Study Questions Homework 5 Introduction to Computational Finance Spring 2023

These are study questions. You are not required to submit solutions (even though the problems are worded like graded assignment problems).

Written Study Exercises

Problem 5.1

Consider the Runge Kutta method called the implicit midpoint rule given by:

$$\hat{y}_1 = y_{n-1} + \frac{h}{2} f_1$$
$$f_1 = f(t_{n-1} + \frac{h}{2}, \hat{y}_1)$$
$$y_n = y_{n-1} + h f_1$$

An alternate form of the the method is given by:

$$y_n = y_{n-1} + hf\left(\frac{t_n + t_{n-1}}{2}, \frac{y_n + y_{n-1}}{2}\right)$$

Show that the two forms are identical.

Problem 5.2

Recall the explicit 2-step Adams-Bashforth method

$$y_n = y_{n-1} + \frac{h}{2}\left(3f_{n-1} - f_{n-2}\right)$$

was derived by integrating from t_{n-1} to t_n the linear polynomial, $p_1(t)$, that interpolates f_{n-1} and f_{n-2}.

The implicit 2-step Adams-Moulton method is derived by integrating from t_{n-1} to t_n the quadratic polynomial, $p_2(t)$, that interpolates f_n, f_{n-1} and f_{n-2}.

(5.2.a) Derive the implicit 2-step Adams-Moulton method.

(5.2.b) Show that the method is consistent.

(5.2.c) Determine the order of the method.

Problem 5.3

Consider the quadratic polynomial, $p_2(t)$, that interpolates y_n, y_{n-1} and y_{n-2}. An integration method can be derived via numerical differentiation, i.e., by setting

$$p_2'(t_n) = f(t_n, y_n)$$

5.3.a. Find the implicit 2-step method described by the derivation above.

5.3.b. Show that the method is consistent.

5.3.c. Determine the order of the method.

Problem 5.4

The interval of absolute stability is the intersection of the region of absolute stability in the complex plane with the real axis. Consider the two Runge Kutta methods: Forward Euler and the Explicit Midpoint. Show that they have the same interval of absolute stability.

Problem 5.5

Linear multistep methods with a constant stepsize can be written

$$\alpha_0 y_n + \alpha_1 y_{n-1} + \cdots + \alpha_k y_{n-k} = h\left[\beta_0 f_n + \beta_1 f_{n-1} + \cdots + \beta_k f_{n-k}\right]$$

$$\sum_{i=1}^{k} \alpha_i y_{n-i} = h \sum_{i=1}^{k} \beta_i f_{n-i}$$

where $f_j = f(t_j, y_j)$. Recall, that some of the coefficients can be 0 and the number of steps used in the method is determined by the oldest index of either the α's or β's, i.e., either $\alpha_k \neq 0$ or $\beta_k \neq 0$ or both are not 0. For example, AB-2

$$y_n - y_{n-1} = h\left(\frac{3}{2} f_{n-1} - \frac{1}{2} f_{n-2}\right)$$

is a $k = 2$ step method with $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0 = 0$, $\beta_1 = 3/2$ and $\beta_2 = -1/2$.

Linear multistep methods are analyzed in terms of two characteristic polynomials

$$\rho(\xi) = \alpha_0 \xi^k + \alpha_1 \xi^{k-1} + \cdots + \alpha_k \xi^0$$
$$\sigma(\xi) = \beta_0 \xi^k + \beta_1 \xi^{k-1} + \cdots + \beta_k \xi^0.$$

It is crucial that you use the correct value of k when defining these polynomials.

A k-step method needs k initial conditions $y_0,\ y_1,\ldots,y_k$ when the first element of the numerical solution's sequence is y_{k+1}. $y_0 = y(t_0)$ is given by the IVP but the others must be estimated by some method when computing the solution.

Applying the numerical method to $y' = \lambda y$ determines the absolute stability of the linear multistep method. The form of y_n for any initial conditions can be determined by solving the associated linear homogeneous constant coefficient k-th order recurrence

$$\sum_{i=1}^{k} \alpha_i y_{n-i} - h\lambda \sum_{i=1}^{k} \beta_i y_{n-i} = 0$$

$$\sum_{i=1}^{k} (\alpha_i - h\lambda\beta_i) y_{n-i} = 0$$

whose characteristic polynomial is

$$p(\xi) = \rho(\xi) - h\lambda\sigma(\xi).$$

The roots of the k-degree polynomial $p(\xi)$ as a function of $h\lambda$ therefore determine the boundedness of $|y_n|$ for the model problem and the absolute stability region of the method.

Letting $|\lambda| \to 0$ means $p(\xi) \to \rho(\xi)$ and defines 0-stability of the method. It is equivalent to applying the numerical method to $y' = 0$ with $y_0 = 0$ and $y_i = \epsilon_i$, $i = 1, \ldots k$ with ϵ_i representing a small perturbation, determines the 0-stability of the linear multistep method. The form of y_n under given these initial conditions and differential equation can be determined by solving the associated limiting linear homogeneous constant coefficient k-th order recurrence defined by the characteristic polynomial $\rho(\xi)$.

A constant stepsize linear multistep method is 0-stable if the roots of $\rho(\xi)$ have magnitude strictly less or equal to one and roots with magnitude one are simple. A consistent linear multistep method is convergent if and only if it is 0-stable method. So convergence can be proven by showing $d_n = O(h^p)$ with $p \geq 1$ and 0-stability.

A 0-stable constant stepsize linear multistep method with a $\rho(\xi)$ that has a simple root at 1 and all other roots with magnitude strictly less than one is called strongly stable. This is the prefered type of linear multistep method for general problems. If a 0-stable constant stepsize linear multistep method is not strongly stable it is called weakly stable. Consider the following linear multistep methods:

1. $y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$
2. AB-2 $y_n = y_{n-1} + h\left(\frac{3}{2}f_{n-1} - \frac{1}{2}f_{n-2}\right)$
3. AM-2 $y_n = y_{n-1} + h\left(\frac{5}{12}f_n + \frac{8}{12}f_{n-1} - \frac{1}{12}f_{n-2}\right)$
4. BDF-2 $y_n - \frac{4}{3}y_{n-1} + \frac{1}{3}y_{n-2} = \frac{2}{3}f_n$

5.5.a. Determine if the methods are 0-stable.

5.5.b. Determine which of these constant step linear multistep methods are convergent.