AF:Small: Linear Differential Equations with a Convergent Integer Series Solution, Project Description

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1 Introduction

The topic in the PI's current grant[1] is: Given a linear homogeneous linear differential equation with polynomial coefficients, decide if it can be solved in terms of special functions, and if so, find such solutions. Much progress on his topic has been made by the PI and his graduate students (Sections 2 and 6). Numerous algorithms were developed that turned out to be very effective.

The PI tested these algorithms on differential equations of order 2 and 3 coming from combinatorics, physics, and the OEIS (Online Encyclopedia of Integer Sequences, oeis.org). Every equation encountered that had a Convergent Integer power Series as a solution turned out to be solvable in terms of hypergeometric functions. That was a surprising observation, because the OEIS is large and contains examples from many different sources. This included dozens of equations that were not expected to be solvable in terms of hypergeometric functions. The unexpected observation leads to the following definition and question:

Definition 1 *A linear homogeneous differential equation will be called a* CIS-equation *if, possibly after a Möbius transformation, it has a CIS-solution. Here CIS is short for a* non-zero Convergent Integer power Series*.*

For example, if a differential equation has a convergent solution $\sum_{n=0}^{\infty} a_n x^n$ with $1728^n \cdot a_n \in \mathbb{Z}$, then a simple transformation $x \mapsto 1728 \cdot x$ suffices to obtain a CIS, a Convergent Integer power Series. Such an equation will

[1] NSF 1017880, project description is available at www.math.fsu.edu/~hoeij/NSF2010

be called a CIS-equation. Of course, CIS-equations are very common in combinatorics, as minimal equations for generating functions. But CIS-equations are surprisingly common in other parts of science as well [8, 9, 28]. Such equations are the main focus in this proposal, in particular the following question.

Question 1 *Does every CIS-equation of order 2 or 3 have a solution expressible (defined in Section 2) in terms of hypergeometric functions?*

Despite having tested this on many examples, this is not yet a conjecture, for two reasons. The first is that the situation changes significantly for order 4 (see Section 4). The second reason is that, despite the fact that CIS-equations were examined that came from numerous sources, they all ended up in the same *class* (defined in Section 3). This indicates that the contexts in which these equations arise have additional properties from which a solution in terms of hypergeometric functions could be obtained. We will discuss this in more details in one context, namely diagonals.

In [39], Bousquet-Mélou and Mishna studied 79 integer sequences that count various types of walks in the quarter plane. They showed that in 23 of these 79 cases, the generating function $y \in \mathbb{Z}[[x]]$ is holonomic (i.e. y satisfies a linear homogeneous differential equation with polynomial coefficients). The way that y was shown to be holonomic is by showing that it equals the *diagonal* (Section 3) of a multivariate rational function. The PI found expressions [40] in terms of hypergeometric functions for all 22 non-algebraic cases[2], and an algebraic expression [3] for the 1 algebraic case. But instead of viewing this as support for Question 1, one could also consider the possibility that diagonals are more than just holonomic; they could have stronger properties that directly lead to a hypergeometric expression, regardless of Question 1.

There are two reasons to search for such properties. The differential equations were generally larger than their hypergeometric-type solutions, which indicates that y is represented more naturally by a hypergeometric-type expression than by a differential equation. Moreover, Section 3 divides hypergeometric functions into classes, but as mentioned, all examples turned out to be in the same class (whether they came from diagonals, the Ising model in physics, or the OEIS).

[2] Current computer algebra systems could solve almost none of these. Changing that is one of the main motivations for this research.

1.1 Goals and timeline

The goals in this proposal are: (1) developing new algorithms to solve differential equations, in particular CIS-equations, (2) constructing tables and theorems that enable the algorithm to prove the non-existence of closed form solutions whenever it finds no solution, (3) study CIS-equations and Question 1, (4) study properties of diagonals of multivariate rational functions, (5) developing a solver for order 3 and (6) equations of order ≥ 4 (in particular, algorithms for Calabi-Yau equations).

