Algorithms for Solving Linear Differential Equations with Rational Function Coefficients

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Introduction

- Closed form solutions of differential equations are solutions that can be written in terms of well-studied functions (exponential, logarithmic, Airy, Bessel, Whittaker, hypergeometric, ...).
- There are powerful algorithms for some type of closed form solutions.
- We are interested to find hypergeometric solutions.
- Hypergeometric solutions are common, but current solvers in computer algebra systems often fail to find such solutions.

Introduction

- Consider the second order linear differential equation

$$L_{\text{inp}}(y) = 0$$

where

$$L_{\text{inp}} = A_2\partial^2 + A_1\partial + A_0 \in \mathbb{Q}(x)[\partial] \qquad \left(\partial = \frac{d}{dx}\right)$$

A **hypergeometric solution** of L_{inp} is a solution of the form

$$S(x) = \exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathsf{F}_1(a_1, a_2; b_1; f) + r_1 \cdot {}_2\mathsf{F}_1'(a_1, a_2; b_1; f)\right)$$

where $f, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$, $a_1, a_2, b_1 \in \mathbb{Q}$.

- We are interested in finding such solutions of a given regular singular $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$.

Introduction

- There is also a conjecture about hypergeometric solutions.

- Conjecture (van Hoeij, Kunwar)
  Every second order globally bounded equation has a hypergeometric solution or an algebraic solution.

  A **globally bounded equation** is a differential equation which (after a simple scaling) admits a **Convergent Integer power Series solution (CIS solution)**.

- Example (Franel Numbers, OEIS A000172)

$$\text{sequence} = 1, 2, 10, 56, 346, 2252, 15184, 104960, \ldots$$

$$y(x) = 1 + 2\,x + 10\,x^2 + 56\,x^3 + 346\,x^4 + 2252\,x^5 + \ldots$$

$$L_{\text{inp}} = x\,(1+x)\,(8\,x-1)\,\partial^2 + \left(24\,x^2 + 14\,x - 1\right)\partial + (2 + 8\,x)$$

$$y(x) = \frac{1}{1-2\,x} \cdot {}_2\mathsf{F}_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27\,x^2}{(1-2\,x)^3}\right)$$

Introduction

- Question How can we find hypergeometric solutions of second order regular singular differential operators?

- There are powerful algorithms for **specific tasks**:
  Fang, van Hoeij (2012)
  Kunwar, van Hoeij (2013)
  Kunwar (2014)
  van Hoeij, Vidunas (2015)

- We want to develop a **general** algorithm.

Introduction

- Contributions

We have developed two (heuristic) effective algorithms to find hypergeometric solutions (if they exist) of second order regular singular differential operators in $\mathbb{Q}(x)[\partial]$.

One of our algorithms is the most general algorithm in the literature.

We have developed fast algorithms to simplify n-th order regular singular differential operators in $\mathbb{Q}(x)[\partial]$.

Simplifying $\approx$ Solving.

- Value to the Scientific Society

Hypergeometric solutions are common in physics and combinatorics.

Our implementations have been already used by physicists.

Example Feynman Diagrams.

One of our algorithms "simplify" a differential operator to another operator which is easier to solve (Simplifying $\approx$ Solving).

Example A 3rd order operator.

Formal Solutions, Exponents

- Let $L_{\text{inp}} = A_2\partial^2 + A_1\partial + A_0 \in \mathbb{Q}(x)[\partial]$.

- $s \in \overline{\mathbb{Q}}$ is a **singularity** if it is a root of A_2, or a pole A_1 or A_0.

- s is a **regular singularity** if $\frac{A_1}{A_2}(x-s)^1, \frac{A_0}{A_2}(x-s)^0$ are analytic at s.
- L_{inp} is **regular singular** if it has only regular singularities.

- **Formal solutions** of L_{inp} at $x = s$:

  $y_1 = (x-s)^{e_1}(1 + \dots)$

  $y_2 = (x-s)^{e_2}(1 + \dots) + ky_1 \log(x-s) \quad$ (k might be 0)

  Exponents of L_{inp} at $x = s$: e_1, e_2.

  Exponent-difference: $\Delta(L_{\text{inp}}, s) = |e_1 - e_2|$.

- Special Case

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ with $\operatorname{ord}(L_{\text{inp}}) = 2$.

