AN EFFICIENT PARTICLE FILTERING TECHNIQUE ON THE GRASSMANN MANIFOLD

IAP VI/4 - DYSCO

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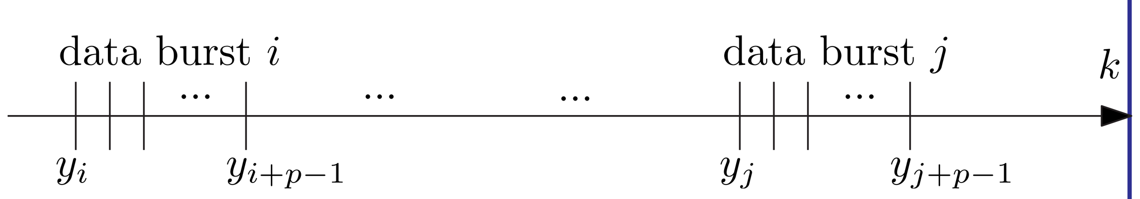
1 Motivation

DOA tracking:

Let us consider p incoming signals arriving on an antenna array with incident angles $\theta_1, ..., \theta_p$.
When a time-division multiple access technique is applied, we measure some data burst matrices $Y_i = [y_i \ \cdots \ y_{i+p-1}]$



where y_i is the vector of the received signal at time i.

Goal: find the time-varying directions of arrival $\theta_1(i), ..., \theta_p(i)$ using the data burst matrices Y_k up to time i

Method: identify the signal subspace and then apply the ESPRIT algorithm to recover the directions of arrival

Object tracking on a video:

Goal: draw a rectangle frame enclosing the object to track

- The object is represented by the dominant subspace of a covariance matrix built from the previous frames.
- Due to object deformations and illumination variations, this subspace must be updated.

$\Longrightarrow$ Filtering techniques for subspaces are required.

2 Geometric viewpoint

The unknown signal subspaces and their measurements col(Y_i) belong to the Grassmann manifold $G(n,p)$, i.e., the set of all p-dimensional subspaces of $\mathbb{R}^n$ or $\mathbb{C}^n$.
A point on $G(n,p)$ can be represented by the column space of an orthogonal $n \times p$ matrix X, i.e., by an element of the Stiefel manifold $St(n,p)$.
The tangent space to $G(n,p)$ at X is represented by:

$$T_X G = \{V \in \mathbb{C}^{n \times p} | X^\top V = 0\}.$$

Thus, a tangent vector is also represented by an $n \times p$ matrix.
$G(n,p)$ is a Riemannian manifold with the corresponding distance function:

$$d(X,Y) = \sum_{i=1}^{p} \sigma_i^2$$

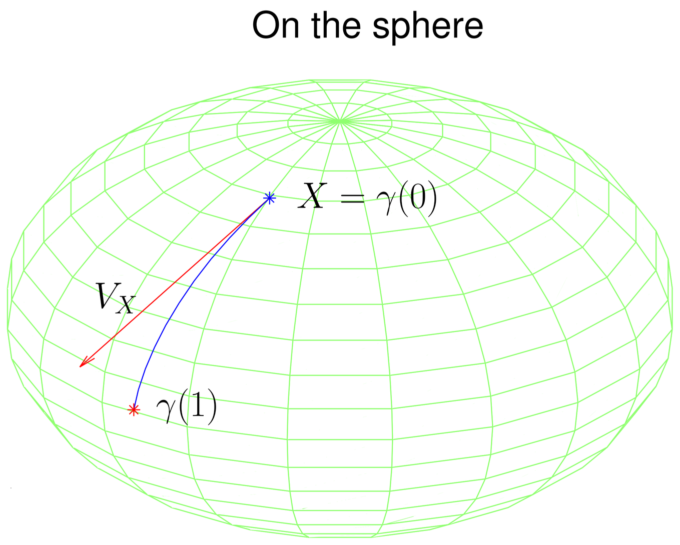
where the σ_i are the principal angles.

