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FINDING ALL BESSEL TYPE SOLUTIONS FOR LINEAR DIFFERENTIAL EQUATIONS WITH RATIONAL FUNCTION COEFFICIENTS

By

QUAN YUAN

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Quan Yuan defended this dissertation on March 19 2012.
The members of the supervisory committee were:

Mark van Hoeij
Professor Directing Dissertation

Robert A. van Engelen
University Representative

Amod Agashe
Committee Member

Ettore Aldrovandi
Committee Member

Paolo Aluffi
Committee Member

The Graduate School has verified and approved the above-named committee members, and certifies that the dissertation has been approved in accordance with the university requirements.

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ABSTRACT

A linear differential equation with rational function coefficients has a *Bessel type solution* when it is solvable in terms of Bessel functions, change of variables, algebraic operations and exponential integrals. For second order equations with rational function coefficients, the function f of change of variables must be a rational function or the square root of a rational function. An algorithm was given by Debeerst, van Hoeij, and Koepf, that can compute Bessel type solutions if and only if f is a rational function. In this thesis we extend this work to the square root case, resulting in a complete algorithm to find all Bessel type solutions. This algorithm can be easily extended to a Whittaker/Kummer solver. Combining the two algorithms, we get a complete algorithm for all ${}_0F_1$ and ${}_1F_1$ type solutions. We also use our algorithm to analyze the relation between Bessel functions and Heun functions.

CHAPTER 1

INTRODUCTION

Ordinary differential equations have many applications. Although there is no general algorithm to solve every equation, there are many methods, such as integrating factors, symmetry method. One way to solve ordinary differential equations is using differential Galois theory.

We consider a differential field K, e.g $K = \mathbb{Q}(x)$ or $K = \mathbb{C}(x)$ and $\partial = \frac{d}{dx}$, and then we write homogeneous differential equations as $Ly = 0$, here

$$L = \sum_{i=0}^{n} a_i \partial^i, \ a_i \in K$$

Now L is an element of the non-commutative ring of differential operators $K[\partial]$. We can study the algebraic properties of a differential operator L to get the solution space of the equation.

If we can find a first order right factor $\partial - r$ of L, then $\exp(\int r)$ is a solution of L. This type of solutions is called hyperexponential solutions. In general, if we can find the solution in terms of algebraic operations and iterations of exponentials and integrals, then the solutions are called Liouvillian solutions. To find Liouvillian solutions, we can use Kovacic's algorithm [22]. For reducible operators, Beke's Algorithm and the algorithm in [35] can factor L into irreducible factors. After factoring, if we have a first order right factor, then we have the corresponding exponential solution, which is a special case of a Liouvillian soultion. For irreducible operators of order 2, we can use Kovacic's algorithm [22].

But not all operators have Liouvillian solutions. For example, the Bessel operator

$$L_{B1} = x^2\partial^2 + x\partial + (x^2 - \nu^2)$$

is irreducible in $K[\partial]$ and has no Liouvillian solution when $\nu \notin \frac{1}{2} + \mathbb{Z}$.

Although some irreducible operators have no Liouvillian solutions, it may correspond to special functions. For example, the solutions of the Bessel operator are Bessel functions. Because many special functions such as Bessel, Airy and Kummer/Whittaker are well studied, it is useful to find the solution in terms of them, along with algebraic operations, and exponential integrals. From this point on, we just consider Bessel type solutions. If we allow a square root in the change of variables, then we can find all ${}_pF_q$ type solutions of second order equations except $(p, q) = (2, 1)$.

We only consider irreducible operators with order 2, because if the second order operator is irreducible, then it has Liouvillian solutions. So we can solve it by Kovacic's algorithm.

If the order is higher than 2, we define Eulerian solution:

Definition 1. *If $L \in \mathbb{C}(x)[\partial]$, and if a solution y can be expressed (using sums, products, field operations, algebraic extensions, integrals, differentiations,* exp*,* ln *and change of variables) in terms of second order equations $L_{2a}, L_{2b}, \ldots \in \overline{\mathbb{C}(x)}[\partial]$, then y is a eulerian solution.*

Singer [28] showed that solving such L can be reduced to solving second order L's; through factoring operators, or reducing operators to tensor products of lower order operators. An algorithm and implementation for such reduction (order 3 to order 2) is given in [37]. Such reduction to order 2 is valuable, *if* we can actually solve such second order equations. That is why we focus on second order operators.

Let $a_0, a_1, a_2 \in \mathbb{C}(x)$ and let $L = a_2\partial^2 + a_1\partial + a_0$ be a differential operator of order two. The corresponding differential equation is $L(y) = 0$, i.e. $a_2y'' + a_1y' + a_0y = 0$. Let $B_\nu(x)$ denote one of the Bessel functions (one of Bessel I, J, K, or Y functions). The question studied in [14, 15] is the following: Given L, decide if there exists a rational function $f \in \mathbb{C}(x)$ such that L has a solution y that can be expressed[1] in terms of $B_\nu(f)$. If so, then find f, ν, and the corresponding solutions of L. The same problem was also solved for Kummer/Whittaker functions, see [14]. This means that for second order L, with rational function coefficients, there is an almost-complete algorithm in [14] to decide if $L(y) = 0$ is solvable in terms of ${}_0F_1$ or ${}_1F_1$ functions, and if so, to find the solutions.

The reason this almost-complete algorithm is not complete is the following: If $B_\nu(f)$ satisfies a second order linear differential equation with rational function coefficients, then either: $f \in \mathbb{C}(x)$, or (*square root case*): $f \notin \mathbb{C}(x)$ but $f^2 \in \mathbb{C}(x)$, see Section 2.3.1.

However, only the $f \in \mathbb{C}(x)$ case was handled in [14, 15], the square-root case was listed in the conclusion of [15] as a task for future work. This meant that [14, 15] is not yet a complete solver for ${}_0F_1$ and ${}_1F_1$ type solutions.

In this thesis, we treat the square-root case for Bessel functions. The combination of this with the treatment of Kummer/Whittaker functions in [14] is then a complete algorithm to find ${}_0F_1$ and ${}_1F_1$ type solutions whenever they exist[2].

The reason why the square-root case was not yet treated in [15] will be explained in the next two paragraphs. If f is a rational function $f = A/B$, then from the generalized exponents at the irregular singularities, we can compute B, as well as $\deg(A)$ linear equations for the coefficients of A, see [15], or see [14] which contains more details and examples. Since a polynomial A of degree $\deg(A)$ has $\deg(A) + 1$ coefficients, this meant that only one more equation was needed to reconstruct A, and in each of the various cases in [14, 15] there was a way to compute such an equation.

[1] using sums, products, differentiation, and exponential integrals (see Definition 19)

[2] Other ${}_0F_1$ and ${}_1F_1$ type functions can be rewritten in terms of Bessel, or Kummer/Whittaker functions. For instance, Airy type functions form a subclass of Bessel type functions (provided that the square-root case is treated!)

In the square-root case, we can not write f as a quotient of polynomials, but we can write $f^2 = A/B$. The same method as in [14, 15] will still produce B, and linear equations for the coefficients of A. The number of linear equations for the coefficients of A is still the same as it was in the $f \in \mathbb{C}(x)$ case. Unfortunately, by squaring f to make it a rational function, we doubled the degree of A, but we do not get more linear equations, which means that in the square-root case the number of linear equations is only $\frac{1}{2}\deg(A)$ (plus an additional ≥ 0 equations coming from regular singularities). So in the worst case, the number of equations is only half of the degree of A. This is why the square-root case was not solved in [15] but only mentioned as a future task.

Our approach is the following: One can rewrite $A = CA_1A_2^d$ where A_1 can be computed from the regular singularities, but A_2 can not. The problem is that while the degree of A_2 is only $\frac{1}{d}$ times the degree of A/A_1, the linear equations on the coefficients of A translate into polynomial equations (with degree d) for the coefficients of A_2. Solving systems of polynomial equations (e.g. with Gröbner basis) can take too much CPU time. However, we discovered that with some modifications, one can actually obtain linear equations for the coefficients of A_2. This means that we only need to solve linear systems. (See section 4.5.) The result is an efficient algorithm that can handle complicated inputs.

This thesis is organized as follows. After introducing some preliminaries in Chapter 2, we will discuss the transformations and the local information we collect to solve the differential equations in Chapter 3. Chapter 4 will give the details of the algorithm case by case along with examples.

We prove a uniqueness theorem for f in Chapter 4. This theorem allows us to conclude that $f^2 \in K$ (instead of $\overline{C_K} \cdot K$), which in turn allow us to discard any candidate that is defined over an algebraic extension of C_K. That in turn speeds up the algorithm significantly.

Chapter 5 will discuss how to extend our algorithm to Airy and Whittaker/Kummer type solutions. Chapter 6 will apply our algorithm to find relations between Heun functions and Bessel functions.

An implementation of the algorithms in this thesis is available online at:

http://www.math.fsu.edu/∼qyuan/Besselsolver.txt.

All examples in this thesis are included in a Maple worksheet at:

http://www.math.fsu.edu/∼qyuan/ExamplesforThesis.mw.

A worksheet for the details of Chapter 6 is available at:

http://www.math.fsu.edu/∼qyuan/Heun.mw

CHAPTER 2

PRELIMINARIES

In this chapter, we will first introduce differential operators, their singularities, solution spaces and corresponding differential equations. Then we will introduce formal solutions and generalized exponents of differential operators. Our methods and algorithms are based on information we collect from generalized exponents. Then we will give some basic properties of Bessel functions. The goal of this thesis is solving differential equations in terms of Bessel functions. Finally we will discuss some maple commands we need. In this chapter, we skip most of the proofs, for more details, see [14].

2.1 Differential Operators

Definition 2. *Let K be a field. A derivation D is a additive map on K such that $D(ab) = D(a)b + aD(b), \forall a, b \in K$. A field K with derivation D is called a differential field.*

Theorem 1. *Let K be a differential field with derivation D, then $C_K := \{a \in K | D(a) = 0\}$ is also a field. We call it the constant field of K.*

Proof. The proof is trivial and can be found in [35] $\square$

Example 1. *Let C_K be an extension of $\mathbb{Q}$, and $D = \partial := \frac{d}{dx}$, then $K := C_K(x)$ is a differential field.*

Definition 3. *Let K be a differential field with derivation ∂, then*

$$L := \sum_{i=0}^{n} a_i \partial^i,\ a_i \in K,$$

is called a differential operator. If $a_n \neq 0$, then n is the order of L, which is denoted by $\deg(L)$*. We also denote $K[\partial]$ as the ring of all differential operators.*

Remark 1. *In general, $K[\partial]$ is not commutative. For example $\partial x = x\partial + 1$. But it is a Euclidean ring. For two operators L_1 and L_2, there are unique operators Q, R, such that $L_1 = QL_2 + R$, where* $\deg(R) < \deg(L_2)$ *or $R = 0$. If $R = 0$, we say L_2 is a right divisor of L_1. Since $K[\partial]$ is not commutative, we can not define the greatest common divisor or the least common multiple in $K[\partial]$. But we can define the greatest common right divisor (notation GCRD) and the least common left multiple (notation LCLM) of differential operators (We do not use GCLD or LCRM).*

In our context, each differential operator corresponds to a homogeneous differential equation $Ly = 0$ and vice versa. In Maple, the command DEtools[diffop2de] will convert a differential operator to a differential equation and DEtools[de2diffop] will convert it back.

Definition 4. *Let L be a differential operator. We say y is a solution of L, if $L(y) = 0$. The vector space of solutions, which is denoted as $V(L)$, is called the solution space of L. $V(L)$ is a subspace of a universal extension of K, see Theorem 3 in section 2.2.*

$K[\partial]$ is a Euclidean ring. To find the solutions of L, one can try to factor L into lower degree operators. If we know a right divisor L_2 of L, then $V(L_2) \subseteq V(L)$. If we have two operators, L_1 and L_2, then $\mathrm{LCLM}(L_1, L_2)$ will give us the operator with minimal order such that all solutions of L_1 and L_2 are solutions of $\mathrm{LCLM}(L_1, L_2)$ as well.

$$V(L_1) + V(L_2) = V(\mathrm{LCLM}(L_1, L_2))$$

And for GCRD, we have

$$V(L_1) \cap V(L_2) = V(\mathrm{GCRD}(L_1, L_2)).$$

Definition 5. *Let K be a differential field and C_K be its constants, $\overline{C_K}$ be the algebraic closure of C_K. We call $p \in \overline{C_K} \cup \{\infty\}$ a singularity of the differential operator $L \in K[\partial]$, if $p = \infty$ or p is a zero of the leading coefficient of L or p is a pole of a coefficient of L. If p is not a singularity, p is* regular.

Remark 2. *To understand the singularity at $x = \infty$, one can always use the change of variables $x \mapsto \frac{1}{x}$ and deal with 0.*

If p is a singularity of a solution of L, then p must be a singularity of L. But the converse is not true. See apparent singularity in Definition 8 below.

Definition 6. *If $p \in \overline{C_K} \cup \{\infty\}$, we define the local parameter t_p as*

$$t_p := \begin{cases} x - p & \text{if } p \neq \infty \\ \frac{1}{x} & \text{if } p = \infty \end{cases}$$

Definition 7. *Let $L \in K[\partial]$ with leading coefficient $a_n = 1$. A singularity p of L is:*

(i) regular singularity ($p \neq \infty$) if $t_p^i a_{n-i}$ is analytic at $x = p$ for $1 \leq i \leq n$.

(ii) regular singularity ($p = \infty$) if $\frac{a_{n-i}}{t_\infty^i}$ is analytic at $x = \infty$ for $1 \leq i \leq n$.

(iii) irregular singularity otherwise.

Definition 8. *We say a singularity is an apparent singularity if all solutions of L are analytic at $x = p$.*

Theorem 2. *$L \in K[\partial]$, and* $\deg(L) = 2$*, then:*

(i) If L is non-singular or apparent singular at $x = p$, then all solutions are analytic at $x = p$. Hence, we can write it as convergent power series $y(x) = \sum_{i=0}^{\infty} b_i t_p^i$.

(ii) If L is regular singular at $x = p$, then there exist the two linearly independent solutions

$$y_1(x) = t_p^{e_1} \sum_{i=0}^{\infty} a_i t_p^i,\ a_0 \neq 0$$

and

$$y_2(x) = t_p^{e_2} \sum_{i=0}^{\infty} b_i t_p^i + c y_1(x) \ln t_p,\ \ b_0 \text{ and } c \text{ are not both } 0$$

where $e_1, e_2, a_i, b_i, c \in \overline{C_K}$ and $c = 0$ if $e_1 - e_2 \notin \mathbb{Z}$.

(iii) If L is irregular singular at $x = p$, two linearly independent solutions are

$$y_1(x) = \exp\left(\int \frac{e_1}{t_p} dt_p\right) \sum_{i=0}^{\infty} a_i t_p^{\frac{i}{m}},\ a_0 \neq 0$$

and

$$y_2(x) = \exp\left(\int \frac{e_2}{t_p} dt_p\right) \sum_{i=0}^{\infty} b_i t_p^{\frac{i}{m}} + c y_1(x) \ln t_p,\ \ b_0 \text{ and } c \text{ are not both } 0$$

where $a_i, b_i, c \in \overline{C_K}$, $e_1, e_2 \in \overline{C_K}[t_p^{-1/m}]$, $c = 0$ if $e_1 - e_2 \notin \mathbb{Z}$ and m is 1 or 2 (because the order of L is 2).

Proof. The proof can be found in [40]. □

Definition 9. *In the previous Theorem, if $c = 0$, then the solutions of L do not contain logarithmic terms. If $c \neq 0$, then we say L has logarithmic solutions at $x = p$.*

Remark 3. *(i) Note that L can only have logarithmic solutions at $x = p$ if $e_1 - e_2 \in \mathbb{Z}$. We will discuss more details in the following chapter.*

(ii) In the regular singular case, the constants e_1 and e_2 are called exponents, they can be found by solving the indicial equation

$$\lambda(\lambda - 1) + p_0 \lambda + q_0 = 0$$

where p_0 resp. q_0 is the constant coefficient of the power series expansion of $t_p p(x)$ resp. $t_p^2 q(x)$ at $x = p$.

(iii) For irregular singularity, e_1 and e_2 are generalized exponents, which will be explained in Section 2.2

2.2 Formal Solutions and Generalized Exponents

In this section, we introduce the idea of generalized exponents. Generally, the generalized exponents give us the asymptotic local information about solutions. We will use the information to reconstruct solutions in Chapter 3. In this section, we consider operators in $\mathbb{C}((x))[\partial]$.

Definition 10. *A universal extension U of K is a minimal differential ring*[1] *in which every operator $L \in K[\partial]$ has precisely* $\deg(L)$ *linearly independent*[2] *solutions.*

We construct a universal extension of $\mathbb{C}((x))$ as follows:
First denote $E := \bigcup_{m\geq 1} \mathbb{C}[x^{-1/m}]$.

Theorem 3. *(i)* $\overline{\mathbb{C}((x))} = \bigcup_{n\geq 1} \mathbb{C}((x^{1/m}))$.

(ii) For each $q \in E$, we introduce a symbol $\mathrm{Exp}(q)$.

(iii) Consider the ring $R := \overline{\mathbb{C}((x))}[ln(x), \{\mathrm{Exp}(q) | q \in E\}]$. Let I be the ideal generated by all $\mathrm{Exp}(q_1)\mathrm{Exp}(q_2) - \mathrm{Exp}(q_1 + q_2)$ and all $\mathrm{Exp}(r) - x^r$ for all $q_1, q_2 \in E$ and all $r \in \mathbb{Q}$.

(iv) Let $V := R/I$, and define $\mathrm{Exp}(q)' := \frac{q}{x}\mathrm{Exp}(q)$ for all $q \in E$.

(v) Then V is a universal extension of $\mathbb{C}((x))$. Which means:

- *the constant field of V is $\mathbb{C}$.*
- *If $L \in \mathbb{C}((x))[\partial]$ has order n, then $V(L) := \ker(L : V \to V)$ is a $\mathbb{C}$-vector space of dimension n.*

---

[1] with field of constant equal to $\overline{C_K}$
[2] over $\overline{C_K}$

Proof. The proof and other details of universal extension can be found in [35]. $\square$

Definition 11. *Let*

$$\begin{aligned} V_e :=& \mathrm{Exp}(e)\mathbb{C}((x))[e, ln(x)] \\ =& \mathrm{Exp}(e)\mathbb{C}((x^{1/\mathrm{ram}(e)}))[ln(x)] \end{aligned}$$

Here $\mathrm{ram}(e)$ *is the ramification index of* e*, it is the smallest postitive integer for which* $e \in \mathbb{C}[x^{-1/\mathrm{ram}(e)}]$.

Define an equivalence relation $\sim$ *on* E*:*

$$e_1 \sim e_2 \Longleftrightarrow e_1 - e_2 \in \frac{1}{\mathrm{ram}(e_1)}\mathbb{Z}$$

Then $e_1 \sim e_2 \Longleftrightarrow V_{e_1} = V_{e_2}$ *(working mod the ideal* I *in Theorem 3 (iii)). Then* $V = \bigoplus_{e \in E/\sim} V_e$ *is the universal extension in Theorem 3.*

Remark 4. V_e *is an indecomposable* $\mathbb{C}((x))[\partial]$*-module (is not a direct sum of smaller modules). For each nonzero* $L \in \mathbb{C}((x))[\partial]$*, there exist a basis* $y_1, \dots, y_n$ *of* $V(L)$ *such that each* y_i *is in* V_{e_i} *for some* $e_i \in E$*, where* e_i *are the generalized exponents (see Definition 12).*

Since V_e *is a* $\mathbb{C}((x))[\partial]$*-module, so it is also a* $\mathbb{C}(x)[\partial]$*-module. In particular, if* $G : V(L_1) \to V(L_2)$ *and* $G \in \mathbb{C}(x)[\partial]$ *and* G *is a bijection from* $V(L_1)$ *to* $V(L_2)$ *(i.e a gauge transformation, see Definition 18), then* L_1 *and* L_2 *have the same generalized exponents up to* $\sim$*.*

Remark 5. *If* $p \in \mathbb{C} \cup \{\infty\}$*, then one can do a similar construction for* $\mathbb{C}((t_p))$*, where* t_p *is the local parameter (see Definition 6) at* $x = p$*. So at each point* p*,* L *has* $\deg(L)$ *linearly independent solutions.*

Definition 12. *We say* $e \in \mathbb{C}[t_p^{-\frac{1}{m}}]$ *is a generalized exponent of* L *at* $x = p$ *if* L *has a solution* $\exp\left(\int \frac{e}{t_p}\, dt_p\right) S$*, with* $S \in R_m$*, and* $S \notin t_p^{\frac{1}{m}} R_m$*, where* $m \in \mathbb{N}$ *(*$m := \mathrm{ram}(e)$*)*

and $R_m = \mathbb{C}[[t_p^{\frac{1}{m}}]][\ln(t_p)]$. *If the solution involves a logarithm, we call it a logarithmic solution. If* $m = 1$, *then* e *is unramified. If* $m > 1$, *then* e *is ramified.*

Remark 6. *Since we only consider second order differential operators,* $\mathrm{ram}(e)$ *is either* 1 *or* 2.

If the order of L is n, then at every point p, counting with multiplicity, there are n generalized exponents $e_1, e_2, ..., e_n$, and the corresponding solutions $\exp\left(\int \frac{e_i}{t_p}\, dt_p\right) S_i$, $i = 1, ..., n$ form a basis of the solution space $V(L)$. If p is regular, then the generalized exponents of L at $x = p$ are $0, 1, ..., n-1$. One can compute generalized exponents with the Maple command DEtools[gen_exp].

2.3 Special functions

In this section, we study the definition and basic properties of Bessel functions. Our goal is to solve differential equations in terms of Bessel functions. Because our algorithm can also deal with Airy functions, Kummer/Wittaker functions, we introduce them as well. These special functions are generalized hypergeometric functions, ${}_0F_1$ and ${}_1F_1$.

2.3.1 Bessel Functions

Definition 13. *The solutions of*

$$L_{B1} := x^2\partial^2 + x\partial + (x^2 - \nu^2)$$

with the constant parameter $\nu \in \mathbb{C}$ *are called Bessel functions. Two linearly independent solutions*

$$J_\nu := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} (\frac{x}{2})^{2k+\nu}$$

and

$$Y_\nu := \frac{J_\nu(x)cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad \text{if } \nu \notin \mathbb{Z}$$

or

$$Y_n := \lim_{\nu \to n} Y_\nu, \quad \textit{if } n \in \mathbb{Z}$$

generate the solution space $V(L_{B1})$. J_ν and Y_ν are called first and second kind Bessel functions respectively.

Similarly solutions of $L_{B2} := x^2\partial^2 + x\partial - (x^2 + \nu^2)$ are called modified Bessel functions. First and second kind modified Bessel functions I_ν and K_ν are two linearly independent solutions which span $V(L_{B2})$. Here

$$I_\nu := (\frac{x}{2})^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} (\frac{x}{2})^{2k}$$

and

$$K_\nu := \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{2\sin(\nu\pi)}$$

Notation: B_ν refers to any element of $\{J_\nu, Y_\nu, I_\nu, K_\nu\}$. For example, the following lemma holds for all four elements:

Lemma 1. *The space*

$$S := \mathbb{C}(x)B_\nu + \mathbb{C}(x)B'_\nu$$

is invariant under the substitution $\nu \mapsto \nu + 1$.

Proof. See [14] Corollary 1.23 □

If $\nu = \frac{1}{2}$ then the modified Bessel operator $L := x^2\partial^2 + x\partial + x^2 - \frac{1}{4}$ can be factored:

$$L = (\partial + \frac{\sqrt{-1}(2x - \sqrt{-1})}{2x})(\partial - \frac{\sqrt{-1}(2x + \sqrt{-1})}{2x})$$

Combined with Lemma 1, this implies that L is reducible if ν is any half-integer (if $\nu \in \frac{1}{2} + \mathbb{Z}$). One can always get the solutions by factoring such operators. We will exclude this case from this thesis.