For goals (1) and (2), the PI's graduate student Vijay Kunwar is currently working on algorithms and tables, with support of NSF 1017880. New topics in this proposal are (3), (4), (5), (6) as well as a part of (2), in particular the use of modular curves proposed in Section 5. The PI's other graduate student Erdal Imamoglu will start with (3) and (4): Can one directly compute a closed form expression for a CIS coming from constructions such as diagonals, without first computing the (potentially much larger) differential equation? Why do these expressions always involve the same class of hypergeometric functions? Topics (4), (5), (6) are planned for the second and third year of this proposal, and will be discussed further in Sections 3, 4, and 5.

Developing algorithms and implementations is time consuming and will likely be the majority of the work for this proposal. The algorithms, and the study of their completeness, touch upon intriguing theoretical questions. These questions should be investigated, not only because it is important to prove completeness results for the algorithms, but also because of the new directions they might lead to. The outcome of the theoretical work is hard to predict, but for the practical work the PI is confident that useful algorithms will continue to emerge.

1.2 Value to research, education, and society

An important benefit of producing good algorithms and implementations is that people benefit from the work even if they did not study it. Researchers can use these implementations to solve their equations, without first having to learn the math behind the algorithms.

Computer algebra systems are an important part of the infrastructure for research and education. Thus, the value of the algorithms to be developed in this project will increase significantly when these algorithms are incorporated into computer algebra systems. To facilitate this, the PI will make implementations available on the web, and will assist to incorporate the algorithms into commercial as well as free computer algebra systems.

Many branches of science have important impacts on society. Differential equations occur in almost every branch of science, and having closed form solutions is very useful in practical applications. Computer algebra systems are widely used and are of great value to society. Within computer algebra, differential equations is one of the areas with the highest overall impact.

2 Notations and current status

A linear differential equation with rational function coefficients can be represented by a differential operator $L \in \mathbb{C}(x)[\partial]$ where $\partial = \mathrm{d}/\mathrm{d}x$. For example, if $L = a_2\partial^2 + a_1\partial + a_0$ for some rational functions $a_2, a_1, a_0 \in \mathbb{C}(x)$, then the corresponding equation $L(y) = 0$ is $a_2y'' + a_1y' + a_0y = 0$. We assume that L has no Liouvillian solutions, otherwise L can be solved quickly with Kovacic's algorithm [60, 64].

Definition 2 *If $S(x)$ is a special function that satisfies a differential operator L_S (called a* base equation*) of order n, then a function y is called a* linear S-expression *if there exist algebraic*[3] *functions $f, r, r_0, r_1, \ldots$ such that*

$$y = \exp(\int r \,\mathrm{d}x) \cdot \Big(r_0S(f) + r_1S'(f) + \cdots + r_{n-1}S^{(n-1)}(f)\Big). \qquad (1)$$

More generally, we say that y can be expressed in terms of S *if it can be written in terms of expressions of the form (1), using sums, products, and integrals.*

The reason higher derivatives of S are not needed is because they are linear combinations of $S, S', \ldots, S^{(n-1)}$.

If $L \in \mathbb{C}(x)[\partial]$ has order 3 or 4, and S is a special function that satisfies a second order equation, then the problem of solving L in terms of S can be reduced, with an algorithm and implementation [63] developed by the PI, to the problem of solving second order equations[4]. This reduction of order motivates a focus on second order equations.

[3] If y is a CIS-function, then $f, r, r_0, r_1, \ldots$ are usually rational functions in $\mathbb{Q}(x)$. To cover the remaining cases, one has to classify the non-rational f's that can occur, which the PI will do by using modular curves, see Section 5.

[4] For non-trivial examples of a CIS whose expression (containing products of hypergeometric functions) was obtained with this algorithm, see the generating functions listed under A135395 or A136045 at oeis.org.

