25. More on bivariate functions: partial derivatives and integrals

Although we said that the graph of photosynthesis versus temperature in Lecture 16 is like a hill, in the real world hills are three-dimensional objects that can be climbed in several directions. So the side of a real hill looks more like Figure 1(a), where the bivariate function P defined by

$$P(x,y) = \frac{9}{250}x^{5/3}y^{-3/2} \tag{25.1}$$

is graphed on $[0.1, 1] \times [0.1, 1]$. As discussed in Lecture 24, $P(x, y)$ represents the power in kilowatts needed to keep an x-kilogram bird with a y-meter wing span in steady flight at the speed that minimizes fuel consumption over a given distance.

In Figure 1, a possible path up the side of the hill includes KL, and a possible path down includes MN. The curve KL is the curve in which surface $z = P(x, y)$ meets vertical plane $y = 0.3$, and so KL has equation

$$z \;=\; P(x,0.3) \;=\; \frac{9}{250}x^{5/3}(0.3)^{-3/2} \;=\; 0.219x^{5/3}. \tag{25.2}$$

Thus KL is the graph $z = f(x)$ of the ordinary function f defined on $[0.1, 1]$ by

$$f(x) \;=\; \frac{9}{250}(0.3)^{-3/2}x^{5/3}, \tag{25.3}$$

which describes how power requirement varies with mass in birds with a 30 cm wing span; see Figure 2(a). To determine its gradient f', we apply (22.31) with $\beta = 5/3$:

$$\begin{aligned} f'(x) &= \frac{9}{250}(0.3)^{-3/2}\frac{d}{dx}\left\{x^{5/3}\right\} \\ &= \frac{9}{250}(0.3)^{-3/2}\,\frac{5}{3}x^{(5/3-1)} \;=\; \frac{3}{50}(0.3)^{-3/2}x^{2/3} \;=\; 0.365x^{2/3}. \end{aligned} \tag{25.4}$$

The graph of f' also appears in Figure 2(a). Note that f is concave up and f' is concave down, in agreement with Exercise 2.2.

Similarly, the curve MN is the curve in which surface $z = P(x, y)$ meets vertical plane $x = 0.8$, and so MN has equation

$$z \;=\; P(0.8,y) \;=\; \frac{9}{250}(0.8)^{5/3}y^{-3/2} \;=\; 0.02482y^{-3/2}. \tag{25.5}$$

Thus MN is the graph $z = g(y)$ of the ordinary function g defined on $[0.1, 1]$ by

$$g(y) \;=\; \frac{9}{250}(0.8)^{5/3}y^{-3/2}, \tag{25.6}$$

which describes how power requirement varies with wing span in 800 gm birds; see Figure 2(b). To determine its gradient g', we apply (22.31) with $\beta = -3/2$:

$$\begin{aligned} g'(y) &= \frac{9}{250}(0.8)^{5/3}\frac{d}{dy}\left\{y^{-3/2}\right\} \\ &= \frac{9}{250}(0.8)^{5/3}\left(-\frac{3}{2}y^{(-3/2-1)}\right) \;=\; -\frac{27}{500}(0.8)^{5/3}y^{-5/2} \;=\; -0.03723y^{-5/2}. \end{aligned} \tag{25.7}$$

The graph of g' also appears in Figure 2(b); again, in agreement with Exercise 2.2, g and g' are concave up and concave down, respectively.

Now, for climbing the side of Figure 1's hill in a direction parallel to the x-axis, there is no special reason why the value of y should be fixed at $y = 0.3$. Any y such that $0.1 \le y \le 1$ would do, say $y = k$. The corresponding curve KL would be that in which surface $z = P(x, y)$ meets vertical plane $y = k$, KL would have equation

$$z = P(x,k) = \frac{9}{250}x^{5/3}k^{-3/2}, \tag{25.8}$$

and KL would be the graph of the ordinary function f defined on [0.1, 1] by

$$f(x) = \frac{9}{250}k^{-3/2}x^{5/3}, \tag{25.9}$$

instead of by (3). Similarly, for descending in a direction parallel to the y-axis, there is no special reason why the value of x should be fixed at $x = 0.8$. Any x such that $0.1 \le x \le 1$ would do, say $x = m$. The corresponding curve MN would be that in which surface $z = P(x, y)$ meets vertical plane $x = m$, MN would have equation