The change of variables $x \to x\sqrt{-1}$ sends $V(L_{B1})$ to $V(L_{B2})$ and vice versa. Since our algorithm will deal with change of variables, as well as two other transformations (see Section 3.1), we only need one of L_{B1}, L_{B2}. We choose L_{B2} and denote $L_B := L_{B2}$.

L_B has only two singularities, 0 and ∞. The generalized exponents are $\pm\nu$ at $x = 0$ and $\pm t_\infty^{-1} + \frac{1}{2}$ at ∞. Note that L_{B1} also has 0 and ∞ as singularities. The generalized exponents are $\pm\nu$ at $x = 0$ and $\pm t_\infty^{-1}\sqrt{-1} + \frac{1}{2}$ respectively. The generalized exponents of L_{B2} are simpler than L_{B1}, that is the reason our algorithm starts with $L_B := L_{B2}$.

After a change of variables $y(x) \to y(\sqrt{x})$, we get a new operator

$$L_B^{\surd} := x^2\partial^2 + x\partial - \frac{1}{4}(x + \nu^2)$$

Note it is still in $\mathbb{Q}(x)[\partial]$. This means for $f \in K$ the function $B_v(\sqrt{f})$ is a solution of an element of $K[\partial]$. So this introduces the square root case. Let $\mathrm{CV}(L, f)$ denote the operator obtained from L by change of variables $x \mapsto f$. For any differential field extension K of $\mathbb{Q}(x)$, if $\nu^2 \in C_K$, and if $f^2 \in K$, then $\mathrm{CV}(L_B, f) \in K[\partial]$ since this operator can be written as $\mathrm{CV}(L_B^{\surd}, f^2)$. The converse is also true:

Lemma 2. *Let K be a differential field extension of $\mathbb{Q}(x)$, let f, ν be elements of a differential field extension of K, and ν be constant. Then*

$$\mathrm{CV}(L_B, f) \in K[\partial] \Longleftrightarrow f^2 \in K \text{ and } \nu^2 \in C_K.$$

Proof. It remains to prove $\Longrightarrow$. Let ν be a constant. Let $\mathrm{monic}(L)$ denote L divided by the leading coefficient of L and

$$M := \mathrm{monic}(\mathrm{CV}(L_B, f)) = \partial^2 + a_1\partial + a_0$$

We have to prove

$$a_0, a_1 \in K \Longrightarrow f^2, \nu^2 \in K$$

and so we assume $a_0, a_1 \in K$. Let $g = f^2$. By computing $M = \mathrm{monic}(\mathrm{CV}(L_B^{\surd}, g))$ we find

$$a_1 = -\mathrm{ld}(\mathrm{ld}(g)), \quad a_0 = \frac{-1}{4}(g + \nu^2)\mathrm{ld}(g)^2$$

where ld denotes the logarithmic derivative, $\mathrm{ld}(a) = a'/a$. Let

$$a_2 := \mathrm{ld}(\mathrm{ld}(a_0) + 2a_1) + \mathrm{ld}(a_0) + 3a_1,$$

$$a_3 := -4a_0/a_2^2, \quad a_4 := a_3(2a_1 + \mathrm{ld}(a_0))$$

which are in K since $a_0, a_1 \in K$. Direct substitution shows that $a_2 = \mathrm{ld}(g)$, $a_3 = g + \nu^2$, and $a_4 = g'$. Hence $g = a_4/a_2 \in K$ and $\nu^2 = a_3 - g \in K$. □

Because of this lemma, when we solve differential equations in terms of Bessel functions, instead of using $B_\nu(f)$, with $f \in K$, we should use $B_\nu(f)$, with $f^2 \in K$.

2.3.2 Airy Functions

Definition 14. *The solutions of*

$$L_A := \partial^2 - x$$

are called Airy functions. Two linearly independent solutions

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + xt)dt$$

and

$$Bi(x) = \frac{1}{\pi}\int_0^\infty (e^{\frac{1}{3}t^3+xt} + \sin(\frac{1}{3}t^3 + xt))dt$$

are called first and second kind Airy functions.

Airy functions can be written in terms of Bessel functions, but only if we allow square roots:

Lemma 3.

$$Ai(x) := -\frac{\sqrt{x}}{3}I_{\frac{1}{3}}(\frac{2}{3}\sqrt{x^3}) + \frac{\sqrt{x}}{3}I_{-\frac{1}{3}}(\frac{2}{3}\sqrt{x^3})$$
$$Bi(x) := -\frac{\sqrt{x}}{3}K_{\frac{1}{3}}(\frac{2}{3}\sqrt{x^3}) + \frac{\sqrt{x}}{3}K_{-\frac{1}{3}}(\frac{2}{3}\sqrt{x^3})$$

Proof. See [3] □

Thus equations solvable in terms of Airy functions will be solved by our Bessel solver.

2.3.3 Hypergeometric Series

Definition 15. *A generalized hypergeometric series ${}_pF_q$ is defined by:*

$$ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k k!} x^k $$

Here $(\alpha)_k$ is defined as:

$$ (\alpha)_k := \begin{cases} 1 & \textit{if} \;\; k = 0 \\ \alpha(\alpha+1)\ldots(\alpha+k-1) & \textit{if} \;\; k > 0 \end{cases} $$

Many special functions are hypergeometric series, for example:

$$ e^x = {}_0F_0(;;x) $$

$$ \cos(x) = {}_0F_1(;\frac{1}{2};\frac{-x^2}{4}) $$

Bessel functions and Airy functions are also hypergeometric series, for example:

$$ J_v(x) = (\frac{x}{2})^{\nu} \frac{1}{\Gamma(\nu+1)} {}_0F_1(;\nu+1;-\frac{x^2}{4}) $$

and

$$ I_v(x) = (\frac{x}{2})^{\nu} \frac{1}{\Gamma(\nu+1)} {}_0F_1(;\nu+1;\frac{x^2}{4}) $$

where $\Gamma(x)$ is the Gamma function:

$$ \Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt $$

Definition 16. *The equation*

$$ x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0 $$

is called the hypergeometric differential equation. The solutions can be expressed by ${}_2F_1$ functions (also called Gauss' Hypergeometric functions)

Definition 17. *The equation*

$$xy'' + (\gamma - x)y' - \alpha y = 0$$

is called the Kummer equation. The solution

$$M(\alpha, \gamma, x) := {}_1F_1(\alpha; \gamma; x)$$

and

$$U(\alpha, \gamma, x) := x^{-\alpha} {}_2F_0(\alpha, 1 + \alpha - \gamma; ; -\frac{1}{x})$$

are called Kummer function and the second Kummer function, respectively.

The ${}_0F_1$ function can be expressed in terms of Kummer functions, with Kummer formula:

$$\exp(-\frac{x}{2}) {}_1F_1(\alpha; 2\alpha; x) = {}_0F_1(; \frac{1}{2} + \alpha; \frac{x^2}{16}).$$

If we have an irreducible second order differential operator L, then the possible hypergeometric series solutions will satisfy $p \leq 2$ and $q \leq 1$, which can be written in terms of ${}_2F_1$ functions, Bessel functions and Kummer functions. Using the method we developed in the thesis and the algorithm in [14], we can get all hypergeometric series solver except for the ${}_2F_1$ case. In that case, the differential operator has only regular singularities. We get more information from irregular singularities than regular singularities, see section 3.3.2.

For more detail and properties of hypergeometric series, see [3].

2.4 Maple Commands

All of my algorithms and code are developed with Maple. All of our examples are illustrated with Maple. So in this section we want to introduce some commands we need in Maple.

In Maple, the DEtools package contains commands that help us work with differential equations. To use this package, we input:

```
>with(DEtools):
```

For DEtools package, every command has a short version or long version. For the long version, you always need to tell maple the variable and the derivation. To use the short version, one needs to tell Maple the symbol for the variable x and the derivation D by using:

```
>_Envdiffopdomain:=[Dx,x]:
```

This command tells Maple that we use x as variable and Dx as derivation. In this thesis, we always assume that the DEtools package is loaded and the differential domain is defined by [Dx,x].

The command diffop2de gives the corresponding differential equation. We use Bessel operators as an example:

```
> L := x^2*Dx^2+x*Dx+x^2-nu^2;
```

$$L := x^2 Dx^2 + x Dx + x^2 - \nu^2$$

```
>eq := diffop2de(L, y(x))
```

$$eq := \left(x^2 - \nu^2\right) y\left(x\right) + x \frac{d}{dx} y\left(x\right) + x^2 \frac{d^2}{dx^2} y\left(x\right)$$

We can use de2diffop to convert the differential equations into differential operators, for example:

```
>de2diffop(eq, y(x));
```

$$x^2 Dx^2 + x Dx + x^2 - \nu^2$$

The command gen_exp will compute generalized exponents of the differential operators. Let us compute the generalized exponents for modified Bessel operators at $x = 0$.

```
>L := x^2*Dx^2+x*Dx-x^2-nu^2;
```

```
>gen_exp(L, T, x = 0);
```

$$[[\nu, T = x], [-\nu, T = x]]$$

So the generalized exponents are $\pm\nu$ at $x = 0$. The local parameter (denoted by T) is simply x here. At ∞, we can compute the generalized exponents and find $\pm t_\infty^{-1} + \frac{1}{2}$.

```
>gen_exp(L, T, x = infinity);
```

$$[[-T^{-1} + 1/2, T = x^{-1}], [T^{-1} + 1/2, T = x^{-1}]]$$

And we can also verify, at any regular point (for example $x = 1$), the generalized exponent are 0,1.

```
>gen_exp(L, T, x = 1);
```

$$[[0, 1, T = x - 1]]$$

If the parameter $\nu = \frac{1}{2}$ then the operator is reducible, we can use DFactor to factor it:

```
>L := subs(nu = 1/2, L);
```

$$L := x^2 Dx^2 + x Dx - x^2 - 1/4$$

```
>DFactor(L);
```

$$[x^2 Dx + 1/2\, x\, (2\, x + 1)\, , Dx - 1/2\, \frac{2\, x - 1}{x}]$$

Maple can also solve it with dsolve command:

```
>dsolve(diffop2de(L, y(x)), y(x));
```

$$y\,(x) = \frac{\_C1\ \sinh\,(x)}{\sqrt{x}} + \frac{\_C2\ \cosh\,(x)}{\sqrt{x}}$$

Maple can also compute the formal solutions and can detect if there are logarithmic solutions. For example, if $\nu = 0$ then Bessel operator has logarithmic solutions.

```
>L := x^2*Dx^2+x*Dx-x^2-nu^2;
```

```
>L := subs(nu = 0, L);
```

$$L := x^2Dx^2 + xDx - x^2$$

```
>formal_sol(L, t, x = 0);
```

$$[[\ln t + \left(-\frac{1}{4} + \frac{1}{4}\ln t\right)t^2 + \left(\frac{1}{64}\ln t - \frac{3}{128}\right)t^4 + O(t^6), 1 + \frac{1}{4}t^2 + \frac{1}{64}t^4 + O(t^6), t = x]]$$

```
>formal_sol(L, `has logarithm?`, x = 0);
```

$$true$$

In the following section, we can see that if the input has logarithmic solution, we need to treat it separately.

CHAPTER 3

TRANSFORMATION AND LOCAL INFORMATION

In this chapter, we will discuss the transformations that preserve order 2 differential operators. Then we can clarify the definition of solving differential equations in terms of solutions of another equation (in this thesis, the Bessel equation). We will also discuss the invariants under the transformations, which we use to reconstruct the transformation from the Bessel operator to the operator we want to solve. The details in this chapter will be the key to our algorithm for solving differential operators in terms of Bessel functions. In this section, we also assume the order of the differential operators is 2, unless stated otherwise.

3.1 Transformation

Definition 18. *A transformation between two differential operators L_1 and L_2 is an onto map from the solution space $V(L_1)$ to $V(L_2)$. For an order 2 operator $L_1 \in K[\partial]$, there are three types of transformations that preserve K and preserve order 2. They are (notation as in [14, 15]):*

(i) change of variables: $y(x) \mapsto y(f(x))$, $\qquad f(x) \in K \setminus C_K$.

(ii) exp-product: $y \mapsto \exp(\int r\, dx) \cdot y$, $\qquad r \in K$.

(iii) gauge transformation: $y \mapsto r_0 y + r_1 y'$, $\qquad r_0, r_1 \in K$.

We denote them by $\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ respectively. We also denote $\longrightarrow_{CEG}$ be any combinations of $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$.

Lemma 4. *$\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ will send a second order irreducible $L_1 \in K[\partial]$ to an L_2 that is again in $K[\partial]$ of order 2.*

Proof. [14, 15] $\square$

With this definition, we can state what we mean by solving equations in terms of Bessel functions.

Definition 19. *Assume y is a solution of a differential operator L_O, we say we can solve differential operator L in terms of y when we can find a transformation $L_O \longrightarrow_{CEG} L$.*

To solve differential equations in terms of the Bessel functions means to find a transformation from the Bessel operator L_B to the operator L. Since we only focus on order 2 differential operators, we only need to find combinations of $\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ which send L_B to L.

Remark 7. *We can consider $\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ as binary relations on $K[\partial]$. Let $L_1, L_2 \in K[\partial]$. Then L_1 is $\longrightarrow_C$ ($\longrightarrow_E$, $\longrightarrow_G$ resp.) related to L_2, if and only if $L_1 \longrightarrow_C L_2$ ($\longrightarrow_E$, $\longrightarrow_G$ resp).*

So it is natural to ask, whether $\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ are equivalence relations and whether the order of the transformations is important.

Lemma 5. *$\longrightarrow_G$ and $\longrightarrow_E$ are equivalence relations, but $\longrightarrow_C$ is not.*

Proof. [14, 15]. See also Remark 4. $\square$

The order of the $\longrightarrow_E$ and $\longrightarrow_G$ is not important, because:

Lemma 6. *If $L_1, L_2, L_3 \in K[\partial]$ such that $L_1 \longrightarrow_G L_2 \longrightarrow_E L_3$, then there exist $M \in K[\partial]$ such that $L_1 \longrightarrow_E M \longrightarrow_G L_3$. Similarly, if $L_1 \longrightarrow_E L_2 \longrightarrow_G L_3$, then $\exists M \in K[\partial]$, such that $L_1 \longrightarrow_G M \longrightarrow_E L_3$*

Proof. [14, 15]. □

According to the lemma, the order of $\longrightarrow_E$ and $\longrightarrow_G$ can be switched, we denote the combinations of $\longrightarrow_E$ and $\longrightarrow_G$ as $\longrightarrow_{EG}$. Since both $\longrightarrow_E$ and $\longrightarrow_G$ are equivalence relation, so is $\longrightarrow_{EG}$. We also define:

Definition 20. *We say $L_1 \in K[\partial]$ is gauge equivalent to L_2 if and only if $L_1 \longrightarrow_G L_2$. And $L_1 \in K[\partial]$ is projectively equivalent to L_2 if and only if $L_1 \longrightarrow_{EG} L_2$.*

If two operators are projectively equivalent, there are several algorithms to compute the map of projective equivalence. The details of the algorithm can be found in [4], [14] and Section 3.2. For $\longrightarrow_C$, we have:

Theorem 4. *([14, 15]) If $L_1 \longrightarrow_{CEG} L_2$, then there exist an operator $M \in K[\partial]$ such that $L_1 \longrightarrow_C M \longrightarrow_{EG} L_2$.*

If $L_B \longrightarrow_{CEG} L$ then L has solution of the form:

$$e^{\int r dx}(r_0 B_\nu(f) + r_1(B_\nu(f))'),$$

here $B_\nu(x)$ is one of the Bessel functions, and $r, r_0, r_1, f \in K$.

For Bessel functions, in Section 2.3.1, we learned that $x \mapsto \sqrt{x}$ sends L_B to an element of $K[\partial]$ namely $L_B^{\surd}$. So instead of starting from L_B, we can apply Theorem 4 to $L_1 = L_B^{\surd}$ and L_2, which is the operator we want to solve. But we still want to use the notations and results in [14]. That means we start with L_B, but we extend change of variables to:

(i)' Change of Variables: $y(x) \to y(f(x))$, $\quad f^2(x) \in K \setminus C_K$.

Also we can extend the definition of solving differential equations in terms of Bessel functions to:

Definition 21. *We say we can solve L in terms of Bessel functions, when $L_B^{\surd} \longrightarrow_{CEG} L$, i.e the solutions of L, can be written as the following form:*

$$e^{\int r dx}(r_0 B_\nu(f) + r_1(B_\nu(f))') \tag{3.1}$$

here $B_\nu(x)$ is one of the Bessel functions, and $r, r_0, r_1, f^2 \in K$.

Remark 8. *Let $K = C_K(x)$ and $L \in K[\partial]$. In section 4.10, we will show that if there exists a solution of the form (3.1) with $r, r_0, r_1, f^2 \in \overline{K}$ then there is also a solution in the form (3.1) with $r, r_0, r_1, f^2 \in K$.*

To summarize, our goal is to solve differential equations in terms of Bessel functions. This means: if $L_B^{\surd} \longrightarrow_{CEG} L$, then find the transformation, hence solve L. So our Main problem is:

Main problem: Let C_K be a field, $C_K \subseteq \mathbb{C}$, and let $K = C_K(x)$. Let $L \in K[\partial]$ be irreducible and of order 2. The question we will solve in this thesis is the following: Does there exist an operator $M \in K[\partial]$ such that

1. L is projectively equivalent to M, and
2. $L_B^{\surd} \xrightarrow{g}_C M$ for some $g \in K$ and some constant ν.

If so, find g, ν and solve L.

Note that $L_B^{\surd} \xrightarrow{g}_C M$ is the same as $L_B \xrightarrow{f}_C M$, where $f^2 = g \in K$. The reason we also use the second form is because we can then use the same notation as in [14] [15].

There are two steps to find Bessel type solutions of L. The first step is to find the middle operator M (i.e the change of variables f). If M (or equivalently f) is known, then the next step is to find the map from M to L, which is a projective equivalence. We will discuss the second the step first, because there are algorithms to find projective equivalence.

3.2 Projective Equivalence

In this section, we will describe an algorithm to find the map of the projective equivalence. There are several algorithms for this [4, 35]. For example the algorithm by Barkatou and Pflügel which is implemented in the ISOLDE package [4]. The idea to find the equivalence maps is to reduce it to a differential equations system.

Theorem 5. *The question whether two operators $L_1, L_2 \in K[\partial]$ are projectively (gauge resp.) equivalent can be reduced to finding hyperexponential (rational resp.) solutions of a system of differential equations.*

Proof. If $L_1 \longrightarrow_{EG} L_2$, then there exist an operator $G = \exp(\int r)(r_1\partial + r_0)$, such that $G(V(L_1)) = V(L_2)$. So L_1 will be a right factor of L_2G. Write $G = s_1\partial + s_0$. By computing the remainder of L_2G divided by L_1, which should equal zero, we can get two differential equations of order two in two unknowns s_0, s_1. By introducing two new variables $\overline{s_0} = s_o', \overline{s_1} = s_1'$, we get a linear system of four order one equations. We can write it in the standard way as $Y' - AY = 0$. The details of the proof can be find in [14].

□

These hyperexponential (rational resp.) solutions can be found with the cyclic vector method, or by a direct method (implemented in ISOLDE).

Definition 22. *Let M be a n-dimensional vector space and $\partial : M \to M$. A vector $v \in M$ is called a cyclic vector of M if*

$$\{v, \partial v, \ldots, \partial^{n-1}v\}$$

is a basis of M.

Definition 23. *Let*

$$L = \sum_{i=0}^{n} a_i\partial^n \in K[\partial]$$

be a differential operator. We define the adjoint operator

$$L^* := (-1)^n \sum_{i=0}^{n} (-\partial)^i a_i$$

The adjoint operator satisfies $L^{**} = L$ and $(L_1L_2)^* = L_2^*L_1^*$.
The cyclic vector method works as follows.

Theorem 6. *Let A be the matrix of the equation $Y' - AY = 0$ and let $\partial y := y' - Ay$. The we can compute a hyperexponential (rational resp.) solution by the following steps:*

(i) Pick a random element $v \in K^4$.

(ii) Check whether v is cyclic, otherwise go back to step (i).

(iii) Compute $L = a_0 + a_1\partial + a_2\partial^2 + a_3\partial^3 + \partial^4$ such that $Lv = 0$.

(iv) Compute a hyperexponential (rational resp.) solution s of L^*, see [13].

(v) Compute R such that $L = (\partial + \frac{s}{s'})(\frac{1}{s}R)$.

(vi) Let $R = y_0 + y_1\partial + y_2\partial^2 + y_3\partial^3$ and $y = y_0v + y_1\partial v + y_2\partial^2 v + y_3\partial^3 v$.

Then y is a hyperexponential (rational resp.) solution of $Y' - AY = 0$.

Proof. The proof can be found in [14] □

Here we also want to mention the p-curvature test for projective equivalence. It is the idea to test whether two operators can be equivalent mod p.
Let $\overline{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. We define the ring $\overline{\mathbb{F}}_p(x)[\partial]$ in the same way as before. The main difference is that the field of constants is $\overline{\mathbb{F}}_p(x^p)$. The most important tool for operators in characteristic p is the p-curvature. We briefly introduce the idea of the p-curvature test here, for more details, see [13, 33].

The differential field $\overline{\mathbb{F}}_p(x)$ is a finite dimensional vector space over its field of constants $\overline{\mathbb{F}}_p(x^p)$:

$$\overline{\mathbb{F}}_p(x) = \bigoplus_{i=0}^{p-1} \overline{\mathbb{F}}_p(x^p)x^i.$$

To any differential operator $L \in \overline{\mathbb{F}}_p(x)[\partial]$, one can associate the differential module:

$$\mathcal{M}_L := \overline{\mathbb{F}}_p(x)[\partial]/\overline{\mathbb{F}}_p(x)[\partial]L$$

Multiplication by ∂ gives a map $\overline{\partial}$ satisfying:

$$\overline{\partial} fm = f'm + f\overline{\partial}m, \ \forall m \in \mathcal{M}_L, \ f \in \overline{\mathbb{F}}_p(x)$$

Definition 24. *Let $L \in \overline{\mathbb{F}}_p(x)[\partial]$. The p-curvature of L is the $\overline{\mathbb{F}}_p(x)$-linear map $\overline{\partial}^p$ acting on the differential module $\mathcal{M}_L$ associated to L.*

An algorithm to compute the p-curvature matrix is given in [33]. Once we get the matrix, it is natural to consider the characteristic polynomial. We denote the characteristic polynomial of the p-curvature of $L \in \overline{\mathbb{F}}_p(x)[\partial]$ as $\chi_p(L)$.

Lemma 7. *Let $L \in \overline{\mathbb{F}}_p(x)[\partial]$ and if $L = L_1L_2$ then $\chi_p(L) = \chi_p(L_1)\chi_p(L_2)$ with* $\deg(\chi_p(L_i)) = \deg(L_i)$.

Proof. See [33]. □

Theorem 7. *Let $K \subseteq \overline{\mathbb{Q}}(x)$. Let $L_1, L_2 \in K[\partial]$ with order n and $L_1 \longrightarrow_G L_2$, then $\chi_p(L_1) = \chi_p(L_2)$ for any prime p.*

Proof. Since $L_1 \longrightarrow_G L_2$, there exist order $< n$ operators R_1, R_2 such that $L_1R_1 = R_2L_2$, so R_1 and R_2 are also gauge equivalent. So $\chi_p(R_1) = \chi_p(R_2)$ by induction. By Lemma 7 we have $\chi_p(L_1)\chi_p(R_1) = \chi_p(L_2)\chi_p(R_2)$. So $\chi_p(L_1) = \chi_p(L_2)$. □

The results can also be extended to algebraic extensions of $\overline{\mathbb{F}}_p(x)$. It is because:

Remark 9. *Let $K_p = \overline{\mathbb{F}}_p(x)[y]/(F(x,y))$ be an algebraic extension of $\overline{\mathbb{F}}(x)$ of degree n, where $n < p$. Let $C_p = \{a \in K_p | a' = 0\}$. Then $[K_p : C_p] = p$ and*

$K_p = C_p + C_p \cdot x + \ldots + C_p x^{p-1}$. So we can use the p-curvature for K_p in the same way in Theorem 7.