If y and S satisfy second order operators, then products of (1) are not needed, and the form reduces to

$$y = \exp(\int r \,\mathrm{d}x) \cdot \big(r_0 S(f) + r_1 S'(f)\big) \tag{2}$$

Although this form still looks technical, it is the most natural form to consider, because it is closed under every known transformation sending irreducible second order operators in $\mathbb{C}(x)[\partial]$ to second order linear operators:

(i) Multiplying all solutions y by $\exp(\int r \,\mathrm{d}x)$
(ii) Gauge transformations $y \mapsto r_0 y + r_1 y'$
(iii) Compositions $y(x) \mapsto y(f)$. Here f is called the *pullback* function.

All three transformations (i),(ii),(iii) send expressions in terms of S to expressions in terms of S. So any solver for finding solutions in terms of S, if it is complete, then it must be able to deal with transformations (i),(ii),(iii). So it must be able to find any solution of the form (2).

If transformation (ii) is omitted; if one searches only for solutions in this restricted form:

$$\exp(\int r \,\mathrm{d}x) \cdot S(f) \tag{3}$$

then the problem is significantly easier. One can then reconstruct f by comparing a quotient of solutions of L with a corresponding quotient of solutions of L_S. This involves a technical issue (one first has to multiply the quotient by a suitable constant), but this can be dealt with. Although this approach is efficient in practice, form (3) is too restrictive in general. Among the L's that can be solved, only a modest subset[5] can be solved in the restricted form (3). Thus, the focus is to develop algorithms that can handle all three transformations, including (ii).

Let $L \in \mathbb{C}(x)[\partial]$ have order 2, and suppose it is S-solvable, where S is any of the classical special functions (Airy, Bessel, Kummer, Legendre, Whittaker, exp, log, hypergeometric functions, etc.). Then the implementation [26] of the PI's graduate student Quan Yuan[6] can solve L except in one case, when S is a hypergeometric ${}_2F_1$ function. Quan's implementation is very efficient so it can handle large complicated inputs. Theory contributes greatly to this efficiency. Quan's Ph.D thesis [26] contains a classification

[5] usually the ones that come from the most symmetric problems, such as the example in subsection 2.1

[6] supported by NSF 1017880 and 0728853, received Ph.D in March 2012

theorem that allows the algorithm to bypass checking many cases; including cases that would have costed the most CPU time (the ones that involve field extensions). This algorithm and its implementation [26] represent great progress, it means that we have a complete algorithm, efficient in practice, to find closed form solutions in terms of all but one of the classical special functions.

Quan's implementation is valuable as a differential solver. It is also a powerful tool to find closed form expressions for holonomic *divergent* integer power series. That is one of the reasons that the current focus is on convergent integer power series, where ${}_2F_1$ will be needed.

2.1 A ${}_2F_1$ example; king walks

Much is known about hypergeometric functions [43, 44]. Here we only mention the definition and differential operator:

$$S(x) := {}_2\mathrm{F}_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\,\right|\, x\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{4}$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$.

$$L_S := x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab \tag{5}$$

The *generalized hypergeometric function* ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ is defined likewise, with $(a)_n(b)_n/(c)_n$ replaced by $(a_1)_n \cdots (a_p)_n/((b_1)_n \cdots (b_q)_n)$. Quan's implementation [26] can solve any second order $L \in \mathbb{C}(x)[\partial]$ that is ${}_pF_q$-solvable with $p \neq q+1$. The case relevant to CIS-equations is $p = q+1$, in which case the ${}_pF_q$ function satisfies an equation of order p. Question 1 involves the functions ${}_2F_1$ and ${}_3F_2$.

Consider an infinite "chess board" $\mathbb{N}^2$ and place one piece, a king, at the origin $(0, 0)$. A walk of length n is a sequence of n moves, starting from $(0, 0)$. In each move, the king must move one step (horizontally, vertically, or diagonally) and stay within $\mathbb{N}^2$. Let a_n denote the number of walks of length n. The sequence $a_0, a_1, \ldots$ is listed as A151331 in the OEIS. It is is one of the 79 sequences considered in [39]. The other sequences in [39] are obtained by taking a subset of the of king-moves.