Two useful mappings:

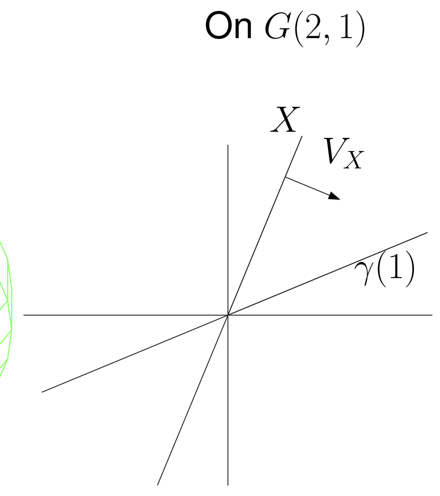
Exponential map:
Let $\gamma(t)$ be the geodesic curve s.t. $\gamma(0) = X$ and $\dot\gamma(0) = V_X$.

$$\exp_X : T_X G \mapsto G, V_X \mapsto \exp_X(V_X) = \gamma(1)$$

On the sphere



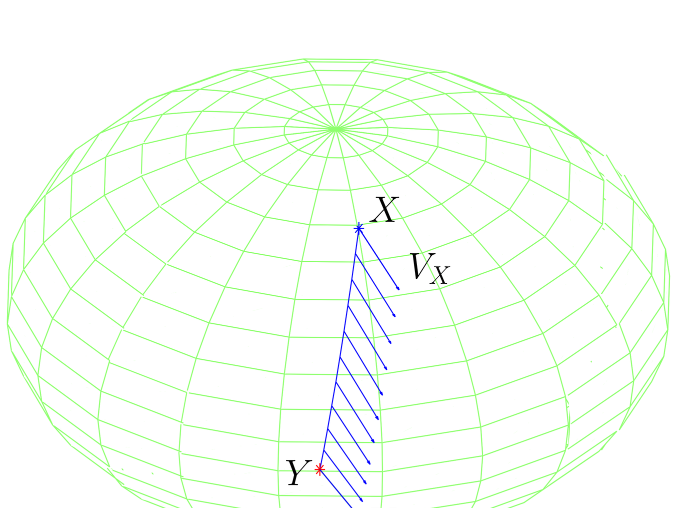
On $G(2,1)$

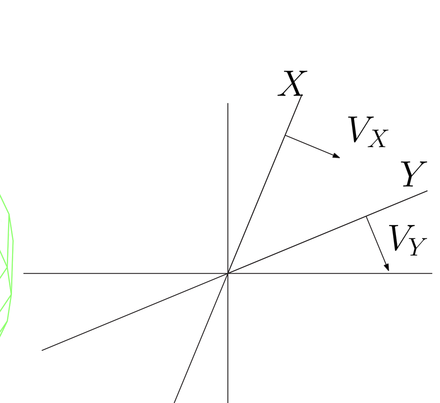


Parallel transport: along the geodesic joining X to Y

$$\Gamma_{X \to Y} : T_X G \mapsto T_Y G, V_X \mapsto V_Y$$

It is an isometry.





3 Filtering problem

Problem: We measure corrupted data col(Y_k) $\in G(n,p)$. The goal is to filter these data to reduce the influence of the noise and the outliers.
Approach: We assume that these data are the outputs of the following dynamical system on $G(n,p)$:
State space model: It is a stochastic piecewise geodesic model.

