15. Making joins smooth. The derivative of a piecewise-smooth function

From Lecture 13, we have two equivalent definitions of the derivative of a smooth function, namely,

$$F'(t) \;=\; \text{that part of } DQ(F,[t,t+h]) \text{ which is independent of } h \qquad (15.1a)$$

and[1]

$$F'(t) \;=\; \lim_{h\to 0} DQ(F,[t,t+h]) \,. \qquad (15.1b)$$

Here we extend the concept from smooth to piecewise-smooth functions. But note that our principal reason for extending the definition is to obtain conditions for a join to be smooth. Thus piecewise-smooth does not imply that a function cannot smooth, only that it need not be smooth.

The simplest kind of piecewise-smooth function is a continuous one with a potential corner at c, say W defined on [a, b] by

$$W(t) \;=\; \begin{cases} F(t) & \text{if } a \le t \le c \\ G(t) & \text{if } c \le t \le b \end{cases} \qquad (15.2)$$

with

$$F(c) \;=\; G(c) \qquad (15.3)$$

(otherwise, (2) does not define a function). Each component of W is smooth on its subdomain; i.e., F is smooth on [a, c], and G is smooth on [c, b]. So, from Lecture 13, F' has domain [a, c), G' has domain [c, b), and we can define W' on [a, b) by

$$W'(t) \;=\; \begin{cases} F'(t) & \text{if } a \le t < c \\ G'(t) & \text{if } c \le t < b. \end{cases} \qquad (15.4)$$

If $F'(c) = G'(c)$ then W' is continuous, implying that W is smooth; whereas if $F'(c) \neq G'(c)$ then W is merely piecewise-smooth (but continuous). Note that, strictly, $F'(c)$ does not actually mean anything until, by analogy with (13.26), we extend the domain of F' from [a, c) to [a, c] by defining

$$F'(c) \;=\; \lim_{t\to c-} F'(t) \,. \qquad (15.5)$$

For example, in Lecture 3, mean testes size in European starlings is defined on [0, 12] by (2) with $a = 0$, $b = 12$, $c = 3.0$,

$$F(t) \;=\; 1.995 + 2.195t - 0.175t^2 \qquad (15.6)$$

and

$$G(t) \;=\; 8.86788 + 0.00934343t - 0.272727t^2 + 0.0209596t^3 \,. \qquad (15.7)$$

From Lecture 13, $F'(t) = 2.195 - 0.35t$ and $G'(t) = 0.00934343 - 0.545455t + 0.0628788t^2$ (why?). So (3) implies that W' is defined on [0, 12) by

$$W'(t) \;=\; \begin{cases} 2.195 - 0.35t & \text{if } 0 \le t < c \\ 0.00934343 - 0.545455t + 0.0628788t^2 & \text{if } c \le t < 12. \end{cases} \qquad (15.8)$$

Both W and W' are graphed in Figure 1. W is continuous, but not smooth, because W' is discontinuous at $t = c$, where growth rate abruptly drops from $F'(c) = 1.14$ to $G'(c) = -1.06$ mm per month: Mean testes size was growing faster than a millimeter per month, when all of a sudden it is decaying faster than a millimeter per month.

---

[1] In effect, (1b) defines the derivative as the limit of a function sequence, successive terms of which are average growth rates of F over smaller and smaller intervals of time. See Appendix 15.

A piecewise-smooth function need not be continuous to have a derivative (e.g., W' in Figure 1 is discontinuous, but it has a derivative). It is therefore expedient to redefine W by

$$W(t) \;=\; \begin{cases} F(t) & \text{if} \quad a \le t < c \\ G(t) & \text{if} \quad c \le t \le b, \end{cases} \tag{15.9}$$

where F is still smooth on [a, c], and G is still smooth on [c, b]. Even if $F(c) \neq G(c)$, (9) defines a piecewise-smooth function, and its derivative W' is still defined by (4).