Assume that we want to find hypergeometric solutions this form:

$$\exp\left(\int r\,dx\right) \cdot {}_2\mathsf{F}_1(a_1, a_2; b_1; f) \qquad (f, r \in \overline{\mathbb{Q}(x)}, a_1, a_2, b_1 \in \mathbb{Q})$$

If they exist, then there exists a **GHDO**

$$L_B = \partial^2 + \frac{b_1 - (a_1 + a_2 + 1)}{x(1-x)}\partial - \frac{a_1 a_2}{x(1-x)} \in \mathbb{Q}(x)[\partial]$$

such that

$$L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{\text{inp}}$$

because

$${}_2\mathsf{F}_1(a_1, a_2; b_1; x) \xrightarrow{f}_C {}_2\mathsf{F}_1(a_1, a_2; b_1; f) \xrightarrow{r}_E \exp\left(\int r\,dx\right) {}_2\mathsf{F}_1(a_1, a_2; b_1; f)$$

Quotient Method

- Problem We know L_{inp}, we do not know L_B (i.e., a_1, a_2, b_1), f, r.

- Assume that we have a (right) candidate L_B.

  Then $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{\text{inp}}$.

  We can compute formal solutions y_i of L_B and Y_i of L_{inp} at a non-removable singularity (order of formal solutions $= a$).

  $$y_i(x) \xrightarrow{f}_C y_i(f(x)) \xrightarrow{r}_E Y_i(x) = \exp\left(\int r\, dx\right) y_i(f(x)) \quad (i = 1, 2)$$

  Let $q(x) = \frac{y_1(x)}{y_2(x)}$, $Q(x) = \frac{Y_1(x)}{Y_2(x)}$.

  This gives us $q(f(x)) = c \cdot Q(x)$ where c is an unknown constant, so

  $$f(x) \equiv q^{-1}(c \cdot Q(x)) \mod x^a$$

Quotient Method

- Problem We do not know c.
- Idea Choose a suitable prime ℓ and compute

$$f(x) \equiv q^{-1}(c \cdot Q(x)) \mod (x^a, \ell)$$

and try $c = 1, \ldots, \ell - 1$.

For each c try rational function reconstruction to find $f \in \mathbb{F}_\ell(x)$.

If this succeeds for at least one c, then try ℓ-adic Hensel lifting and rational number reconstruction to find $f \in \mathbb{Q}(x)$.

- Compute M s.t. $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{inp}$ and find r (there is a formula).
- Problem We do not know L_B (i.e., $a_1, a_2, b_1 \in \mathbb{Q}$).

Quotient Method

- Question How to find candidates for L_B (i.e., $a_1, a_2, b_1 \in \mathbb{Q}$)?
- $\alpha_0 = |1 - b_1|$, $\alpha_1 = |b_1 - a_1 - a_2|$, $\alpha_\infty = |a_2 - a_1|$.

- Assume that $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{\text{inp}}$.

	L_B	L_{inp}
singularities	$0, 1, \infty$	$s_1, \ldots, s_r$
exponent-differences	$\alpha_0, \alpha_1, \alpha_\infty$	$\Delta(L_{\text{inp}}, s_1), \ldots, \Delta(L_{\text{inp}}, s_r)$

$$\alpha_0, \alpha_1, \alpha_\infty \in \left\{ \frac{a}{kb} : \; a \in \{1, \Delta(L_{\text{inp}}, s_i)\}, \; 1 \leq b \leq d_f, \; 1 \leq k \leq a_f \right\}$$

where $d_f = \deg(f)$ and $a_f = [\mathbb{Q}(x, f) : \mathbb{Q}(x)]$.

- We have a bound on d_f and we consider $a_f \leq 2$.
- Problem This set might have too many elements.

Quotient Method

- Theorem (Special Case of Riemann-Hurwitz Formula for DEs)
  If $f : X \longrightarrow \mathbb{P}^1$ and $L_B \xrightarrow{f}_C M \xrightarrow{r}_E L_{\text{inp}}$, then

$$-2 + \sum_{s_i} (1 - \Delta(L_{\text{inp}}, s_i)) = \frac{d_f}{a_f} \left(-2 + \sum_{i \in \{0,1,\infty\}} (1 - \alpha_i) \right)$$

  where $d_f = \deg(f)$ and $a_f = [\mathbb{C}(x, f) : \mathbb{C}(x)]$.