Proof. Consider the following diagram:

$$
\begin{tikzcd}
C_p \arrow[r, dash, "\neq"] \arrow[d, dash, "\cup\!|"'] & K_p \arrow[dl, dash, "a"'] \arrow[dr, dash, "n"] & \\
\overline{\mathbb{F}}_p(x^p, y^p) \arrow[dr, dash, "b"'] & & \overline{\mathbb{F}}_p(x) \arrow[dl, dash, "p"] \\
& \overline{\mathbb{F}}_p(x^p) &
\end{tikzcd}
$$

where a, b, n, p are the indices of the algebraic extensions.
Since the extension of $[K_p : \overline{\mathbb{F}}_p(x^p, y^p)]$ is purely inseparable (note that $(K_p)^p = \overline{\mathbb{F}}_p(x^p, y^p)$), so a is a p-power. From the diagram, we have $a \cdot b = n \cdot p$.
We assumed $p > n$ and p is a prime number. Then p divides $n \cdot p = a \cdot b$ only once. So $a = p$ or 1. But $a \neq 1$, because $x \notin C_p$. So $a = p$. Then $b = n$ and $C_p = \overline{\mathbb{F}}_p(x^p, y^p)$. It follows that $K_p = C_p + C_p \cdot x + \ldots + C_p x^{p-1}$. □

Remark 10. *$\longrightarrow_E$ changes $\chi_p(L)$ but not its discriminant.*

Assume we have candidates for f and ν, then we compute $L_B \xrightarrow{f}_C M$. We can pick some prime number (in our implementation, we choose 3 and 5), and compute $\chi_p(L)$ and $\chi_p(M)$. If their discriminants do not match, then according to Theorem 7 and Remark 10, L and M can not be projectively equivalent. We call this process the p-curvature test, it will increase the speed of our algorithm, because it quickly eliminate most of candidates for (f, ν)
We summarize this section in Algorithm 1 for projective equivalence and Algorithm 2 for gauge equivalence:

Input : two differential operator L_1 and L_2
Output: an first order operator $G \in K[\partial]$ and $r \in K$, such that $\exp(\int r)G(V(L_1)) = V(L_2)$, if L_1 and L_2 are projectively equivalent

for some prime number p, compute $\chi_p(L_1)$ and $\chi_p(L_2)$;
if *discriminant*($\chi_p(L_1)$)$\neq$ *discriminant*($\chi_p(L_2)$) **then**
  output 0;
else
  Use methods introduced in Theorem 6 to find **exponential** solutions of the equation in Theorem 5;
  output the solution;
end

Algorithm 1: Projective Equivalence

Input : two differential operator L_1 and L_2
Output: an first order operator $G \in K[\partial]$, such that $G(V(L_1)) = V(L_2)$, if L_1 and L_2 are gauge equivalent

for some prime number p, compute $\chi_p(L_1)$ and $\chi_p(L_2)$;
if $\chi_p(L_1) \neq \chi_p(L_2)$ **then**
  output 0;
else
  Use methods introduced in Theorem 6 to find **rational** solutions of the equation in Theorem 5;
  output the solution;
end

Algorithm 2: Gauge Equivalence

3.3 Local Information: The Exponent Difference

3.3.1 Exponent Difference

According to Section 3.1 and Section 3.2, if we want to solve the differential operator L, we need to recover the change of variable $\xrightarrow{f}_C$. If we have f then we can compute the middle operator M by computing $L_B \xrightarrow{f}_C M$. If M and L are projectively equivalent, then we can compute the transformation from L_B to L, hence solve L in terms of Bessel functions. If M and L are not projectively equivalent for all candidates (f, ν), then we can conclude that L can not be solved in terms of

Bessel functions. So the remaining problem is how to reconstruct f. Notice that we only have L and not M. However, M should be projectively equivalent to L. So for computing f, the only information retrieved from L that we can use is information invariant under projective equivalence. The invariant we use is the difference of the generalized exponents of L.

Definition 25. *If L is an order 2 differential operator, then at each singularity p, there are two generalized exponents e_1 and e_2 of L at $x = p$. The difference $e_1 - e_2$ is called the* exponent difference *of L at $x = p$. We denote it by $\Delta(L, p) = \pm(e_1 - e_2)$.*

Remark 11. *Note that our definition contains $\pm$, because the generalized exponents are not ordered.*

This following lemma shows that $\Delta(L, p)$ is invariant under projective equivalence.

Lemma 8. *The exponent difference $\Delta(L, p)$ modulo $\frac{1}{m}\mathbb{Z}$ is invariant under projective equivalence, where m is the ramification index. Here $m = 1$ if the generalized exponents are unramified, and $m = 2$ if they are ramified.*

Proof. [14] gives the proof of the unramified case. We can just repeat the process to proove the ramified case just replacing integers with half integers. See also Remark 4 $\square$

Definition 26. *A singularity p of $L \in K[\partial]$ is:*

(i) removable singularity *if and only if $\Delta(L, p) \in \mathbb{Z}$ and L is not logarithmic at $x = p$. This is equivalent to saying that L has a basis of solutions y_1, y_2 for which y_1/y_2 is analytic at $x = p$.*

(ii) non-removable regular singularity *if and only if $\Delta(L, p) \in \mathbb{C} \setminus \mathbb{Z}$ or L is logarithmic at $x = p$.*

(iii) irregular singularity *if and only if $\Delta(L, p) \in \mathbb{C}[t_p^{-\frac{1}{2}}] \setminus \mathbb{C}$.*

We also denote the set of non-removable regular singularities resp. irregular singularities by S_{reg} resp. S_{irr}.

Proof. The definition is compatible with Definition 7. See [14]. $\square$

Note: removable singularities are neither in S_{reg} nor S_{irr}.

Definition 27. *Let $g = f^2 \in K$. If g has a root resp. pole at $x = p$ of order $k \in \mathbb{N}$, then we say that f has a root resp. pole at p of order $m_p := \frac{k}{2}$.*

3.3.2 An Example of Change of Variables

To illustrate the relations between local information, exponent difference, change of variables and Bessel Parameter, we start with an example. Let $L_B = \partial^2 + \frac{1}{x}\partial - \frac{x^2+\nu^2}{x^2}$. Take for example $\nu = \frac{1}{3}$ and $f = \frac{x}{(x-2)^2}\sqrt{\frac{(x-1)^3}{x+2}}$, and $L_B \xrightarrow{f}_C M$. Then

Example 2.

$$M = \partial^2 + \frac{(6x^5 + 39x^4 - 60x^3 - 44x^2 + 64x - 32)}{x(x-1)(x-2)(x+2)(3x^2+14x-8)}\partial - \frac{1}{36}\frac{(10x^5 - 33x^4 + 35x^3 + 7x^2 - 48x + 32)(3x^2+14x-8)^2}{x^2(x-1)^2(x+2)^3(x-2)^6}$$

We will illustrate how the singularities of M are related to f.

First, we can expect 0, 1 are singularities because f maps them to a singularity of L_B, namely 0. And we can also expect ± 2 to be singularities, because f maps them to ∞. But we also notice that by applying the change of variables, we introduce new singularities, the roots of $3x^2 + 14x - 8$. If we compute the generalized exponent at those points,

```
>gen_exp(M, T, x = RootOf(3*x^2+14*x-8));
```

$$[[0, 2, T = x - RootOf(3\_Z^2 + 14\_Z - 8)]]$$

So they are removable singularity. Removable singularities do not give us useful information because they can disappear under $\longrightarrow_{EG}$. If we choose a random

gauge transformation $\longrightarrow_G$, then it is likely that these singularities (the roots of $3x^2 + 14x - 8$) disappear, and that new removable singularities will appear.
We also consider ∞ as possible singularity. The computation gives us:

```
>gen_exp(M, T, x = infinity);
```

$$[[0, 1, T = \frac{1}{x}]]$$

Also L is not logarithmic at $x = \infty$:

```
>formal_sol(L, 'has logarithm?', x = infinity);
```

$$false$$

So ∞ is a regular point.
Since f maps 0 and 1 to 0 (0 is the only regular singularity of L_B, so we expect 0 and 1 to be regular singularities of M too.

```
>gen_exp(M, T, x = 0);
```

$$[[\frac{1}{3}, T = x], [-\frac{1}{3}, T = x]]$$

The exponent difference is $\pm\frac{2}{3}$ which is the 2 times of the Bessel parameter ν and note the multiplicity of f at $x = 0$ is 1. At 1:

```
>gen_exp(M, T, x = 1);
```

$$[[-\frac{1}{2}, T = x - 1], [\frac{1}{2}, T = x - 1]]$$

The exponent difference is ± 1 which is equal 2ν times the multiplicity $\frac{3}{2}$ of f at $x = 1$. But since the exponent difference is an integer, it is a removable singularity. The local information at $x = 1$ is not useful after $\longrightarrow_{EG}$, because after $\longrightarrow_{EG}$ this point $x = 1$ can become a regular point. For example if we apply the exp-product $\exp(\int \frac{1}{2(x-1)}dx)$ to M and $M \overset{\frac{1}{2(x-1)}}{\longrightarrow}_E M_2$, then

$$\begin{aligned}M_2 =& \partial^2 + \frac{3x^4+28x^3-12x^2+32}{x(x-2)(x+2)(3x^2+14x-8)}Dx \\ &- \frac{1}{36(x-2)^6x^2(x+2)^3(3x^2+14x-8)}(27x^{11}+216x^{10} \\ &- 1701x^9+5607x^8+21474x^7-40852x^6-41528x^5 \\ &+ 167584x^4-132992x^3-92672x^2+151552x-16384)\end{aligned}$$

But now 1 is not a singularity of M_2. So if we want to find solutions of M_2, we can not get information at $x = 1$. But $x = 1$ is a root of f. This example also shows that a root of f can be a removable singularity, which can disappear under $\longrightarrow_{EG}$. This is the case we refer to as a 'disappearing' singularity in Section 3.3.4. Because of those 'disappearing' singularities, it is hard to reconstruct f because not all roots of f and be read from the singularities. We will discuss it in Section 4.5.

Now we will analyze the irregular singularities. L_B has only one irregular singularity $x = \infty$ and f maps ± 2 to ∞. So we expect ± 2 to be irregular singularities of M. At $x = 2$:

```
>gen_exp(M, T, x = 2);
```

$$[[-\frac{2}{T^2}-\frac{15}{8T}+1, T=x-2], [\frac{2}{T^2}+\frac{15}{8T}+1, T=x-2]]$$

Note the exponent difference is $\pm(\frac{4}{t_2^2}+\frac{15}{4t_2})$. According to Definition 26, it is an irregular singularity. The exponent difference is in the form of a series in t_p, we can write f as a series in t_p to see the relation. We use Maple series command to find the Laurent series of f at $x = 2$:

```
>series(sqrt(x^2*(x-1)^3/((x-2)^4*(x+2))), x = 2,1)
```

$$\frac{1}{(x-2)^2}+\frac{15}{8(x-2)}+O((x-2)^0)$$

To see the relation (See Theorem 8 for exact formula), for t_2^{-2}, the coefficient of exponent difference is 2 times of the product of the degree -2 and the coefficient of Laurent series of f at $x = 2$. For t_2^{-1}, a similar relation also holds. So in essence, the generalized exponent difference gives us the polar parts of Laurent series of f at $x = 2$.

At $x = -2$, the situation is more complicated. The Maple computation gives us:

```
>gen_exp(M, T, x = -2);
```

$$g := [[\frac{1}{T} + \frac{1}{4}, -\frac{27}{256}T^2 = x + 2]]$$

So the two generalized exponents are $\pm\frac{16i}{3\sqrt{3}t_{-2}^{1/2}} + \frac{1}{4}$, and the exponent difference at $x = -2$ is $\pm\frac{32i}{3\sqrt{3}t_{-2}^{1/2}}$. Now we have a series with half integer order. It still satisfies the formula from Theorem 8. But to avoid a half integer pole, we will consider the Laurent series of $g = f^2 \in K$, which is a rational function. We can find the match in this way. First, the exponent difference is $\pm\frac{2}{T}$, we square that we get $\frac{4}{T^2} = -\frac{27}{64t_{-2}}$, which should match the Laurent series of g after applying Theorem 8 (ii) in Section 3.3.4.

```
>series(x^2*(x-1)^3/((x-2)^4*(x+2)), x = -2,1)
```

$$-\frac{27}{64}\frac{1}{(x+2)} + O((x+2)^0)$$

Here we get a match of the polar part. In this example, we get all of the polar part of f at $x = -2$, this is only because the polar part of g has only one term. In general, the generalized exponents give half (round up) of the terms of the polar part of g. The details will be explained later in Remark 12

So to summarize, by analyzing the exponent differences at each regular and irregular singularities, we can get 0 or more zeros of f, half (round up) of the polar part of $g = f^2$ and some information about Bessel parameter ν. We will use this information to reconstruct the change of variable f. This reconstruction is not unique; we will find a finite set of candidates for (f, ν)

3.3.3 An Analogy

An analogy between solving polynomials in terms of radical and solving differential equations in terms of Bessel is helpful to understand the nature of our problem.

1. Solving $x^6 - 12$ in terms of radicals (give one solution) is trivial. $x = \sqrt[6]{12}$ is by definition a solution. In analogy, if we solve differential equations in terms of Bessel allowing only $\longrightarrow_C$, then it is easy. See example 2.

2. Solving $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x - 11$ is not obvious but it is also trivial, because the coefficient of x^5 suggests $x \mapsto x + 1$, that leads to $x^6 - 12$. So we have a solution $1 + \sqrt[6]{12}$. In analogy, if we solve using only $\longrightarrow_C$ and $\longrightarrow_E$, that is a similar (i.e easy) problem.

3. $x^6 - 24x^3 - 108x^2 - 72x + 132$ also has a solution in $\mathbb{Q}(\sqrt[6]{12})$, but it is less obvious than case 1 and 2. Note: once we know it has a solution in $\mathbb{Q}(\sqrt[6]{12})$, then we are almost done[1]. Likewise, we solve differential equations under $\longrightarrow_{CEG}$. If we know $\longrightarrow_C$, then we are almost done.

3.3.4 Change of Variables and exponent difference

By studying the example, we have an intuitive idea how $\Delta(L, p)$ and f are related. In this section, we will give the precise link between $\Delta(L, p)$ with f and in next section, the relation with ν.

Lemma 9. *If $f^2 \in K$, and $f = \sum_i a_i t_p^i$, where $i \in \frac{1}{2}\mathbb{Z}$ and t_p are local parameter at $x = p$, then the set $\{i | a_i \neq 0\}$ is either a subset of $\mathbb{Z}$ or a subset of $\frac{1}{2} + \mathbb{Z}$.*

[1]there exists a program to compute roots over a number field

Proof. Assume the initial term is $a_i t_p^i$ and then check the coefficient of each terms of f^2 and use $f^2 \in K$. $\square$

Theorem 8. *Let $K = C_K(x)$, and $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$, where $f^2 \in K$. (Note: L is the input to our algorithm, and f and M are to be computed.)*

(i) if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an removable singularity or $p \in S_{reg}$, and $\Delta(M,p) = 2m_p\nu$.

(ii) p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(M,p) = 2\sum_{i<0} i f_i t_p^i$.

If $p \in S_{reg}$, then $\Delta(L,p) \equiv \Delta(M,p)$ mod $\mathbb{Z}$ which means that we can compute $2m_p\nu$ mod $\mathbb{Z}$.
If $p \in S_{irr}$, then $\Delta(L,p) \equiv \Delta(M,p)$ mod $\frac{1}{m}\mathbb{Z}$, where $m = 1$ or 2 is the ramification index. Then $\sum_{i<0} f_i t_p^i$ can be computed from $\Delta(L,p)$ by dividing coefficients by $2i$ (the congruence only affects the t_p^0-term of Δ, but that term does not depend on f when $p \in S_{irr}$).

Proof. We can use the same proof in [14], we just need to replace integers with half integers and combine it with Lemma 9. $\square$

Definition 28. *Let $f = \Sigma_{i=N}^{\infty} a_i x^i$, $N \in \mathbb{Z}$, $a_N \neq 0$. We say that we have a k-term truncated power series for f when the coefficient of $x^N, ..., x^{N+k-1}$ are known.*

Remark 12. *If a k-term truncated series for f is known, then we can compute a k-term truncated series for f^2.*

According to Theorem 8 (ii), from $\Delta(M,p)$, we can get a $\lceil m_p \rceil$-term truncated series of f at $x = p$. In [15], f was assumed to be in K, in which case the truncated series is exactly the polar part of f at $x = p$. But in this thesis, we have to compute $g = f^2 \in K$. Theorem 8 (ii) gives us the polar part of f, i.e. a truncated series for

f. We square it to obtain a truncated series of g. But this truncated series for g has $\lceil m_p \rceil$ terms (the same number of terms as the one for f, see Remark 12). So it is only half (rounded up) of the polar part of g. For instance, if f has a pole of order 3 at $x = 0$, then from $\Delta(L, p)$ we can obtain a truncated series $\Sigma_{i=-3}^{-1} a_i x^i$ of f at $x = 0$. Squaring this series, we can get the coefficients of x^{-6}, x^{-5}, x^{-4} of g, but not more. So we have:

Corollary 1. *If $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$ and $g = f^2$ then:*

(i) if $p \in S_{reg}$ then p is a zero of g.

(ii) $p \in S_{irr}$ if and only if p is a pole of g. We can also get a $\lceil m_p \rceil$-term truncated series of g from $\Delta(L, p)$, where m_p is the pole order of f.

Remark 13. *For ∞, we can apply $x \mapsto \frac{1}{x}$ and then study the point $x = 0$, or just treat it like other points. If $g = f^2 = \frac{A}{B}$ and A, B are polynomials with no common factors and B is monic, then:*

(i) if $\infty \in S_{irr}$ this means that $\deg(A) > \deg(B)$, *and the truncated series, which we get from the exponent difference with the same method that as for the other points, gives half of the terms of the quotient of A divided by B.*

(ii) if $\infty \in S_{reg}$ this means that $\deg(A) < \deg(B)$, *and if the multiplicity of f at $x = \infty$ is m_∞, then the multiplicity of g at $x = \infty$ is $2m_\infty$, which means that* $\deg(B) - \deg(A) = 2m_\infty$.

(iii) if ∞ is a removable singularity then $\deg(A) \leq \deg(B)$.

(iv) if $\infty \notin S_{reg} \cup S_{irr}$, then it is either an removable singularity or a 'disappearing' singularity (see section 3.3.2 and section 4.5).

The goal in this section is to collect local information for f from exponent differences. If we assume $f^2 = g = \frac{A}{B}$ and A, B are polynomials with no common factors and B is monic, Corollary 1 gives the poles of g as well as pole order, hence we get B. Remark 13 tells us we know the degree of A or at least a bound for the degree of A. To be precise:

Lemma 10. *We can retrieve B from S_{irr}.*

Proof. According to Theorem 8 (ii), if $p \in S_{irr}$ then p is a pole of f. Let $m_p \in \frac{1}{2}\mathbb{Z}^+$ be the pole order of $\Delta(M,p)$. g has a pole order $2m_p$. Theorem 8 implies $B = \prod_{p\in S_{irr}\setminus\{\infty\}}(x-p)^{2m_p}$. □

Lemma 11. *Let*

$$d_A = \begin{cases} \deg(B) + 2m_\infty & \text{if } \infty \in S_{irr} \\ \deg(B) & \text{otherwise} \end{cases}$$

(i) If $\infty \in S_{reg}$ then $\deg(A) < d_A$*;*

(ii) if $\infty \in S_{irr}$ then $\deg(A) = d_A$*;*

(iii) otherwise $\deg(A) \le d_A$*.*

Proof. According to Corollary 1 (i), if $\infty \in S_{reg}$ then $\deg(A) < \deg(B)$. If $\infty \in S_{irr}$ with pole order m_∞, then $\deg(A) = \deg(B) + 2m_\infty$ (see Corollary 1 (ii)). If $\infty \notin S_{irr}$ then f does not have a pole at $x = \infty$, so that $\deg(A) \le \deg(B)$. □

Corollary 1, Lemma 10 and Lemma 11 will be used to reconstruct f in Chapter 4.

3.4 The Parameter ν

The exponent difference is also associated with the Bessel parameter ν. From the example, we can see it is related to S_{reg}. We have:

Theorem 9. *If $L_B^{\checkmark} \longrightarrow_{CEG} L$, then*

(i) if $S_{reg} = \emptyset$ then $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

The following holds for any $p \in S_{reg}$:

(ii) L logarithmic at $x = p$ if and only if $\nu \in \mathbb{Z}$.

(iii) *if* $\Delta(L,p) \in \mathbb{Q}$ *then* $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

(iv) $\Delta(L,p) \in C_K \setminus \mathbb{Q}$ *if and only if* $\nu \in C_K \setminus \mathbb{Q}$.

(v) $\Delta(L,p) \notin C_K$ *if and only if* $\nu \notin C_K$.

Proof. Follows directly from Theorem 8(i). Details can be found in [14]. □

We will divide our algorithm into different cases by different situations in Theorem 9. We call (ii) *logarithmic case.* (i) and (iii) *rational case*, and (iv) and (v) *irrational case.* We also have *easy case* which will be defined in Section 4.2 (for easy case, ν can be any number, but as the name suggest, it is easy to get solutions). With the properties of Bessel functions, we can give more restrictions on ν. For the logarithmic case, we have

Remark 14. *If any $p \in S_{reg}$ is logarithmic then by Theorem 9 (ii), $\nu \in \mathbb{Z}$, then by again Theorem 9 (ii), every $p \in S_{reg}$ must be logarithmic. If not, then L has no Bessel type solutions. Also by the fact $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to -\nu$, for the logarithmic case, we can let $\nu = 0$.*

For the rational case:

Remark 15. *$\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to -\nu$. Those transformations are special case of a gauge transformations. Since the last step for our Algorithm is to compute projective equivalence which includes gauge transformations. It suffices to compute ν up to $< \nu \mapsto \nu + 1, \nu \mapsto -\nu >$. Hence we may assume $Re(\nu) \in [0, \frac{1}{2}]$. If $\nu = \frac{1}{2}$, then the operator will be reducible, it is easy to solve the operator by factoring. So we do not consider $\nu = \frac{1}{2}$.*

According to Theorem 8 (i), ν is related to the multiplicity at each regular singularity. So if we know f, then we know exactly the multiplicity at each point in S_{reg} so that we can get a finite list of ν. We have:

Lemma 12. *Let Z be the set of all zeroes of f, for $p \in Z$ let m_p be the multiplicity at $x = p$.*

(i) If $\Delta(L, p) \in C_K$, then let

$$\mathcal{N}'_p := \left\{ \frac{\Delta(L,p) + i}{2m_p} \mid 0 \leq i \leq 2m_p - 1, i \in \mathbb{Z} \right\}$$

We can make the rational part of each element in $\mathcal{N}'_p$ belong to $[0, \frac{1}{2}]$ by using maps $< \nu \mapsto \nu + 1, \nu \mapsto -\nu >$. Let the new set be $\mathcal{N}_p$. Then $\nu \in \mathcal{N} := \cap_{p \in S_{reg}} \mathcal{N}_p$.

(ii) If $\Delta(L, p) \notin C_K$, we can write $\Delta(L, p)$ as $a_1\sqrt{k} + a_2$ where $k \in C_K$ and a_1, $a_2 \in C_K$. Then $\nu = \frac{a_1\sqrt{k}}{2m_p}$ (if for different p, we get different ν then there are no Bessel type solutions.)

Proof. The lemma follows from the fact that we know the number $\Delta(M, p) \equiv 2m_p\nu \mod \mathbb{Z}$ (See Theorem 8), the fact that $\nu^2 \in C_K$ (See Lemma 2), and the fact that $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to -\nu$. $\square$

In chapter 4, we use this lemma to get a list of ν for the easy, logarithmic and irrational cases. For the rational case, the list of candidate ν's is constructed along with the list of candidate f's.

