The H_∞-norm calculation for large sparse systems

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Model Reduction Project

- Rice University – A. Antoulas and D. Sorensen
- Purdue University – A. Sameh, A. Grama, C. Hoffmann
- FSU/UCL – P. Van Dooren, K. Gallivan

Develop efficient and effective techniques and codes for large scale model reduction for applications in:

- simulation
- control

Problem Statement

Given a system

$$G(z) = C(zI_n - A)^{-1}B + D,$$

define the H_∞-norm by

$$\gamma^* := \|G(z)\|_\infty := \sup_\omega \|G(e^{j\omega})\|_2.$$

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Conditions:

- discrete time, stable
- A is an $(n \times n)$, non-symmetric large sparse matrix
- $n \gg m,\ p$ where p inputs and m outputs
- $O(m\alpha^2)$ where $\alpha = m$ or $\alpha = p$ computations per step is acceptable

Motivation

Large system $T(z)$ system and reduced order approximation $\hat{T}(z)$ approximate H_∞-norm of

$$G(z) = T(z) - \hat{T}(z)$$

to evaluate global quality. Needed when $\hat{T}(z)$ is generated by:

- large scale Approximate Balanced Truncation methods (Antoulas and Sorensen) – driven by global error considerations
- Rational and Tangential Interpolation methods (Grimme, Vandendorpe, Van Dooren, and Gallivan) – driven by local error considerations
- Passive Reduced System methods via rational interpolation at selected spectral zeros (Antoulas and Sorensen) – driven by preserving structure
- Hybrid methods

Level Set Methods

Consider the para-hermitian transfer functions and their spectra

$$\Phi(z) = G_*(z)\, G(z) \qquad \Phi_\gamma(z) = \gamma^2 I_m - \Phi(z)$$

where $\quad G_*(z) := D^T + zB^T(I - zA^T)^{-1}C^T$

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$$\Phi_\gamma(e^{j\omega}) \succ 0 \Leftrightarrow \gamma > \gamma^* = \|G(z)\|_\infty \quad \text{for } \; \omega \in [-\pi, +\pi]$$

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Define eigenvalue bounds for $\Phi(e^{j\omega})$

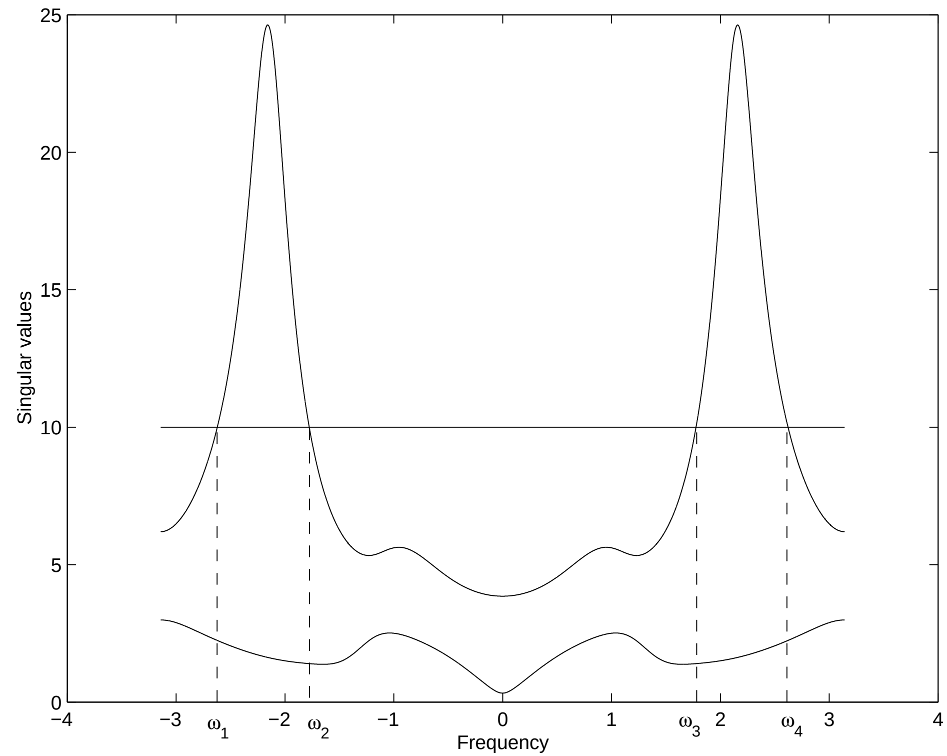
$$\lambda_* \doteq \min_{\omega\in[-\pi,+\pi]} \lambda_{min}(\Phi(e^{j\omega})), \quad \lambda^* \doteq \max_{\omega\in[-\pi,+\pi]} \lambda_{max}(\Phi(e^{j\omega})).$$