The definition of derivative for a join is readily extended to any finite number of components. In particular, if S is defined on [a, b] by

$$S(t) \;=\; \begin{cases} F(t) & \text{if} \quad a \le t < c_1 \\ H(t) & \text{if} \quad c_1 \le t < c_2 \\ G(t) & \text{if} \quad c_2 \le t \le b \end{cases} \tag{15.10}$$

with F smooth on [a, c_1], G smooth on [c_1, c_2] and H smooth on [c_2, b], then S' is defined on [a, b) by

$$S'(t) \;=\; \begin{cases} F'(t) & \text{if} \quad a \le t < c_1 \\ H'(t) & \text{if} \quad c_1 \le t < c_2 \\ G'(t) & \text{if} \quad c_2 \le t < b. \end{cases} \tag{15.11}$$

For example, in Lecture 3, smoothed mean testes size is defined on [0, 12] by

$$S(t) = \begin{cases} 1.995 + 2.195t - 0.175t^2 & \text{if} \quad 0 \le t < 3 \\ H(t) & \text{if} \quad 3 \le t < 3.002 \\ 8.86788 + 0.00934343t - 0.272727t^2 + 0.0209596t^3 & \text{if} \quad 3.002 \le t \le 12 \end{cases} \tag{15.12}$$

with

$$H(t) \;=\; 445147.2481 - 446647.4823t + 149384.5255t^2 - 16654.01947t^3, \tag{15.13}$$

so that $c_1 = 3$ and $c_2 = 3.002$ in (10). S is continuous, because F(3) = H(3) = 7.005 and H(3.002) = G(3.002) = 7.0052. From (11) and Lecture 13, its derivative is defined by

$$S'(t) \;=\; \begin{cases} 2.195 - 0.35t & \text{if} \quad 0 \le t < 3 \\ H'(t) & \text{if} \quad 3 \le t < 3.002 \\ 0.00934343 - 0.545455t + 0.0628788t^2 & \text{if} \quad 3.002 \le t \le 12 \end{cases} \tag{15.14}$$

where

$$H'(t) = -446647.4823 + 298769.051t - 49962.05842t^2. \tag{15.15}$$

S' is continuous, because $F'(3) = H'(3) = 1.145$ and $H'(3.002) = G'(3.002) = -1.06145$. Both S and S' are graphed in Figure 2. Rate of mean growth falls very rapidly from 1.145 to –1.06 mm per month; nevertheless, the change is continuous. Together, Figures 1-2 illustrate the general result that although a piecewise-smooth function may have a discontinous derivative, a smooth function always has a continuous one. Indeed that is precisely what makes a smooth function smooth.

In practice, we can determine the continuity or smoothness of joins without having different names for different components, provided that we first introduce some general notation for the different values that

$$W(t) = \begin{cases} F(t) & \text{if } a \le t < c \\ G(t) & \text{if } c \le t \le b \end{cases} \tag{15.16}$$

and

$$W'(t) = \begin{cases} F'(t) & \text{if } a \le t < c \\ G'(t) & \text{if } c \le t < b. \end{cases} \tag{15.17}$$

may approach as $t \to c$ from below (i.e., with $t < c$) or from above (i.e., with $t > c$). Accordingly, we use $W(c-)$ to denote the limit of $W(t)$ as $t \to c$ from below and $W(c+)$ to denote the limit of $W(t)$ as $t \to c$ from above; correspondingly, we use $W'(c-)$ to denote the limit of $W'(t)$ as $t \to c$ from below and $W'(c+)$ to denote the limit of $W'(t)$ as $t \to c$ from above. That is, we set

$$W(c-) = F(c), \qquad W(c+) = G(c) \tag{15.18}$$

and

$$W'(c-) = F'(c), \qquad W'(c+) = G'(c) \tag{15.19}$$

where $F'(c)$ is defined by (5).

With our new notation we can write conditions for W to be smooth in terms of W alone. By virtue of (18), the condition that W be a continuous join becomes $W(c-) = W(c+)$. Similarly, by virtue of (19), the condition that W' be a continuous join becomes $W'(c-) = W'(c+)$. Thus conditions for W to be a smooth join, namely $F(c) = G(c)$ *and* $F'(c) = G'(c)$, become the **continuity condition**

$$W(c-) = W(c+) \tag{15.20a}$$

and the **smoothness condition**

$$W'(c-) = W'(c+). \tag{15.20b}$$

If both are satisfied, then W is smooth; whereas if the first is satisfied without the second, then W has a corner at $t = c$. (If, on the other hand, the second is satisfied without the first, then it doesn't mean a thing.)