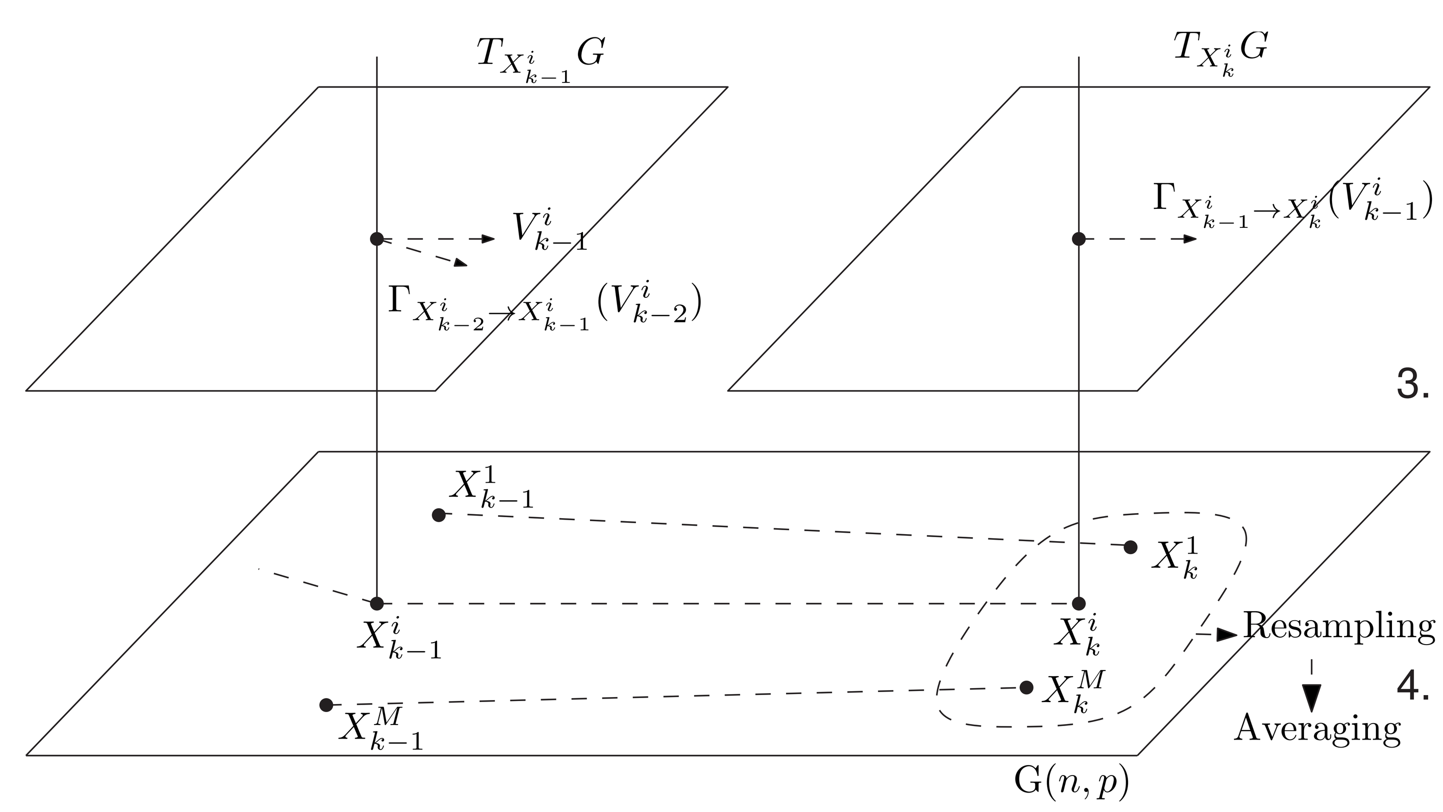
$$X_k = \exp_{X_{k-1}}(V_{k-1})$$
$$V_k = \Gamma_{X_{k-1} \to X_k}(V_{k-1}) + \Omega_k$$ where $\Omega_k \in T_{X_k}G$ is an i.i.d. Gaussian vector of mean 0 and variance σ^2_{model}

Observation model: $Y_k = \exp_{X_k}(U_k)$ where $U_k \in T_{X_k}G$ is an i.i.d. Gaussian vector of mean 0 and variance σ^2_{data}

Associated distribution: $p(Y_k|X_k) = Ce^{-\frac{d(X_k,Y_k)^2}{\sigma^2_{\text{data}}}}$

4 Main steps of the particle filtering technique

1. Initialization: choose a set of M initial particles $X_1^1, ..., X_1^M$ and associated velocities $V_1^1 \in T_{X_1^1}G, ..., V_1^M \in T_{X_1^M}G$



2. Prediction: for each particle, go in the direction V^i_{k-1} following the geodesic curve passing through X^i_{k-1}: $X^i_k = \exp_{X^i_{k-1}}(V^i_{k-1})$ generate M samples $V^1_k, ..., V^M_k$ of speed according to the prior model: $V^i_k = \Gamma_{X^i_{k-1} \to X^i_k}(V^i_{k-1}) + \Omega^i_k$
3. Resampling: compute the importance weights β^i_k for $i = 1, ..., M$ using the observation model and resample the set $S_k = \{(X^1_k, V^1_k), ..., (X^M_k, V^M_k)\}$ with respect to the β^i_k
4. Averaging: compute the Karcher mean $\mu(X^1_k, ..., X^M_k) = \arg\min_{Y \in G} \frac{1}{2M} \sum_{i=1}^{M} d(X^i_k, Y)^2$ and iterate

$\Longrightarrow$ An efficient way to simulate the state space model is needed.

