) - \ell(y)| \leq 1$.

Proof. For contradiction suppose $\ell(x) \geq \ell(y) + 2$. Then

$$\ell(x) \geq \ell(u') + 3 \geq \ell(v') + 4,$$

so $\ell(v) \geq \ell(v') + 3$, a contradiction. Similarly if $\ell(y) \geq \ell(x) + 2$.

Lemma 4. *Let u, v, u', v' be distinct nodes of a finite ideal S of T such that u is adjacent to v, u' is adjacent to v', and*

$$\max\{|\ell(u) - \ell(u')|, |\ell(v) - \ell(v')|\} \leq 2.$$

Then

$$|\ell(S(u, v)) - \ell(S(u', v'))| \leq 2.$$

Proof. By Lemma 1, $\ell(u) \neq \ell(v)$ and $\ell(u') \neq \ell(v')$. There are four possible cases.

Case 1: $\ell(v) < \ell(u)$ and $\ell(v') < \ell(u')$. Then by Lemma 2, $\ell(S(u, v)) = \ell(v) + 1$, $\ell(S(u', v')) = \ell(v') + 1$, and

$$|\ell(S(u, v)) - \ell(S(u', v'))| = |\ell(v) - \ell(v')| \leq 2.$$

Case 2: $\ell(v) < \ell(u)$ and $\ell(v') < \ell(u')$ is similar.

The remaining two cases are covered by Lemma 3.

3. Proof of Theorem 1

Let α denote the golden mean $(\sqrt{5} - 1)/2 = 0.61803...$. Define the irrational rotation $R_\alpha : S^1 \to S^1$ by $R_\alpha(x) = x + \alpha \mod 1$. Define $R : \mathbf{Z} \to S^1$ by

$$R(n) = R_\alpha^n(0) = n\alpha \mod 1.$$

The following theorem will be proved in Section 4:

Theorem 2. *There is a bijection $h : \mathbf{Z} \to T$ with the same circular ordering as the bi-infinite sequence $\{R(n) : n \in Z\}$, and with the property that, for all n,*

$$|\ell(h(n)) - \ell(h(n + 1))| \leq 2.$$

Corollary. *There is a an order preserving homeomorphism $g : T \to T$ with rotation number α such that $T = \{g^n(r) : n \in \mathbf{Z}\}$ and*

$$|\ell(g(u)) - \ell(u)| \leq 2$$

for all $u \in T$.

Proof of Corollary. Define $g : T \to T$ by $g = h\sigma h^{-1}$, where h is the function given in Theorem 2, and $\sigma : \mathbf{Z} \to \mathbf{Z}$ is the shift $\sigma(n) = n+1$. Then g is an order-preserving bijection of T, and since its orbit has the same ordering as the R-orbit of 0, it must have rotation number α. The rest of the Corollary follows immediately.

Our desired function f will be the extension to S^1 of the natural lift of g to the intervals comprising the complement of K_S in S^1. To make this precise we use the function τ defined in the previous section. Recall $\mathcal{I}$ is the collection of connected components of $S^1 \setminus K_S$. For each $I \in \mathcal{I}$, define $f|_I$ to be the unique orientation-preserving affine map taking I onto $\tau \circ g \circ \tau^{-1}(I)$. In this way, f is defined on all of $S^1 \setminus K_S$. Since g is order-preserving on T, f is also order-preserving. Since $\cup\mathcal{I}$ is dense in S^1, f extends to a unique continuous function, also called f, on S^1. By its contruction, f permutes the intervals of $\mathcal{I}$, is a homeomorphism of the circle, and $f(K_S) = K_S$.

Let β denote the (constant) derivative of S. Then the length of any interval at level k is simply $(b-a)/\beta^k$. Since g changes the level of any node by at most two, this means that f can expand or contract any $I \in \mathcal{I}$ by at most a factor β^2. That is, on $S^1 \setminus K_S$,

$$1/\beta^2 \leq f' \leq \beta^2.$$

Since K_S has measure zero and $f(K_S) = K_S$, f is absolutely continuous on S^1. The bound on f' therefore yields a global Lipschitz constant of β^2 for f and f^{-1}.

Since f is a homeomorphism with an invariant Cantor set and irrational rotation number, it must be a Denjoy counterexample, and $\Gamma_f \subset K_S$. If x is any endpoint of K_S, then since h is surjective, $\{f^n(x) : n \in \mathbf{Z}\}$ is dense in K_S, and therefore $\Gamma_f = K_S$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

The proof is by induction. Recall the sequence of denominators of best approximations to the golden mean (the Fibonnaci sequence): $q_0 = 1$, $q_1 = 1$, $q_n = q_{n-1} + q_{n-2}$. (See, for example, Hardy and Wright [2] as a general reference for Diophantine approximation.)