- This formula eliminates vast majority of the candidates:

$$\alpha_0, \alpha_1, \alpha_\infty \in \left\{ \frac{a}{kb} : \ a \in \{1, \Delta(L_{\text{inp}}, s_i)\}, \ 1 \leq b \leq d_f, \ 1 \leq k \leq a_f \right\}$$

Quotient Method

- Algorithm Outline (`find_2f1`)
  - Compute candidates L_B's (if any).
  - Compute quotients of formal solutions at a non-removable singularity.
  - Use modular reduction, Hensel lifting, rational reconstruction to find f (if any).
  - Find r.
  - Return a basis of hypergeometric solutions (if they exist) or an empty list.

- Algorithm `find_2f1` is very effective to find solutions of type

$$\exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathsf{F}_1(a_1,a_2;b_1;f) + r_1 \cdot {}_2\mathsf{F}_1'(a_1,a_2;b_1;f)\right)$$

  where $r_1 = 0$.
- Question What if $r_1 \neq 0$?
- Idea Simplifying $\approx$ Solving

Integral Bases

- Example (Simplifying ≈ Solving)

  Consider the following number field

  $$\mathbb{Q}[x]/(f_1)$$

  where $f_1 = 98818x^6 - 800756x^5 + 3495803x^4 - 8505211x^3 + 15375943x^2 - 17721960x + 7848261.$

  We can reduce this to

  $$\mathbb{Q}[x]/(f_1) \cong \mathbb{Q}[x]/(f_2)$$

  where $f_2 = x^6 - 5x^4 - 21x^3 - 12x - 2.$

- Key step to find an isomorphic number field is computation of an **integral basis**.

Integral Bases

- Let $A = \{f \in \mathbb{Q}[x] \mid f \text{ is irreducible}\}$.

$$f_1 \sim f_2 \iff \mathbb{Q}[x]/(f_1) \cong \mathbb{Q}[x]/(f_2) \qquad (f_1, f_2 \in \mathbb{Q}[x])$$

Goal Given f_1, find $f_2 \in A$ with small bit-size such that $f_1 \sim f_2$.

f_2 = a **standard form** of f_1

Solution `POLRED` algorithm (Cohen and Diaz Y Diaz, 1991).

- Compute a basis for the algebraic integers of $\mathbb{Q}[x]/(f_1)$.
- Apply `LLL` (Lenstra, Lenstra, and Lovasz, 1982) to this basis.

Application Reduce computations in $\mathbb{Q}[x]/(f_1)$ to computations in $\mathbb{Q}[x]/(f_2)$.

Integral Bases

- Let $D = \mathbb{Q}(x)[\partial]$ and $A = \{L \in D \mid L \text{ is irreducible}\}$.

$$L_1 \sim L_2 \iff D/DL_1 \cong D/DL_2 \text{ as } D\text{-modules} \qquad (L_1, L_2 \in D)$$

Goal Given L_1, find $L_2 \in A$ with small bit-size such that $L_1 \sim L_2$.

$L_2 =$ a **standard form** of L_1

Solution Idea Imitate POLRED.

Application Reduce solving L_1 with many **apparent singularities** to solving L_2 with few apparent singularities.

Integral Bases

- Computation of integral bases for (algebraic) number fields (algebraic) function fields are well studied:

  Trager (1984)
  Cohen and Diaz Y Diaz (1991)
  van Hoeij (1994)
  de Jong (1998)
  Montes (1999)
  van Hoeij and Stillman (2015)

- Differential analogue of integral bases is new (Kauers and Koutschan, 2015).
- We give an integral basis algorithm that is much **faster**. In addition, we **normalize the basis at infinity**, which is the analogue of phase 2 in `POLRED`.

Integral Bases

- $L \in \mathbb{C}(x)[\partial]$ be regular singular.

  L has a basis of formal solutions at $x = s$ in the form

  $$y = t_s^{\nu_s} \sum_{i=0}^{\infty} P_i t_s^i \qquad t_s = \begin{cases} x - s, & \text{if } s \neq \infty \\ \frac{1}{x}, & \text{if } s = \infty \end{cases}$$

  where $\nu_s \in \mathbb{C}$ and $P_i \in \mathbb{C}[\log(t_s)]$ with $\deg(P_i) < \operatorname{ord}(L)$ and $P_0 \neq 0$.