3.5 Algorithm to collect local information from p

This following algorithm summarizes this chapter: (The combination of 'singInfo' and 'singSeries' in my online code is an implementation of this algorithm)

Input : A differential operator L
Output: S_{reg} with exponent differences, S_{irr} with truncated series, B, d_A and information about ν

Find all singularities of L;
foreach *singularity* p *of* L **do**
    Compute exponent difference $\Delta(L,p)$;
    if $\Delta(L,p) \in \mathbb{Z}$ *and* L *is not logarithmic at* $x = p$ **then**
        p is removable singularity or 'disappearing' singularity;
    else if $\Delta(L,p)$ *is a constant or* L *is logarithmic at* $x = p$ **then**
        Add p to the list of S_{reg} with $\Delta(L,p)$;
        if L *is logarithmic at* $x = p$ **then**
            Store information: "$\nu = 0$";
        else if $\Delta(L,p) \in \mathbb{Q}$ **then**
            Store information: "$\nu \in \mathbb{Q} \setminus \{0\}$";
        else
            Store information: "$\nu \notin \mathbb{Q}$";
        end
    else
        Adjust the coefficients of $\Delta(L,p)$ with the method in Corollary 1 (ii) to get a truncated series ;
        Add p to S_{irr} with the truncated series;
    end
    Compute B resp. d_A with Lemma 10 resp. Lemma 11 ;
end
Output S_{reg} with exponent differences, S_{irr} with truncated series, B, d_A and the information about ν. (If conflicting information about ν was stored, then there are no Bessel type solutions.)

Algorithm 3: Local Information

CHAPTER 4

SOLVING DIFFERENTIAL EQUATIONS IN TERMS OF BESSEL FUNCTIONS

In this chapter, we will discuss the details of the algorithm to find Bessel type solutions. In the last chapter, we have given an algorithm to compute the local information from a differential operator L. In this chapter, our main goal is using the local information to find a list of candidates of the change of variables f and the Bessel parameter ν. Note f and ν are related. If we can fix f then we can get a finite list of candidates of ν by Lemma 12. On the other hand, if we know ν, we can identify 'disappearing' singularities and it will give us the information about multiplicities of factors of f. So in our algorithm, in different cases, we fix one parameter first, and use that information to determine the other parameter. So finally, we can get a a finite list of the pairs (f, ν). In this chapter, for the input differential operator L, we assume $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$. We want to get information about f (hence M) from L, so that we can compute the projective equivalence between M and L. If such relation exist, we find Bessel type solutions. So the main question is how to find f. Since f might not be in K, but $g = f^2$ is in K, we can assume $g = \frac{A}{B}$, $A, B \in C_K[x]$, B is monic and $\gcd(A, B) = 1$. We want to get information about A, B from L. In the last chapter, Algorithm 3 gives us a method to compute local information about f and B and degree of A or at least a bound for the degree of A (denoted by d_A). We will use the information to reconstruct f and ν. For the rational case, we also assume $A = CA_1A_2^d$ where C is a constant, A_1 represents the zeroes from S_{reg} and A_2^d represents the part contributed by 'disappearing' singularities. Also we will use

m_p as the multiplicity (resp. pole order) of f at $x = p$ if $p \in S_{reg}$ (resp. $p \in S_{irr}$). We will use these notations throughout this chapter.

4.1 Linear equations

To reconstruct f, we need to find A. In this section, we will introduce a straightforward way to find linear equations satisfied by the coefficients of A from the information of Algorithm 3 in section 3.5. For each $p \in S_{reg}$, by Corollary 1 (i) in section 3.3.4, we can get a zero of A, and for each $p \in S_{irr}$, we can get the truncated series of g at $x = p$. Our linear equations are based on this information.
First we know a bound for the degree of A which is computed in Algorithm 3, and is denoted by d_A. So we can write $A = \Sigma_{i=0}^{d_A} a_i x^i$. Then we have $d_A + 1$ unknowns.

Lemma 13. *Assume $p \in C_K$, if $p \in S_{reg}$, we will get one linear equation for the coefficients of A. If $p \in S_{irr}$ with m_p as pole order of $\Delta(L, p)$, we will get $\lceil m_p \rceil$ linear equations.*

Proof. According to Corollary 1 (i), if $p \in S_{reg}$, p is a zero of A. Then we will get a linear equation of $\{a_i\}_{i=0,\ldots,d_A}$ by setting $\mathrm{rem}(A, x - p) = 0$.
In addition, for each $p \in S_{irr}$ with pole order m_p, by Corollary 1 (ii) we will have a $\lceil m_p \rceil$-term truncated series of g at $x = p$. Then we can get the truncated series of $A = gB$. On the other hand, we can rewrite $A = \Sigma_{i=0}^{d_A} a_i x^i$ as a truncated series at $x = p$ (by Taylor or Laurent series). Since the terms in a Taylor series or Laurent series depend linearly on the coefficients of A, by comparing the coefficients, each term will give a linear equation about a_i's. $\square$

Example 3. [1] [2]

[1] the data is from examples at http://www.math.fsu.edu/∼qyuan

[2] the example will be continued in example 10 in section 4.5

$$L = \partial^2 + \frac{10x^3 - 21x^2 + 12x - 4}{x(x-1)(x-2)(5x-2)}\partial$$
$$- \frac{1}{36}\frac{(28x^4 - 89x^3 + 105x^2 - 59x + 16)(5x-2)^2}{x^2(x-1)^2(x-2)^6}$$

Then we can compute $S_{reg} = \{0\}$, $S_{irr} = \{2\}$ and the truncated series of g at $x = 2$ is $6t_2^{-4} + 21t_2^{-3} + O(t_2^{-2})$, so $B = (x-2)^4$ and $d_A = 4$. We assume $A = \Sigma_{i=0}^{4} a_i x^i$. Then $rem(A, x) = a_0 = 0$ gives us one linear equation. And since we can rewrite $\frac{A}{B} = (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4)t_2^{-4} + (a_1 + 4a_2 + 12a_3 + 32a_4)t_2^{-3} + O(t_2^{-2})$. By comparing the coefficient of two truncated series, we can get 2 linear equations $a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 6$ and $a_1 + 4a_2 + 12a_3 + 32a_4 = 21$. For this example, we have 5 unknowns and we only have 3 linear equations. But we can still solve it (see Example 10).

For $p \in \overline{C_K}$, $p \notin C_K$, we have:

Lemma 14. *If $p \notin C_K$, let $l(x)$ be the minimal polynomial of p over C_K. If $p \in S_{reg}$, we will have $\deg(l)$ linear equations. If $p \in S_{irr}$, we will have $\deg(l) \cdot \lceil m_p \rceil$ linear equations.*

Proof. If $p \in S_{reg}$, then p is a zero of g. Then all the conjugates in $\overline{C_K}$ of p are zeroes of g. There are $\deg(l)$ conjugate zeroes and by setting $\mathrm{rem}(A, l(x)) = 0$, we will get $\deg(l)$ linear equations with coefficient in C_K.

If $p \in S_{irr}$ with m_p as pole order of $\Delta(L, p)$, we can first consider it in the field $C_K(p)$. Then according to lemma 13 we will get $\lceil m_p \rceil$ linear equations with coefficients in $C_K(p)$. Let $c + \sum_{i=0}^{n} c_i a_i = 0$ be such an equation, where $\{a_i\}$ are unknowns and $\{c_i\}$ are coefficients in $C_K(p)$. We can rewrite the equation as $\sum_{i=0}^{\deg(l)-1} e_i p^i = 0$ where $\{e_i\}$ are linear functions with coefficients in C_K. Now p is algebraic over C_K of degree $\deg(l)$, so $1, p, ..., p^{\deg(l)-1}$ are linearly independent over C_K. Hence the e_i's are 0; we get $\deg(l)$ linear equations over C_K. We can do this for all $\lceil m_p \rceil$ linear equations. Then we get $\deg(l) \cdot \lceil m_p \rceil$ equations. $\square$

Example 4. *Suppose* $\sqrt{2} \in S_{reg}$*. If* $\sqrt{2} \in C_K$*, we get one equation by* $rem(A, x - \sqrt{2}) = 0$*. If* $\sqrt{2} \notin C_K$*, we get two equations by* $rem(A, x^2 - 2) = 0$*.*
Suppose $\sqrt{2} \in S_{irr}$*, and that one of the* $\lceil m_{\sqrt{2}} \rceil$ *linear equations is* $3+(1-\sqrt{2})a_1+(1+\sqrt{2})a_2 = 0$*. If* $\sqrt{2} \notin C_K$*, we can rewrite that equation as* $(3+a_1+a_2)+(a_2-a_1)\sqrt{2} = 0$*. Then we can get two equations* $\{3 + a_1 + a_2 = 0, a_2 - a_1 = 0\}$*.*

The straightforward method to get linear equations can be used to solve the easy case, the irrational case and the logarithmic case.

4.2 the Easy Case

So far, we get at least $\sharp S_{reg} + \frac{1}{2}d_A$ linear equations for the coefficients of A by the method introduced in Section 4.1. If this number is greater than d_A, then we can solve them and find A, then we can find ν by Lemma 12. We call this case the *easy case*. This case is very similar to the case in [15]. We summarize the easy case as Algorithm 4. ('sqrtEasy' of my codes is an implementation of this algorithm.)

```
Input  : S_reg, S_irr with truncated series, B, d_A
Output: a list of candidates of (f, ν)

Find all linear equations described in Lemma 13 and Lemma 14;
Solve linear equations to find f;
if there is no solution then
 | output ∅
else
 | Use Lemma 12 to get a list N of candidate ν's
end
foreach ν ∈ N do
 | Add (f, ν) to output list
end
```

Algorithm 4: the Easy Case

Let us illustrate it by an example:

Example 5.

$$L := \partial^2 + \frac{(-1 + x^8 - x^7 - 21x^6 - 16x^5 + 2x^4 + 19x^3 - 3x^2 - 16x)\partial}{(x+1)(x-1)(x^3+x+1)(x^4-4x^2-1-2x)}$$
$$- \frac{1}{144(x^3+x+1)^2(x-1)^3(x+1)^3(x^4-4x^2-1-2x)}(36x^{15} - 5x^{14}$$
$$- 450x^{13} - 574x^{12} + 678x^{11} + 1281x^{10} + 1170x^9 - 779x^8 - 2310x^7$$
$$- 5183x^6 - 7124x^5 - 6960x^4 - 3720x^3 - 817x^2 - 66x - 41)$$

We can input L into Algorithm 3, we get the following information:

1. The roots of $x^4 - 4x^2 - 1 - 2x$ are removable singularities.
2. The roots of x^3+x+1 are non-removable regular singularities with the exponent difference $\frac{1}{3}$.
3. $S_{irr} = \{1, -1, \infty\}$ and the truncated series are $\{\frac{1}{t_\infty}, \frac{3}{2t_1}, \frac{1}{2t_{-1}}\}$ respectively.
4. $B = (x-1)(x+1)$, and degree of A equals $d_A = \deg(B) + 1 = 3$.

Since $d_A = 3$, we can rewrite $A = \sum_{i=0}^{3} a_i x^i$. So we have 4 unknown variables to solve. But according to Lemma 13 and Lemma 14, we can have 6 linear equations from S_{reg} and S_{irr}, enough to solve $\{a_i\}$'s.

For S_{reg}, we use Lemma 14. The remainder of A divided by $x^3 + x + 1$ equals $a_2x^2 + (a_1 - a_3)x + a_0 - a_3$. Take it to zero, then we have three equation $\{a_2 = 0, a_1 - a_3 = 0, a_0 - a_3 = 0\}$.

For S_{irr}, at $x = 1$ the truncated series is $\frac{3}{2t_1}$ and the truncated series of $\frac{A}{B}$ is $\frac{a_1+a_2+a_3+a_4}{2t_1}$ so by comparing the coefficients we get a linear equation $a_1+a_2+a_3+a_4 = 3$. Similarly, at $x = 1$, we will get $a_1 - a_0 + a_3 - a_2 = 1$. and at $x = \infty$, the quotient A divided by B is $a_3x + a_2$, comparing with truncated series $\frac{1}{t_\infty} = x$, we have a equations $a_3 = 1$. Now we can solve $\{a_i\}$'s, and the solutions is $A := x^3 + x + 1$. so $f = \sqrt{\frac{x^3+x+1}{x^2-1}}$

We still need to figure out the value of ν. By Lemma 12, we can compute ν from S_{reg} and exponent differences. Since we have three zeroes and each has exponent difference $\frac{1}{3}$. So the only possible ν in $(0, \frac{1}{2})$ is $\frac{1}{3}$. Now we can compute $L_B \xrightarrow{f}_C M$, then we find the projective equivalence. we will find the general solutions is:

$$C_1 \frac{I_{\frac{1}{3}}\left(\sqrt{\frac{x^3+x+1}{x^2-1}}\right)}{(x-1)^{\frac{1}{4}}} + C_2 \frac{K_{\frac{1}{3}}\left(\sqrt{\frac{x^3+x+1}{x^2-1}}\right)}{(x-1)^{\frac{1}{4}}}$$

Remark 16. *For the easy case, ν can be an integer, rational or irrational. We do not use information from exponent differences of S_{reg} to compute ν. We compute ν by lemma 12, since it is easy to compute f first in this case.*

Remark 17. *In some unlikely events, the linear equations from S_{reg} are not independent from linear equations from S_{irr}. In that case, even if we have enough equations, we still can not get f. We need to run algorithms for other cases depend on the value of ν.*

Example 6.

$$L := \partial^2 + \frac{3x^4 - 12x^2 + 5}{x(x-1)(x+1)(3x^2-5)}\partial - \frac{1}{4}\frac{(x^2 - 1 + 2x^5)(3x^2-5)^2}{(x-1)^2(x+1)^2x^7}$$

In this case, we have $S_{reg} = \{1, -1, \infty\}$ and $S_{irr} = \{0\}$. The truncated series of g at $x = 0$ is $-t_0^{-5} + t_0^{-3} + O(t_0^{-2})$. So we have three equations from S_{reg} and three equations from S_{irr}. If we solve the linear system, then we find $g = \frac{a_4x^4-(a_4-1)x^2-1}{x^5}$, here a_4 is free. But we only have solutions when $a_4 = 1$. Since $\nu \notin \mathbb{Q}$, we can run Algorithm 6 for the irrational case (in section 4.4) to get two independent solutions: $I_{\sqrt{2}}\left(\frac{x^2-1}{x^5}\right)$ and $K_{\sqrt{2}}\left(\frac{x^2-1}{x^5}\right)$

4.3 the Logarithmic Case

For the logarithmic case, we have:

Lemma 15. *In the logarithmic case we know all zeroes of A.*

Proof. By Theorem 8 (i) in section 3.3.4, a change of variables can transfer a regular singularity to a removable singularity only if $\nu \in \mathbb{Q} \setminus \mathbb{Z}$. So in the logarithmic case, S_{reg} contains all zeroes. $\square$

After we know all zeroes, we have to do a combinatorial search: try all possible combinations of multiplicities of zeroes of A. After that the only unknown is the leading coefficient of A. We have enough equations to find it.
Once we get f, we can set $\nu = 0$ by Lemma 12 and Remark 14 in section 3.4.
We can summarize the process to Algorithm 5. ('sqrtLog' of my codes is an implementation of this algorithm.)

```
Input  : S_reg, S_irr with truncated series, B, d_A
Output: list of (f, ν)
if not every singularity p ∈ S_reg is logarithmic then
 | output ∅
else
 | Let ν = 0, A = aΠ_{p∈S_reg\{∞}}(x − p)^{m_p};
 | if ∞ ∈ Sreg then
 |  | m_∞ = 0;
 | else
 |  | m_∞ ≥ 1 is an integer;
 | end
 | foreach {m_p} such that Σ_{p∈S_reg\{∞}} m_p = d_A − m_∞, m_p ≥ 1 are integers do
 |  | Use linear equations described in Lemma 13 to solve a ;
 |  | if the solution exists then
 |  |  | Add (A/B, 0) to output list
 |  | end
 | end
end
```

Algorithm 5: the Logarithmic case

Example 7.

$$Dx^2 + \frac{(9x^3+55x^2-35-5x)Dx}{2(x+5)(x-1)(x+1)x} + \frac{-9x^6-181x^5-236x^4-144x^3+740x^2+200x-500+45x^8+325x^7}{16(x+5)(x-1)(x+1)x^7}$$

Input this operator to Algorithm 3, we can get the following information.

1. $S_{reg} = \{1, -1\}$ they both have logarithmic solutions locally. (Note since they are logarithmic, the exponent difference will be integer, which is not useful for our algorithm.)

2. $S_{irr} = \{0\}$ and the truncated series is $-t_0^{-5} + t_0^{-4} + 2t_0^{-3} + O(t_0^{-2})$.

3. ∞ and -5 are removable singularities.

4. $B = x^5$ and since ∞ is removable, so $\deg(A) = \deg(B)$. So $d_A = 5$.

If we assume $A = \sum_{i=0}^{5} a_i x^i$, then we have 6 unknowns, but from local information we only have 5 linear equations. But according to Lemma 15, S_{reg} contain all the zeroes and we know $\deg(A) = 5$. (Note here since ∞ is not in S_{reg}, so we do not need to try A with degree less than 5. If ∞ is in S_{reg} then, for combinatorial search we need to try degree of A from 2 to 5). So the candidates of A are $a_5(x-1)(x+1)^4$, $a_5(x-1)^2(x+1)^3$, $a_5(x-1)^3(x+1)^2$, $a_5(x-1)^4(x+1)$. By comparing the truncated series of $\frac{A}{B}$ at $x = 0$. We get $a_5 = 1$ and $f = \sqrt{\frac{(x-1)^3(x+1)^2}{x}}$ is the only candidate. Then we can compute $L_B \xrightarrow{f}_C M$ and compute the projective equivalence. We get the general solutions are:

$$\frac{C_1}{x^{\frac{5}{4}}} I_0\left(\frac{(x-1)^3(x+1)^2}{x^3}\right) + \frac{C_2}{x^{\frac{5}{4}}} K_0\left(\frac{(x-1)^3(x+1)^2}{x^3}\right)$$

4.4 the Irrational Case

For the Irrational Case, we have:

Lemma 16. *In the irrational case, we know all zeroes and their multiplicities as well.*

Proof. By Theorem 8 (i) in section 3.3.4, a change of variables can transfer a regular singularity to a removable singularity only if $\nu \in \mathbb{Q} \setminus \mathbb{Z}$. So in the irrational cases, S_{reg} contains all zeroes.
For each $p \in S_{reg}$, let a_p be the coefficient of the irrational part of the exponent difference. Since $\longrightarrow_{EG}$ only change the exponent difference by integers, so $\longrightarrow_{EG}$ does not change the irrational part of exponent difference. Then there exists k, such that $k\Sigma_{p\in S_{reg}} a_p = d_A$. Then $\frac{a_p}{k}$ will give the multiplicity of p. □

After we get all zeroes and their multiplicities, there is only one unknown coefficient, the leading coefficient of A. But we have $\frac{1}{2}d_A$ linear equations, enough to get A.

Once we get f, we can get a list of candidate ν's by Lemma 12 and Remark 14 in section 3.4.

Example 8.

$$\begin{aligned}\partial^2 &+ \frac{(x^4-6x^3+22x^2-32x+16)\partial}{x(x-2)(x-1)(x^2-8x+8)}\\ &+ \frac{1}{4((x-1)^2(x-2)^2x^6(x^2-8x+8)}(2x^{10}+11x^9-235x^8+1008x^7\\ &- 660x^6-5408x^5+17312x^4-25600x^3+21760x^2-10240x+2048)\end{aligned}$$

To analyze the generalized exponents, we get the following information:

1. $S_{reg} = \{1, 2, \infty\}$ and the exponent differences are $\{-\sqrt{2}, -2\sqrt{2}, \sqrt{2}\}$ respectively.

2. $S_{irr} = \{0\}$ and the truncated series is $-4t_0^{-4} + 8t_0^{-3} - O(t_0^{-2})$.

3. so $B = x^4$ and $d_A = 4$.

Since the coefficient of the irrational part of exponent differences at $x = 1$,$x = 2$, $x = \infty$ are 1, 2, 1 resp. The only possible multiplicities such that $m_1 : m_2 : m_\infty = 1 : 2 : 1$ and $m_1+m_2+m_\infty = d_A$ is $m_1 = m_\infty = 1$ and $m_2 = 2$. (Note the multiple of ∞ equal

m_∞ means that $\deg(A) + m_\infty = \deg(B)$). So $g = \frac{a(x-1)(x-2)^2}{x^4} = -\frac{4a}{x^4} + \frac{8a}{x^3} + O(x^{-2})$. By comparing the coefficients, we get $a = 1$.

After we get f, we can use Lemma 12 to compute that ν can be either $\sqrt{2}$ or $\sqrt{2}+\frac{1}{2}$. By apply the projective equivalence, we get the general solutions is:

$$C_1 \frac{I_{\sqrt{2}}\left(\sqrt{\frac{(x-1)(x-2)^2}{x^4}}\right)}{x} + C_2 \frac{K_{\sqrt{2}}\left(\sqrt{\frac{(x-1)(x-2)^2}{x^4}}\right)}{x}$$

The Algorithm ('sqrtIrrat' is an implementation of my code) is described as:

Input : S_{reg}, S_{irr} with truncated series, B, d_A **Output**: list of (f, ν) Use Lemma 15 find all zeroes and multiplicities; Use linear equations given by Lemma 13 to get the leading coefficient; Use Lemma 12 to get a list of candidates for ν's; Add solutions to output list;

Algorithm 6: the Irrational case

Remark 18. *If we only apply $\longrightarrow_C$, then Lemma 2 proved that $\nu^2 \in C_K$. But if we apply $\longrightarrow_{EG}$ as well, then the exponent differences will shift by integers. Then it is possible that $\nu^2 \notin C_K$. By analyzing the conjugates, we can get $(\nu - \frac{n}{2})^2 \in C_K$ for some $n \in \mathbb{Z}$. (if $\frac{n}{2} \in \mathbb{Z}$ we may assume $\nu^2 \in C_K$)*

Example 9.

$$L := \partial^2 + \frac{x+2}{x(x+1)}\partial - \frac{7}{16} \cdot \frac{4 + 7x^2 + 7x^3}{(x+1)x^2}$$

Using our implementation and input "BesselSolver(L)" produces two independent solutions:

$$\frac{-(9+4\sqrt{2})x + (4-8\sqrt{2})}{x} I_{\sqrt{2}-\frac{1}{2}}\left(\frac{7}{4}x\right) + 7I_{\frac{3}{2}-\sqrt{2}}\left(\frac{7}{4}x\right)$$

and

$$(9+4\sqrt{2})K_{\sqrt{2}-\frac{1}{2}}\left(\frac{7}{4}x\right) - 7K_{\frac{1}{2}+\sqrt{2}}\left(\frac{7}{4}x\right)$$

Note in this example, $L \in \mathbb{Q}(x)[\partial]$. But $\nu = \sqrt{2} - \frac{1}{2}$ and $\nu^2 \notin \mathbb{Q}$.

4.5 Linear Equations for the Rational Case

The hardest case is the rational case. This case is hard because in the rational case, if the multiplicity at $x = p$ of g is multiple of denominator of ν, then the change of variable will send regular singularities to removable singularities. Removable singularities may become regular under $\longrightarrow_{EG}$. Those 'disappearing' singularities will not give us any useful local information. Note those 'disappearing' singularities is not a trouble in [14] because it just consider the unramified case, in which f itself is a rational functions. In that case, we will have enough information from S_{irr}. But for more general cases, we only have $g = f^2$ is a rational function. To reconstruct g we need double information than [14]. So roughly S_{irr} only give us about half information we need to construct g. For example, in Example 3, we do not have enough information to construct g. So we have to consider other method.

The first thing we notice is that the denominator of ν (denoted by $d = \mathrm{denom}(\nu)$) is important, because d along with the multiplicities will determine whether the singularities 'disappear'. We also notice that $d > 2$ because the case $\nu \in \mathbb{Z}$ has already been treated (the logarithmic case) and if $\nu \in \frac{1}{2}\mathbb{Z}$ then L_B is reducible, which can be solved by factoring the operator. We will compute a finite set of possible values of d. (see Lemma 17 and 18 on next page)

Now assume ν is found. Let p be a root of A and $\Delta(L,p) \equiv 2m_p\nu \bmod \mathbb{Z}$. If $d \mid 2m_p$, change of variables $x \mapsto f$ will send p to a removable singularity. Again this is hard because if p is removable, then $p \notin S_{reg}$, which means that not all roots of A are known (not all roots of A are in S_{reg}). But if a zero p of A becomes an removable singularity, the multiplicity[3] $2m_p$ must be a multiple of d. So we can rewrite $A = CA_1A_2^d$, where $A_1, A_2 \in C_K[x]$ and $C \in C_K$, A_1 is monic and the roots of A_1 are the known roots of A (the elements of S_{reg}).