First we need purely number-theoretic lemma.

Lemma 5. *Let $n \geq 4$ be a positive integer and suppose k, k' are integers satisfying*

$$q_{n-2} \leq k < q_{n-1} \leq k' < q_n.$$

Let

$$Q = \{i \in \mathbf{Z} : -k \leq i \leq q_n - k - 1\} \text{ and}$$
$$Q' = \{i \in \mathbf{Z} : -k' \leq i \leq q_{n+1} - k' - 1\}.$$

Then $Q \subset Q'$. Moreover, $R(Q)$ partitions S^1 into q_n open intervals, and each such interval contains at most one point of $R(Q')$.

Equivalently, the the two nearest neighbors in $R(Q')$ of any point of $R(Q' \setminus Q)$ lie in $R(Q)$.

In particular, if n is odd, for each $i = -k', \ldots, -k-1$,

$$R(q_n + i) \prec R(i) \prec R(q_{n-1} + i),$$

and for each $i = q_n - k, \ldots, q_{n+1} - k' - 1$,

$$R(i - q_{n-1}) \prec R(i) \prec R(i - q_n).$$

In each case the three points are nearest neighbors in $R(Q')$.

If n is even the reverse inequalities hold.

Proof of Lemma 5. A standard fact of the theory of Diophantine approximation is the following:

For n odd and $x \in S^1$,

$$x + R(q_n) \prec x \prec x + R(q_{n-1})$$

and the interval

$$\{y \in S^1 : x + R(q_n) \prec y \prec x + R(q_{n-1})\}$$

contains no other points of the set

$$\{x + R(i) : i = 1, \dots, q_{n+1} - 1\}.$$

For n even the same is true but the inequalities are reversed.

Now fix $i \in \{-k', \dots, -k-1\}$. From the definitions of k and k', it is straightforward to check that $i + q_n \in Q$ but $i + q_{n+1} \notin Q'$.

This means that the nearest neighbors in $R(Q')$ to $R(i)$ are $R(i + q_n)$ and $R(i + q_{n-1})$, both belonging to $R(Q)$. Also

$$R(i + q_n) \prec R(i) \prec R(i + q_{n-1}).$$

A similar argument works for $i \in \{q_n - k, \dots, q_{n+1} - k' - 1\}$ using the fact that $i - q_n \in Q$ and $i - q_{n+1} \notin Q'$.

To prove Theorem 2, what we will actually prove is the following: for every $n \geq 4$, there is a positive integer k_n such that

$$q_{n-2} \leq k_n \leq q_{n-1} - 1,$$

and a function $h_n : Q_n := \{-k_n, -k_n + 1, \dots, 0, \dots, q_n - k_n - 1\} \to T$ such that

(i_n) the values of h_n have the same circular ordering as $\{R(i) : i \in Q_n\}$,

(ii_n) $|\ell(h_n(i)) - \ell(h_n(i+1))| \leq 2$ for $i = -k_n, \dots, q_n - k_n - 2$,

(iii_n) $|\ell(h_n(-k_n)) - \ell(h_n(q_n - k_n - 1))| \leq 1$,

(iv_n) $h_n(Q_n)$ is an ideal in T, and

(v_n) $h_n|_{Q_{n-1}} = h_{n-1}$ $(n > 4)$.

It will follow from the proof that each h_n is injective and the union of the sets $h_n(Q_n)$ is the whole tree T. Then (v) will imply that the h_n's define a well-defined function h on $\mathbf{Z}$, and conditions (i) and (ii) yield the conclusions of the theorem.

(Because of (v_n), we will henceforth write h instead of h_n for convenience.)

To start, set $k_4 = 2$ and $h(0) = r*$, $h(1) = r$, $h(2) = r(0)$, $h(-1) = r(01)$, and $h(-2) = r(1)$. This defines h on the set $Q_4 = \{-2, -1, 0, 1, 2\}$, and the circular ordering of these points in T is

$$h(2) \prec h(-1) \prec h(1) \prec h(-2) \prec h(0)$$

which coincides with the circular ordering of $\{R^i(0) : i = -2, \dots, 2\}$. Conditions (i) - (iv) are easily verified. (See Figure 2)

[figure 2]

Now assume by induction that $(i_n) \dots (v_n)$ hold. We wish to define k_{n+1} so that

$$q_{n-1} \leq k_{n+1} \leq q_n - 1,$$

and extend h to $Q_{n+1} = \{-k_{n+1}, -k_{n+1} + 1, \dots, 0, \dots, q_{n+1} - k_{n+1} - 1\}$ so that $(i_{n+1}) \dots (v_{n+1})$ hold.