- The **valuation** of y at $x = s$ is

  $$v_s(y) = \operatorname{Re}(\nu_s)$$

Integral Bases

- Fix $L \in \mathbb{C}(x)[\partial]$, $\operatorname{ord}(L) = n$, and let $G \in \mathbb{C}(x)[\partial]$.

  G is called **integral for L at s** if

  $$v_s(G) = \inf\{v_s(G(y)) \mid y \text{ is a solution of } L \text{ at } x = s\} \geq 0$$

  G is called **integral for L** if $v_s(G) \geq 0$ for all $s \in \mathbb{C}$.

- Consider the $\mathbb{C}[x]$-module $\mathcal{O}_L$

  $$\mathcal{O}_L = \{G \in \mathbb{C}(x)[\partial] \mid G \text{ is integral for } L \text{ and } \operatorname{ord}(G) < n\}$$

  A basis of $\mathcal{O}_L$ is called an **(global) integral basis** for L.

Integral Bases

- Let $L \in \mathbb{C}(x)[\partial]$, $\operatorname{ord}(L) = n$, and $P \in \mathbb{C}[x]$.

  The set $\{b_1, \ldots, b_n\}$ is a **local integral basis** for L at P when

  $$\left\{ \frac{A_1}{B_1} b_1 + \cdots + \frac{A_n}{B_n} b_n \,\middle|\, A_i, B_i \in \mathbb{C}[x] \text{ and } \gcd(P, B_i) = 1 \right\}$$

  $$=$$

  $$\{G \,|\, G \text{ is integral for } L \text{ at every root of } P\}$$

- A local integral basis at a finite singularity $s \in \mathbb{C}$ is the local integral basis at $P = x - s$.

Integral Bases

- Question How to compute a local integral basis for an operator at one point?

- Theorem

  $\{b_1, \ldots, b_n\}$ is a local integral basis for $L \in \mathbb{C}(x)[\partial]$ at $x = 0$

  $\Longleftrightarrow$

1 $\forall i, j \in \{1, \ldots, n\}$ we have

$$v_0(b_i(y_j)) \geq 0 \quad (y_j = \text{a solution of } L \text{ at } x = 0)$$

and

2 $\forall (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus (0, \ldots, 0)$ there exists $j \in \{1, \ldots, n\}$ such that

$$v_0((c_1 b_1 + \cdots + c_n b_n)(y_j)) < 1$$

Integral Bases

- Let $L_{\mathrm{inp}} \in \mathbb{Q}(x)[\partial]$ be regular singular.

- Algorithm Outline (`local_basis_at_0`)

- Compute $v_j = v_0(y_j)$ of the all formal solutions y_j of L_{inp} at $x = 0$.
- Let $m = -\lfloor \min(v_j) \rfloor$.
- Let $b_1 = x^m \partial^0$.
- For i from 2 to n do:
  1. Let $b_i = x \cdot \partial \cdot b_{i-1}$
  2. Make the ansatz $\mathfrak{A} = \frac{1}{x}(u_1 \cdot b_1 + \cdots + u_{i-1} \cdot b_{i-1} + b_i)$.
  3. Evaluate $\mathfrak{A}(y_j)$ and equate coefficients of the non-integral terms to 0.
  4. If there is a solution, find it, update $b_i = \mathfrak{A}$ and return to Step 2.
- Return $b_1, \ldots, b_n$.

Integral Bases

- Improvements (Apparent singularities and algebraic singularities)

1. Apparent Singularities

   A local integral basis at an apparent singularity is given by

   $$\{b_1 = \partial^{e_1}, \ldots, b_n = \partial^{e_n}\}$$

   If $e_i \geq \operatorname{ord}(L_{\text{inp}})$, then $b_i = \operatorname{Rem}(\partial^{e_i}, L_{\text{inp}})$.

   Our implementation only checks for apparent singularities of the most common type where

   $$e_1, e_2, \ldots, e_{n-1}, e_n = 0, 1, \ldots, n-2, n$$

2 Algebraic Singularities

Let s be an algebraic singularity with minimal polynomial P.

We compute a local integral basis $\{b_1, \ldots, b_n\}$ for L at $x = s$.