These conditions readily extend to joins of more than two components, e.g., S defined on [a, b] by

$$S(t) = \begin{cases} F(t) & \text{if } a \le t < c_1 \\ H(t) & \text{if } c_1 \le t < c_2 \\ G(t) & \text{if } c_2 \le t \le b \end{cases} \tag{15.21}$$

with F smooth on $[a, c_1]$, G smooth on $[c_1, c_2]$, H smooth on $[c_2, b]$ and W' defined by

$$S'(t) = \begin{cases} F'(t) & \text{if } a \le t < c_1 \\ H'(t) & \text{if } c_1 \le t < c_2 \\ G'(t) & \text{if } c_2 \le t < b. \end{cases} \tag{15.22}$$

In terms of the new notation, continuity conditions $F(c_1) = H(c_1)$ *and* $H(c_2) = G(c_2)$ for S become

$$S(c_1-) = S(c_1+) \quad \textit{and} \quad S(c_2-) = S(c_2+). \tag{15.23a}$$

Similarly, smoothness conditions $F'(c_1) = H'(c_1)$ *and* $H'(c_2) = G'(c_2)$ for S become

$$S'(c_1-) = S'(c_1+) \quad \textit{and} \quad S'(c_2-) = S'(c_2+). \tag{15.23a}$$

In terms of (20) and (23), testes size W is continuous on [0, 12] with a corner at $t = c$ because $W(c-) = W(c+) = 7.0$ but $1.14 = W'(c-) \neq W'(c+) = -1.06$. In contrast, testes size S is smooth on [0, 12] because $S(3-) = S(3+) = 7.005$, $S(3.002-) = S(3.002+) = 7.0052$, $S'(3-) = S'(3+) = 1.145$ and $S'(3.002-) = S'(3.002+) = -1.06145$.

ELAPSED TIME (YEARS)	PROPORTION DECEASED	ELAPSED TIME (YEARS)	PROPORTION DECEASED	ELAPSED TIME (YEARS)	PROPORTION DECEASED
1	0.652	4	0.953	7	0.992
2	0.840	5	0.965	8	0.996
3	0.930	6	0.988	9	1.000

Table 15.1 Proportions deceased at various times among melanoma patients. Source: Table 5.3

To illustrate how smoothness conditions are applied in practice, we now take a fresh look at our epidemiological data on patients admitted to the M.D. Anderson Tumor Clinic between 1944 and 1960. Define W on $[0, \infty)$ by

$$W(t) \;=\; \text{proportion deceased at time } t. \tag{15.24}$$

Then W is the function we sampled in Lecture 5. If P_n denotes proportion deceased at time $t = n$ then, from Table 1, $P_1 = 167/256 = 0.652$, $P_2 = 215/256 = 0.84$, and so on. The sequence $\{P_n\}$ is the same as in Figure 5.3.

We expect the function W defined by (24) to have the following four properties:

$$W(0) \;=\; 0 \tag{15.25}$$

$$\lim_{t\to\infty} W(t) \;=\; 1 \tag{15.26}$$

$$\text{"}W(t) \text{ varies smoothly"} \tag{15.27}$$

$$W(n) \text{ is close to } P_n \text{ for } n = 1, .., 9. \tag{15.28}$$

What kind of function might work? From Figure 3, we see that

$$W(t) \;=\; 0.884766t - 0.232422t^2 \tag{15.29}$$

satisfies (25), (27) and (28) for $n \le 2$ (solid curve), although it satisfies neither (26) nor (28) for $n \ge 3$ (dashed curve). On the other hand,

$$W(t) \;=\; 1 - \frac{0.643589}{t^2} \tag{15.30}$$

satisfies (26), (27) and (28) extremely well for $n \ge 3$ (solid curve), although it satisfies neither (25) nor (28) for $n \le 2$ (dashed curve). The graphs of these two functions cross at, say, $t = \xi$, where Figure 3 suggests $\xi \approx 2$.[2] So a decent model results from using (29) at times earlier than $t = \xi$ and (30) at later times. That is,

$$W(t) \;=\; \begin{cases} 0.884766t - 0.232422t^2 & \text{if } 0 \le t \le \xi \\ 1 - \dfrac{0.643589}{t^2} & \text{if } \xi \le t < \infty. \end{cases} \tag{15.31}$$

Then, from Exercise 13.13,

$$W'(t) \;=\; \begin{cases} 0.884766 - 0.464844t & \text{if } 0 \le t < \xi \\ \dfrac{1.28718}{t^3} & \text{if } \xi \le t < \infty. \end{cases} \tag{15.32}$$

Both W and W′ are plotted in Figure 3.