$$z = P(m,y) = \frac{9}{250}m^{5/3}y^{-3/2}, \tag{25.10}$$

and MN would be the graph of the ordinary function g defined on [0.1, 1] by

$$g(y) = \frac{9}{250}m^{5/3}y^{-3/2}, \tag{25.11}$$

instead of by (6). To obtain the gradient along KL, all we need do is to replace 0.3 by k in (4); and to obtain the gradient along MN, all we need do is to replace 0.8 by m in (7). Thus we obtain

$$f'(x) = \frac{3}{50}k^{-3/2}x^{2/3} \tag{25.12}$$

in place of (4), and

$$g'(y) = -\frac{27}{500}m^{5/3}y^{-5/2} \tag{25.13}$$

in place of (7).

Comparing (12)-(13) with (8)-(11), we observe that $f'(x)$ is dz/dx for KL and $g'(y)$ is dz/dy for MN. Use of the symbol dz/dx implies that z depends only on x, whereas use of the symbol dz/dy implies that z depends only on y. Isn't this a contradiction? The answer is no, not really, because even though z depends on both x and y in general, on KL the value of y is fixed, so that z depends only on x; whereas on MN the value of x is fixed, and so z depends only on y. Nevertheless, there is potential for confusion, and so we introduce some new notation. For a bivariate function P, we use $\partial z/\partial x$ to denote the derivative of $z = P(x, y)$ with respect to x alone, i.e., with y held fixed; and we use $\partial z/\partial y$ to denote the derivative of $z = P(x, y)$ with respect to y alone, i.e., with x held fixed. That is, by analogy with (16.1b), we define

$$\frac{\partial z}{\partial x} = \lim_{h\to 0}\frac{P(x+h,y)-P(x,y)}{h} \tag{25.14}$$

and

$$\frac{\partial z}{\partial y} = \lim_{h\to 0}\frac{P(x,y+h)-P(x,y)}{h}. \tag{25.15}$$

We refer to $\partial z/\partial x$, i.e., the gradient of z in a direction parallel to the x-axis, as the **partial derivative** of z with respect to x; and to $\partial z/\partial y$, i.e., the gradient of z in a direction parallel to the y-axis, as the partial derivative of z with respect to y. For example, in the case of Figure 1, where k denotes an arbitrary y-value and m an arbitrary x-value, we can replace k by y in (12) to obtain

$$\frac{\partial z}{\partial x} = \frac{3}{50}y^{-3/2}x^{2/3}, \tag{25.16}$$

and we can replace m by x in (13) to obtain

$$\frac{\partial z}{\partial y} = -\frac{27}{500}x^{5/3}y^{-5/2}. \tag{25.17}$$

The partial derivatives of $z = P(x, y)$ yield two new bivariate functions, say Q and R.[1] That is,

$$Q(x,y) = \lim_{h\to 0}\frac{P(x+h,y)-P(x,y)}{h} \tag{25.18}$$

$$R(x,y) = \lim_{h\to 0}\frac{P(x,y+h)-P(x,y)}{h}. \tag{25.19}$$

For example, in the case of Figure 1, we have

$$Q(x,y) = \frac{3}{50}x^{2/3}y^{-3/2} \tag{25.20}$$

and

$$R(x,y) = -\frac{27}{500}x^{5/3}y^{-5/2} \tag{25.21}$$

from (16)-(17). An important special case occurs when the function P is **separable**, i.e., when ordinary functions F and G exist such that

$$P(x,y) = F(x)G(y) \tag{25.22}$$

for any relevant x or y. Then (18) implies

$$\begin{aligned} Q(x,y) &= \lim_{h\to 0}\frac{F(x+h)G(y)-F(x)G(y)}{h} \\ &= \left\{\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\right\}G(y) = F'(x)G(y) \end{aligned} \tag{25.23}$$

and (19) implies

$$\begin{aligned} R(x,y) &= \lim_{h\to 0}\frac{F(x)G(y+h)-F(x)G(y)}{h} \\ &= F(x)\lim_{h\to 0}\frac{G(yx+h)-G(y)}{h} = F(x)G'(y) \end{aligned} \tag{25.24}$$

or, in mixed notation,

$$\frac{\partial}{\partial x}\{F(x)G(y)\} = F'(x)G(y), \qquad \frac{\partial}{\partial y}\{F(x)G(y)\} = F(x)G'(y). \tag{25.25}$$