then

$$\gamma^* = \|G(z)\|_\infty = \sqrt{\lambda^*}$$

Level Set Methods

Eigenvalues of $\Phi(e^{j\omega})$ and a level line at $\lambda > 0$



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- $0 < \lambda < \lambda_*$ no intersections
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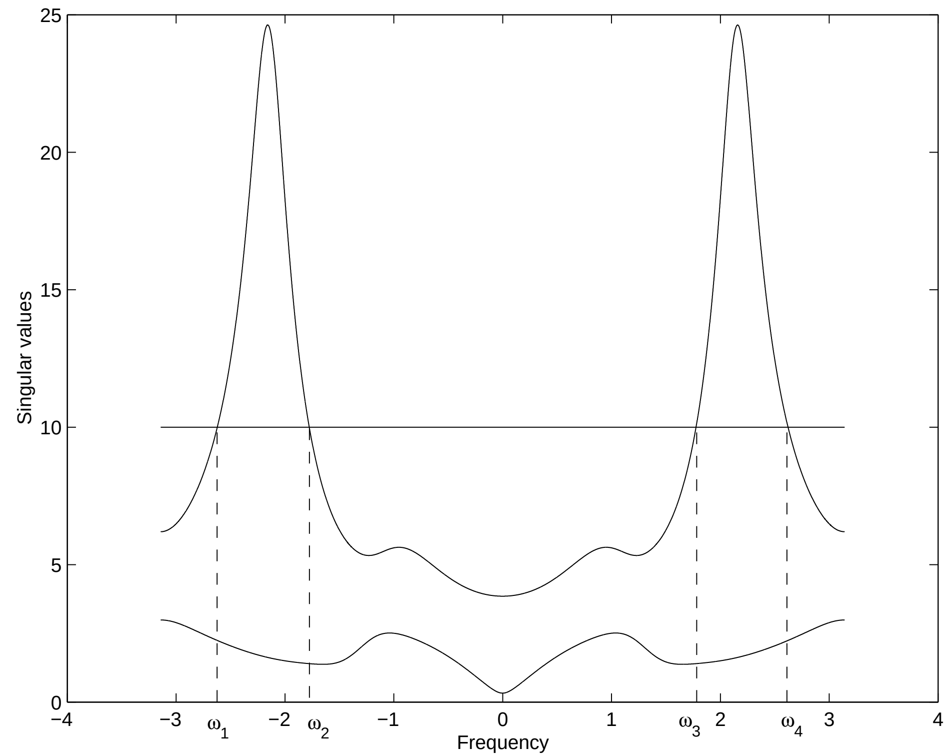
- $\lambda_* < \lambda < \lambda^*$ intersections