[3]If m_p is multiplicity of f at $x = p$, then $2m_p$ is multiplicity of A.

For $S_{reg} = \emptyset$, since we don't know any singularities we can let $A_1 = 1$ and find the list of d by the following lemma:

Lemma 17. *If $S_{reg} = \emptyset$, then $d \mid d_A$.*

Proof. Since $S_{reg} = \emptyset$, if p is a root of g, then it is a disappearing singularity. So $d \mid 2m_p$. So $A = C \cdot A_2^d$. $\square$

For $S_{reg} \neq \emptyset$, we have:

Lemma 18. *If $S_{reg} \neq \emptyset$, we can find a list of candidate pairs (d, A_1) by solving Diophantine equations.*

Proof. We assume $N = \#S_{reg}$, $S_{reg} = \{p_1, ..., p_N\}$ and $\Delta(L, p)$ is the exponent difference at $x = p$. Let $A_1 = \Pi_{i=1}^{N}(x - p_i)^{m_{p_i}}$, $1 \leq m_{p_i} < d$ and $d_p = \text{denom}(\Delta(L, p))$. For each point $p \in S_{reg}$, $d_p \mid d$. So we have $l \mid d$ where $l := \text{lcm}_{p \in S_{reg}} d_p$. So d can only be a multiple of l, and it must be less or equal than d_A. So there are $\lfloor d_A/l \rfloor$ possibilities for d. Once we fix d, then for each $p \in S_{irr}$ we have $\frac{d}{d_p} \mid m_p$. So solve $(\sum_{i=1}^{N} m_{p_i}) + \deg(A_2)d = d_A$, $1 \leq m_{p_i} < d$ and $\frac{d}{d_{p_i}} \mid m_{p_i}$. It will give finitely many candidates for A_1. $\square$

After we fixed A_1, we need to compute C. Assume $g = \frac{CA_1A_2^d}{B}$. By remark 24 in section 4.10, $g \in K$. By the construction of A_1 and B, $\frac{A_1}{B} \in C_K(x)$. Now $CA_2^d \in C_K[x]$. It does not imply that $A_2 \in C_K[x]$. To get $A_2 \in C_K[x]$ we choose $C \in C_K$ by the following methods.

Case 1: $(C_K \cup \{\infty\}) \cap S_{irr} \neq \emptyset$

Let $p \in (C_K \cup \{\infty\}) \cap S_{irr}$. Then $\Delta(L, p) \in C_K(t_p)$. From $\Delta(L, p)$, we can compute a truncated series for $f^2 = \frac{CA_1A_2^d}{B}$. From it, we can compute a truncated series for f^2B/A_1 (which equals CA_2^d). Let C be *the coefficient of the first term of this series.* Now the first term of truncated series of A_2^d is 1 at $x = p$. $1 \in C_K$ is always a dth root of 1. When we try to find truncated series of A_2 at $x = p$, we can let the first term be 1 (or other roots of 1 in C_K) and construct other terms by Hensel Lifting. By the way of the construction, A_2 will be in $C_K[x]$.

To proof the method to choose C is correct, we still need to prove that if we choose other $\widehat{C}$ such that A_2 is in $C_K[x]$, then it will lead to the same candidates of g.

Definition 29. *We say C_1 and C_2 are equivalent ($C_1 \sim C_2$), if $C_1 = C_2 \cdot c^d$ where $c \in C_K$.*

Let $C_1 = c^d C_2$. Suppose $g = \frac{C_1 A_1 A_2^d}{B}$ is a candidate we find from the local data. Then $g = \frac{C_2 A_1 (cA_2)^d}{B}$. So if we can get g from C_1 we can also get it from C_2. So if $C_1 \sim C_2$ then they lead to the same candidates.

Lemma 19. *Assume $(C_K \cup \{\infty\}) \cap S_{irr} \neq \emptyset$. A_1 and B, which are monic and in $C_K[x]$, are fixed. Let $p \in (C_K \cup \{\infty\}) \cap S_{irr}$ and C and A_2 be computed by the method we introduced in case 1 above and Theorem 10 below. Then if $g = \frac{\widehat{C} A_1 \widehat{A_2}^d}{B}$, $\widehat{C} \in C_K$ and $\widehat{A_2} \in C_K[x]$, then $C \sim \widehat{C}$.*

Proof. According to the assumptions, $CA_2^d = \widehat{C}\widehat{A_2}^d$. So $\frac{C}{\widehat{C}} = \frac{\widehat{A_2}^d}{A_2^d} \in C_K$. Then the difference between A_2 and $\widehat{A_2}$ is a scale multiplication. Assume $A_2 = c \cdot \widehat{A_2}$ and $c \in C_K$, then $C \cdot c^d = \widehat{C}$. So $C \sim \widehat{C}$. □

Since all C's in this case are equivalent. So our method is sufficient in this case.

Case 2: $(C_K \cup \{\infty\}) \cap S_{irr} \neq \emptyset$

In that case, we can temporarily extend the field C_K to $C_K(p)$, for some $p \in S_{irr}$. Recompute the local data over the new field and compute all candidate $g \in C_K(p)(x)$ as in case 1. Then discard all g's that do not simplify to an element of $C_K(x)$. (See Remark 24 in Section 4.10).

Remark 19. *In case 2, sometimes we can still use the way in case 1 to guess the value of C. But it might not lead to the correct candidates, because C might not be* *unique up to $\sim$. So we need to introduce the algebraic extensions. More details and examples are discussed in section 4.7.*

Now the only unknown part of A is A_2. We can assume $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$. Since $\deg(A_2) \leq \frac{1}{d} d_A \leq \frac{1}{3} d_A$, we have

Lemma 20. *For the rational case, we only need $\frac{1}{3}d_A + 1$ equations to recover A.*

We can not get the equations by the same methods as in Lemma 13 and [14, 15]. If we do so, the equations we get for $\{b_i\}$ will not be linear, because we need to evaluate the dth power (See Example 10) The solution to this problem is as follows:

Theorem 10. *In the rational case, for $A = CA_1A_2^d$, and $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$, for each $p \in S_{irr}$ with m_p as the pole order of g at $x = p$, we will get $\lceil m_p \rceil$ linear equations of $\{b_i\}$.*

Proof. Since the exponent difference at $x = p$ will give a $\lceil m_p \rceil$-term truncated series of $g = \frac{A}{B}$ at $x = p$, we can also write B and CA_1 as a series at $x = p$. Then we can get the $\lceil m_p \rceil$-term truncated series of $A_2^d = \frac{gB}{CA_1}$. We assume the series is $\sum_{m_p < i \leq 2m_p} c_i t_p^{-i}$ where t_p is the local parameter at $x = p$. We can rewrite the series as $c_{2m_p} t_p^{-2m_p} S$, where S is a power series with the initial term 1. Let $S_{1/d}$ be a power series with first term 1 such that $S_{1/d}^d = S$. Write $S_{1/d} = 1 + \Sigma_{i>0} a_i t_p^i$ where $a_1, ..., a_{\lceil m_p \rceil - 1}$ are computed by Hensel lifting. Let $\mu_d = \{r \mid r \in C_K, r^d = 1\}$. By the method to construct C there should be a dth root of c_{2m_p} in C_K. Let c be such a root. Then for each $r \in \mu_d$, let $S_r = ct_p^{-2m_p/d} r S_{1/d}$. Then S_r is a truncated series at $x = p$ whose dth power is the truncated series of $\frac{gB}{CA_1}$ at $x = p$. Then we can also rewrite $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$ as a truncated series at $x = p$. By comparing the coefficients of S_r and A_2, we will get $\lceil m_p \rceil$ linear equations. Doing this for every $p \in S_{irr}$ provides enough linear equations to find A. Note that we have to try all combinations of $r \in \mu_d$ at every $p \in S_{irr}$. $\square$

Remark 20. *If $p \notin C_K$, we can use the results from Lemma 14 to get equations. So we can always obtain $\geq \frac{1}{2}d_A$ linear equations, while $\lfloor \frac{1}{3}d_A \rfloor + 1$ equations are sufficient. So we always get enough linear equations.*

Remark 21. *If we get a candidate (f, d), then $\{f\} \times \{\frac{a}{d} \mid \gcd(a, d) = 1, 1 \leq a < \frac{1}{2}d\}$ is a list of candidates for (f, ν).*

Example 10. *Continue with Example 3. We know $S_{reg} = \{0\}$, $S_{irr} = \{2\}$ with the truncated series of g is $\Delta = 6t_2^{-4} + 21t_2^{-3} + O(t_2^{-2})$, $B = (x-2)^4$ and $d_A = 4$. Lemma 13 did not provide sufficiently many equations. But for this case the only possible situation is $A = CxA_2^3$, and $A_2 = a_0 + a_1 x$. The truncated series of $\frac{A}{B}$ is*

$$\frac{2C(a_0 + 2a_1)^3}{t_2^4} + \frac{6C(a_0 + 2a_1)^2 a_1 + C(a_0 + 2a_1)^3}{t_2^3} + O(t_2^{-2})$$

By comparing the coefficient, we have equations $\{2C(a_0+2a_1)^3 = 6, 6C(a_0+2a_1)^2 a_1 + C(a_0 + 2a_1)^3 = 21\}$, but it is not linear. But the truncated series at $x = 2$ of CA_2^3 is the series of $\Delta \cdot (x-2)^4/x$ at $x = 2$, which is $3 + 9t_2 + O(t_2^2)$. So we can let $C = 3$. Then series of $\frac{gB}{CA_1}$ is $S = 1 + 3t_2$. Since $K = \mathbb{Q}(x)$, the only 3rd root of 1 is 1. So the only possible truncated series which is 3rd root of S is $1 + t_2 + O(t_2^2)$. And comparing it with $a_0 + a_1 x = a_0 + 2a_1 + a_1 t_2$, we get two linear equations $a_0 + 2a_1 = 1$ and $a_1 = 1$. Solve them we get $a_0 = -1, a_1 = 1$. So $g = \frac{3x(x-1)^3}{(x-2)^4}$.

4.6 the Rational Case

This is an example to illustrate the case when $S_{reg} = \emptyset$

Example 11. *$L = \partial^2 + 2 - 10x + 4x^2 - 4x^4$. $K = \mathbb{Q}(x)$*

$S_{reg} = \emptyset$ and $Sirr = \{\infty\}$ with the truncated series of g at $x = \infty$ is $\frac{4}{9}t_\infty^{-6} - \frac{4}{3}t_\infty^{-4} + O(t_\infty^{-3})$. So $d_A = 6$ and B=1.

It is the rational case with $S_{reg} = \emptyset$. d can only be the factor of d_A, so $d \in \{3, 6\}$. Since we do not know zeroes for A, let $A_1 = 1$. We can write $A = CA_2^d$. If $d = 3$

then $A = CA_2^3$, $A_2 = a_0 + a_1x + a_2x^2$. Since $B = 1$, then the truncated series of gB is the same as g.

There is only one singularity and the coefficient of the term with lowest degree is $\frac{4}{9}$. So we can let $C = \frac{4}{9}$. Since ∞ is the only singularity, so C is the leading coefficient of A.

The truncated series of $A_2^3 = gB/C$ is $t_\infty^{-6} + 3t_\infty^{-4} + O(t_\infty^{-3}) = t_\infty^{-6}(1 - 3t_\infty^2 + O(t_\infty^3))$. Since the only 3rd root of 1 in C_K is 1, then the only 3rd root of $1 - 3t_\infty^2 + O(t_\infty^3)$ is $1 - t_\infty^2 + O(t_\infty^3)$. So by comparing coefficients of $t_\infty^{-2}(1 - t_\infty^2 + O(t_\infty^{-3}))$ and $A_2 = a_0 + a_1t_\infty^{-1} + a_2t_\infty^{-2}$, we can get $A_2 = x^2 - 1$ and then $g = \frac{4}{9}(x^2 - 1)^3$.

We can do this process for $d = 6$, in this case, there are no solutions. So we have $(\frac{2}{3}\sqrt{x^2 - 1)^3}, \frac{1}{3})$ as the only possible candidate.

We compute $L_B \xrightarrow{f}_C M$, and then the projective equivalence from M to L. Combining these transformations produces the following solutions of L:

$$\begin{aligned}
&C_1\left(\frac{2(2x^4 + x^3 - 3x^2 + x + 2)}{\sqrt{x^2 - 1}} I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{(x^2 - 1)^3}\right)\right. \\
&\left. + 2(2x + 1)(x^2 - 1) I_{\frac{4}{3}}\left(\frac{2}{3}\sqrt{(x^2 - 1)^3}\right)\right) \\
&+ C_2\left(\frac{2(2x^4 + x^3 - 3x^2 + x + 2)}{\sqrt{x^2 - 1}} K_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{(x^2 - 1)^3}\right)\right. \\
&\left. - 2(2x + 1)(x^2 - 1) K_{\frac{4}{3}}\left(\frac{2}{3}\sqrt{(x^2 - 1)^3}\right)\right)
\end{aligned}$$

Here is an example to illustrate how to find d and A_1 if $S_{reg} \neq \emptyset$.

Example 12. *Consider the operator:*

$$L := \partial^2 - \frac{15x^4 - 30x^3 + x^2 + 8x - 4}{x(x-1)(15x^3 - 10x^2 + 9x - 4)}\partial - \frac{1}{36x^2(15x^3 - 10x^2 + 9x - 4)(x-1)^2}(30375x^{20} - 212625x^{19} + 733050x^{18} - 170595x^{17} + 3034305x^{16} - 435055x^{15} + 5166936x^{14} - 5172228x^{13} + 4401369x^{12} - 3189159x^{11} + 1962738x^{10} - 1016622x^9 + 434943x^8 - 149229x^7 + 38844x^6 - 3933x^5 - 4554x^4 + 3789x^3 - 1612x^2 + 432x - 64).$$

$S_{reg} = \{1, 0\}$, with the exponent difference $\frac{5}{3}$ and $\frac{4}{3}$ respectively. We also have $S_{irr} = \{\infty\}$ and the truncated series of g at $x = \infty$ is $t_\infty^{-15} - 5t_\infty^{-14} + 13t_\infty^{-13} - 25t_\infty^{-12} + 38t_\infty^{-11} - 46t_\infty^{-10} + 46t_\infty^{-9} - 38t_\infty^{-8} + O(t_\infty^{-7})$. So $B = 1$ and $d_A = 15$.

We can easily verify that this is a rational case. The exponent difference at $x = 0$ and $x = 1$ both have denominator 3, so d is a multiple of 3. If $d = 3$ then $A = Cx^2(x-1)A_2^d$ or $A = Cx(x-1)^2A_2^d$. If $d = 6$, then the multiplicity at both 1 and 0 should be a multiple of $\frac{6}{3} = 2$. Then the degree of A is even which will contradict with $\deg(A) = 15$. If $d = 9$ then the multiplicity at both 1 and 0 should be a multiple of $\frac{9}{3} = 3$. Then the only possible case is $A = Cx^3(x-1)^3A_2^9$. Similar to $d = 6$, if $d = 12$, there are no possible cases. If $d = 15$, then $A = Cx^5(x-1)^{10}$ and $A = Cx^{10}(x-1)^5$ are candidates as well. So the list of candidates of A is:

$$\{Cx^2(x-1)A_2^3,\ \ Cx(x-1)^2A_2^3,\ \ Cx^3(x-1)^3A_2^9,\ \ Cx^5(x-1)^{10},\ \ A = Cx^{10}(x-1)^5\}$$

Next we compute each candidate by the method in Theorem 10 to find C and linear equations to solve A_2. Finally, we find $f = \sqrt{x^4(x-1)^5(x^2+1)^3}$ and $\nu = \frac{1}{3}$ is the only remaining candidate .

Let $L_B \xrightarrow{f}_C M$. Now M is already equal to L. So the general solution is:

$$C_1 I_{\frac{1}{3}}\left(\sqrt{x^4(x-1)^5(x^2+1)^3}\right) + C_2 K_{\frac{1}{3}}\left(\sqrt{x^4(x-1)^5(x^2+1)^3}\right)$$

If we assume $d = \text{denom}(\nu)$ and $f^2 = g = \frac{CA_1A_2^d}{B}$. Algorithm 7 gives the sketch for the rational case. (See Appendix 1 for corresponding codes)

Input : S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
if $S_{reg} = \emptyset$ **then**
  Let the list of candidates for d be the set of factors of d_A;
  Let $A_1 = 1$;
else
  Use Lemma 18 to get a list of candidates for d and A_1
end
foreach *candidate* (d, A_1) **do**
  Fix C by method on Page 53;
  Use linear equations given by Theorem 10 to compute A_2 ;
  If a solution exists, add $\{f\} \times \{\frac{a}{d} \mid \gcd(a, d) = 1, 1 \leq a < \frac{1}{2}d\}$ to output list
end

Algorithm 7: the Rational case

4.7 Algebraic Extension

So far, all of our examples have base field $K = \mathbb{Q}(x)$. This section we will discuss how to deal with algebraic extension, in the previous section, we have already discussed some methods to deal with algebraic extension. In this section, we will discuss the more details with examples.

There are three types of algebraic extensions we need to deal with. The easiest one is the case that the singularities or Bessel Parameter ν is not in the base field. Those two cases cause no trouble. If $\nu \notin C_K$, we simply enter the irrational case. The algorithm will work, because we only use coefficient of irrational part of ν to find f. If a singularity p is not in C_K, there is no trouble because by using Lemma 14, we can still get the linear equations we need.

The second type of algebraic extension is $C_K \neq \mathbb{Q}(x)$. In this scenario, we use the same algorithm because our algorithm works for all C_K of characteristic 0. But we might need more computations in the rational case. A example is when

$\xi_3 = \frac{1}{2} + \frac{\sqrt{-3}}{2} \in K$ and $\nu = \frac{1}{3}$. When we try to find the 3rd roots of truncated series, instead of one series, we will get three different series, which will triple the number of candidates. But the idea of the algorithm is the same.

Example 13. *Consider the operator:*

$$L := x\partial^2 + (1 - \sqrt{-6}x - 2x^2)\partial + \frac{36x^2 + 10\sqrt{-6}x - 1}{36x}$$

There is one regular singularity 0 with difference $\frac{1}{3}$ and one irregular singularities at $x = \infty$ with the power series, $\frac{1}{2}(\sqrt{-6}x + x^2)^2$. $B = 1$ and degree of A is 4. So the only possible change of variable is $g = f^2 = Cx(a_1x + a_0)^3$. Since $\xi_3 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ is not in C_K. So we only get one series when we take the 3rd root. Finally we find $g = \frac{1}{4}(\frac{2}{3}\sqrt{-6} + x)^3 x$ and we find the general solution:

$$C_1 \frac{e^{\frac{1}{2}x^2 + \frac{1}{2}\sqrt{-6}x}}{\sqrt{2\sqrt{-6} + 3x}} \Bigg(-3(3x^2 + 3\sqrt{-6} - 8) I_{\frac{2}{3}}\left(\frac{1}{2}\sqrt{\left(\frac{2}{3}\sqrt{-6} + x\right)^3 x}\right) + \sqrt{3}\sqrt{(2\sqrt{-6} + 3x)^3 x}\, I_{\frac{3}{4}}\left(\frac{1}{2}\sqrt{\left(\frac{2}{3}\sqrt{-6} + x\right)^3 x}\right)\Bigg)$$

$$C_2 \frac{e^{\frac{1}{2}x^2 + \frac{1}{2}\sqrt{-6}x}}{\sqrt{2\sqrt{-6} + 3x}} \Bigg(-3(3x^2 + 3\sqrt{-6} - 8) K_{\frac{2}{3}}\left(\frac{1}{2}\sqrt{\left(\frac{2}{3}\sqrt{-6} + x\right)^3 x}\right) + \sqrt{3}\sqrt{(2\sqrt{-6} + 3x)^3 x}\, K_{\frac{3}{4}}\left(\frac{1}{2}\sqrt{\left(\frac{2}{3}\sqrt{-6} + x\right)^3 x}\right)\Bigg)$$

This example shows how algebraic extension affect the behavior of the solver.

Example 14.

$$
\begin{aligned}
L := {} & Dx^2 \\
& - \frac{2}{(2x^2+2x-1+\sqrt{-3})(-2x^3+4x+x^2+3+2x\sqrt{-3})(x+1)(x-1)}(-6x^6 \\
& - 2x^5+16x^4+x^4\sqrt{-3}+4x^3-16x^2-4\sqrt{-3}x^2-10x-8x\sqrt{-3}+2-\sqrt{-3})Dx \\
& - \frac{1}{9}\frac{1}{(2x^2+2x-1+\sqrt{-3})^2(-2x^3+4x+x^2+3+2x\sqrt{-3})(x+1)^4(x-1)^4}(477 \\
& + 10185x^7\sqrt{-3}+72\sqrt{-3}-462x^{11}\sqrt{-3}+756x^{12}\sqrt{-3}-3129x^4\sqrt{-3} \\
& - 2502x^2\sqrt{-3}-144x^{15}-4278x^9\sqrt{-3}-4947x^4+531x^2-3578x^{11}+1468x^{13} \\
& + 2281x^7+216x^{13}\sqrt{-3}+16804x^8-10214x^{10}+963x-20831x^5+2982x^{12} \\
& + 2413x^3+4857x^8\sqrt{-3}-4497x^{10}\sqrt{-3}-4527x^3\sqrt{-3}+4443x^6\sqrt{-3} \\
& - 1143x^5\sqrt{-3}+9x\sqrt{-3}-21545x^6-216x^{14}+2452x^9)
\end{aligned}
$$

After collecting the local information, we find that there are no regular singularities and there are three irregular singularities at $x=\infty$, at $x=\pm 1$, the truncated series are $2t_\infty$, $-\frac{1}{2}t_{-1}^2$, and $\frac{10+9\sqrt{-3}}{2}t_1^2$ respectively. $B=(x-1)^2(x+1)^2$ and $d_A=6$. Since $d_A=6$ and no regular singularities, so d (denominator of ν) can only be 3 or 6. Let us look at the case $d=3$. We notice here the 3rd unit root is in the field. So if we take the third root of the power series, we will get three instead of one power series. For example, we assume $A=CA_1A_2^3$ and $A_2=a_2x^2+a_1x_1+a_0$. We can fix $C=2$ and $A_1=1$. Then we can get the at $x=\infty$, the truncated power series of A_2^3 is t_∞^6, when we take the 3rd root of this series, we can get three series: t_∞^2, $-\frac{1}{2}+\frac{\sqrt{-3}}{2}t_\infty^2$ and $-\frac{1}{2}-\frac{\sqrt{-3}}{2}t_\infty^2$. We have 3 irregular singularities, at each point, we will have 3 series as well. So we will have to try 27 combinations to obtain the list of candidate of f when $\nu=\frac{1}{3}$ (there are no solutions when denominator of ν equal 6):

$$
\frac{2(x^2+x+\frac{1}{2}+\frac{\sqrt{-3}}{2})^3}{(-1+x^2)^2}, \frac{2(\frac{1}{2}+\frac{\sqrt{-3}}{2}x+x^2)^3}{(-1+x^2)^2}, \frac{2(-1-\sqrt{-3}+(-\frac{1}{2}-\frac{\sqrt{-3}}{2})x+x^2)^3}{(-1+x^2)^2},
$$

$$
\frac{2(-1-\frac{\sqrt{-3}}{2}+(-\frac{1}{2}-\sqrt{-3})x+x^2)^3}{/(-1+x^2)^2}, \frac{2(-\frac{7}{4}-\frac{2\sqrt{-3}}{4}+(\frac{1}{4}-\frac{3\sqrt{-3}}{4})x+x^2)^3}{(-1+x^2)^2},
$$

$$
\frac{2(-\frac{7}{4}+\frac{\sqrt{-3}}{4}+(-\frac{5}{4}+\frac{3\sqrt{-3}}{4})x+x^2)^3}{(-1+x^2)^2}, \frac{2(-\frac{7}{4}+\frac{3\sqrt{-3}}{4}+(-\frac{5}{4}+\frac{\sqrt{-3}}{4})x+x^2)^3}{(-1+x^2)^2},
$$

$$\frac{2(-\frac{5}{2}+(\frac{\sqrt{-3}}{2}+(-\frac{1}{2}+\frac{\sqrt{-3}}{2})x+x^2)^3}{(-1+x^2)^2}, \frac{2(-\frac{1}{4}+\frac{\sqrt{-3}}{4}+(\frac{7}{4}+\frac{\sqrt{-3}}{4})x+x^2)^3}{(-1+x^2)^2}$$

We can see the algebraic extension makes the problem more complicated, we need check more candidate. If we try this list, we find the general solution is:

$$C_1 I_{\frac{1}{3}}\left(\frac{\sqrt{2\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}x+x^2\right)^3}}{(x^2-1)}\right)+C_2 K_{\frac{1}{3}}\left(\frac{\sqrt{2\left(\frac{1}{2}+\frac{\sqrt{-3}}{2}x+x^2\right)^3}}{(x^2-1)}\right)$$

The third type of the algebraic extension is that the algebraic extension is introduced by the algorithm itself. It happens when C is not unique (up to multiplication by a dth power) over C_K (only happened in case 2 on page 54, all the irregular singularities are not in the base field). Although it is possible to compute all C's up to $\sim$ directly, it is a complicated algebraic problem. In stead, to find proper C's in that case, we add one irregular singularity p into C_K. So now we work over $C_K(p)$. This action required that we compute the exponent difference over the new field $C_K(p)$, because some algebraic singularities might factor now. Once we extend the base field, we only need to deal with the other two types of algebraic extension. And C is unique (up to multiplication by a dth power) in the new field according to Lemma 19.

