Solving Linear Differential Equations in terms of Hypergeometric Functions

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October 16th, 2012

Differential operator and differential equation

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be a differential operator, with $a_n, a_{n-1}, \cdots, a_1, a_0 \in \mathbb{C}(x)$ and n positive integer. The corresponding differential equation is

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We are interested in finding the ▸ Closed Form Solution of such second order differential equations.

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Defined Operations Set

{field operations, algebraic extensions, compositions, differentiation and $\int dx\}$

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$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \, x \right) $$

which is represented by the hypergeometric series:

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

Traditional Methods of Solving Differential Operator L

- Direct solving by the existing techniques.
- Factor L as a product of lower order differential operators, then solve L by solving the lower order ones.
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Question: For the equations that we can't solve by the above techniques, what should we do?

Overview of the methods

We consider to reduce the differential operator L, if possible, to another differential operator $\tilde{L}$ that is easier to solve (with same order, but with fewer true singularities) by using the 2-descent method or other descent methods.

1. If the above 2-descent exists, we find $\tilde{L}$.
2. If the number of true singularities of $\tilde{L}$ drops to 3, we find its ${}_2F_1$-type solutions, furthermore, find the ${}_2F_1$ solution of L in terms of $\tilde{L}$'s.
3. If the number of true singularities of $\tilde{L}$ drops to 4, we can decide if $\tilde{L}$, furthermore L, $\exists$ ${}_2F_1$-type solutions by building a large table that covers the differential operators with 4 true singularities.

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There are three types of transformations that preserve order 2:

1. change of variables: $y(x) \to y(f(x))$, $f(x) \in \mathbb{C}(x) \setminus \mathbb{C}$.
2. exp-product: $y \to e^{\int r\,dx} \cdot y$, $r \in \mathbb{C}(x)$.
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Given $L_1, L_2 \in \mathbb{C}(x)[\partial]$ with order 2:

If $L_1 \xrightarrow{2\&3} L_2$, then $L_1 \sim_p L_2$ (projectively equivalent)

If $L_1 \xrightarrow{3} L_2$, then $L_1 \sim_g L_2$ (gauge equivalent). Example 1

Example 1

$$L = x^2(36x^2-1)(4x^2-1)(12x^2-1)\partial^2 +$$
$$4x(2x-1)(1296x^5+576x^4-144x^3-72x^2+x+1)\partial +$$
$$2(5184x^6-864x^5-1656x^4+48x^3+162x^2+6x-1)$$

Question: How to find the ${}_2F_1$ solution of L as follows:

$$y_1 = r_1 \cdot {}_2F_1\left(\begin{array}{c} 1/4, 1/4 \\ 3/2 \end{array} \middle| \frac{144x^4+24x^2+1}{64x^2}\right)$$
$$+ r_2 \cdot {}_2F_1\left(\begin{array}{c} 5/4, 5/4 \\ 5/2 \end{array} \middle| \frac{144x^4+24x^2+1}{64x^2}\right)$$
$$(\text{with } r_1, r_2 \in \mathbb{C}(x))$$
$$y_2 = \cdots$$

Informal definition for 2-descent

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Relation to prior Work

- Compoint, van Hoeij, van der Put reduced the problem of 2-descent to another problem, which involved in trivializing a 2-cocycle.
  - No explicit algorithms are given.
- van Hoeij proposed that we first compute the symmetric product of L and $\sigma(L)$, and then factor it to the product of a first order equation and third order equation and then use another method to find the equivalent second order differential equation of the third order factor.
  - The method here involves to calculate the point on a conic. Algorithms were only given when the conic is defined over $\mathbb{Q}$ or the transcendental of $\mathbb{Q}$. NO algorithms are given for the general ground field.

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Main Goal

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Given a second order differential operator L, our goal is to give an explicit algorithm to decide if L has 2-descent, and if so, find this descent.

Formal definition for 2-descent

Given a second order differential operator L defined over $\mathbb{C}(x)$, we say that L has 2-descent if $\exists f \in \mathbb{C}(x)$ with degree(f)$= 2$, and $\exists \tilde{L} \in \mathbb{C}(f)[\partial_f]$ such that $L \sim_p \tilde{L}$.