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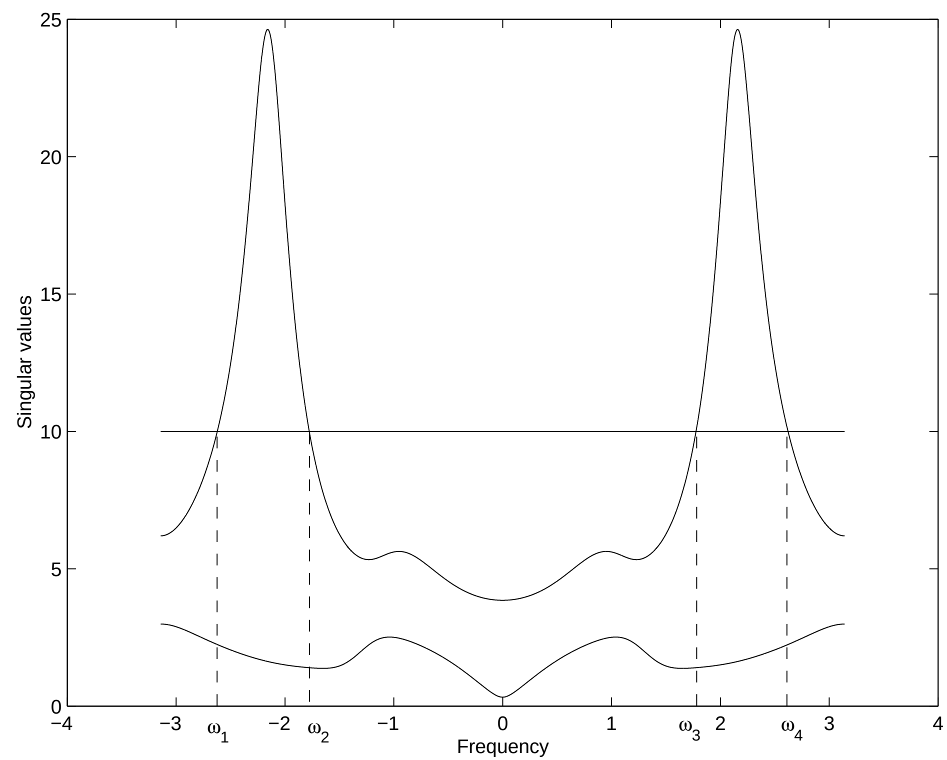
- identify interval
- $0 < \lambda < \lambda_*$ is easily avoided
- move toward γ^*



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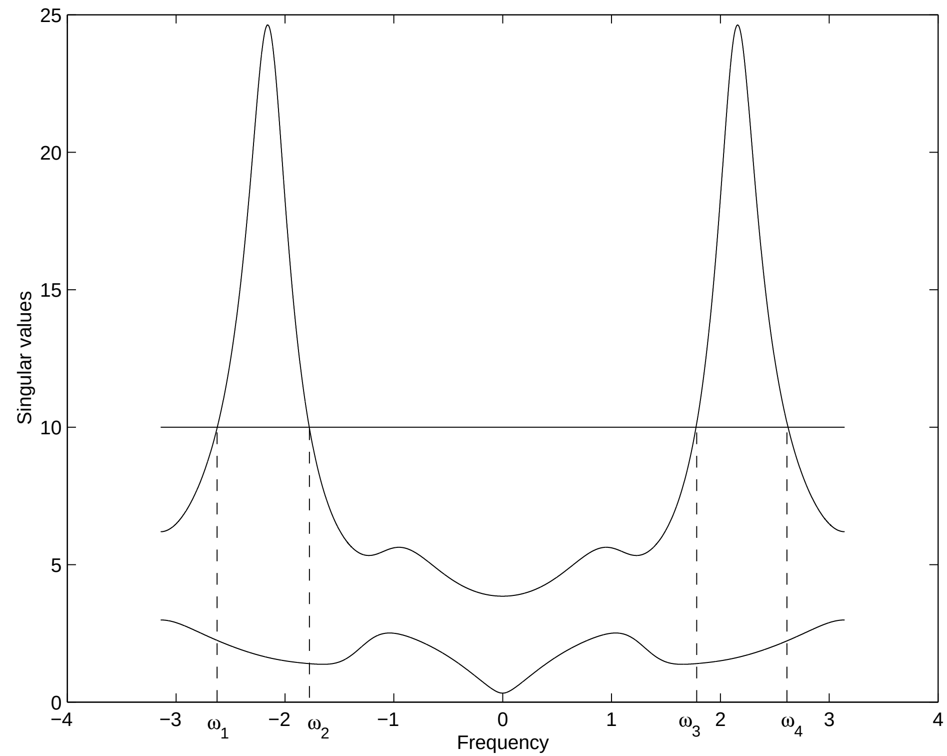
- repeat until $\sqrt{\lambda} \approx \gamma^*$
- fast methods need to find **all** intersections for a given λ