We see that (31) fits the data extremely well. In fact, the sum of squares of the errors is given by

[2] In fact, $\xi = 2.00359$; see Exercise 1.

$$\sum_{n=1}^{9}\{W(n)-P_n\}^2 \quad = \quad 3.02\times10^{-4}, \qquad (15.33)$$

which is practically zero. Unfortunately, however, (31) fails to satisfy (27). Although $W(\xi+) = 0.839679 = W(\xi-)$, making W continuous, Figure 3 reveals a a corner at $t = \xi$, where W' abruptly increases by 0.2066 from $W'(\xi-) = -0.0466$ to $W'(\xi+) = 0.160$. Thus W is merely piecewise-smooth.

What can we do to make W smooth? Taking our cue from (31), we change the model to

$$W(t) \quad = \quad \begin{cases} At + Bt^2 & \text{if } 0 \le t \le 2 \\ 1 - \dfrac{C}{t^2} & \text{if } 2 \le t < \infty \end{cases} \qquad (15.34)$$

where A, B and C are parameters and, to make the arithmetic easier, we have set $\xi = 2$, precisely. W satisfies (25)-(26) for arbitrary A, B or C, and we choose these parameters to make W satisfy (27) as well. From (4), Exercise 13.11 and Exercise 13.13,

$$W'(t) \quad = \quad \begin{cases} A + 2Bt & \text{if } 0 \le t < 2 \\ \dfrac{2C}{t^3} & \text{if } 2 \le t < \infty. \end{cases} \qquad (15.35)$$

So $W(2-) = A\cdot 2 + B\cdot 2^2 = 2A + 4B$, $W(2+) = 1 - C/2^2 = 1 - C/4$, $W'(2-) = A + 2B\cdot 2 = A + 4B$ and $W'(2+) = 2C/2^3 = C/4$, and (20) with $c = 2$ yields continuity condition

$$2A + 4B \; = \; 1 - C/4 \qquad (15.36a)$$

and smoothness condition

$$A + 4B \; = \; C/4. \qquad (15.36b)$$

These equations are easily solved to yield $B = (1-3A)/8$ and $C = 2(1-A)$; see Exercise 3. Then

$$W(t) \quad = \quad \begin{cases} At \; + \; \dfrac{1}{8}\{1-3A\}t^2 & \text{if } 0 \le t \le 2 \\ 1 - \dfrac{2(1-A)}{t^2} & \text{if } 2 \le t < \infty, \end{cases} \qquad (15.37)$$

which satisfies (25)-(27) for any value of A. In particular, it satisfies (27) because $W(2-) = (1+A)/2 = W(2+)$ and $W'(2-) = (1-A)/2 = W'(2+)$.

Because A remains at our disposal, we can choose it to satisfy (28). From (37) and Table 1 we find that $W(1) - P_1 = 0.625A - 0.527$, $W(2) - P_2 = 0.5A - 0.34$, etc. By squaring and adding nine terms, we eventually find that the sum of squared errors is

$$\sum_{n=1}^{9}\{W(n)-P_n\}^2 \quad = \quad 0.429145 \; - \; 1.10374A \; + \; 0.718371A^2, \qquad (15.38)$$

a quadratic function of A. It is plotted in Figure 4, from which you can see that the error is least between $A = 0.76$ and $A = 0.8$. In fact, the minimum error is 0.52×10^{-2}, which occurs where $A = 0.768$ (Exercise 4). The corresponding W and W' are plotted in the upper half of Figure 5. I think you'll agree that this W provides a more useful model of the data than (31), even though 0.52×10^{-2} is a larger error than 3.02×10^{-4}.