We want to scale b_i in such a way that

- $\{c_1 b_1, \ldots, c_n b_n\}$ is still an integral basis at $x = s$,
- $\{c_1 b_1, \ldots, c_n b_n\}$ have valuations ≥ 1 at all roots of $\frac{P}{x-s}$.

Let for $i = 1, \ldots, n$

$$c_i = \left(\frac{P}{x-s}\right)^{v_s(b_1)+i}$$

and then

$$\{\mathrm{Tr}(c_1 b_1), \ldots, \mathrm{Tr}(c_n b_n)\}$$

will be a local integral basis at P.

Integral Bases

- Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be regular singular.

- Algorithm Outline (`global_integral_basis`)

- Let P_{sing} be polynomial whose roots are singularities of L_{inp}.
- For each irreducible factor P of P_{sing}, compute a local integral bases for L_{inp} at P.
- Combine all local integral bases to form a global integral basis for L_{inp}.

Integral Bases

- Timings

  Comparison of timings of Kauers' and Koutschan's integral basis algorithm and our integral basis algorithm (in seconds) on a computer with a 2.5 GHZ Intel Core i5-3210M CPU and 8 GB RAM.

Example	Kauers-Koutschan	Our algorithm
1	0.185	0.111
2	0.863	0.156
3	0.233	0.182
4	0.592	0.226
5	12.351	0.294
6	66.537	0.377
7	124.197	0.499
8	151.942	0.515
9	175.580	0.569
10	157.484	0.596
11	145.185	0.602
12	230.897	0.688
13	1609.865	0.699
14	$>$ 1600	0.918
15	$>$ 1600	1.133
16	$>$ 1600	1.156
17	$>$ 1600	1.251

Integral Bases

- Now, we know how to compute a global integral basis for L_{inp}.
- We have control on finite singularities of L_{inp}.

- Question What about the point $x = \infty$?

- Normalization

  Let $L_{\text{inp}} \in \mathbb{C}(x)[\partial]$ with $\operatorname{ord}(L_{\text{inp}}) = n$.

  The set $\{b_1, \ldots, b_n\}$ is called **normalized at** s, if $\exists r_i \in \mathbb{C}(x)$ such that $\{r_1 b_1, \ldots, r_n b_n\}$ is a local integral basis for L_{inp} at s.

- We want to normalize a global integral basis at $x = \infty$.

- Normalization of an integral basis at $x = \infty$ for an algebraic function was introduced (Trager, 1984).

Integral Bases

- Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be regular singular and let

$$\{B_1, \ldots, B_n\}$$

be a global integral basis for L_{inp}.

- Algorithm Outline (`normalization_at_infinity`)
- Compute a local integral basis $\{b_1, \ldots, b_n\}$ at ∞.
- Compute change of basis matrix and follow Trager's method.

- Basis elements of a normalized integral basis for L_{inp} gives us gauge transformations to simplify L_{inp} (to simplify it to its standard form). Example `Linp[9]`.

Applications

- Lets return to our problems:
  1. Simplify a given differential operator (Simplifying $\approx$ Solving).
  2. Find hypergeometric solutions (generalize `find_2f1`).

1 Simplify a given differential operator:

Goal Given L_1, find L_2 with small bit-size such that L_1 is gauge equivalent to L_2.

Solution Idea Imitate `POLRED`.

Solution `standard_forms` algorithm:

- Compute a normalized integral basis for L_1.
- If necessary, search for a better basis element to simplify L_1.

Applications

2 Generalize `find_2f1`:

Let $L_{\text{inp}} \in \mathbb{Q}(x)[\partial]$ be a second order regular singular operator. We want to find hypergeometric solutions of L_{inp} of the form

$$\exp\left(\int r\,dx\right)\left(r_0 \cdot {}_2\mathsf{F}_1(a_1,a_2;b_1;f) + r_1 \cdot {}_2\mathsf{F}_1'(a_1,a_2;b_1;f)\right)$$

- Algorithm Outline (`hypergeometricsols`)

- Try `find_2f1`.
- If `find_2f1` returns an empty list, use `standard_forms` to simplify L_{inp} to another operator $\tilde{L}_{\text{inp}}$.
- Feed `find_2f1` with $\tilde{L}_{\text{inp}}$.
- If `find_2f1` solves $\tilde{L}_{\text{inp}}$, then obtain solutions of L_{inp} from solutions of $\tilde{L}_{\text{inp}}$.

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