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$e^{j\omega_i}$ are also the symplectic generalized eigenvalues on the unit circle of

$$\left(\begin{bmatrix} 0 & A + BM_\gamma^{-1}D^TC \\ -I_n & -C^TC - C^TDM_\gamma^{-1}D^TC \end{bmatrix}, \begin{bmatrix} BM_\gamma^{-1}B^T & -I_n \\ A^T + C^TDM_\gamma^{-1}B^T & 0 \end{bmatrix} \right),$$

where $M_\gamma := \gamma^2 I_m - D^TD$.

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where $\quad M_\gamma := \gamma^2 I_m - D^TD$.

$O(n^3)$ complexity per step to find all ω_i.

Strategies to Update γ

- Bisection (Enns and Glover)

$$\gamma_{lb} = \sigma_1 \leq \|G(z)\|_\infty \leq 2\sum_i \sigma_i = \gamma_{ub}$$

where σ_i are Hankel singular values.

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- Bisection (Boyd et al.)

$$\begin{aligned}\gamma_{lb} &= \sqrt{\frac{Trace(\mathcal{G}_c\mathcal{G}_o)}{n}} \\ \gamma_{ub} &= 2\sqrt{n\, Trace(\mathcal{G}_c\mathcal{G}_o)}\end{aligned}$$

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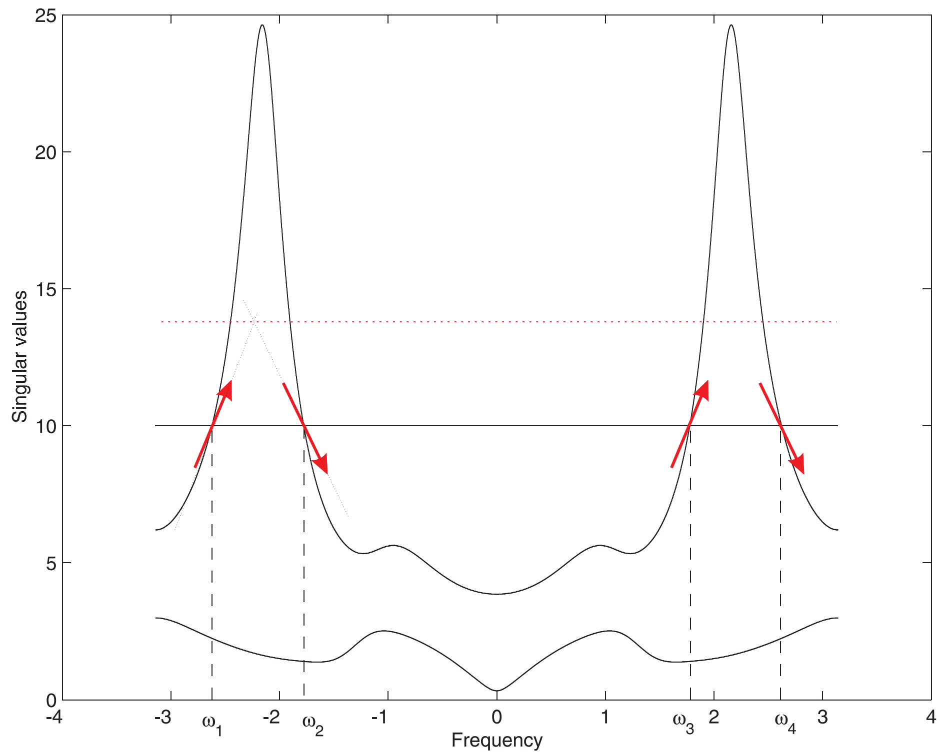
- Level set methods (J. Sreedhar, P. Van Dooren, and A. L. Tits):
  - midpoint rule
  - cubic interpolation
  - rational interpolation

Level set method



Converges very quickly

Sparse Method

- given a γ, identify which of three intervals
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  - dense matrix operations of complexity $O(m\alpha^2)$ where $\alpha = m$ or $\alpha = p$