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Two steps to achieve the main goal

1. Finding the subfield $\mathbb{C}(f)$ with $[\mathbb{C}(x) : \mathbb{C}(f)] = 2$, i.e. finding $f \in \mathbb{C}(x)$ of degree 2.
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The automorphisms of $\mathbb{C}(x)$ over $\mathbb{C}$ are Möbius transformations:

$$x \mapsto \frac{ax+b}{cx+d}$$

Requirements for σ

Necessary Requirements for σ

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For each such σ, we compute a candidate subfield $\mathbb{C}(f) \subseteq \mathbb{C}(x)$. To determine σ, basically, we need find 2 equations of variables a, b, c and then verify if it satifies the requirements mentioned above.

Example 2

Let $C = \mathbb{Q}$, and

$$L = \partial^2 + \frac{(44x^4-7)}{x(2x^2-1)(2x^2+1)}\partial + \frac{8(24x^6-14x^4-3x^2+1)}{x^2(2x^2+1)(2x^2-1)^2}$$

- The set of true singularities is $S = \{\infty, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{-2}}, \frac{1}{\sqrt{-2}}\}$
- and $S_C^{\text{type}} = \{(\infty, 0), (x, 0), (x^2 + \frac{1}{2}, 0), (x^2 - \frac{1}{2}, 0)\}$.

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Analyze example 2, we get the set of candidates for σ is:

$$\{-x, -\frac{1}{2x}, \frac{1}{2x}\}$$

The corresponding subfields set is:

$$\{\mathbb{C}(x^2), \mathbb{C}(x - \frac{1}{2x}), \mathbb{C}(x + \frac{1}{2x})\}$$

Theoretical support

The following σ and $\mathbb{C}(f)$ represent the Möbius transformation found previously and the corresponding fixed field, respectively. Suppose L descends to $\tilde{L} \in \mathbb{C}(f)[\partial_f]$, we have

$$L \sim_p \tilde{L} = \sigma(\tilde{L}) \sim_p \sigma(L), \text{ and so } L \sim_p \sigma(L)$$

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Question: How to compute $\tilde{L}$ from it?

Case A

Case A is when $L \sim_g \sigma(L)$, in other words, there exists $G = r_0 + r_1 \partial \in \mathbb{C}(x)[\partial]$ with $G(V(L)) = V(\sigma(L))$. Then $\exists\ \tilde{L} \in \mathbb{C}(f)[\partial_f]$ with $\tilde{L} \sim_g L$.

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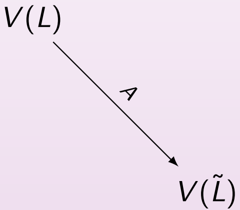

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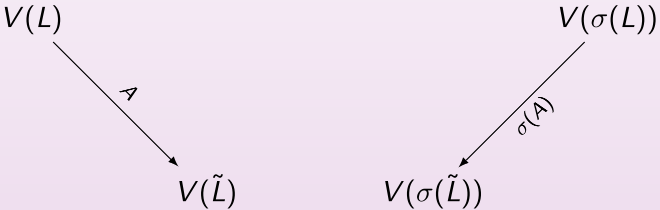

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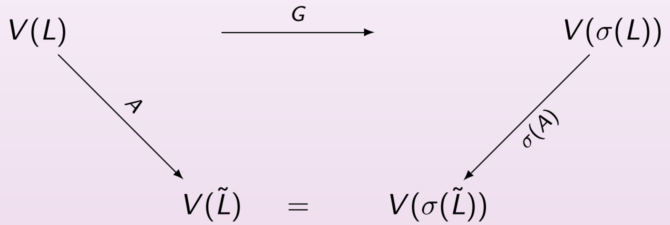

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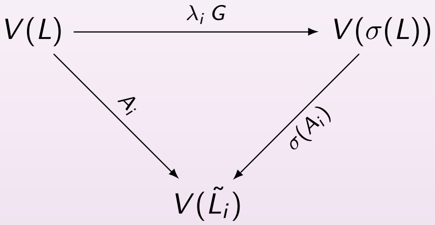
Theorem

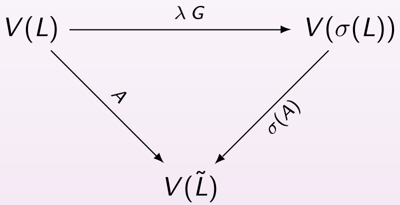
Let L and σ be as before, and $G : V(L) \to V(\sigma(L))$ be a gauge transformation. Suppose $\tilde{L_1}, \tilde{L_2} \in \mathbb{C}(f)[\partial_f]$ and $A_i : V(L) \to V(\tilde{L_i})$ are gauge transformations. Then:

1. For each $i = 1, 2$, there is exactly one $\lambda_i \in \mathbb{C}^*$ such that
   ▸ the following diagram commutes
2. If $\tilde{L_1} \sim_g \tilde{L_2}$ over $\mathbb{C}(f)$, then $\lambda_1 = \lambda_2$; Otherwise, $\lambda_1 = -\lambda_2$.
3. In particular, $\{\lambda_1, -\lambda_1\}$ depends only on (L, σ, G).

Diagram



Finding $\tilde{L}$ in Case A



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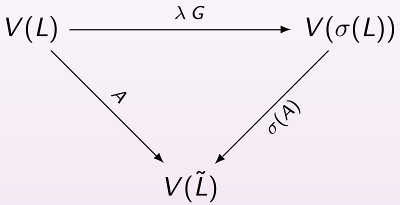

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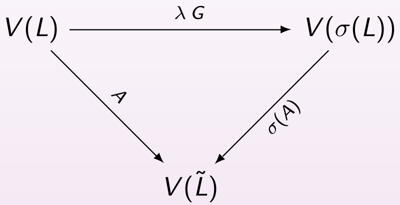

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$A - \sigma(A)\lambda G$ right divided by L, and this gives us 4 equations for coefficients of A.

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$L = \partial^2 + \frac{8(8x+1)}{(4x+1)(4x-1)}\partial + \frac{4(8x+1)}{x(4x-1)(4x+1)}$.
One of the candidates we found for σ is $-x$ and
$G = \frac{x(4x-1)}{4x+1}\partial + \frac{12x+1}{2(4x+1)}$.
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$$\{2a_{01} - 16\lambda a_{00} + \lambda a_{01} - 64\lambda a_{10} + 32fa_{10} + 48f\lambda a_{01} + 16a_{00}, 16fa_{01} + 2a_{00} + 32fa_{00} + 64f\lambda a_{11} - \lambda a_{00} - 48f\lambda a_{00} + 16f\lambda a_{01}, 16\lambda a_{10} + 2\lambda a_{00} + 32fa_{11} + 48f\lambda a_{11} - 32f\lambda a_{00} + 16a_{10} + \lambda a_{11} + 2a_{11}, -16f\lambda a_{11} + 2a_{10} + 32f^2\lambda a_{01} + 16fa_{11} - 48f\lambda a_{10} - \lambda a_{10} + 32fa_{10} - 2f\lambda a_{01}\}.$$

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$$A = (\frac{4}{3}x^2 - \frac{1}{12})\partial + \frac{4x}{3} + 1$$

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Solution

After multiplying solution of L by a suitable $e^{\int s}$, we can reduce this case to Case A.

Sketch of the Main Algorithm

Main Algorithm:
Input: A second order differential operator L;
Output: Another second order differential operator $\tilde{L}$.

1. Compute the set of true singularities of L, and their exponent-difference mod $\mathbb{Z}$.
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To decide $\tilde{L}$, we first compute λ and then a set of linear equations to determine $A = (a_{10} + a_{11}x)\partial + (a_{00} + a_{01}x)$.

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Disadvantage

When we compute A, we select one $(a_{00}, a_{01}, a_{10}, a_{11})$ from a vector space of dimension 2, that means our output $\tilde{L}$ is just one member of a 2-dimensional set of possible outcomes. We can't expect $\tilde{L}$ to have the optimal size.

What is improved in the new Algorithm

The improved algorithm will avoid computing a set of possible $\tilde{L}$s and apt to give a smaller output.

Main Idea for the Improved Case A algorithm

We consider the following algorithm, here we denote $L_4 := \mathrm{LCLM}(L, \sigma(L)) \in C(f)[\partial_f]$ then $V(L_4) = V(L) + V(\sigma(L))$. The order of L_4 is 4 except if $V(L) = V(\sigma(L))$.