This example give us the idea why we need to introduced the algebraic extension.

Example 15.

$$L := Dx^2 - \frac{3(4x^{10}-2x^8+11x^6-13x^4-5x^2+1)Dx}{x(4x^6-6x^4-5x^2+3)(x^4+1)} + \frac{x^2(12x^{18}+18x^{16}-83x^{14}+83x^{12}+534x^10-714x^8-299x^6+315x^4+224x^2-58)}{(4x^6-6x^4-5x^2+3)(x^4+1)^4}$$

Let ξ_8 be one of the roots of x^4+1. L has no regular singularities and 4 irregular singularities at four root of x^4+1. Since every point in S_{irr} is not in $\mathbb{Q}$, it is case 2 on page 54. So we temporarily extend the constant field to $\mathbb{Q}(\xi_8)$. Now it reduced to case 1 on page 53. At $x=\xi_8$, we have the truncated power series $-\frac{\xi_8^2}{2}t_{\xi_8}$. $B=(x^4+1)^2$ and $d_A=8$. So the denominator of ν can be 4 or 8. Let us assume $\nu=\frac{1}{4}$. We can compute the truncated series of $A = CA_1A_2^4$ at $x = \xi_8$ is $-8t_{\xi_8}^0$. So let $C = -8$. Then we find the following candidates of g:

$$\left\{ -\frac{8}{(x^4+1)^2}, \frac{8x^4}{(x^4+1)^2}, -\frac{8x^8}{(x^4+1)^2}, \frac{2(x^2-1)^4}{(x^4+1)^2}, \frac{2(x^2+1)^4}{(x^4+1)^2}, \right.$$

$$-\frac{1}{2}\cdot\frac{(x^2+(\xi_8^3-\xi_8)x-1)^4}{(x^4+1)^2}, -\frac{1}{2}\cdot\frac{(x^2-(\xi_8+\xi_8^3)x+1)^4}{(x^4+1)^2},$$

$$\left. -\frac{1}{2}\cdot\frac{(x^2+(\xi_8-\xi_8^3)x-1)^4}{(x^4+1)^2}, -\frac{1}{2}\cdot\frac{(x^2+(\xi_8+\xi_8^3)x+1)^4}{(x^4+1)^2} \right\}$$

Note here we have $C \sim 2 \sim -\frac{1}{2} \sim 8$ in $\mathbb{Q}(\xi_8)$.

Using Remark 24 in Section 4.10, we can discard those candidates not defined in $\mathbb{Q}(x)$. So we get the list of candidates of g:

$$\left\{ -\frac{8}{(x^4+1)^2}, \frac{8x^4}{(x^4+1)^2}, -\frac{8x^8}{(x^4+1)^2}, \frac{2(x^2-1)^4}{(x^4+1)^2}, \frac{2(x^2+1)^4}{(x^4+1)^2} \right\}$$

Then we compute it under $\longrightarrow_{EG}$. We find the general solutions:

$$C_1\sqrt{\frac{1}{(x^2-1)}}\left(I_{\frac{1}{4}}\left(\frac{\sqrt{2}(x^2-1)^2}{(x^4+1)}\right)+\sqrt{2}(x^2-1)I_{\frac{5}{4}}\left(\frac{\sqrt{2}(x^2-1)^2}{(x^4+1)}\right)\right)$$
$$+C_2\sqrt{\frac{1}{(x^2-1)}}\left(K_{\frac{1}{4}}\left(\frac{\sqrt{2}(x^2-1)^2}{(x^4+1)}\right)+\sqrt{2}(x^2-1)K_{\frac{5}{4}}\left(\frac{\sqrt{2}(x^2-1)^2}{(x^4+1)}\right)\right)$$

If we do not introduce the algebraic extension and work the problem over $\mathbb{Q}(x)$ instead of $\mathbb{Q}(\xi_8)(x)$, then we will get fewer candidates from $C = -8$.[4] Since $C \not\sim 2$ over $\mathbb{Q}$ then we will not find the solutions from $C = -8$. If we can find $C = 2$ when we work over $\mathbb{Q}$, then we can find the solutions. But there is no obvious way to find such C. The example also suggests that $C = 8$ might lead to solutions. If we work over $\mathbb{Q}$ and we want to find the solutions, then we need to find all candidates of C, i.e $\{-8, 8, 2\}$ in this example, and we need to find which candidate of C will lead to the final solutions. It is not a easy problem.

[4]this was what an old version of the program did, it can only find the candidates $-\frac{8}{(x^2+1)^4}$ and $-\frac{8x^8}{(x^2+1)^4}$

The reason we choose algebraic extension instead of compute C direclty is the following:
If we apply the algebraic extension we reduce our problem to a relatively easy case, which has been handled in our program. And we can discard the candidates which not in C_K by remark 24 in section 4.10. By contrast, if we deal with the algebraic problem directly, we will meet the problem with uncertain complexity.

4.8 The Algorithm

The input of the algorithm is a differential operator L of order 2. We want to find whether there exist solutions can be represented in terms of bessel functions. If they exist, then find the solutions. Otherwise the algorithm outputs $\emptyset$. Algorithm 8 on next page gives the sketch.('BesselSolver' is an implementation of my code.)

Here we also want to compare our algorithm with the old Bessel solver in [14]. Any operator which is solvable in old algorithm can be solved in our algorithm. The old solver can not solve the square root case. For those cases that can be solved by both solver, it is natural that we want to choose the faster one. So we can compare our algorithm and the old algorithm for those cases. To find the list of f's, the old one need to try all the possible sign of each exponent difference, since we square the exponent differences, we do not need to deal with the signs, instead, we need to find the dth root of power sires. In general, if no algebraic extension exist, both algorithm run very fast. If we need algebraic extension, the old solver works a little bit faster.

Example 16. *We can find Example 15 can be solved in both solver. In my computer, the old one runs 1.81 second and our solver runs 1.93s*

Although the old code can be fast, we should always run the new code, because even if the generalized exponents are not ramified, it might only be solved in the square root case. Here is the example:

Input : an irreducible differential operator L
Output: solutions represented in terms of Bessel functions if they exist

Use Algorithm 3 to compute local information;
Compute the number of linear equations N;
$(N \geq \#S_{reg} + \frac{1}{2}d_A)$;
if $N > d_A$ **then**
| go to **the easy case**, Algorithm 4
else if *L logarithmic at some $p \in S_{reg}$* **then**
| go to **the logarithmic case**, Algorithm 5
else if *there is $p \in S_{reg}$ with $\Delta(L,p) \notin \mathbb{Q}$ (i.e $\nu \notin \mathbb{Q}$)* **then**
| go to **the irrational case**, Algorithm 6
else
| go to **the rational case**, Algorithm 7
end
`/* It will give us a list of candidates for` (f,ν)`, where` f `is the function of the change of variables, and` ν `is the parameter of Bessel functions */`
foreach *(f,ν) in list of candidates* **do**
Compute an operator $M_{(f,\nu)}$ such that $L_B \xrightarrow{f}_C M_{(f,\nu)}$;
Use Algorithm 1 to compute whether $M_{(f,\nu)} \longrightarrow_{EG} L$ and compute the transformation;
if *such transformation exists* **then**
| Add the solution to Solutions List
end
end
Output the solutions list;

Algorithm 8: Main Algorithms

Example 17. *The generalized exponent of Example 11 are not ramified. The only singularity is ∞ with truncated series $\frac{4}{9}t_\infty^{-6} - \frac{4}{3}t_\infty^{-4} + O(t_\infty^{-3})$. It is not ramified. But it can not be solved by the old solver. It can be solved by our solver.*

4.9 Examples

Here we list more examples:

This example is very short but have complicated solutions.

Example 18.

$$L := \partial^2 + \frac{4+3x+x^2}{3x}\partial - \frac{x+1}{9x};$$

It has one regular singularity at $x = \infty$ with truncated series $\frac{1}{144}t_\infty^{-4} + 0t_\infty^3$ and one regular singularity at 0 with exponent difference $\frac{1}{3}$. Download our program and input 'BesselSolver(L)' produces

$$C_1\frac{e^{-\frac{1}{12}x^2-\frac{1}{2}x}}{\sqrt{4+x}x^{\frac{1}{6}}}\left((x^2+6x+16)I_{\frac{1}{3}}\left(\frac{1}{12}\sqrt{(4+x)^3x}\right)+I_{\frac{4}{3}}\left(\frac{1}{12}\sqrt{(4+x)^3x}\right)\right)$$
$$+C_2\frac{e^{-\frac{1}{12}x^2-\frac{1}{2}x}}{\sqrt{4+x}x^{\frac{1}{6}}}\left((x^2+6x+16)K_{\frac{1}{3}}\left(\frac{1}{12}\sqrt{(4+x)^3x}\right)+K_{\frac{4}{3}}\left(\frac{1}{12}\sqrt{(4+x)^3x}\right)\right)$$

Example 19. *The example occurred in [16], was solved in [17]. It shows how to solve higher order equations in terms of Bessel functions:*

$$L := x\partial^4 + 2\partial^3 - \frac{9M+8x^2}{xM}\partial^2 - \frac{-9M+8x^2}{x^2M}\partial - \frac{\lambda^2x(\lambda^2M+8)}{M}$$

We can use Maple command DFactorLCLM to factor L into two factor L_1 and L_2, so $L = LCLM(L_1, L_2)$. Here

$$L_1 := \partial^2 + \frac{(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-48M)\partial}{x(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-16M)}$$
$$-\frac{4M^3\lambda^4+32\lambda^2M^2+16x^2\lambda^4M^2+80\lambda^2x^2M+128x^2+\lambda^6x^2M^3}{M(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-16M)}$$

$$L_2 := \partial^2 + \frac{(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-48M)\partial}{x(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-16M)}$$
$$-\frac{4M^3\lambda^4+32\lambda^2M^2+16x^2\lambda^4M^2+80\lambda^2x^2M+128x^2+\lambda^6x^2M^3)}{(M(x^2\lambda^4M^2+8\lambda^2x^2M+16x^2-16M))}$$

Now both L_1 and L_2 are order 2 operator so we can call BesselSolver: We solve L_1, we get:

$$\frac{C_1}{x}\left(\sqrt{\frac{\lambda^2M+8}{M}}x(\lambda^2M+4)I_0\left(\sqrt{\lambda^2+\frac{8}{M}}x\right)+(-2\lambda^2M-16)I_1\left(\sqrt{\lambda^2+\frac{8}{M}}x\right)\right)$$
$$\frac{C_2}{x}\left(\sqrt{\frac{\lambda^2M+8}{M}}x(\lambda^2M+4)K_0\left(\sqrt{\lambda^2+\frac{8}{M}}x\right)+(-2\lambda^2M-16)K_1\left(\sqrt{\lambda^2+\frac{8}{M}}x\right)\right)$$

We solve L_2, we get:

$$\frac{C_1}{x}(\lambda x(\lambda^2 M+4)J_0(\lambda x)-2J_1(\lambda x)\lambda^2 M)$$
$$+\frac{C_2}{x}(\lambda x(\lambda^2 M+4)Y_0(\lambda x)-2Y_1(\lambda x)\lambda^2 M)$$

$V(L_1)$ and $V(L_2)$ will generate the solution space $V(L)$.

4.10 Proof of Uniqueness

In this section, we want to prove that the change of variables $f=\sqrt{g}$ is unique. In this section, we prove it only for $K=\overline{\mathbb{Q}}(x)$ and $\overline{K}=\overline{\mathbb{Q}(x)}$ and $\nu\in\overline{\mathbb{Q}}$. We also exclude the case $\nu=\frac{1}{2}+n$, $n\in\mathbb{Z}$, because L_B is reducible in that case.

Theorem 11. *If $L_B^{\surd}$ has a solution $\exp(\int r)(r_0B_\nu(\sqrt{g})+r_1(B_\nu(\sqrt{g}))')$ where $r,r_0,r_1,g\in\overline{\mathbb{Q}(x)}$ and $\nu\in\overline{\mathbb{Q}}$, then $g=x$, $r_1=0$ and $\exp(\int r)r_0$ is a constant.*

Remark 22. *Using a standard argument, the theorem implies a similar statement with $\mathbb{Q}$ replaced by $\mathbb{C}$.*

To prove Theorem 11, we need several lemmas:

Lemma 21. *Let r,r_0,r_1,g be as in Theorem 11. Then $\exp(\int r)\in\overline{\mathbb{Q}(x)}$. Thus we can rewrite the solution in Theorem 11 as $\tilde{r}_0B_\nu(\sqrt{g})+\tilde{r}_1(B_\nu(\sqrt{g}))'$, $\tilde{r}_0,\tilde{r}_1\in\overline{\mathbb{Q}(x)}$.*

Proof. The Wronskian of two functions y_1,y_2 is $W(y_1,y_2):=y_1y_2'-y_2y_1'$. Define the Wronskian of an operator L as the Wronskian of two independent solutions of L. Then if y_1,y_2 are independent solutions of $L:=\partial^2+a_1\partial+a_0$, then $W(y_1,y_2)$ satisfies the differential equation $W'=-a_1W$. (See [35] and [32]).
If $L_B^{\surd}$ has solutions $\exp(\int r)(r_0B_\nu(\sqrt{g})+r_1(B_\nu(\sqrt{g}))')$, then we have

$$L_B^{\surd}\xrightarrow{g}_C M_1\xrightarrow{r_0,r_1}_G M_2\xrightarrow{r}_E L_B^{\surd} \tag{4.1}$$

One can compute formulas for the effect of $\xrightarrow{g}_C$, $\xrightarrow{r_0,r_1}_G$ and $\xrightarrow{r}_E$ on the Wronskian. Applying those to (4.1) gives $\exp(\int r)\in\overline{\mathbb{Q}(x)}$.

Let $\tilde{r}_0 = \exp(\int r)r_0$ and $\tilde{r}_1 = \exp(\int r)r_1$. $L_B^{\surd}$ has solutions $\tilde{r}_0 B_\nu(\sqrt{g}) + \tilde{r}_1(B_\nu(\sqrt{g}))'$, where $\tilde{r}_0, \tilde{r}_1 \in \overline{\mathbb{Q}(x)}$. $\square$

So from this point, it is sufficient to consider the form $r_0 B_\nu(\sqrt{g}) + r_1(B_\nu(\sqrt{g}))'$. To continue our proof, we need to use the p-curvature. (see Section 3.2 and [35]).

Lemma 22. *For $u \in \overline{\mathbb{F}_p}(x)$:*

(i) $\chi_p(\partial - u) = T - u^{(p-1)} - u^p$, *where $u^{(n-1)}$ is the derivative of order $p-1$.*

(ii) $u^{(p-1)} + u^p = 0$ *if and only if u is a logarithmic derivative of an element of $\overline{\mathbb{F}}_p(x)$.*

Proof. See [33]. $\square$

Lemma 23. *The minimal operator of $I_{\frac{1}{2}+n}(x)$ (an element of $\mathbb{Q}(x)[\partial]$) has two independent solutions: $y_1 = \frac{e^{-x}}{x^{1/2}} P_n(\frac{1}{x})$ and $y_2 = \frac{e^{x}}{x^{1/2}} P_n(-\frac{1}{x})$, where $P_n(x)$ are Bessel Polynomials satisfying $P_0(x) = 1$, $P_1(x) = 1 + x$, $P_2(x) = 1 + 3x + 3x^2$ and $P_n(x) = P_{n-2}(x) + (2n-1)xP_{n-1}(x)$*

Proof. See A001498: Coefficients of Bessel polynomials in [42]. $\square$

Lemma 24. *If $\nu \in \overline{\mathbb{Q}}$, then $\chi_p(L_B) = T^2 - 1$ for infinitely many primes p.*

Proof. By Chebotarev's density theorem, there are infinity many p, such that $\nu \equiv \frac{1}{2} + n \mod p$, for some $n \in \mathbb{F}_p$.

According to lemma 23, for such p,

$$L_B \equiv \mathrm{LCLM}\left(\partial + 1 - \frac{1}{2x} - \frac{(P_n(1/x))'}{P_n(1/x)}, \partial - 1 - \frac{1}{2x} - \frac{(P_n(-1/x))'}{P_n(-1/x)}\right) \mod p.$$

Note that $\frac{(P_n(1/x))'}{P_n(1/x)}$ is a logarithmic derivative, and so is $\frac{1}{2x} \mod p$. According to Lemma 22, these parts have no effect on the p-curvature. So:

$$\chi_p(L_B) = \chi_p(\mathrm{LCLM}(\partial + 1, \partial - 1)) = T^2 - 1$$

$\square$

Lemma 25. *If $\nu \in \overline{\mathbb{Q}}$, and $r_0 B_\nu(f) + r_1(B_\nu(f))'$ is a solution of L_B where $r_0, r_1, f \in \overline{\mathbb{Q}(x)}$, then $f = \pm x + c$, where c is a constant.*

Proof. Let $p > \deg(f)$ be some prime (not 2), for which ν reduces to an element of $\mathbb{F}_p$. According to Chebotarev's density theorem, there are infinite many such p. According to the assumptions in the lemma, we have $L_B \xrightarrow{f}_C M \xrightarrow{r_0,r_1}_G L_B$. According to Theorem 7 and Remark 9 in section 3.2, $M \xrightarrow{r_0,r_1}_G L_B$ implies $\chi_p(L_B) = \chi_p(M)$. According to Lemma 24, we have

$$\chi_p(M) = \chi_p(L_B) = \chi_p(\mathrm{LCLM}(\partial + 1, \partial - 1)) = T^2 - 1. \tag{4.2}$$

We can also compute $\chi_p(M)$ from $L_B \xrightarrow{f}_C M$, and find

$$\begin{aligned}
\chi_p(M) &= \chi_p\left(\mathrm{LCLM}\left(\frac{d}{df} + 1, \frac{d}{df} - 1\right)\right) \\
&= \chi_p\left(\mathrm{LCLM}\left(\frac{1}{f'}\frac{d}{dx} + 1, \frac{1}{f'}\frac{d}{dx} - 1\right)\right) \\
&= \chi_p\left(\mathrm{LCLM}\left(\partial + f', \partial - f'\right)\right) \\
&= \chi_p(\partial + f')\chi_p(\partial - f') \\
&= (T + (f')^p + (f')^{(p-1)})(T - (f')^p - (f')^{(p-1)}) \\
&= (T - (f')^p)((T + (f')^p)) \\
&= T^2 - (f')^{2p}
\end{aligned} \tag{4.3}$$

Here $f^{(p)} = 0$ for any f, see Remark 9 in section 3.2.

Comparing $\chi_p(M)$ in (4.2) and (4.3) gives $(f')^{2p} = 1$, thus $(f')^p = \pm 1$ for infinitely many p. So for infinitely many p, $f' = \pm 1$. So $f = \pm x + c$, for some constant c. $\square$

Lemma 26. *The constant c in Lemma 25 is equal 0.*

Proof. We only prove for $f = x + c$, same proof can be used for $f = -x + c$. Suppose $L_B \xrightarrow{x+c}_C M \xrightarrow{r_0,r_1}_G L_B$.

Case 1: If $r_0, r_1 \in \overline{\mathbb{Q}}(x)$, according to Lemma 8, $\xrightarrow{r_0,r_1}_G$ only shift generalized exponents by integers. If $\nu \in \mathbb{Z}$, then L_B has only one regular singularity $x = 0$ and it is logarithmic at $x = 0$. M has only one regular singularity $x = c$ and it is logarithmic at $x = c$, so $c = 0$. If $\nu \notin \mathbb{Z}$ (the assumption also exclude the case that $\nu \in \frac{1}{2}\mathbb{Z}$), L_B has only one regular singularity and $\Delta(L_B, 0) = 2\nu \notin \mathbb{Z}$. Since $\longrightarrow_{EG}$ only shift exponent difference by integer so $\Delta(M, 0) \notin \mathbb{Z}$. So 0 is a regular singularity of M. But M has only one regular singularity at $x = c$. So $c = 0$.

Case 2 (general case): If $r_0, r_1 \in \overline{\mathbb{Q}(x)}$, then let G be Galois group $\overline{\mathbb{Q}(x)}$ over $\overline{\mathbb{Q}}(x)$. Note that $x + c$ is invariant under G. We claim the map $r_0 + r_1\partial : V(M) \to V(L_B)$ is unique up to scalar multiplication. If it is not, then $r_0 + r_1\partial$ has a 1-dimensional eigenspace[5], which gives us an order one factor of M in $\overline{\mathbb{Q}(x)}$. But it is not possible, according to differential Galois theory[6], L_B and M are irreducible even over $\overline{\mathbb{C}(x)}$. Hence $r_0 + r_1\partial$ only changes by scalar multiplication under the action of G. So r_0, r_1 can only change by the same scalar factor under the action of G. Assume $r_0 \neq 0$. Then $\tilde{r}_1 := \frac{r_1}{r_0}$ is G-invariant and hence in $\overline{\mathbb{Q}}(x)$.

Now $r_0 + r_1\partial = r_0(1 + \tilde{r}_1\partial)$. Since $1 + \tilde{r}_1\partial$ is G-invariant, so r_0 can only change by scalar multiplication under G. So $r := \frac{r_0'}{r_0}$ is G-invariant. So $r \in \overline{\mathbb{Q}}(x)$ and $\exp(\int r) = r_0$.

Now the solution can be written as $\exp(\int r)\,(1 + \tilde{r}_1\partial)$ and $r, \tilde{r}_1 \in \overline{\mathbb{Q}}(x)$. It is reduced to case 1. So we get $c = 0$.

If $r_1 \neq 0$, then we can use $r := \exp(\int r_1)$ and $\tilde{r}_0 := \frac{r_0}{r_1}$. We can use the same argument to show $c = 0$. $\square$

Proof of Theorem 11. Since $\exp(\int r)(r_0 B_\nu(\sqrt{g}) + r_1(B_\nu(\sqrt{g}))')$ is a solution of $L_B^{\checkmark}$, $g, r, r_0, r_1 \in \overline{\mathbb{Q}(x)}$ and $L_B^{\checkmark} \xrightarrow{x^2}_C L_B$, so $\exp(\int r(x^2)2xdx)(r_0(x^2)B_\nu(\sqrt{g(x^2)}) + r_1(x^2)(B_\nu(\sqrt{f(x^2)}))')$ is a solution of L_B. According to Lemma 25 and Lemma 26, We have $\sqrt{g(x^2)} = \pm x$. So $g(x^2) = x^2$. Hence, $g = x$.

[5] After a choice of basis of $V(M)$ and $V(L_B)$ as $\overline{\mathbb{Q}}$-vector space, the map $r_0 + r_1\partial$ corresponds to a 2x2 matrix.