The concept of second and higher derivatives extends from smooth to piecewise-smooth functions in the obvious way. For example, the function W defined by (37) not only has the piecewise-smooth derivative W' defined by

$$W'(t) \quad = \quad \begin{cases} A \ + \ \dfrac{1}{4}\{1-3A\}t & \text{if } 0 \le t < 2 \\ \dfrac{4(1-A)}{t^3} & \text{if } 2 \le t < \infty, \end{cases} \tag{15.39}$$

but also piecewise-smooth second and third derivatives defined by

$$W''(t) \quad = \quad \begin{cases} \dfrac{1}{4}\{1-3A\} & \text{if } 0 \le t < 2 \\ \dfrac{12(A-1)}{t^4} & \text{if } 2 \le t < \infty. \end{cases} \tag{15.40}$$

and

$$W'''(t) \quad = \quad \begin{cases} 0 & \text{if } 0 \le t < 2 \\ \dfrac{48(1-A)}{t^5} & \text{if } 2 \le t < \infty \end{cases} \tag{15.41}$$

These higher derivatives are plotted in Figure 5 for $A = 0.768$.

As remarked in Lecture 13, a third derivative is of relatively little use, but a second derivative is often useful for finding inflection points. See Exercises 7-10.

Exercises 15

15.1* Use Mathematica's **FindRoot** command to find exactly where the graphs defined by (29) and (30) intersect.

15.2 Verify that (2) and (A6) imply (A7), and that (3) makes w_n continuous.

15.3 Solve (36) for B and C in terms of A.

15.4 Verify (38) and find its minimum value.

15.5 Show that V defined by Appendix 2B is smooth on [0, 0.9].

15.6 Show that f defined by Appendix 2B is continuous on [0, 0.9], but not smooth.

15.7 Show that the inflection point marked on the graph of W in Figure 1 is at $t = 4.337$ (where W' has a minimum of -1.174).

15.8 Show that f defined by Figure 1.3 or Appendix 2B has inflection points at $t = 7/30$, $t = 23/36$ and $t = 77/90$.

15.9 (i) Does W in Figure 1 have an inflection point at $t = c = 3.00103$? Why, or why not? Hint: Determine the sign of both $W''(c-)$ and $W''(c+)$.
(ii) Does the graph of W in Figure 3 have an inflection point at $t = 2$? Why, or why not? Hint: Use Exercise 13.17 to determine W'' on $[2, \infty)$.

15.10* An alternative model to Figure 3 for patient survival is W defined by

$$W(t) \;=\; \begin{cases} 1.09207t + 0.543188t^2 + 0.103465t^3 & \text{if } 0 \le t \le 2 \\ 1 - \dfrac{0.643589}{t^2} & \text{if } 2 \le t < \infty. \end{cases}$$

(i) Show that W is smooth on $[0, \infty)$, whereas W' is merely piecewise-smooth.
(ii) W has precisely two inflection points. Locate them precisely.
(iii) The sum of squared errors equals 3.02×10^{-4}, by a calculation similar to that which yielded (40). This error is lower than in Figure 5, where the model is also smooth. The error is the same as in Figure 3, where the model has a corner. Does this mean that W yields a better model of patient survival than either Figure 3 or Figure 5? Why, or why not?

Hint: Begin by using Mathematica to plot the graphs of W and W'.

15.11 What is the largest subdomain of [0, 3] on which the function g defined in Exercise 12.2 is smooth? What is the largest subdomain of [0, 4] on which g defined in Exercise 12.3 is smooth?

15.12* The function R is defined on $[0, \infty)$ by

$$R(t) \;=\; \begin{cases} At + Bt^2 & \text{if } 0 \le t < 2 \\ \dfrac{1}{t} & \text{if } 2 \le t < \infty. \end{cases}$$

What must be the values of A and B if R is smooth on $[0, \infty)$? Using Mathematica or otherwise, sketch the graphs of R and R', one above the other.
Hint: You may assume the result you obtained in Exercises 13.11 and 13.12.