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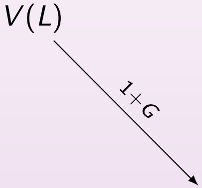

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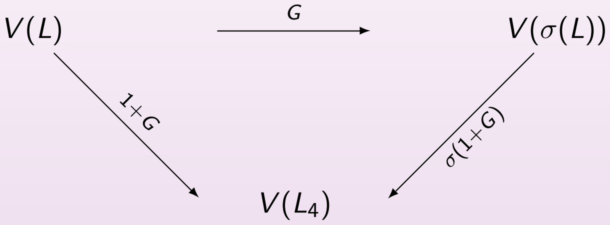

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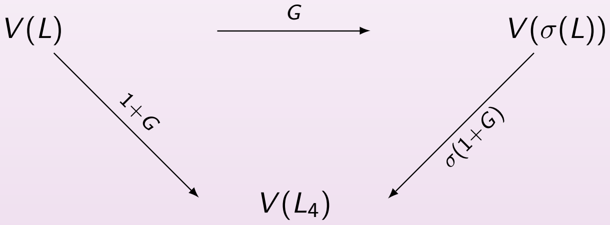

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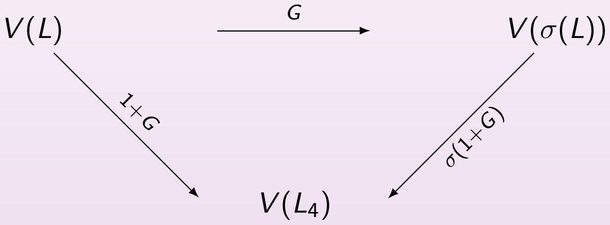

Question: Is this a commutative diagram?

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Given a second order irreducible differential operator L and second order automorphism σ as in Section 3.4, and a gauge transformation $G : V(L) \to V(\sigma(L))$, then there exist a constant λ such that the following diagram commutes.

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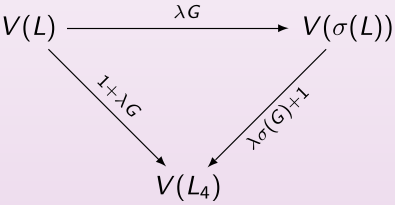

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Given a second order differential operator L. σ, G, λ are as stated previously in the diagram. Then there exists a second order differential operator $\tilde{L}$ such that $\tilde{L}$ is invariant under σ and $(1 + \lambda G)V(L) = V(\tilde{L})$.

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Computing $\tilde{L}$

1. Compute $M :=\text{LCLM}(L, 1+\lambda G)$.
2. Compute the $\tilde{L}$ such that $M = \tilde{L}(1+\lambda G)$.
3. Verify $V(\tilde{L}) \subseteq V(L_4)$ and $V(\sigma(\tilde{L})) = V(\tilde{L})$

Application for Fourth Order Differential Equation

$$L := \partial^4 + \frac{(7x^4 - 68x^3 - 114x^2 + 52x - 5)}{(x+1)(x^2-10x+1)(x-1)x}\partial^3 + \frac{2(5x^5 - 55x^4 - 169x^3 + 149x^2 - 28x + 2)}{(x+1)x^2(x^2-10x+1)(x-1)^2}\partial^2 + \frac{2(x^4 - 13x^3 - 129x^2 + 49x - 4)}{(x+1)x^2(x^2-10x+1)(x-1)^2}\partial - \frac{3(x+1)^2}{(x-1)^2x^3(x^2-10x+1)}$$

L has 4 regular true singularities:

Result after 2-descent Algorithm, Case A

$$
\begin{aligned}
\tilde{L_1} :=\, & 16x_1^4(x_1+3)(5x_1^2+10x_1+1)(9x_1^8+1008x_1^7-31820x_1^6+264480x_1^5 \\
& -14194x_1^4+162992x_1^3-8156x_1^2+18368x_1+529)(x_1-1)^4\partial^4 \\
& +32x_1^3(-7935-358000x_1-3502550x_1^2-24264785x_1^4-1520720x_1^3 \\
& -12737440x_1^5-13562976x_1^7-20800372x_1^6-905046x_1^{10}+20706063x_1^8 \\
& +28080x_1^{11}+6593808x_1^9+225x_1^{12})(x_1-1)^3\partial^3 \\
& +8x_1^2(2250x_1^{13}+312135x_1^{12}-12439492x_1^{11}+134614866x_1^{10} \\
& -42449802x_1^9-470021643x_1^8+267358792x_1^7-102361428x_1^6+163767350x_1^5 \\
& +221768417x_1^4-11134724x_1^3+48114210x_1^2+3717898x_1+77763)(x_1-1)^2\partial^2 \\
& +8x_1(x_1-1)(1350x_1^{14}+230355x_1^{13}-10741153x_1^{12}+169118578x_1^{11} \\
& -503407892x_1^{10}+340703465x_1^9+768939585x_1^8-411403540x_1^7 \\
& +839007558x_1^6-333028107x_1^5-52500447x_1^4+44391810x_1^3-43359960x_1^2 \\
& -2602385x_1-42849)\partial+\cdots
\end{aligned}
$$