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- given a γ, identify which of three intervals
- move toward γ^*
- Only use efficient large scale primitives
  - sparse matrix times k vectors in $\mathcal{R}^n$ where $k \leq m$
  - dense matrix operations of complexity $O(m\alpha^2)$ where $\alpha = m$ or $\alpha = p$
- need effective techniques to accelerate interval detection
- need effective techniques to update γ from accumulated information

Basis of Method

$$\gamma \quad > \quad \gamma^* = \|G(z)\|_\infty$$

$$\Updownarrow$$

$$\Phi_\gamma(e^{j\omega}) \quad \succ \quad 0 \quad for \;\; \omega \in [-\pi, +\pi]$$

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$$\Phi_\gamma(e^{j\omega}) \succ 0 \quad for \;\; \omega \in [-\pi, +\pi]$$

$$\Updownarrow$$

$$\begin{aligned} \exists P &\succeq 0 \\ \tilde{R} &= \gamma^2 I_m - B^T P B - D^T D \succ 0 \\ K &= B^T P A + D^T C \end{aligned}$$

$$P - A^T P A - C^T C + K^T \tilde{R}^{-1} K = 0$$

Background

Bounded Real Lemma

$\gamma \geq \gamma^* \Leftrightarrow \exists P \succeq 0$ that solves of the LMI

$$H(P) := \begin{bmatrix} P - A^T P A - C^T C & -A^T P B - C^T D \\ -B^T P A - D^T C & \gamma^2 I_m - B^T P B - D^T D \end{bmatrix} \succeq 0.$$

Background

Bounded Real Lemma

$\gamma \geq \gamma^* \Leftrightarrow \exists P \succeq 0$ that solves of the LMI

$$H(P) := \begin{bmatrix} P - A^T PA - C^T C & -A^T PB - C^T D \\ -B^T PA - D^T C & \gamma^2 I_m - B^T PB - D^T D \end{bmatrix} \succeq 0.$$

Riccati Matrix Inequality (Willems)

$$\begin{aligned} \exists P &\succeq 0 \\ \tilde{R} &= \gamma^2 I_m - B^T PB - D^T D \succeq 0 \\ K &= B^T PA + D^T C \end{aligned}$$

$$P - A^T PA - C^T C + K^T \tilde{R}^{-1} K \succeq 0$$

Chandrasekhar Iterations

Consider the discrete-time algebraic Riccati equation

$$P = A^T P A + C^T C - K^T R^{-1} K.$$

Solved using an iterative scheme Chandrasekhar equations to exploit sparsity

$$P_{i+1} = A^T P_i A + C^T C - K_i^T R_i^{-1} K_i,$$

where $K_i := B^T P_i A + D^T C$, and $R_i = B^T P_i B + D^T D - \gamma^2 I_m$.
(Hassibi, Sayed, and Kailath)

The difference matrices $\delta P_i := P_{i+1} - P_i$ satisfy

$$\delta P_{i+1} = A^T \delta P_i A - K_{i+1}^T R_{i+1}^{-1} K_{i+1} + K_i^T R_i^{-1} K_i.$$

Chandrasekhar Iteration

If we consider $\delta P_i = L_i^T \Sigma_2 L_i, \quad R_i = S_i^T \Sigma_1 S_i,$ and $K_i = S_i^T \Sigma_1 G_i.$

Then there exists a J-orthogonal matrix Q, i.e., $Q^T J Q = J$, such that

$$Q \begin{bmatrix} S_i & G_i \\ L_i B & L_i A \end{bmatrix} = \begin{bmatrix} S_{i+1} & G_{i+1} \\ 0 & L_{i+1} \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}.$$

The feedback matrix is $F_i := R_i^{-1} K_i = S_i^{-1} G_i$.