A king has 8 moves, so $a_n \leq 8^n$. So the generating function $y := \sum a_n x^n = 1 + 3x + 18x^2 + \cdots$ has a radius of convergence of at least (actually, equal to) $1/8$. Hence y is a CIS. Bostan and Kauers [38] computed minimal differential operators for every holonomic case in [39]. Let $L \in \mathbb{Q}(x)[\partial]$ be the minimal operator for y. It factors[7] as as $L_2 \circ L_1$ where L_2 is an operator of order 2, and $L_1 = \partial + 1/x$. With the implementation ([27], see also subsection 2.2) of the PI's graduate student Tingting Fang (supported by NSF 1017880), the operator L_2 can be solved quickly and one finds:

$$y = \frac{1}{x}\int_0^x (1+4t)^{-3} \cdot {}_2\mathrm{F}_1\left(\left.\begin{matrix}3/2, 3/2\\ 2\end{matrix}\,\right|\, \frac{16t(1+t)}{(1+4t)^2}\right) \mathrm{d}t \qquad (6)$$

(the $\frac{1}{x}\int$ comes from L_1). This example is one of the easiest and most symmetric cases. Its solution is compact; the part inside the integral has the restricted form (3).

Most examples from [39] are less symmetric and have large minimal operators (computed in [38]). Their closed form expressions (listed at [40]) have form (2) but not the restricted form (3). Even though most of the closed form expressions at [40] are substantially larger than (6), they are generally still much smaller then their minimal operators from [38].

2.2 2-descent

Although a relatively simple expression such as (6) could also be found with an ad-hoc search, there are several reasons why systematic methods are needed. In more complicated cases an ad-hoc approach would become time-consuming, and would not help prove the non-existence of closed form solutions in case none was found.

Solving a second order equation in terms of S means finding a combination of transformations (i),(ii),(iii) that sends L_S to L. With support of NSF 1017880, Tingting Fang and the PI developed an efficient algorithm and implementation [27] for *2-descent*. This algorithm solves the equation whenever the pullback function f in form (2) is a rational function of degree 2. More generally, if f allows a decomposition $f = g(h)$ where h has degree 2, then Tingting's implementation reduces the problem of solving L to another problem where the degree of f is reduced in half (this also works for other special functions, and for order > 2 as well). This often solves L, especially when combined an algorithm (currently being developed with graduate student Vijay Kunwar, supported by NSF 1017880) for solving L once the degree of f has been reduced to 3.

[7] The PI's widely used algorithm [53] for factoring differential operators is implemented in Maple's DEtools package, and the source code is available from the PI's webpage

3 Diagonals and class(es?) of hypergeometric functions

Let S and $\tilde{S}$ be a ${}_2F_1$ functions, as in equation (4), with parameters a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$. We say that S, $\tilde{S}$ are in the same *class* when $\tilde{S}$ can be written in terms of S in the form (2). This is an equivalence relation (if f is an algebraic function, then so is its inverse under composition). Our "classes of ${}_2F_1$ functions" correspond to so-called "commensurability classes of triangle groups", for which we can use Takeuchi's classification [41, Section 4].

The PI encountered many dozens of examples that turned out to be solvable in terms of ${}_2F_1$. All turned out to be in the same class! This class corresponds to Diagram (I) in [41, Section 4], so we will denote it as class (I). One of the members of Diagram (I) is denoted as $[2, 3, \infty]$ in [41]. This entry represents every hypergeometric equation whose exponent-differences are of the form $1/2 + n_2$, $\pm 1/3 + n_2$, $0 + n_3$ (for some $n_1, n_2, n_3 \in \mathbb{Z}$). The most frequently occurring member of this collection is the hypergeometric function with $a, b, c = 5/12, 1/12, 1$. It arises in the integration of periods of *elliptic curves*, which in turn is connected to many areas of mathematics and physics [8, 9].