Result after Improved 2-descent Algorithm, Case A

$$\tilde{L_2} := \partial^4 + \frac{77x_1^6 - 1709x_1^5 - 11250x_1^4 - 11530x_1^3 + 10377x_1^2 - 2457x_1 + 108}{(x_1-1)x_1(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial^3 + \frac{220x_1^7 - 6063x_1^6 - 46066x_1^5 - 40985x_1^4 + 71024x_1^3 - 30225x_1^2 + 3078x_1 - 135}{2(x_1^2 - 2x_1 + 1)x_1^2(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial^2 + \frac{22x_1^6 - 931x_1^5 - 10011x_1^4 - 12590x_1^3 + 15680x_1^2 - 3039x_1 + 117}{(x_1^2 - 2x_1 + 1)x_1^2(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)}\partial - \frac{3(121x_1^5 + 175x_1^4 - 166x_1^3 + 1118x_1^2 - 227x_1 + 3)}{16(x_1^2 - 2x_1 + 1)x_1^4(11x_1^4 - 248x_1^3 - 506x_1^2 + 240x_1 - 9)}$$

Where x_1 represents x^2. $\tilde{L_2}$ has length 635.

Things we should consider

After implementing 2-decent, we may end up with $\tilde{L}$ with 3 true singularities. If so, we can solve such $\tilde{L}$ in terms of hypergeometric functions, further more L.

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(a) Three true regular singularities, located at $0, 1, \infty$.

(b) No apparent singularities.

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To solve $\tilde{L}$ in terms of hypergometric functions, we need to apply two types of transformations:

(a) A Möbius transformation (a change of variables) to move $p1, p2, p3$ to $0, 1, \infty$.

(b) A projective equivalence $\sim_p$ to eliminate all apparent singularities.

Classification of Gauss Hypergeometric Equations

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Theorem

Let L_1, L_2 be two Gauss hypergeometric differential operators. Assume the exponent difference set of L_1 at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of L_2 at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If

1. $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$
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Then $L_1 \sim_p L_2$.

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Then $L_1 \sim_p L_2$.

Corollary

Let L_1, L_2 be two Gauss hypergeometric differential operator. Assume the exponent difference set of L_1 at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of L_2 at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If $\frac{1}{2} + \mathbb{Z}$ appears in $\{e_0, e_1, e_\infty\}$ and $\{d_0, d_1, d_\infty\}$, then L_1 is projectively equivalent to L_2 if $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$.

Possible Hypergeometric Equations corresponding to $\tilde{L}$

Lemma

Suppose L is projectively equivalent to a hypergeometric equation. suppose that the exponent-differences of L at $0, 1, \infty$ are d_0, d_1, d_∞. Let L_1 be a hypergeometric equation with exponent-differences: d_0, d_1, d_∞ and L_2 be a hypergeometric equation with exponent-differences: $d_0 + 1, d_1, d_\infty$. Then $L \sim_p L_1$ or $L \sim_p L_2$ (both are true if $\{d_0, d_1, d_\infty\} \bigcap \{\frac{1}{2} + \mathbb{Z}\} \neq \emptyset$).

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With these exponent-differences d_0, d_1, d_∞ at $0, 1, \infty$, we construct the gauss hypergeometric equations by the following fomular:

$$x(x-1)\partial^2 - (-2x + xd_0 + xd_1 + 1 - d_0)\partial + \frac{(d_0 - 1 + d_1 + d_\infty)(d_0 - 1 + d_1 - d_\infty)}{4}$$

Algorithm for finding ${}_2F_1$ Solutions

Having these theorems, we have evidences to find the ${}_2F_1$ solution of our $\tilde{L}$.