For example, P defined by (1) is separable; and so, from $z = 9x^{5/3}y^{-3/2}/250$, we calculate

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(x^{5/3}\frac{9}{250}y^{-3/2}\right) = \frac{d}{dx}\left(x^{5/3}\right)\frac{9}{250}y^{-3/2} \\ &= \frac{5}{3}x^{2/3}\frac{9}{250}y^{-3/2} = \frac{3}{50}x^{2/3}y^{-3/2} \end{aligned} \tag{25.26}$$

in agreement with (16), and

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[1] If P has domain $[a_1, b_1] \times [a_2, b_2]$, then the domains of Q and R are, respectively, $[a_1, b_1) \times [a_2, b_2]$ and $[a_1, b_1] \times [a_2, b_2)$, by (13.25); i.e., $x = b_1$ is, strictly speaking, excluded from the domain of Q, whereas $y = b_2$ is excluded from the domain of R. In practice, however, both domains can be extended to $[a_1, b_1] \times [a_2, b_2]$ by analogy with (13.26), i.e., by agreeing that $Q(b_1, y)$ is the limit of $Q(x, y)$ as $x \to b_1$ and $R(x, b_2)$ is the limit of $R(x, y)$ as $y \to b_2$.

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{9}{250}x^{5/3}y^{-3/2}\right) = \frac{9}{250}x^{5/3}\frac{d}{dy}\left(y^{-3/2}\right) \\ &= \frac{9}{250}x^{5/3}\left(-\frac{3}{2}y^{-5/2}\right) = -\frac{27}{500}x^{5/3}y^{-5/2}\end{aligned} \tag{25.27}$$

in agreement with (17).

Furthermore, because Q is itself a bivariate function, it has its own partial derivatives, which are **second partial derivatives** of z. We write

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \lim_{h\to 0}\frac{Q(x+h,y)-Q(x,y)}{h} \tag{25.28}$$

and

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \lim_{h\to 0}\frac{Q(x,y+h)-Q(x,y)}{h}. \tag{25.29}$$

For example, in the case of Figure 1, (25)-(26) imply

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{3}{50}x^{2/3}y^{-3/2}\right) = \frac{d}{dx}\left\{x^{2/3}\right\}\frac{3}{50}y^{-3/2} \\ &= \frac{2}{3}x^{-1/3}\frac{3}{50}y^{-3/2} = \frac{1}{25}x^{-1/3}y^{-3/2},\end{aligned} \tag{25.30}$$

whereas (25) and (27) imply

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(\frac{3}{50}x^{2/3}y^{-3/2}\right) = \frac{3}{50}x^{2/3}\frac{d}{dy}\left\{y^{-3/2}\right\} \\ &= \frac{3}{50}x^{2/3}\left\{-\frac{3}{2}y^{-5/2}\right\} = -\frac{9}{100}x^{2/3}y^{-5/2}.\end{aligned} \tag{25.31}$$

Likewise, because R is a bivariate function, we have second partial derivatives

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \lim_{h\to 0}\frac{R(x+h,y)-R(x,y)}{h} \tag{25.32}$$

and

$$\frac{\partial^2 z}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \lim_{h\to 0}\frac{R(x,y+h)-R(x,y)}{h}. \tag{25.33}$$