The question why no other classes occur in the OEIS is not the right question, after all, one could simply add an integer sequence to the OEIS with a generating function that is not in class (I), such as ${}_2\mathrm{F}_1\left(\begin{matrix} 1/2, 3/10 \\ 1 \end{matrix} \middle| \, 400x\right)$. Instead, the question is:

Question 2 *Let S be a hypergeometric function in Class (I). If L is the minimal operator of a diagonal (Definition 3 below) of a bivariate rational function, must every irreducible factor of L order 2 be solvable in terms of S, and every factor of order 3 be solvable in terms of S^2, with a solution in the form (1)?*

Algebraic solutions of L, or more generally, Liouvillian solutions of the form $\exp(\int r)$ where r is an algebraic function, are not considered here because they are of form (1) for any S. So the question is, of all possible classes of non-Liouvillian functions that might occur as solutions of differential operators of order < 4, is it true that only one actually does occur when the differential equation comes from a diagonal? One can ask the same question for other constructions as well, including:

Definition 3 *Let $F \in \mathbb{Q}(x, t) \bigcap \mathbb{Q}((x))((t))$, which means one can write F as A/B for some $A, B \in \mathbb{Q}[x, t]$, but also as $\sum_{j=N}^{\infty}(\sum_{i=M_j}^{\infty} a_{ij}x^i)t^j$ for some*

$N, M_j \in \mathbb{Z}$. *The* diagonal *of* F *is* $\sum_i a_{ii}x^i$. *The* non-negative part *of* F *is* $\sum_{i,j\geq 0} a_{ij}x^i t^j$.

These constructions are known to give holonomic functions, i.e., functions that satisfy a differential operator $L \in \mathbb{C}(x,t)[\partial]$. However, L can be surprisingly large, even if it has a closed form solution of modest size. For example, the minimal equation for the complete generating function for Gessel walks is estimated in [3, section 3.3] to have 750 million terms. Nevertheless, the PI computed a complete closed form expression for this generating function that fits on 1/4 of a page in [3, appendix]. Note that the Gessel generating function is algebraic; its minimal equation was a multivariate polynomial instead of a differential equation. But the same phenomenon, a closed form expression that is significantly smaller than its minimal equation, occurs in most large examples.

Techniques such as *creative telescoping* to prove the correctness of a minimal equation can be computationally non-trivial (e.g. [3, 7, 33]). The PI proposes to directly compute a closed form expression without first computing a minimal equation. There are three reasons to expect this to be possible: (1) The large size of many minimal equations suggests that closed form expressions are more natural. (2) The fact that in non-Liouvillian examples one always ends up in the same class of hypergeometric functions would be difficult to explain if there did not exist a more direct connection. (3) Hypergeometric functions in this class correspond to integration of periods on elliptic curves; one can thus search for a link to the integration and residue analysis used in creative telescoping.

4 Comparison to related concepts

In [34] Dwork conjectured that *globally nilpotent* [37] operators are solvable in terms of hypergeometric functions. If this conjecture were true, then Question 1 would be true as well, because every irreducible CIS-operator is automatically globally nilpotent. Alas, a counter example to Dwork's conjecture was given by Krammer in [65].

Question 1 asks if Dwork's conjecture becomes true when we restrict to the most interesting (and in practice, most common) globally nilpotent operators, namely the CIS-operators. Although CIS-operators resp. CIS functions form a proper subset of globally nilpotent operators resp. G-functions [30, 31], they share interesting properties. For example, if $L \in \mathbb{Q}(x)[\partial]$ is CIS, then so is any operator in $\mathbb{Q}(x)[\partial]$ obtained from L through operations[8]

[8]This means that for a solver to be complete for CIS-equations, it must necessarily be

(ii) and (iii). So CIS-operators are just as natural as the much more studied globally nilpotent operators.