Define k_{n+1} to be largest integer k in $\{q_{n-1}, \dots q_n - 1\}$ such that $\ell(h(q_n - k)) > \ell(h(q_{n-1} - k))$.

For convenience of notation in the rest of this proof, we will write k for k_{n+1}.

To extend h to the new domain Q_{n+1}, by induction we need only define new values for h on $Q_{n+1} \setminus Q_n = \{-k, \dots, -k_n - 1\} \cup \{q_n - k_n, \dots, q_{n+1} - k - 1\}$.

Notice that $h(Q_n)$ divides the tree T into q_n intervals. Similarly the set $R(Q_n)$ divides the circle into q_n intervals. Each such interval contains at most one point of the set $R(Q_{n+1} \setminus Q_n)$ by Lemma 5.

For $j \in Q_{n+1} \setminus Q_n$, if $R(j)$ falls into the interval with endpoints $R(j'), R(j'')$, $j', j'' \in Q_n$, we then define $h(j)$ to be the unique node (Lemma 2) of $T \setminus h(Q_n)$ between $h(j')$ and $h(j'')$ of least possible level. (This is known as the standard tree insertion.)

This will guarantee properties (i),(iv), and (v) for $n+1$; we need only verify (ii) and (iii). To do this, we need to be more explicit about where the new values of h lie with respect to the nodes of $h(Q_n)$. Assume n is odd (otherwise the argument is similar with inequalities reversed).

By Lemma 5, for $i = q_n - k_n, \dots q_{n+1} - k - 1$, we have

$$h(i - q_{n-1}) \prec h(i) \prec h(i - q_n), \tag{1}$$

and for $i = -k, \dots, -k_n - 1$,

$$h(q_n + i) \prec h(i) \prec h(q_{n-1} + i). \tag{2}$$

By Lemma 4, it follows from (ii_n) that for

$$i = q_n - k_n, \dots, q_n - k - 2 \text{ and } -k, \dots, -k_n - 2,$$

we have

$$|\ell(h(i)) - \ell(h(i+1))| \leq 2.$$

To complete the verification of (ii_{n+1}), it remains to show that

$$|\ell(h(-k_n - 1)) - \ell(h(-k_n))| \leq 2 \tag{3}$$

and

$$|\ell(h(q_n - k_n - 1)) - \ell(h(q_n - k_n))| \leq 2. \tag{4}$$

We will prove (3), leaving the similar proof of (4) to the reader. Now the nearest neighbors of $h(-k_n - 1)$ in $h(Q_n)$ are $u := h(q_n - k_n - 1)$ and $v := h(q_{n-1} - k_n - 1)$ by Lemma 5. When the value u was assigned at the previous stage, its nearest neighbors in $h(Q_{n-1})$ were v and $h(q_{n-2} - k_n - 1)$, again by Lemma 5. Therfore by Lemma 2, $\ell(u) > \ell(v)$. This means, again by Lemma 2, that

$$\ell(h(-k_n - 1)) = \ell(u) + 1.$$

But by (iii_n), we have

$$|\ell(u) - \ell(h(-k_n))| \leq 1.$$

These last two statements imply (3).

Finally, we verify (iii_{n+1}). From the order properties (1) and (2) we have the following:

$$
\begin{aligned}
&h(0) \prec h(q_n - q_{n-2}) \prec h(-q_{n-2}) \\
&h(1) \prec h(q_n - q_{n-2} + 1) \prec h(-q_{n-2} + 1) \\
&\ldots \\
&h(q_n - k - 1) \prec h(q_{n+1} - k - 1) \prec h(-q_{n-2} + q_n - k - 1) \\
&h(q_n - k) \prec h(-k) \prec h(q_{n-1} - k) \\
&\ldots \\
&h(q_n - q_{n-1}) \prec h(-q_{n-1}) \prec h(0).
\end{aligned}
$$

It follows from our definition of k that $\ell(h(q_n - k)) > \ell(h(q_{n-1} - k))$ and $\ell(h(q_n - k - 1)) < \ell(h(-q_{n-2} + q_n - k - 1))$. Lemmas 2 and 3 therefore give us property (iii_{n+1}). This completes the proof of Theorem 2.

5. Final Remarks

In the end, the use of trees here is mainly as a convenient device for keeping track of scales in a Cantor set. These results are clearly the tip of a large iceberg. For example, we suspect that similar constructions would work for other rotation numbers – we have only done the simplest case. The problem of handling affine but nonlinear Cantor sets is untouched. One could also imagine using trees other than the complete binary tree to model certain Cantor sets, and some trees could be very much better adapted to irrational rotation than the standard binary tree. To retreat to the motivating question of this work, we note that a good geometric intrinsic characterization of Denjoy minimal sets is still unavailable.

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