In particular, in the case of Figure 1, (25)-(26) imply

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(-\frac{27}{500}x^{5/3}y^{-5/2}\right) = \frac{d}{dx}\left\{x^{5/3}\right\}\left\{-\frac{27}{500}y^{-5/2}\right\} \\ &= \frac{5}{3}x^{2/3}\left\{-\frac{27}{500}y^{-5/2}\right\} = -\frac{9}{100}x^{2/3}y^{-5/2},\end{aligned} \tag{25.34}$$

whereas (25) and (27) imply

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(-\frac{27}{500}x^{5/3}y^{-5/2}\right) = -\frac{27}{500}x^{5/3}\frac{d}{dy}\left\{y^{-5/2}\right\} \\ &= -\frac{27}{500}x^{5/3}\left\{-\frac{5}{2}y^{-7/2}\right\} = \frac{27}{200}x^{5/3}y^{-7/2}\end{aligned} \tag{25.35}$$

by Exercise 1. Note that (31) and (34) imply

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial^2 z}{\partial x\partial y}. \tag{25.36}$$

This result holds not only for z in Figure 1, but also in general.[2] Thus, although in principle a bivariate function has four second derivatives, defined by (28), (29), (32) and (33), in practice there are only three, because (36) makes (29) and (32) equal.

Now, from Lecture 24, if a bivariate function P has a global minimum on the interior of its domain at $(\hat{x}, \hat{y})$, then $\hat{x}$ must minimize the ordinary function f defined by $f(x) = P(x, \hat{y})$, whose graph is a vertical section through the graph of P along $y = \hat{y}$; and $\hat{y}$ must minimize the ordinary function g defined by $g(y) = P(\hat{x}, y)$, whose graph is a vertical section through the graph of P along $x = \hat{x}$. Thus the partial derivative of P with respect to x along $y = \hat{y}$ must change sign from negative to positive at $x = \hat{x}$, and the partial derivate of P with respect to y along $x = \hat{x}$ must change sign from negative to positive at $y = \hat{y}$. In other words,

$$Q(\hat{x}, \hat{y}) \quad = \quad 0 \quad = \quad R(\hat{x}, \hat{y}), \tag{25.37}$$

where Q and R are defined by (18)-(19); or, in differential notation,

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=\hat{x}\\y=\hat{y}}} \quad = \quad 0 \quad = \quad \left.\frac{\partial z}{\partial y}\right|_{\substack{x=\hat{x}\\y=\hat{y}}}. \tag{25.38}$$

This pair of equations is sometimes useful for finding global minima. For example, in Appendix 2A we required the global minimum of S defined by

$$S(\alpha,\beta) = 3433.26 - 15599\alpha + 18006.5\alpha^2 - 275.6\beta + 667\alpha\beta + 7\beta^2. \tag{25.39}$$

Here, with $z = S(\alpha, \beta)$, we have

$$\begin{aligned}
\frac{\partial z}{\partial \alpha} &= \frac{\partial}{\partial \alpha}(3433.26) - 15599\frac{\partial}{\partial \alpha}(\alpha) + 18006.5\frac{\partial}{\partial \alpha}(\alpha^2) - 275.6\frac{\partial}{\partial \alpha}(\beta) + 667\frac{\partial}{\partial \alpha}(\alpha\beta) + 7\frac{\partial}{\partial \alpha}(\beta^2)\\
&= 0 - 15599 \times 1 + 18006.5 \times 2\alpha - 0 + 667\beta + 0\\
&= -15599 + 36013\alpha + 667\beta
\end{aligned} \tag{25.40}$$

and

$$\begin{aligned}
\frac{\partial z}{\partial \beta} &= \frac{\partial}{\partial \beta}(3433.26) - 15599\frac{\partial}{\partial \beta}(\alpha) + 18006.5\frac{\partial}{\partial \beta}(\alpha^2) - 275.6\frac{\partial}{\partial \beta}(\beta) + 667\frac{\partial}{\partial \beta}(\alpha\beta) + 7\frac{\partial}{\partial \beta}(\beta^2)\\
&= 0 - 0 + 0 - 275.6 \times 1 + 667\alpha + 7 \times 2\beta\\
&= -275.6 + 667\alpha + 14\beta,
\end{aligned} \tag{25.41}$$

so that (38) yields $36013\alpha + 667\beta = 15599$ and $667\alpha + 14\beta = 275.6$. This pair of equations is readily solved to yield $\hat{\alpha} = 172804/296465 = 0.5829$ and $\hat{\beta} = -2396751/296465 = -8.084$, yielding a global minimum of $S(\hat{\alpha}, \hat{\beta}) = 1.1095$. It is well to remember, however, that (38) isn't always so useful; for example, it wouldn't have helped much in Lecture 24.