15.13* The function R is defined on $[0, \infty)$ by

$$R(t) \;=\; \begin{cases} At + Bt^3 & \text{if } 0 \le t < 1 \\ \dfrac{1-t}{1+t} & \text{if } 1 \le t < \infty. \end{cases}$$

What must be the values of A and B if R is smooth on $[0, \infty)$? Using Mathematica or otherwise, sketch the graphs of R and R', one above the other.
Hint: You may assume the result you obtained in Exercise 13.16.

15.14 The function W is defined on [0, 2] by

$$W(t) \;=\; \begin{cases} 5t - t^2 & \text{if } 0 \le t < 1 \\ t^3 + 3 & \text{if } 1 \le t \le 2 \end{cases}$$

(i) What is the largest subdomain of [0, 2] on which W is smooth?
(ii) Show that Int(W, [0, 2]) = 107/12 (= 13/6 + 27/4).
(iii) Use Mathematica to sketch the graphs of W and W', one above the other.

15.15 The function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} 2t + 3t^3 - 7 & \text{if } 0 \le t < 2 \\ 7t^2 - 6t + 5 & \text{if } 2 \le t < 5 \\ 9t^2 - 18t + 15 & \text{if } 5 \le t < \infty \end{cases}$$

(i) What is the largest subdomain of $[0, \infty)$ on which W is smooth?
(ii) Calculate Int(W, [1, 6])
(iii) Use Mathematica to sketch the graphs of W and W', one above the other.

15.16 (i) The function F is defined on [0, 1] by

$$F(t) = \frac{1}{4}t(a - t),$$

where a (> 0) is a constant. **By extracting the leading term** of the difference quotient DQ(F, [t, t+h]), find an expression for $F'(t)$.

(ii) The function G is defined on $[1, \infty)$ by

$$G(t) = \frac{t}{b + t},$$

where b (>0) is a constant. **By finding the limit** as $h \to 0$ of the difference quotient DQ(G, [t, t+h]), find an expression for $G'(t)$.

(iii) A smooth function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} F(t) & \text{if } 0 \le t < 1 \\ G(t) & \text{if } 1 \le t < \infty \end{cases}$$

What must be the values of a and b? Hint: Obtain b first (by subtracting two relevant equations to eliminate a).

(iv) Use Mathematica to sketch the graphs of W and W', one above the other.

15.17 (i) The function F is defined on [0, 2] by $F(t) = at^3$, where a (> 0) is a constant. **By extracting the leading term** of the difference quotient DQ(F, [t, t+h]), find an expression for $F'(t)$.

(ii) The function G is defined on [0, b) by

$$G(t) = \frac{t^2}{b - t}$$

where b (> 0) is a constant. **By finding the limit** as $h \to 0$ of the difference quotient DQ(G, [t, t+h]), find an expression for $G'(t)$.

(iii) A smooth function W is defined on [0, 3] by

$$W(t) = \begin{cases} F(t) & \text{if } 0 \le t < 2 \\ G(t) & \text{if } 2 \le t \le 3 \end{cases}$$

What must be the values of a and b?

(iv) Use Mathematica to sketch the graphs of W and W', one above the other.

15.18 (i) The function F is defined on [0, 3] by $F(t) = At - Bt^2$, where A and B are positive constants. **By extracting the leading term** of the difference quotient DQ(F, [t, t+h]), find an expression for $F'(t)$.

(ii) The function G is defined on $[3, \infty)$ by

$$G(t) = \frac{16t}{t+1}.$$

By finding the limit as $h \to 0$ of the difference quotient DQ(G, [t, t+h]), find an expression for $G'(t)$.

(iii) A smooth function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} F(t) & \text{if } 0 \le t < 3 \\ G(t) & \text{if } 3 \le t < \infty \end{cases}$$

What must be the values of A and B?

15.19 (i) The function F is defined on [0, 1] by $F(t) = At - Bt^2$, where A and B are positive constants. **By extracting the leading term** of the difference quotient DQ(F, [t, t+h]), find an expression for $F'(t)$.

(ii) The function G is defined on $[1, \infty)$ by

$$G(t) = \frac{9t}{t+2}.$$

By finding the limit as $h \to 0$ of the difference quotient DQ(G, [t, t+h]), find an expression for $G'(t)$.