For our conditions we have

- $P_0 = 0$
- $\Sigma_1 = -I_m$ and $\Sigma_2 = I_p$
- constant signature is necessary and sufficient for convergence (Hassibi, Sayed, and Kailath)

Intervals

$\gamma > \sqrt{\lambda^*} = \gamma^*$

- convergence with $\rho(A - BF_\gamma) < 1$
- ρ can be estimated
- $\Sigma_1 = -I_m$ and $\Sigma_2 = I_p$ constant

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$\sqrt{\lambda_*} < \gamma < \sqrt{\lambda^*}$

- non-convergent and $\rho(A - BF_i)$ may be erratic
- Σ_1 and Σ_2 not constant nor consistent with $\gamma > \gamma^*$

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- convergence with $\rho(A - BF_\gamma) < 1$
- $\Sigma_1 = I_m$ and $\Sigma_2 = I_p$ constant but not consistent with $\gamma > \gamma^*$
- can be avoided with use of γ_{lb}
- easily detected via signature on first step

Dense numerical examples

Random dense stable discrete-systems $n = 300$, $p = m = 2$.

$\rho(A)$	0.95	0.9	0.7	0.5
Matlab	14.2180	9.4695	4.6672	5.2757
SB	14.2207(561)	9.4695(523)	4.6672(426)	5.2758(411)
SLM	14.2196(134)	6.7198(105)	4.5590(49)	3.3535(124)
SLC	14.2196(120)	6.7198(103)	4.5590(48)	3.3535(76)
SLR	14.2196(93)	6.7198(104)	4.5590(49)	3.3535(122)
CB	15.3416(41)	9.3256(59)	4.7523(23)	5.3436(33)
ACB	14.1142(12)	9.5516(11)	4.6509(10)	5.3327(11)

Legend:

$\hat{\gamma}^*$(cputime S)

SB: symplectic bisection

SLM: set level with midpoint rule

SLC: set level with cubic interpolation

SLR: set level with rational interpolation

CB: Chandrasekhar iterations combined with bisection

ACB: Chandrasekhar iterations combined with bisection with signature check

Sparse numerical example

Random sparse stable discrete-system $n = 800$, $p = m = 2$.

	$\hat{\gamma}^*$(cputime s)	iterations
Matlab	7.7193(865)	–
SB	–	–
SLM	–	–
SLC	–	–
SLR	–	–
CB	7.6923(120)	40
ACB	7.7166(78)	42

Enhancements

Update of γ

- Currently bisection used
- extrapolation schemes with $\rho(\gamma_i)$ for $\gamma_i > \gamma^*$ estimates are being investigated
- structure of problem and the behavior when $\sqrt{\lambda_*} < \gamma < \sqrt{\lambda^*}$ are also being investigated

Adaptation of γ
Suppose it is detected that $\gamma_i > \gamma^*$ and γ_{i+1} chosen

- Rather than restart the iteration, use a variable coefficient Chandrasekhar iteration to reduce the time required to detect convergence with γ_{i+1}
- if $\gamma_{i+1} < \gamma^*$ iteration data must be reset to data present when γ_i was modified

Variable Coefficient Iteration

Let $\eta^2 := \gamma_i^2 - \gamma_{i+1}^2$ it follows that $(\varsigma = i+1)$

$$\begin{bmatrix} R_\varsigma & K_\varsigma \\ K_\varsigma^T & \delta P_\varsigma + K_\varsigma^T R_\varsigma^{-1} K_\varsigma \end{bmatrix} = \begin{bmatrix} B^T \delta P_i B + R_i + \eta^2 I_m & B^T \delta P_i A + K_i \\ A^T \delta P_i B + K_i^T & A^T \delta P_i A + K_i^T R_i^{-1} K_i \end{bmatrix},$$

with the additional correction

$$Q^T \begin{bmatrix} \Sigma_1 & 0 \\ 0 & I_m \end{bmatrix} Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & I_m \end{bmatrix}, \quad Q \begin{bmatrix} \hat{S}_i \\ \eta I_m \end{bmatrix} = \begin{bmatrix} S_\varsigma \\ 0 \end{bmatrix}.$$

Conclusion and Future Work

- Initial evidence suggests that the method for estimation of the H_∞-norm is efficient for large sparse systems and may be competetive for dense systems
- Evaluation with large scale industrial problems
- Further development of γ update extrapolation methods
- Further investigation of variable coefficient implementation
- Handle the case where m and/or p get large
- Continuous time systems
- Inclusion in hybrids with other MR methods