1. Compute the exponent-difference at the three singularities of $\tilde{L}$ module $\mathbb{Z}$. Denote them as e_1, e_2, e_3.
2. Find the two Gauss hypergeometric equations L_1, L_2 by the formula and theorem.
3. Find the Möbius transformation $m(x)$ between p_1, p_2, p_3 and $0, 1, \infty$.
4. Call **equiv** to check if L_1 or L_2 (after change of variable) is projectively equivalent to $\tilde{L}$, if so, go to next step. Denote the equivalence as G
5. Find the Gauss hypergeometric solutions of $Sol := C_1 y_1(m(x)) + C_2 y_2(m(x))$ if $e_1 \neq 0$, otherwise, compute $Sol := C_1 y_1(m(x)) + C_2 y_2'(m(x))$.
6. Compute the ${}_2F_1$-type solution of $\tilde{L}$ by computing $G(Sol)$.

Final solving by 2-descent

Input: A second order irreducible differential operator $L \in C(x)[\partial]$ and the field C.

Output: ${}_2F_1$-type solution, if it exists..

1. Call **Algorithm 2-descent** in Chapter 3 to Compute the 2-descent of L, $\tilde{L}$, if it exists.
2. Compute the true singularities of $\tilde{L}$.
3. If $\tilde{L}$ has 3 true regular singularities, then call **Algorithm finding ${}_2F_1$-type solution with 3 singularities** and find the solution sol; Otherwise, stop and return NULL.
4. Apply the Change of variable $x \mapsto f$ to $\tilde{L}$, Sol, we get $\tilde{L}'$ and its ${}_2F_1$ solution Sol'.
5. Call **equiv** to Compute the equivalence G between $\tilde{L}'$ and L.
6. Compute the ${}_2F_1$-type solution of L by computing $G(Sol')$.

Example 5

Let

$$L = \partial^2 + \frac{28x-5}{x(4x-1)}\partial + \frac{144x^2+20x-3}{x^2(4x-1)(4x+1)}$$

Step 1: Compute the 2-descent of L from Section 3.7, we have

$$\tilde{L} := (16x-1)x\partial^2 + (32x-2)\partial + 4$$

step 2: Compute the true singularities of $\tilde{L}$, we found it has 3 true regular singularities: $0, \frac{1}{16}, \infty$.

step 3: Call Algorithm finding ${}_2F_1$-type solution with 3 singularities, we found the ${}_2F_1$ solution of $\tilde{L}$ as

$$Sol := C_1(64x-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;16x) - C_2(64x-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x)$$

Example 5, continued...

Step 4: From Section 3.7, we know that $f = x^2$, so the change of variable would be $x \mapsto x^2$. Apply transformation to $\tilde{L}$ and Sol, we have

$$\tilde{L}' := x(4x+1)(4x-1)\partial^2 + (12x-3)(4x+1)\partial + 16x$$

$$Sol' := C_1(64x^2-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;16x^2) - C_2(64x^2-4){}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x^2)$$

Step 5: Compute the equivalence between $\tilde{L}'$ and L, we have

$$G := \frac{1}{x(4x-1)}$$

Step 6: Compute $G(Sol')$, we have the final solution as

$$C_1\frac{16x+1}{x}{}_2F_1(\frac{3}{2},\frac{3}{2};2;16x^2) - C_2\frac{16x+1}{x}{}_2F_1(\frac{3}{2},\frac{3}{2};2;1-16x^2)$$

Conclusion

We focus on finding the hypergeometric solutions of second order linear equations. Contributions of this theis are:

1. Developed 2-descent algorithms to reduce our differential equation to one with fewer true singularities.
2. Improved the 2-descent algorithm to produce shorter output, which is helpful for finding the ${}_2F_1$ solutions.
3. Finding the ${}_2F_1$ solutions.

Work may be done in future:

1. Extend the 2-descent algorithm to bigger descent, for example: $C_2 \times C_2$, D_n, A_4, S_4, or A_5.
2. Extend the 2-descent to 3-descent, for which the index of the descent subfield is 3.

Thank Dr. van Hoeij for the continuous support, encouragement, the guide of research and interest.

Thank every committee member for devoting time to reading this dissertation and giving me suggestions.