(iii) A smooth function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} F(t) & \text{if } 0 \le t < 1 \\ G(t) & \text{if } 1 \le t < \infty \end{cases}$$

What must be the values of A and B?

Appendix 15: The derivative as limit of a function sequence

To reinterpret the derivative as the limit of a function sequence, let n be a nonnegative integer, and define

$$h \;=\; 2^{-n} \tag{15.A1}$$

Thus $n = 0$ corresponds to $h = 1$ (the largest value of h we consider), $n = 1$ to $h = 0.5$, $n = 2$ to $h = 0.25$, and so on. As n gets larger and larger, h gets smaller and smaller, until eventually $h \to 0$ while $n \to \infty$. Define a sequence of functions $\{f_n\}$ by

$$f_n(t) \;=\; DQ\big(F,[t,t+2^{-n}]\big), \quad n \geq 0. \tag{15.A2}$$

Then

$$\begin{aligned}\lim_{n\to\infty} f_n(t) \;&=\; \lim_{n\to\infty} DQ\big(F,[t,t+2^{-n}]\big)\\ &=\; \lim_{h\to 0} DQ\big(F,[t,t+h]\big)\\ &=\; F'(t),\end{aligned} \tag{15.A3}$$

by (1). So F' is defined as the limit of a function sequence by

$$F'(t) \;=\; \lim_{n\to\infty} \frac{F(t+2^{-n})-F(t)}{2^{-n}} \tag{15.A4}$$

To illustrate this convergence, consider the smooth function G defined on [c, 12] by (7). In Figure 6, solid curves show graphs of g_n defined on $[c, 12-2^{-n}]$ by

$$g_n(t) \;=\; \frac{G(t+2^{-n})-G(t)}{2^{-n}} \tag{15.A5}$$

for $n = 0, 1, \dots, 5$. The dashed curve is the graph of G', to which $\{g_n\}$ converges as $n \to \infty$; it is virtually indistinguishable from the graph of g_n if $n > 5$. Why does g_n have domain $[c, 12-2^{-n}]$, as opposed to [c, 12]?

Similarly, the derivative of the piecewise-smooth function W on [a, b] defined by (2) is the limit as $n \to \infty$ of the function sequence $\{w_n\}$, defined on $[a, b-2^{-n}]$ by

$$w_n(t) \;=\; \frac{W(t+2^{-n})-W(t)}{2^{-n}}. \tag{15.A6}$$

In other words (Exercise 2), W' is the limit of the function sequence defined by

$$w_n(t) \;=\; \begin{cases} 2^n\{F(t+2^{-n})-F(t)\} & \text{if } a \leq t \leq c-2^{-n}\\ 2^n\{G(t+2^{-n})-F(t)\} & \text{if } c-2^{-n} \leq t \leq c\\ 2^n\{G(t+2^{-n})-G(t)\} & \text{if } c \leq t \leq b-2^{-n}.\end{cases} \tag{15.A7}$$

In the special case of W defined on [0, 12] by (6)-(7), w_n on $[0, 12 - 2^{-n}]$ is graphed in Figure 7 as a solid curve for $n = 0, 1, \dots, 5$. The dashed curve is the graph of W', to which $\{w_n\}$ converges as $n \to \infty$. The dashed curve is identical to the solid curve in Figure 1(b). Although W' is discontinous at $t = c$, w_n is continuous for any finite value of n (Exercise 2). Thus Figure 7 illustrates the important point that a sequence of continuous functions may converge to a discontinuous one.

Finally, Figure 8 illustrates the convergence to S' as $n \to \infty$ of $\{s_n\}$ defined by

$$s_n(t) \;=\; \frac{S(t+2^{-n})-S(t)}{2^{-n}}, \tag{15.A8}$$

where S is defined by (12). The solid curves are the graphs of s_n for $n = 0, 1, \dots, 5$; the dashed curve is the graph of S', and is identical to the solid curve in Figure 2.

Answers and Hints for Selected Exercises

15.3 Adding (36a) and (36b) yields $2A + 4B + A + 4B = 1 - C/4 + C/4 = 1$ or $3A + 8B = 1$, so that $B = (1-3A)/8$. Subtracting (36b) from (36a) yields $2A + 4B - (A + 4B) = 1 - C/4 - C/4 = 1 - C/2$ or $A = 1 - C/2$, so that $C = 2(1-A)$ on rearranging.