Calabi-Yau equations [28, 29] are particularly interesting CIS-equations. Many Calabi-Yau equations of order 4 are not ${}_pF_q$-solvable[9] (${}_4F_3$, ${}_3F_2$, or ${}_2F_1$) so Question 1 does not extend to order 4.

The PI will investigate the following questions in the second or third year of this proposal. How many classes (similar definition as in Section 3) of Calabi-Yau equations are there in the database from [28]? To answer this, it is necessary to develop a good algorithm that can find (if it exists) a transformation (composed of (i), (ii), and (iii)) between fourth order operators. Such an algorithm would also be needed to develop a solver for 4'th order equations. Another question is if the techniques used to construct the Calabi-Yau equations could be used to construct a counter example to Question 1 of order 3 (a counter example of order 2 is not likely; the PI tested many examples of order 2).

5 Hypergeometric functions, tables, and modular curves

The strategy proposed for NSF 1017880 to solve differential equations was to develop algorithms to solve generic cases and to build tables to cover special cases (when the pullback function f has high degree, or, when f is not a rational function). Denote d as the number of *non-removable singularities* (points that can not become regular points under transformations (ii) and (iii)). It turns out that for each d, the tables needed for the proposed strategy are indeed finite. So for a fixed d, a complete solver can be developed.

For $d = 4$, the rational pullback functions f that need to be tabulated are so-called *Belyi maps*. The PI has done this classification jointly with Raimundas Vidunas in [22]. This work is by far the largest classification of Belyi maps and their corresponding *dessins d'enfants*, consisting of 872 Belyi maps in 366 Galois orbits. Although the table is finite, it is very

able to deal with these transformations. This is why the PI develops solvers that are not restricted to form (3).

[9]The equations in [29] of order < 4 are ${}_2F_1$-solvable. Page 1 of [29] mentions that it took a long time to find a proof that the sequence A_n defined there is an integer sequence. However, the differential equation given there has a ${}_2F_1$ type solution that one can use to quickly prove that the A_n are integers. So a lot of time could have been saved if current computer algebra systems had been able to find the ${}_2F_1$ type solution, which they could not. Addressing this is an important goal of this proposal.

large, which suggests that tabulating cases could become problematic for larger d. However, over 95% of the $d = 4$ table consists of cases that do not correspond to CIS-equations. Most differential equations from practical applications will fall in remaining $\leq 5\%$ of the table.

By restricting to CIS cases, the table becomes much smaller, and $d = 5$ becomes feasible. The PI is working on the $d = 5$ case with graduate student Vijay Kunwar with support of NSF 1017880. The table for $d = 5$, restricted to CIS cases, is smaller than the $d = 4$ table without restrictions. However, to build this table it is necessary to implement new algorithms, because unlike $d = 4$, for $d = 5$ there are many non-trivial near-Belyi maps (each near-Belyi map is a parameterized family of pullback functions f, finding and processing them requires new algorithms).

The PI will finish the work of building tables and solvers for $d = 4, 5$ with Vijay Kunwar at the end of 2013. Completeness for $d = 4, 5$ also implies that our algorithms will also cover any equation that, after a number of rounds of 2-descent, ends up with < 6 non-removable singularities. Inputs where the programs can neither find a closed form solution, nor construct a non-existence proof, should become rare by then, *if* the following issue is addressed as well:

So far, the work on the above mentioned tables has only involved rational functions for the pullback function f. To fully accomplish the stated goal of finding every closed form solution (of the form (2) when the order is 2) it is also necessary to build a table of all algebraic functions f that can occur as a pullback. Let S be a hypergeometric function. If $L \in \mathbb{C}(x)[\partial]$ can be solved in terms of $S(f)$ for some algebraic function f, then it can also be solved when f is replaced by one of its conjugates. If $\tilde{f} \neq f$ is one of these conjugates, then $S(\tilde{f})$ must be of the form (2) for some algebraic functions r_0, r_1, r. The PI found a proof that r_1 must then be 0.