We conclude by broaching the concept of a partial integral, which is important in probability theory. Partial integration is the opposite of partial differentiation, in the sense that a partial derivative is the derivative of a bivariate function with respect to one variable, whereas a partial integral is the integral of a bivariate function with respect to one variable. Now, you know from Lecture 12 that the result of integrating $P(x)$ between $x = a_1$ and $x = b_1$ is independent of x, and so the result of integrating $P(x, y)$ between $x = a_1$ and $x = b_1$ must also be independent of x. The difference is simply that if P is ordinary then $\mathrm{Int}(P, [a_1, b_1])$ depends only on a_1 and b_1, whereas if P is

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[2] More precisely, whenever z represents the height of a smooth surface, which is usually the case in practice. The result is sometimes called Clairaut's Theorem. See Exercise 2 for an important special case.

bivariate then Int(P, $[a_1, b_1]$) can also depend on y. So denote it by W(y). Then, by analogy with (12.20), we define the partial integral of P with respect to x by

$$W(y) \;=\; \int_{a_1}^{b_1} P(x,y)\,dx \;=\; \lim_{\delta x \to 0} \sum_{[a_1,b_1]} P(x,y)\,\delta x. \tag{25.42}$$

Correspondingly, the partial integral of P with respect to y is defined by

$$U(x) \;=\; \int_{a_2}^{b_2} P(x,y)\,dy \;=\; \lim_{\delta y \to 0} \sum_{[a_2,b_2]} P(x,y)\,\delta y. \tag{25.43}$$

Both U and W are ordinary functions, and in Appendix 28C we will require an expression for their derivatives. It is simplest to obtain if we assume that P is separable. Then, by (22), ordinary functions F and G exist such that P(x, y) = F(x)G(y). If we are integrating with respect to x, then G(y) behaves like a constant, and so from (42) and (12.25) with u = F, k = G(y) we have

$$W'(y) \;=\; \frac{d}{dy}\left\{\int_{a_1}^{b_1} F(x)G(y)\,dx\right\} \;=\; \frac{d}{dy}\left\{G(y)\int_{a_1}^{b_1} F(x)\,dx\right\}. \tag{25.44}$$

But if we are differentiating with respect to y, then Int(F, $[a_1, b_1]$) behaves like a constant, and so from (44) and (16.17) with k = Int(F, $[a_1, b_1]$) we have

$$W'(y) \;=\; G'(y)\int_{a_1}^{b_1} F(x)\,dx \;=\; \int_{a_1}^{b_1} G'(y)F(x)\,dx \tag{25.45}$$

after applying (12.25) again, but this time with k = G′(y), which behaves like a constant if we are integrating with respect to x. Now, applying (25) to (45), we obtain

$$W'(y) \;=\; \int_{a_1}^{b_1} F(x)G'(y)\,dx \;=\; \int_{a_1}^{b_1} \frac{\partial}{\partial y}\{F(x)G(y)\}\,dx \;=\; \int_{a_1}^{b_1} \frac{\partial}{\partial y}\{P(x,y)\}\,dx. \tag{25.46}$$

In other words, on using (42),

$$\frac{d}{dy}\left\{\int_{a_1}^{b_1} P(x,y)\,dx\right\} \;=\; \int_{a_1}^{b_1} \frac{\partial}{\partial y}\{P(x,y)\}\,dx. \tag{25.47a}$$

A parallel argument establishes that

$$\frac{d}{dx}\left\{\int_{a_2}^{b_2} P(x,y)\,dy\right\} \;=\; \int_{a_2}^{b_2} \frac{\partial}{\partial x}\{P(x,y)\}\,dy \tag{25.47b}$$

(see Exercise 2). Indeed (47a) and (47b) are in essence the same result; and although we have established it only for the case where P is separable, it can be shown to hold more generally.

Exercises 25

25.1 Establish (36) for the special case of a separable function, i.e., for z = F(x)G(y).

25.2 Establish (47b) for the special case where P is separable, i.e., P(x, y) = F(x)G(y).
Hint: Differentiate U in (43) with respect to x and use (12.25) and (16.17).