[6] If the Kovacic's algorithm [22] find no solution of a second order $L \in \mathbb{C}(x)[\partial]$, then L is irreducible in $\overline{\mathbb{C}(x)}[\partial]$

Now $\exp(\int r)(r_0+r_1\partial)$ gave us a map from $V(L_B^{\checkmark})$ to itself. The map is unique up to scalar multiplication. Otherwise we can get a 1-dimension eigenspace of $V(L_B^{\checkmark})$, i.e a first order factor of $L_B^{\checkmark}$. But $L_B^{\checkmark}$ is irreducible over $\overline{\mathbb{C}(x)}$. Since $id : V(L_B^{\checkmark}) \to V(L_B^{\checkmark})$ is a such map, we get our conclusion. $\square$

Theorem 12. *(Uniqueness) If* L *has a solution* $\exp(\int r)(r_0B_\nu(f_1)+r_1(B_\nu(f_1))')$ *and* $\exp(\int \hat{r})(\hat{r}_0B_\nu(f_2)+\hat{r}_1(B_\nu(f_2))')$ *where* $r, r_0, r_1, \hat{r}, \hat{r}_0, \hat{r}_1, g_1 = f_1^2, g_2 = f_2^2 \in \overline{\mathbb{Q}}(x)$*, then* $f_1 = \pm f_2$.

Proof. Let $g \in \overline{\mathbb{Q}}(x)$ and $g = \frac{N}{D}$, where $N, D \in \overline{\mathbb{Q}}[x]$. Then we define an inverse (denote by g^{-1}) of g over $\overline{\mathbb{Q}(x)}$ as a solution in $\overline{\mathbb{Q}(x)}$ of $N(T) - D(T) \cdot x \in \overline{\mathbb{Q}}(x)[T]$. According to the assumptions, we have

$$L_B^{\checkmark} \xrightarrow{g_1}_C M_1 \longrightarrow_{EG} L$$

and

$$L_B^{\checkmark} \xrightarrow{g_2}_C M_2 \longrightarrow_{EG} L.$$

Since $\longrightarrow_{EG}$ are equivalence relations, we have

$$L_B^{\checkmark} \xrightarrow{g_1}_C M_1 \longrightarrow_{EG} M_2 \xrightarrow{g_2^{-1}}_C L_B^{\checkmark}.$$

If we write the solution space of $L_B^{\checkmark}$ following the diagram, its solutions can be written as

$$\exp\left(\int r(g_2^{-1})(g_2^{-1})'dx\right)\left(r_0(g_2^{-1})B_\nu(g_2^{-1}g_1) + r_1(g_2^{-1})(B_\nu(g_2^{-1}g_1))'\right)$$

According to Theorem 11 we have $g_2^{-1}g_1 = x$. So $g_1 = g_2$, thus $f_1 = \pm f_2$. $\square$

Remark 23. *Theorem also holds if we replace* $\overline{\mathbb{Q}}(x)$ *by* $\overline{\mathbb{Q}(x)}$*. The proof is essentially the same, but definition of* g^{-1} *is more technical.*

Remark 24. *Let* $\mathbb{Q} \subseteq C_K \subseteq \overline{\mathbb{Q}}$*. Let* $K = C_K(x)$ *and* $L \in K[\partial]$ *order 2. Suppose* L *has a solution of the form* $\exp(\int r)(r_0B_\nu(f) + r_1(B_\nu(f))')$*, where* $r, r_0, r_1, f \in \overline{\mathbb{Q}(x)}$*.*

The uniqueness of f^2 implies $f^2 \in K$. In the algorithm any candidates g for f^2 that is not defined over K can thus be discarded without further computation. In some complicated examples, it significantly reduces CPU time.

Although some steps of the algorithm involve filed extension of C_K, we may keep track of the original field C_K so that we can use Remark 23 to eliminate as many as possible candidate g's

CHAPTER 5

SOLVING DIFFERENTIAL EQUATIONS IN TERMS OF ${}_0F_1$ AND ${}_1F_1$

In this chapter, we extend our algorithm to Airy and Kummer/Whittaker function. So we get a complete solver for all ${}_0F_1$ and ${}_1F_1$ functions (defined in Section 2.3.3)

5.1 Airy Functions

According to Lemma 3 or Example 20, Airy functions are a special case of Bessel function with square root. So we do not need a another solver for Airy functions, we can just add some restrictions to our original solver to obtain an Airy solver. If an operator L can be solved by Airy functions, then it can be solved by Bessel functions as well. But the solution might shorter in terms of Airy than Bessel.

Example 20. *If we input Airy operator, $L_A = \partial^2 - x$ into Algorithm 8, we find*

$$C_1\sqrt{x}I_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}) + C_2\sqrt{x}K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})$$

Suppose L has Bessel type solutions $B_\nu(f)$. If it is also Airy solvable, then ν must be in $\frac{1}{3}+\mathbb{Z}$ or $\frac{3}{2}+\mathbb{Z}$ and each factor in $f^2 = g$ must have multiplicity 3 or a multiple of 3. So for each candidate pair (f, ν) in Algorithm 8 in Section 4.8, we can check if it satisfies these two conditions. If so, then we can apply the change of variable to $L_A \xrightarrow{\sqrt[3]{g}}_C M$ and then compute $\longrightarrow_{EG}$ transformation.

Example 21.

$$
\begin{aligned}
L = \partial^2 - & \frac{(3x^4+3x^3+5x^2+5x+8)\partial}{x(x-1)(x+1)(x+3x^2+2)} \\
& + \frac{1}{4x^8(x-1)(x+1)(x+3x^2+2)}(108x^{19} - 216x^{18} - 288x^{17} + 688x^{16} \\
& + 192x^{15} - 464x^{14} - 176x^{13} - 560x^{12} + 568x^{11} + 783x^{10} - 359x^9 \\
& - 137x^8 - 345x^7 - 38x^6 + 368x^5 - 16x^4 - 4x^3 - 40x^2 - 48x + 32)
\end{aligned}
$$

Use Bessel solver, we get the candidate $\nu = \frac{1}{3}$, $f := \frac{2}{3}\frac{(x-1)^{\frac{9}{2}}(x+1)^3}{x^3}$. The denominator of ν is $\frac{1}{3}$ and $f^{\frac{2}{3}} = \frac{(x-1)^3(x+1)^2}{x^2}$. So L may be solvable in terms of Airy functions. So we apply $L_A \xrightarrow{f^{\frac{2}{3}}}_C M$, then compute the projectively equivalence between M and L, and find:

$$
C_1 \frac{Ai\left(\frac{(x-1)^3(x+1)^2}{x^2}\right)}{\sqrt{x}} + C_2 \frac{Bi\left(\frac{(x-1)^3(x+1)^2}{x^2}\right)}{\sqrt{x}}
$$

Example 22. $L = \partial^2 + 2 - 10x + 4x^2 - 4x^4$.

This is also Example 11. If we solve it in terms of Bessel functions, we get $\nu = \frac{1}{3}$ and $f = (x^2-1)^{\frac{3}{2}}$. So L should be Airy-solvable. So we apply $L_A \xrightarrow{x^2-1}_C M$, then compute the projectively equivalence. We find:

$$
\begin{aligned}
& C_1((x+1)(2x-1)Ai(x^2-1) + (1+2x)Ai(x^2-1)) \\
& + C_2((x+1)(2x-1)Bi(x^2-1) + (1+2x)Bi(x^2-1))
\end{aligned}
$$

This solution is shorter than the one given in Example 11.

We summarize it as the following algorithm

Input : an irreducible differential operator L
Output: solutions represented in terms of Airy functions if they exist

Use Algorithm 3 to compute local information S_{reg} with exponent difference, S_{irr} with truncated series and B, d_A ;
Enter different cases to compute Bessel pair (ν, f);
if *at any point* $\text{denom}(\nu) \neq 3$ *or* $f^{\frac{2}{3}}$ *is not rational* **then**
  drop that case;
end
foreach (ν, f) *left* **do**
  compute $Ai \xrightarrow{f^{\frac{2}{3}}}_C M$;
  Use algorithm described in [4] to compute whether $M \longrightarrow_{EG} L$ and compute the transformation;
  if *such transformation exists* **then**
    Add the transformation along with $f^{\frac{2}{3}}$ to Solutions List
  end
end
Output the solution list;

Algorithm 9: Airy Solver

5.2 Kummer\Whittaker Functions

The Bessel solver can be used to solve differential equation in terms of Whittaker functions. The same method can be found in [15] section 3.5. We include it here fore completeness.

The Whittaker operator is:

$$L_W := \partial^2 - \frac{1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4} - \nu^2}{x^2}$$

It has two independent solutions:

$$M_{\mu,\nu}(x) := \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}+\nu} M\left(\frac{1}{2} + \nu - \mu, 1 + 2\nu, x\right)$$

$$W_{\mu,\nu}(x) := \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}+\nu} U\left(\frac{1}{2} + \nu - \mu, 1 + 2\nu, x\right)$$

Here $M(\mu,\nu,x)$ and $U(\mu,\nu,x)$ are the Kummer functions.
Whittaker operator has two singularities, $x = 0$ and $x = \infty$. The generalized exponents are $\frac{1}{2} \pm \nu$ at $x = 0$ and $\pm(\frac{1}{2}t_\infty^{-1} + \mu)$ at $x = \infty$. Recall that

$$M(\alpha, \gamma, x) := {}_1F_1(\alpha; \gamma; x)$$

$$\exp(-\frac{x}{2})_1F_1(\alpha; 2\alpha; x) = {}_0F_1(; \frac{1}{2} + \alpha; \frac{x^2}{16}).$$

$$J_v(x) = (\frac{x}{2})^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1(; \nu+1; -\frac{x^2}{4})$$

and

$$I_v(x) = (\frac{x}{2})^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1(; \nu+1; \frac{x^2}{4})$$

With these equations, we can prove that with projective equivalence, any ${}_0F_1$ and ${}_1F_1$ functions can be written in terms of Whittaker functions or Bessel functions (the latter with square root).
Now we can build a similar algorithm for Whittaker solver. Recall that in Bessel case, we have only one parameter ν and We can only find ν modulo $\mathbb{Z}$. For Whittaker case, we have:

Lemma 27. *Suppose some combination of the following transformation:*

(i) $\nu \mapsto \nu + 1$

(ii) $\mu \mapsto \mu + 1$

(iii) $\mu \mapsto \mu + \frac{1}{2}$ *and* $\nu \mapsto \nu + \frac{1}{2}$

changes L_W *to* L*, then* L_W *and* L *are projectively equivalent.*

Hence it is sufficient to compute one parameter modulo $\frac{1}{2}\mathbb{Z}$ and the other modulo $\mathbb{Z}$. Now we can study the generalized exponents to find the parameters and change of variables, like we did for Bessel. Although Whittaker has two parameters, when we fix μ, its generalized exponents behaviors like Bessel functions. In fact, we have similar results for Whittaker function as Theorem 8 for Bessel functions.

Theorem 13. *Let $K = C_K(x)$, and $L_W \xrightarrow{f}_C M \longrightarrow_{EG} L$, where $f \in K$. Note: Similar as Bessel case, L is the input to our algorithm, and f and M are to be computed.*

(i) if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an apparent singularity or $p \in S_{reg}$, and $\Delta(M,p) = 2m_p\nu$.

(ii) p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(M,p) = 2m_p\mu + \sum_{i<0} i f_i t_p^i$.

Proof. The proof is analogous to the proof of Theorem 8. The only difference is at $x = \infty$. We just need to adjust the generalized exponents of Bessel function to Whittaker function. We will get the extra term $2m_p\mu$. □

The only difference of generalized exponents between Bessel case and Whittaker case is the constant terms $2m_p\mu$ of exponent differences of irregular singularities. The parameter μ only appears in the constant terms of exponent differences of irregular singularities. So we can find a potential list of μ modulo $\frac{1}{2}\mathbb{Z}$. Once we get μ then we can remove $2m_p\mu$ for the exponent difference so that we have exactly the same exponent differences as Bessel case. We then can run Algorithm 8 or the algorithm in [14] for Bessel case to find ν and f and the projective equivalence. For μ, we have:

Lemma 28. *Let $L_W \xrightarrow{f}_C M \longrightarrow_{EG} L$ and S_{irr} be the set of all irregular singularities of L, for $p \in S_{irr}$ let m_p be pole order at $x = p$ and let c_p be the constant term of the exponent difference $\Delta(L,p)$ at $x = p$. then let*

$$\mathcal{N}'_p := \left\{ \frac{c_p + i}{2m_p} \mid 0 \le i \le 2m_p - 1, i \in \mathbb{Z} \right\}$$

We can make the rational part of each element in $\mathcal{N}'_p$ belong to $[0, \frac{1}{2}]$ by the transformation $\mu \mapsto \mu + \frac{1}{2}$. Let the new set be $\mathcal{N}_p$. Then $\mu \in \mathcal{N} := \cap_{p \in S_{irr}} \mathcal{N}_p$.

Proof. The lemma follows from the fact that we know the number $c_p = 2m_p\mu \bmod \frac{1}{2}\mathbb{Z}$, and Lemma 27. $\square$

For the Whittaker case, we do not need the square root case. So it is substantially easier than the Bessel case. If $f \notin K$ and $L_W \xrightarrow{f}_C M \longrightarrow_{EG} L$, then $L \notin K[\partial]$. For operators with ramified exponent difference, there are no Whittaker type solutions. So we can use the algorithm in [14] for the Whittaker case, because it runs a little bit faster.

Example 23.

$$L := \frac{\partial^2}{(6x^2+5)^2} - \frac{12x\partial}{(6x^2+5)^3} + \frac{1}{4}\frac{-4x^6 - 20x^4 - 7x^3 - 25x^2 - \frac{35}{2}x - \frac{17}{18}}{(2x^3+5x+3)^2}$$

There is only one irregular singularity at $x = \infty$, the exponent difference is $\Delta(L, \infty) = -6t_\infty^{-3} - 5t_\infty^{-1} + \frac{15}{4}$. So the constant term is $c_\infty = \frac{15}{4}$ and $m_\infty = 3$. According to Lemma 28, the candidate μ's are:

$$\mathcal{N} = \{\frac{5}{8}, \frac{5}{24}, \frac{1}{24}, \frac{1}{8}, \frac{7}{24}, \frac{11}{24}\}.$$

So we can try each μ and then remove the constant term in the exponent difference. Apply the data to Algorithm 8 or (the algorithm in [14]) to get a list of ν and f. We try all combinations, and find:

$$C_1 M_{\frac{5}{8},\frac{1}{3}}(2x^3+5x+3) + C_2 W_{\frac{5}{8},\frac{1}{3}}(2x^3+5x+3)$$

Example 24. *This example is from W. Koepf's and M. Foupouagigni's research about orthogonal polynomials:*

$$L := (4x^4 - 12x^2 + 3)\partial^2 - 2x(4x^4 + 4x^2 - 21)\partial + 64x^4 - 96x^2 + 8nx^4 - 24nx^2 + 6n$$

They suspected the existence of a closed form solution but could not find it. Downloading our implementation at http://math.fsu.edu/∼qyuan and typeing BesselSolver(L) produces:

$$
\begin{aligned}
&C_1((-9n-33+4nx^4+4x^4-4nx^2-4x^2)M_{-2-\frac{1}{2}n,\frac{1}{2}}(x^2)\\
&-4(2x^2-3)(4+n)M_{-1-\frac{1}{2}n,\frac{1}{2}}(x^2))+\\
&+C_2((-9n-33+4nx^4+4x^4-4nx^2-4x^2)U_{-2-\frac{1}{2}n,\frac{1}{2}}(x^2)\\
&-4(2x^2-3)(4+n)U_{-1-\frac{1}{2}n,\frac{1}{2}}(x^2))
\end{aligned}
$$

Where M and U are Kummer functions.

We summarize the Whittaker solver in the following algorithm:

Input : an irreducible differential operator L
Output: solutions represented in terms of Whittaker functions if they exist

Use Algorithm 3 to compute local information S_{reg} with exponent difference, S_{irr} with truncated series and B, d_A ;
For each $s \in S_{irr}$, find a possible list of μ, then find a possible set for μ according to Lemma 28;
For each μ in list, use Algorithm 8 (Bessel Solver), to find a list of (ν, f) ;
foreach (μ, ν, f) **do**
    compute $L_W \xrightarrow{f}_C M$;
    Use algorithm described in [4] to compute whether $M \longrightarrow_{EG} L$ and compute the transformation;
    if *such transformation exists* **then**
        Output the solution
    end
end

Algorithm 10: Whittaker Solver

CHAPTER 6

HEUN FUNCTIONS

As an application of our Bessel solver, we investigate which Heun function can be written in terms of Bessel functions. The relations we find that involve a nontrivial $\longrightarrow_G$ gauge transformation are new. Previously, only $\longrightarrow_E$, $\longrightarrow_C$ solvable examples were known.

6.1 Heun Operator and functions

The general Heun operator is:

$$H_G := \partial^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\partial + \frac{\alpha\beta x - q}{x(x-1)(x-a)}$$

where $\epsilon = \alpha + \beta + 1 - \gamma - \delta$. The solutions of the general Heun operator are Heun functions. The general Heun operator has no Bessel type solutions, because it has no irregular singularities. We can try to solve it in terms of ${}_2F_1$ functions, but that is beyond this thesis. The other four Heun equations are confluent cases, obtained from the general Heun equation above through confluence processes. The Heun Confluent operator:

$$H_C := \partial^2 + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} - \epsilon\right)\partial + \left(\frac{q-\alpha\beta}{x-1} - \frac{q}{x}\right)$$

has regular singularities at $x = 0, x = 1$ and one irregular singularity at $x = \infty$. The solutions of confluent operators are Heun confluent functions denoted by $HeunC(\alpha, \beta, \gamma, \delta, \epsilon; x)$.

The Biconfluent operator:

$$H_B := \partial^2 + \left(-2x - \beta + \frac{1+\alpha}{x}\right)\partial + \left(\gamma - \alpha - 2 - \frac{1}{2}\frac{(1+\alpha)\beta + \delta}{x}\right)$$

has one regular singularity at $x = 0$ and an irregular singularities at $x = \infty$. The solutions of Biconfluent operators are Heun Biconfluent functions denoted by $HeunB(\alpha, \beta, \gamma, \delta; x)$.

The Doubleconfluent operator:

$$H_D := \partial^2 - \frac{(\alpha + 2x + x^2\alpha - 2x^3)\partial}{(z+1)^2(z-1)^2} + \frac{\delta + (2\alpha + \gamma)x + \beta x^2}{(z-1)^3(z+1)^3}$$

has no regular singularities and two irregular singularities at $x = \pm 1$. The solutions of Doubleconfluent operators are Heun Doubleconfluent functions denoted by $HeunD(\alpha, \beta, \gamma, \delta; x)$.

The Triconfluent operators:

$$H_T := \partial^2 + (-\gamma - 3x^2)\partial + (\alpha + \beta x - 3x)$$

has no regular singularities and one irregular singularities at $x = \infty$. The solutions of Triconfluent operators are Heun Triconfluent functions denoted by $HeunT(\alpha, \beta, \gamma; x)$.

Those four operator are solvable in terms of Bessel functions, if the parameters satisfy some conditions. For example, Example 11, Example 13 are also Heun operators, but they can be solved in terms of Bessel functions.

Example 25. *(See also example 11.)* $L = \partial^2 + 2 - 10x + 4x^2 - 4x^4$.

We know it is solvable in term of Bessel. But it is solvable in terms of

$$HeunT\left(\frac{6^{2/3}12}{(-1+\sqrt{-3})^4}, -15/2, -\frac{6^{1/3}4}{(-1+\sqrt{-3})^2}, \left(-6^{-1/3}\sqrt{-2+2\sqrt{-3}}\right)x\right)$$

It was not previously known that a $HeunT$ function with such parameter values can be expressed in terms of Bessel functions.

Example 26. *(See also example 13.)*

$$L := x\partial^2 + (1 - \sqrt{-6}\cdot x - 2x^2)\partial + \frac{36x^2 + 10\sqrt{-6}\cdot x - 1}{36x}$$

This operator is Heun Biconfluent operator with $\alpha = \frac{1}{3}$, $\beta = \sqrt{-6}$, $\gamma = 3$ and $\delta = -\frac{14}{9}\sqrt{-6}$. We saw in example 13 that L can be expressed in terms of Bessel functions. And this relation was not previously known either.

6.2 Solving Heun operator in terms of Bessel Functions

6.2.1 Method to find relations

In this section, we introduce our method of finding Heun functions that can be written in terms of Bessel functions in a non-trivial way (involving $\longrightarrow_G$). Our method works well for Triconfluent Heun function and Biconfluent Heun functions. We use Heun Triconfluent functions to illustrate the idea: If $HeunT$ function can be expressed in terms of Bessel functions, then we have:

$$L_B \xrightarrow{f}_C M \longrightarrow_{EG} H_T \tag{6.1}$$

Here L_B has parameter ν and H_T has parameter α, β, and γ.

Before we introduce our method to find relation, we can consider the analogy to a linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0 \end{cases} \tag{6.2}$$

Does (6.2) have a non-zero solution? For generic values of a_{ij} the answer is no. But if a relation (in this case $\det(a_{ij}) = 0$) is satisfied, then the answer is yes. Likewise, for $HeunT(\alpha, \beta, \gamma; x)$ to be written in terms of Bessel functions, whether (6.1) exists depends on equations in the parameters α, β, and γ. (6.1) does not hold for generic values α, β, and γ. But for some specific values, (6.1) holds similar to the way (6.2) depends on determinant. However, the number of equations in (6.2) was fixed, but for (6.1), the number of equations also depends on the parameters (in particular, the parameters must satisfy not only polynomial conditions similar to $\det(a_{ij}) = 0$ in (6.2), but must also satisfy a Diophantine condition).

Our idea to find the relation between $HeunT(\alpha, \beta, \gamma; x)$ and Bessel functions is to treat α, β, and γ as unknown variables not as transcendental constant[1]. Then we break down Algorithm 8 step by step to find equations for parameters.

[1]if we input H_T into Algorithm 8 directly, then it will try to solve it over $\mathbb{Q}(\alpha, \beta, \gamma)$

Let us find parameters of $L_B \xrightarrow{f}_C M$ first. We can input H_T into Algorithm 8 in section 4.8, follow it step to step, but treat α, β, and γ as unknown variables. Then Algorithm 8 will produce candidates for ν, f in terms of α, β, and γ (hence it produces the middle operator M), and some conditions of α, β, and γ (this only happens for H_B). We can also get one or more (depends on how many irregular singularities we have) Diophantine equations from generalized exponents, because of Lemma 29 and Lemma 31 below:

Lemma 29. *We denote const(e) as the constant term of series e. If a Heun operator H (one of H_C, H_B, H_D, H_T) is solvable in terms of Bessel functions, then for each irregular singularity s of H, const$(\Delta(H, s)) \in \mathbb{Z}$.*

Proof. It follows directly from Theorem 8 in Section 3.3.4. We have const$(\Delta(H, s)) \in \frac{1}{m}\mathbb{Z}$, where $m = 1$ or 2 is ramification index. But if we compute the generalized exponents of all four cases of Heun operators, none of them are ramified. So $m = 1$. $\square$

Once we find ν and f (hence M), we can try to find some specific values for α, β, and γ such that (6.1) holds. We can determine whether M is projectively equivalent to H_T in two steps. First we compute possible $\longrightarrow_E$ (this process will remove some apparent singularities), $M \longrightarrow_E \widetilde{M}$. Then we need to determine whether $\widetilde{M}$ is gauge equivalent ($\longrightarrow_G$) to H_T. This can be reduced to computing rational solutions of a 4th order operator L_s (namely the symmetric product of H_T and the adjoint of $\widetilde{M}$, see Lemma 30). L_s still contains the parameters of H_T.