To build a table of algebraic pullback functions f, the idea is to find every possible polynomial relation R such that $R(f, \tilde{f}) = 0$ implies that $S(\tilde{f})$ can be written in form (2). For hypergeometric functions in class (I), take say $a, b, c = 5/12, 1/12, 1$, the answer to this task appears to be given by the *modular curve*[10] $X_0(N)$.

If R is the polynomial equation of the curve $X_0(N)$ for some $N = 2, 3, \ldots$ and if $R(1728/f,\ 1728/\tilde{f}) = 0$ then $S(\tilde{f})$ can be written in terms of $S(f)$ in the form (2). (As a side remark, this observation provides a new method to rapidly compute an equation for $X_0(N)$.) The converse appears to be

[10]Modular curves are a topic that have been studied extensively, they were key to the proof of Fermat's last theorem

true as well. The PI wants to prove this using techniques from number theory and reduction modulo primes (such techniques were key to prove an important classification theorem for the solver in Quan's thesis [26].) Then a table for non-rational f's can be constructed, at least for hypergeometric functions in class (I).

5.1 Higher order

Much of the focus so far has been on second order equations, and one might assume that higher order will be more difficult. But this is not necessarily the case. Most of the work in finding a closed form solution of the form (1) revolves around constructing f (with a combination of algorithms and tables). This pullback function f has to be reconstructed from so-called exponent-differences. For order 2 one has two exponents e_1, e_2 at each singularity, and hence one exponent-difference $e_1 - e_2$ which, due to transformation (ii), is only known modulo the integers (and up to a $\pm$ sign). But for order 3, a singularity carries more information, we have exponent-differences $e_1 - e_2$ and $e_2 - e_3$. In general this extra data makes it easier to reconstruct f. Of course, if $e_1 - e_2 = e_2 - e_3$ then we do not obtain more information, but if this happens at every singularity then it will likely correspond to a case that can be reduced to order 2 with the PI's algorithm [63].

In order to make good use of the extra data for order > 2 it will be necessary to develop new algorithms. The PI will do this with graduate student Erdal Imamoglu.

5.2 Long term goal

The PI's long term goal is to develop algorithms that will solve essentially every $L \in \mathbb{C}(x)[\partial]$ with closed form solutions that one could realistically encounter in research. But what if no closed form solution is found? Non-existence proofs do exist in the literature [65], and one could consider trying to turn those into algorithms.

But there is a more interesting strategy if one focuses on the most interesting equations, CIS-equations, or more generally, globally nilpotent equations. Say L is globally nilpotent, and no solution in terms of hypergeometric functions was found. Rather than computing a non-existence proof, one could instead search for a solution inside a larger class of functions, such as A-hypergeometric functions [35, 36].

6 Results from prior NSF support.

The PI was awarded two NSF grants during the past five years. In addition to supporting the PI and 19 publications, these grants have provided valuable research opportunities for seven research graduate students in the form of RA funding and travel support so they could present their research at large conferences. NSF 1017880 currently supports two RA's and will be depleted in August.

Title: Closed Form Solutions for Linear Differential and Difference Equations. NSF 0728853, 09/01/07 – 08/31/10, $275,000

Title: AF:Small: Solving Linear Differential in terms of Special Functions. NSF 1017880, 09/01/10 – 08/31/13, $396,085

Supported Graduate students:
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Publications supported:
Journal publications: [1, 2, 3, 4, 5, 6, 7, 8, 9]
Refereed conference publications: [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]
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Ph.D theses: [23, 24, 25, 26, 27]
These publications are available at: www.math.fsu.edu/∼hoeij/papers.html

Overview of the results:

1. Solving linear differential equations.

   The majority of the PI's research in recent years has been on differential equations. To solve Heun equations in terms of hypergeometric functions whenever possible, the PI and Raimundas Vidunas determined a table of all rational Belyi maps that can occur as a pullback function between a Heun and a hypergeometric equation. The result is the largest (by far) table of Belyi functions and dessins d'enfants. It contains functions f up to degree 60, which is about 3 times higher than what standard methods for computing Belyi functions can reach. To find them it was necessary to develop new algorithms. The table and preprint is available online at [22].