5 Computation

Using the matrix exponential	Our approach (as in [2])
state representation: $(X_k, X_{\perp,k}, A_k)$ with $V_k = X_{\perp,k}A_k$ size: $n^2 + (n-p)p$	state representation: (X_k, V_k) size: $2np$

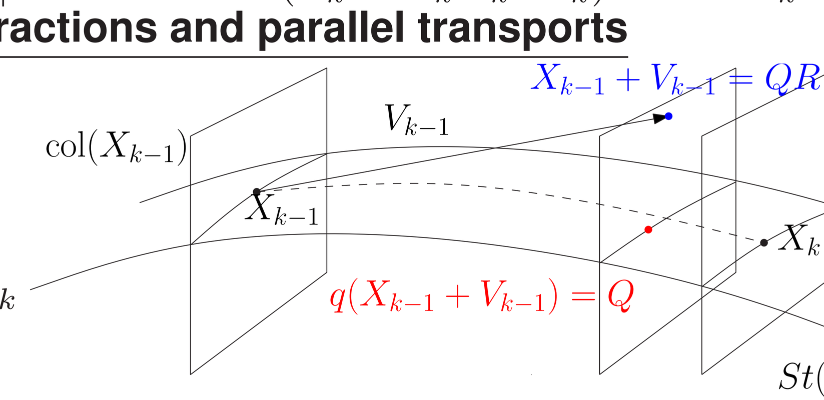
$$[X_k | X_{\perp,k}] = [X_{k-1} | X_{\perp,k-1}] e^{\begin{bmatrix} 0 & A_{k-1}^\top \\ A_{k-1} & 0 \end{bmatrix}}$$
$$A_k = A_{k-1} + N_k$$

where N_k is a Gaussian $(n-p) \times p$ matrix

$$V_{k-1} = U_{k-1}\Sigma_{k-1}W_{k-1}^\top \text{ (compact svd)}$$
$$D_{k-1} = X_{k-1}W_{k-1}$$
$$X_k = (D_{k-1}\cos\Sigma_{k-1} + U_{k-1}\sin\Sigma_{k-1})\,W_{k-1}{}^\top$$
$$V_k = (-D_{k-1}\sin\Sigma_{k-1} + U_{k-1}\cos\Sigma_{k-1})\,\Sigma_{k-1}W_{k-1}{}^\top + (\Omega_k - X_kX_k{}^\top\Omega_k)$$ where Ω_k is an $n \times p$ matrix

Approximation of the state space equations using retractions and parallel transports

$$X_k = q(X_{k-1} + V_{k-1})$$
$$V_k = \frac{(I_n - X_kX_k{}^\top)V_{k-1}\|V_{k-1}\|_F}{\|(I_n - X_kX_k{}^\top)V_{k-1}\|_F} + (I_n - X_kX_k{}^\top)\Omega_k$$



Computational time comparison (in percent of the computational time spent when the 'expm' function of Matlab is used)

	our approach	approximation
$n = 100, p = 5$	7 %	10 %
$n = 100, p = 25$	34 %	19 %
$n = 100, p = 50$	97 %	37 %

- our approach is more efficient if $n >> p$
- the approximation is interesting when p is close to $\frac{n}{2}$

Quality of the approximation

$\|A_0\|_F$	mean error	mean error with the approximation
0.1118	0.0186	0.0190
0.2236	0.0231	0.0250
0.3354	0.0269	0.0358
0.4472	0.0342	0.1147

- for a stochastic piecewise geodesic trajectory on $G(4,2)$ with $\sigma_{\text{model}} = 0.05\|A_0\|_F$
- the error is the distance between the data and the filtered data
- the approximation gives similar results if the speed $\|A_0\|_F$ is small

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