Definition 30. *We say L_s is the symmetric product of L_1 and L_2 (denoted by $L_1 \circledS L_2$), if L_s is a monic operator with lowest order, such that, if $y_1 \in V(L_1)$ and $y_2 \in V(L_2)$, then $y_1 y_2 \in V(L_s)$.*

Lemma 30. *$\{G \in K[\partial] | G(V(L_1)) \subseteq V(L_2)\}$ correspond to solutions of $L_s := L_1^* \circledS L_2$. If L_1 and L_2 are projectively equivalent, then L_s has an exponential solution. If L_1 and L_2 are gauge equivalent, then L_s has a rational solution. See [35].*

According to the following lemma, finding an exponential solution (i.e finding a first order right hand factor) of L_s can be reduced to a diophantine equation (6.3) and computing polynomial solutions (denoted Q in Lemma 31) of an other operator. Computing polynomial solutions reduces to solving a set of linear equations for the coefficients of the polynomial Q in Lemma 31. Similar to (6.2), this produces on or more polynomial equations for the Heun parameters.

Lemma 31. *Let $r \in \overline{\mathbb{Q}}(x)$. If $\partial - r$ is a first order right hand factor of $L \in \overline{\mathbb{Q}}[\partial]$, and let $const(e)$ denote the constant term of series e. Then for all each singularity s_i, there exists a generalized exponents $e_{s_i} \in \overline{\mathbb{Q}}[t_{s_i}^{-1}]$ of L such that*

$$r = S + \frac{Q'}{Q} \text{ where } S = \sum_{s_i} \frac{e_{s_i}}{t_{s_i}} - t_\infty const(e_\infty) \text{ and } Q \in \overline{\mathbb{Q}}[x]$$

and

$$\deg(Q) + \sum_{s_i} const(e_{s_i}) = 0 \tag{6.3}$$

See [13].

We will illustrate this process by an example in next section. For more details, see http://www.math.fsu.edu/~qyuan/Heun.mw

6.2.2 Triconfluent Heun functions

We start with Triconfluent Heun functions because it has fewer parameters.
If H_T is solvable in Bessel functions, then we have a relation $L_B \xrightarrow{f}_C M \longrightarrow_{EG} H_T$. So we need to find some ν, f for L_B to compute M and then find when M and H_T are equivalent.
First, let us find f. We can find f by examining the generalized exponents. There

is only one singularity ∞ of H_T. If we compute the generalized exponent at $x = \infty$, we get $-\frac{1}{3}\beta + 1$ and $-3t_\infty^3 - \gamma t_\infty + 1 + \frac{1}{3}\beta$. According to Lemma 29, if it is Bessel Solvable, then the constant term of generalized exponents should be an integer. Here the constant term is $\frac{2}{3}\beta$, it must be an integer. Assume it is. Then we can use Algorithm 8 to find candidates of f and ν. We find the only candidate is that $\nu = \frac{1}{3}$ and $f = \frac{1}{2}\sqrt{(\frac{2}{3}\gamma + x^2)^3}$.
So we apply the transformation $L_B \xrightarrow{f}_C M$ and find

$$M := \partial^2 - \frac{-3x^2 + 2\gamma}{x(2\gamma + 3x^2)}\partial - \frac{3}{4}\frac{(8\gamma^3 + 36x^2\gamma^2 + 54x^4\gamma + 27x^6 + 12)x^2}{(2\gamma + 3x^2)^2}$$

Notice that there are several apparent singularities of M. We can compute[2] $\widetilde{M} = M \circledS (\partial - \frac{3x}{3x^2+2\gamma} - \frac{3}{2}x^2 - \frac{\gamma}{2})$. $\widetilde{M}$ is an equivalent operator with fewer apparent singularities.

$$\widetilde{M} := \partial^2 - \frac{\gamma x + 3x^3 + 1}{x}\partial + \frac{x\gamma^2 + 2\gamma - 6x^2}{4x}$$

Now let us consider when $\widetilde{M}$ will be gauge equivalent to H_T. If so, there exist a rational solution for $L_S := \widetilde{M}^* \circledS H_T$. Hence some generalized exponents of the symmetric product are integers. We compute the generalized exponents at $x = \infty$, two of which contains t_∞, which can not be integers. The other two are $\pm\frac{1}{3}\beta + \frac{3}{2}$. So $\pm\frac{1}{3}\beta + \frac{3}{2}$ must be an integer, combine it with $\frac{2}{3}\beta$ is integer. We get:

Lemma 32. *If $HeunT(\alpha, \beta, \gamma; x)$ can be written in terms of $B_\nu(f)$. Then we have $\beta = 3k + \frac{3}{2}$, $k \in \mathbb{Z}$, $\nu = \frac{1}{3}$ and $f = \frac{1}{2}\sqrt{(\frac{2}{3}\gamma + x^2)^3}$.*

Note that it is not a sufficient condition. We need restrictions for γ and α to get the $\longrightarrow_G$ equivalence. Here we show how we can get some specific examples:
Let k in Lemma 32 be 1. So $\beta = \frac{9}{2}$. Then L_S has generalized exponents 0 and -3 at $x = \infty$. So a polynomial with degree 3 might be a solution of L_S. Write $Q(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Computing $L_S(Q)$ and equaling it to 0 produces some polynomial equations for γ, α and the a_i's. There exist two solutions of those

[2] the symmetric product also compute $\longrightarrow_E$

equations. One is trivial[3]. The other is $\gamma = 0, \alpha = 0$[4]. So we plug it into H_T. We get

$$L = \partial^2 - 3x^2\partial + \frac{3}{2}x$$

So this Heun equation L must be Bessel solvable. Our Bessel solver implementations find:

$$\begin{aligned}&\frac{C_1}{x}\exp\left(\frac{1}{2}x^3\right)\left((-3x^3+4)I_{\frac{1}{3}}\left(\frac{1}{2}x^3\right)+3I_{\frac{4}{3}}\left(\frac{1}{2}x^3\right)x^3\right)\\ +&\frac{C_2}{x}\exp\left(\frac{1}{2}x^3\right)\left((-3x^3+4)K_{\frac{1}{3}}\left(\frac{1}{2}x^3\right)+3K_{\frac{4}{3}}\left(\frac{1}{2}x^3\right)x^3\right)\end{aligned}$$

We could also choose other values for k, for example, If $k = 2$ then $L = \partial^2 - 3x^2\partial - \frac{15}{2}x$; If $k = 3$ then $L := \partial^2 + \frac{1}{10}(5+3\sqrt{5})(-5+4x^2+3\sqrt{5})\partial - \frac{1}{4}(5+3\sqrt{5})(4x-\sqrt{5}+3)$; If $k = 4$ then $L := \partial^2 - 3x^2\partial + \frac{21}{2}x$. They are solvable in terms of Bessel functions and also HeunT functions. One can make more examples in the same way.

6.2.3 Heun Doubleconfluent function

We use the same idea in section 6.2.2, we want to find relation $L_B \xrightarrow{f}_C M \longrightarrow_{EG} H_D$. The Heun Doubleconfluent operator

$$H_D := \partial^2 - \frac{\alpha + 2x + x^2\alpha - 2x^3}{(x+1)^2(x-1)^2}\partial + \frac{\delta + 2x\alpha + x\gamma + \beta x^2}{(x-1)^3(x+1)^3}$$

has two irregular singularities ± 1. At $x = 1$ the constant term of generalized exponents is $\frac{\gamma}{2\alpha} + \frac{\beta+\delta}{2\alpha}$. And at $x = -1$, it is $\frac{\gamma}{2\alpha} - \frac{\beta+\delta}{2\alpha}$. They should both be integers. So we have $\frac{\gamma}{\alpha}$ and $\frac{\beta+\delta}{\alpha}$ must be integers for H_D to be Bessel solvable.
Then we use Algorithm 8 in Section 4.8 to find ν and f. We find the only candidate of nu is $\nu = \frac{1}{4}$ and f can be either $f_1 = \frac{1}{2}\sqrt{\frac{\alpha^2}{(x^2-1)^2}}$ or $f_2 = \frac{1}{2}\sqrt{\frac{\alpha^2 x^4}{(x^2-1)^2}}$.

[3]all a_i's equal zero
[4]the value of a_i's are not important as long as they are not all zeroes

Then we can apply the change of variable f_1 and $\nu = \frac{1}{4}$ to L_B. Then we can compute $\longrightarrow_E$ to remove apparent singularities. We get

$$\widetilde{M} = \partial^2 - \frac{-3x^3 + 3x + x^2\alpha + \alpha}{(x-1)^2(x+1)^2}\partial + \frac{1}{4}\frac{\alpha(-2x^3 + x^2\alpha + 6x - \alpha)}{(x-1)^3(x+1)^3}$$

Then we take the symmetric product of M and the adjoint of H_D. We should have some integer exponents at both ± 1. The constant exponents at $x = 1$ are $\frac{7}{4} \pm (\frac{\gamma}{4\alpha} + \frac{\beta+\delta}{4\alpha})$ and at $x = -1$ are $\frac{7}{4} \pm (\frac{\gamma}{4\alpha} - \frac{\beta+\delta}{4\alpha})$. then, we have one of $\frac{\gamma}{\alpha}$ and $\frac{\beta+\delta}{\alpha}$ must be even, and the other odd.

If we apply f_2, we will get something similar. So we have

Lemma 33. *If $HeunT(\alpha, \beta, \gamma; x)$ can be written in terms of $B_\nu(f)$. Then $\frac{\gamma}{\alpha}$ and $\frac{\beta+\delta}{\alpha}$ are integers. One of them is odd and the other is even. $\nu = \frac{1}{4}$ and $f = \frac{1}{2}\sqrt{\frac{\alpha^2}{(x^2-1)^2}}$ or $\frac{1}{2}\sqrt{\frac{\alpha^2 x^4}{(x^2-1)^2}}$*

6.2.4 Heun Biconfluent equations

There exists a relation between Heun Biconfluent functions and Bessel functions. [26]

Lemma 34.

$$HeunB(\alpha, \beta, \gamma, \delta; x) = \frac{I_{\frac{1}{4}\alpha}(\frac{x^2}{2})\Gamma(1 + \frac{1}{4}\alpha)2^{\frac{1}{4}\alpha}e^{\frac{1}{2}x^2}}{(\frac{x^2}{2})^{\frac{1}{4}\alpha}}$$

If $\beta = \gamma = \delta = 0$.

But we can find more cases beyond this lemma, especially with square root case. We also use the same idea as in previous sections. The Heun Biconfluent operator has one irregular singularity at $x = \infty$ and one regular singularity at $x = 0$. At $x = \infty$, the constant term of generalized exponents is $-\gamma$. So γ must be integer.

The exponent difference at $x = 0$ is α, so ν is only related to α and not to other parameters. We can separate the cases depending on whether there is disappearing singularities.

If there are no disappearing singularities, then roots of f are regular singularities. Since we only have one regular singularity at $x = 0$, then $f = \sqrt{cx^4}$. Then we can run Algorithm 8[5]. We get $c = \frac{1}{4}$ and $\beta = 0$. Then we can find several examples by the same method as in sections 6.2.2. What we find in this case is either a reducible operator, or a case related to Lemma 34. So we skip the examples of this case.

Now we assume there are disappearing singularities. In that case, since ν is only related to α, so α must be rational. Since the degree of the numerator (d_A in the algorithm) is 4, the disappearing part can only have degree 3. So the denominator of α (also ν) can only be 3. To find some specific examples, we will take $\alpha = \frac{1}{3}$.

Now we can insert this new information into Algorithm 8. We get $\nu = \frac{1}{3}$ and $f = \frac{1}{2}\sqrt{x(x+\frac{2}{3}\beta)^3}$. Then we compute $\longrightarrow_C$, $\longrightarrow_E$ and L_s as in Section 6.2.2. Finally we get the constant exponents at $x = \infty$ are $\pm\frac{\gamma}{2} + \frac{1}{2}$. So γ must be odd. (This condition is only for $\alpha = \frac{1}{3}$. For example, if $\alpha = \frac{4}{3}$, then γ must be even). We find:

Lemma 35. *If $HeunB(\alpha, \beta, \gamma; x)$ can be written in terms of $B_\nu(f(x))$. then one of the following cases holds:*

(i) $\beta = 0$, γ is integer, and $f = \frac{x^2}{2}$

(ii) $\alpha = \frac{1}{3}$ (up to some integer), γ is integer and $f = \frac{1}{2}\sqrt{x(x+\frac{2}{3}\beta)^3}$

Let $\gamma = 3$, according to the exponents, we might have a degree 2 polynomial as a solution of L_s. Write $Q(x) = a_0 + a_1x + a_2x^2$. Then $L_s(Q) = 0$ gives polynomial equations and we find two solutions, one of which is not trivial. It is $\alpha = \frac{1}{3}$, $\beta = \sqrt{-6}$, $\gamma = 3$ and $\delta = \frac{14}{9}\sqrt{-6}$. We plug it into the H_B. We get:

$$\partial^2 + \frac{\frac{4}{3} - x\sqrt{-6} - 2x^2}{x}\partial + \frac{1}{2}\frac{\frac{4}{3}x + \frac{2}{9}\sqrt{-6}}{x}$$

[5]treat $\alpha, \beta, \gamma, \delta$ as unknown variable instead of transcendental constant

Algorithm 8 in Section 4.8 (which can be download from www.math.fsu.edu/∼qyuan) produces solution:

$$
\begin{aligned}
&C_1 \frac{\exp(\frac{1}{2}x^2 + \frac{1}{2}x\sqrt{-6})}{x^{\frac{1}{6}}\sqrt{2\sqrt{-6}+3x}(6x+\sqrt{-6})} \Bigg(\\
&\quad - 3(6x+\sqrt{-6})(3x^2+3x\sqrt{-6}-8) I_{\frac{1}{3}}\left(\frac{1}{2}\sqrt{(\frac{2}{3}\sqrt{-6}+x)^3 x}\right) \\
&\quad + \sqrt{3(2\sqrt{-6}+3x)^3 x}(6x+\sqrt{-6}) I_{\frac{4}{3}}\left(\frac{1}{2}\sqrt{(\frac{2}{3}\sqrt{-6}+x)^3 x}\right)\Bigg) \\
&+ C_2 \frac{\exp(\frac{1}{2}x^2 + \frac{1}{2}x\sqrt{-6})}{x^{\frac{1}{6}}\sqrt{2\sqrt{-6}+3x}(6x+\sqrt{-6})} \Bigg(\\
&\quad - 3(6x+\sqrt{-6})(3x^2+3x\sqrt{-6}-8) K_{\frac{1}{3}}\left(\frac{1}{2}\sqrt{(\frac{2}{3}\sqrt{-6}+x)^3 x}\right) \\
&\quad + \sqrt{3(2\sqrt{-6}+3x)^3 x}(6x+\sqrt{-6}) K_{\frac{4}{3}}\left(\frac{1}{2}\sqrt{(\frac{2}{3}\sqrt{-6}+x)^3 x}\right)\Bigg)
\end{aligned}
$$

For more examples (by taking different γ) see the Maple worksheet at: http://www.math.fsu.edu/∼qyuan/Heun.mw

6.2.5 Heun Confluent Equations

There exist relations between Heun confluent functions and Bessel functions.

Lemma 36. *[26]*

$$HeunC(\alpha,\beta,\gamma,\delta,\eta;x) = \frac{-I_{\frac{1}{2}\beta}(-\frac{1}{2}x)\Gamma(1+\frac{1}{2}\beta)2^{\frac{1}{2}\beta}\exp(-\frac{1}{2}x)}{\frac{1}{2}x^{\frac{1}{2}\beta}(x-1))}$$

We can following the idea from the previous section, but since HeunC are more general functions, most of cases are reducible or related to easy identities. We still have the following:

Lemma 37. *If $HeunC(\alpha,\beta,\gamma,\delta,\eta;x)$ can be written in terms of $B_\nu(f)$. Then $\nu=\beta$, $f=\frac{\alpha}{2}\sqrt{x(x-1)}$ and $\frac{2\delta}{\alpha}$, $\frac{\gamma}{2}+\frac{3\beta}{2}+\delta\alpha$ are integers.*

CHAPTER 7

CONCLUSION

In this thesis, we developed an algorithm to solve differential equations in terms of ${}_0F_1$ and ${}_1F_1$ functions. Given irreducible second order differential operator $L \in K[\partial]$, we found solutions of the form

$$\exp\left(\int r\right)\left(r_0 B_\nu(\sqrt{f}) + r_1 (B_\nu(\sqrt{f}))'\right)$$

or

$$\exp\left(\int r\right)(r_0 F(f) + r_1 F'(f))$$

where B_ν is a Bessel function with parameter ν, $F(x)$ is Airy function, or Kummer/Whittaker functions with parameter ν, μ and $f, r_0, r_1, r \in K$. Note that our algorithm does not deal with reducible case (for example, Bessel case with $\nu = \frac{1}{2}$), because in that case, we can find solutions by factoring. Our algorithm does not deal with operators with order more than 2, because in that case, if the operator L has ${}_0F_1$ or ${}_1F_1$ type solutions, it must have irreducible order 2 right factor or another type of reduction in [37], then we can apply our algorithm to that factor.

We start with our algorithm for Bessel functions. Given operator L, we study its local information from generalized exponents. From that we can get partial information about zeroes and poles of f. We reconstruct a possible list of f and Bessel parameter ν from that partial information. Then we study if the relation

$$L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$$

exists, here L_B is the modified Bessel operator. If we find such relation, then we can give the solutions.

We also discussed the algorithm for Bessel functions can be extend to Airy and Whittaker functions, which incudes solutions in terms of Kummer functions. So we get a complete solver for ${}_0F_1$ and ${}_1F_1$ functions.

APPENDIX

PROGRAM DESCRIPTION

We have developed all algorithms in this thesis in Maple. The programme can be download at http://www.math.fsu.edu/∼qyuan. In this chapter, we will give the description of important functions implemented in the programme. We will indicate the corresponded Algorithm in thesis.

BesselSolver\BesselSolver_doit

Implementation of Algorithm 8, the main programe

Input:

(i) A differential operator L and the differential domain [x,Dx];

or

(ii) A differential equation Eq and the dependent variable y.

Output: 0 or solution with type $e^{\int r}(r_0F(x) + r_1F'(x))$, where F(x) are bessel, Whittaker, Kummer or Airy functions.

singInfo

Implementation of part of Algorithm 3, collect information of Singularities.

Input: Differential Operator L and Base field ext

Output: S_{reg}, S_{irr} with exponents differences, and determine if it is logarithmic, rational or irrational case.

findWhittaker

Implementation of Algorithm 10, find Kummer/Whittaker type solutions.

Input: Differential operator L with corresponding S_{reg}, S_{irr} ,base field *ext* and which case we will meet.

Output: The solution space of L if it can be written in terms of Whittaker, Kummer functions.

Kummerequiv

Input: Operator $L \in K[\partial]$, two parameter ν, μ, a rational function $f \in K$

Output: $M \in K[\partial]$, $[y_1, y_2]$, where y_1 and y_2 are Kummer functions of the first and second kind with change of variable $x \mapsto f$ and $M(y)$ is a solution of L. or 0 if such solution does not exist.

Whittakerequiv

Input: Operator $L \in K[\partial]$, two parameter ν, μ, a rational function $f \in K$

Output: $M \in K[\partial]$, $[y_1, y_2]$, where y_1 and y_2 are Whittaker functions of the first and second kind with change of variable $x \mapsto f$ and $M(y)$ is a solution of L. or 0 if such solution does not exist.

findBesselvf

Find ν and f for Bessel non-square-root case. It is used for Whittaker solver.

Input: S_{reg}, S_{irr}, base filed *ext* and which case we meet

Output: A list of pair (ν, f)

findBesselvfirrat \findBesselvfK \findBesselvfln \findBesselvfrat \findBesselvfint

Those are the implementations of different cases of Bessel non-square-root case. They are only used by Whittaker/Kummer Solver.

Input: S_{reg}, S_{irr}, base filed ext

Output: A list of pair (ν, f)

findBessel

Input: Differential operator L with corresponding S_{reg}, S_{irr} ,base field ext and which case we will meet.

Output: The solution space of L if it can be written in terms of Bessel functions.

BesselSqrtequiv

Input: Operator $L \in K[\partial]$, a parameter ν, a function $f \in K$

Output: $M \in K[\partial]$, $[y_1, y_2]$, where y_1 and y_2 are (modified) Bessel functions of the first and second kind with change of variable $x \mapsto \sqrt{f}$ and $M(y)$ is a solution of L. or 0 if such solution does not exist.

Airyequiv

Input: Operator $L \in K[\partial]$, a parameter ν, a function $f \in K$

Output: $M \in K[\partial]$, $[y_1, y_2]$, where y_1 and y_2 are Airy functions of the first and second kind with change of variable $x \mapsto f$ and $M(y)$ is a solution of L. or 0 if such solution does not exist.

testAiry

This functions combine with Airyequiv is an implementation of Algorithm 9.

Input: f, ν and base field ext

Output: A boolean variable indicate if $B_\nu(f)$ can be written Airy functions.

singSeries

Combine this with singInfo is an implementation of Algorithm 3.

Input: S_{reg} and S_{irr} with exponents differences, base field ext.

Output: S_{reg} and S_{irr} with truncated power series, denominator of possible change of variable B, the boundary of degree of numerator dA and a boolean indicate if it is easy case.

sqrtEasy

This is an implementation of Algorithm 4, find solutions for easy case.

Input: S_{reg}, S_{irr} with truncated series, B, dA, ext

Output: List of pairs (f, ν) (note, there is only possible f in this case).

findnueasyIrrat

given f and the condition $\nu \notin \mathbb{Q}$, find all possible ν for easy case.

Input: f, S_{reg}

Output: List of pairs (f, ν).

findnueasyrat

given f and the condition $\nu \in \mathbb{Q}$, find all possible ν for easy case.

Input: f, S_{reg}

Output: List of pairs (f, ν).

findnuLog

given f and we have logarithmic solutions, find all possible ν for easy case and logarithmic case.

Input: f, S_{reg}

Output: List of pairs (f, ν).

sqrtLog

This is an implementation of Algorithm 5, find solutions for logarithmic case.

Input: S_{reg}, S_{irr} with truncated series, B, dA, ext

Output: List of pairs (f, ν).

searchKnlog

For logarithmic case, Try all possible multiplicities for zeroes. *Input*: S_{reg}, dA

Output: List of f.

sqrtIrrat

This is an implementation of Algorithm 6, find solutions for irrational case.

Input: S_{reg}, S_{irr} with truncated series, B, dA, ext

Output: List of pairs (f, ν).

sqrtRat

This is an implementation of Algorithm 7, find solutions for rational case.

Input: S_{reg}, S_{irr} with truncated series, B, dA, ext

Output: List of pairs (f, ν).

findnuRat

Given f, find ν for rational case.

Input: f, up to d disappearing Singularities, S_{reg}, B

Output: List of pairs (f, ν).

searchA1k

write $f = CA_1A_2^d$, search possible A_1 and d

Input: S_{reg}, dA the upper bound of disappearing singularities.

Output: List of pairs (A_1, d).

searchA1k

write $f = CA_1A_2^d$, given d and degree of A_1, search possible A_1

Input: S_{reg}, de the degree of A_1, d the upper bound of disappearing singularities.

Output: List of pairs (A_1, d).

findSqrtf

find possible f for rational case. *Input*: S_{irr}, B, up to d disappearing singularities, possible list of A_1, base field ext.

Output: List of f.

equiv

Input: Two operators L_1 and $L_2 \in K[\partial]$ of degree two

Output: An operator M such that $My \in V(L_2)$ for every $y \in V(L_1)$ if such operator exists.

poly_to_powerseries

Input: $P \in C_K[x]$, point p, accuracy acc

Output: Power series of P at $x = p$ with acc accurate term.

dthroot_powseries

Input: Base field ext, power series L with local parameter t_p and an integer d

Output: All possible Power series of dth root of L with the same accuracy of L.

pow_time

Input: Power series L_1 and L_2 with local parameter t_p

Output: Power series of L_1L_2 with the accuracy the minimum of L_1 and L_2.

pow_reciprocal

Input: Power series L with local parameter t_p

Output: Power series of $1/L_1$ with the same accuracy of L.

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BIOGRAPHICAL SKETCH

Quan Yuan

I was born in Kunming, China on June 7 1981. I completed my Bachelor degree in Mathematics at Northwest University, Xi'an and my Master degree in applied mathematics at Peking University. I started my Ph.D starting under the advisement of Professor Mark van Hoeij at Florida State University in 2006.