The two other main results were mentioned in Section 2 because they are relevant for describing this proposal, so we mention them only briefly here. The first is *2-descent* [16, 27] (a key technique to reduce a differential equation to equations with fewer non-removable singularities, which greatly reduces the tabulation work). The other is [14, 15, 26], the combination of which fulfills one of the PI's most important long term goals: It solves every second order equation whose solutions can be expressed in terms of Airy/Bessel/Kummer/Whittaker functions (i.e. ${}_0F_1$ and ${}_1F_1$ type functions). It is implemented [26], efficient, and proven complete.

Solving equations lead to three application papers [7, 8, 9] plus one in progress (giving closed form solutions of all 23 equations from [38]).

2. Recurrence relations

   (a) The PI and former graduate student Yongjae Cha have developed algorithms to compute local data for recurrence relations. The first application was to compute Liouvillian solutions. The resulting paper [17] won an award at the ISSAC'2009 conference for best student co-authored paper. But the main application is [19]. This algorithm solves a very large portion of the second order non-Liouvillian recurrence relations in the OEIS (oeis.org). An implementation is available at sites.google.com/site/yongjaecha.

   (b) Giles Levy, one of the PI's former graduate students, gives three new algorithms in his Ph.D thesis (December 2009, [24]). The first algorithm decides if a recurrence relation can be solved in terms of the sequence $u(n) = {}_2F_1(a+n, b; c; z)$ for some constants a, b, c, z.

       Next, an algorithm [18] for Liouvillian solutions is given. This algorithm treats only order 2, but it is the fastest Liouvillian solver for order 2. The third algorithm checks if an input equation can be reduced to an equation whose solution appears in the OEIS. The implementations are very useful in practice, and can be downloaded from [24].

       Giles has also written an implementation that uses techniques from [54] to factor recurrence operators. No paper has been written on this yet, but the strategy works well in practice.

   (c) The paper [2] gives results on holonomic sequences that are useful for a number of algorithms, such as definite summation.

3. Subfields.

   The paper [6] gives a new algorithm for computing subfields of algebraic number fields. The algorithm improves the theoretical complexity of computing subfields, and is efficient in practice as well (examples and CPU timings are available at www.math.fsu.edu/∼hoeij/subfields). It has been incorporated in the Magma computer algebra system.

   Computation of subfields can be used to obtain smaller expressions for algebraic expressions. The PI used this in the Appendix of [3]. The general principle that subfields can lead to simpler expressions was also a key for 2-descent [16], where a differential equation is reduced to a subfield of index 2, which reduces the number of non-removable singularities.

4. The complexity of factoring polynomials.

   For 25 years there was a significant gap [1] between the best theoretical complexity and the complexity of the best practical algorithm [55]. This gap has now been closed with two recent papers. First, the PI and his former graduate student Andrew Novocin have proven a new complexity result [10]. Second, an implementation [12] has been written that (a) follows [10] so that it has the best theoretical complexity, and (b) is just as fast as the best practical implementations in the worst case, while being faster in common cases. This speedup arises from our complexity analysis, which showed that a short-cut strategy called early-termination could be used to speed up the running time for most cases, without harming the theoretical worst-case complexity.

5. Projects with other students.

   The PI worked with Vivek Pal (then an undergraduate student at Florida State University) on computing isomorphisms between number fields. This resulted in a publication [4]. The PI also worked with James Fullwood (a graduate student of Paolo Aluffi) on computing Hirzebruch invariants. This lead to a joint publication [13] and a preprint [21].

   The PI has a joint project with Maarten Derickx on the gonality of modular curves. The paper is not yet finished, but the data is available online www.math.fsu.edu/∼hoeij/files/X1N and is already being used: www.math.harvard.edu/∼chaoli/doc/MazurTalk2.pdf

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