15.7 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Assignment B, #4)

15.12 From (4) and Exercises 13.11 and 13.12, we have

$$R'(t) \quad = \quad \begin{cases} A + 2Bt & \text{if} \quad 0 \le t < 2 \\ -\dfrac{1}{t^2} & \text{if} \quad 2 \le t < \infty. \end{cases}$$

Continuity requires $R(2-) = R(2+)$ or $2A + 4B = 1/2$. Smoothness requires in addition that $R'(2-) = R'(2+)$ or $A + 4B = -1/4$. Solving these equations readily yields $A = 3/4$ and $B = -1/4$.

15.13 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Assignment B, #5)

15.14 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Second Test, #3)

15.15 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Mock Test 2), #2)

15.16 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Second Test, #1)

15.17 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Mock Test 2), #1)

15.18 (i) From $F(t) = At - Bt^2$ we have $F(t+h) = A(t+h) - B(t+h)^2$ and hence

$$DQ(F,[t,t+h]) = \frac{F(t+h)-F(t)}{h} = \frac{A(t+h)-B(t+h)^2 - At + Bt^2}{h}$$
$$= A - 2Bt - Bh = A - 2Bt + O[h],$$

after simplification. Extracting the leading term, we have $F'(t) = A - 2Bt$.

(ii) From $G(t) = 16t/(t+1)$, we have $G(t+h) = \frac{16(t+h)}{t+h+1}$. Therefore

$$DQ(G,[t,t+h]) = \frac{1}{h}\{G(t+h)-G(t)\} = \frac{1}{h}\left\{\frac{16(t+h)}{t+h+1} - \frac{16t}{t+1}\right\}$$
$$= \frac{1}{h}\,\frac{16(t+h)(t+1) - 16t(t+h+1)}{(t+h+1)(t+1)} = \frac{16}{(t+h+1)(t+1)}.$$

implying

$$G'(t) = \lim_{h\to 0} DQ(G,[t,t+h]) = \frac{16}{(t+1)^2}$$

(iii) From above,

$$W'(t) = \begin{cases} F'(t) & \text{if} \quad 0 \le t < 3 \\ G'(t) & \text{if} \quad 3 \le t < \infty \end{cases} = \begin{cases} A - 2Bt & \text{if} \quad 0 \le t < 3 \\ \dfrac{16}{(t+1)^2} & \text{if} \quad 3 \le t < \infty \end{cases}$$

For W to be smooth, we require $W(3-) = W(3+)$ or $3A - 9B = 12$ and $W'(3-) = W'(3+)$ or $A - 6B = 1$. So $A = 7$ and $B = 1$.

15.19 (ii) From $G(t) = 9t/(t+2)$, we have $G(t+h) = \frac{9(t+h)}{t+h+2}$. Therefore

$$DQ(G,[t,t+h]) = \frac{1}{h}\{G(t+h)-G(t)\} = \frac{1}{h}\left\{\frac{9(t+h)}{t+h+2} - \frac{9t}{t+2}\right\}$$
$$= \frac{1}{h}\,\frac{9(t+h)(t+2) - 9t(t+h+2)}{(t+h+2)(t+2)} = \frac{18}{(t+h+2)(t+2)}.$$

implying

$$G'(t) = \lim_{h\to 0} DQ(G,[t,t+h]) = \frac{18}{(t+2)^2}$$

(iii) From above,

$$W'(t) = \begin{cases} F'(t) & \text{if} \quad 0 \le t < 1 \\ G'(t) & \text{if} \quad 1 \le t < \infty \end{cases} = \begin{cases} A - 2Bt & \text{if} \quad 0 \le t < 1 \\ \dfrac{18}{(t+2)^2} & \text{if} \quad 1 \le t < \infty \end{cases}$$

For W to be smooth, we require $W(1-) = W(1+)$ or $A - B = 3$ and $W'(1-) = W'(1+)$ or $A - 2B = 2$. So $A = 4$ and $B = 1$.