Arithmetic identities characterising Heun functions reducible to hypergeometric functions

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This paper demonstrates that interesting arithmetic identities $a+b=c$ in integers or algebraic integers can be obtained from the t-parameter of Heun differential equations reducible to hypergeometric equation by a pullback transformation. The pullback coverings are usually Belyi maps, and they are expected to degenerate only a few small primes. Correspondingly, the numbers a,b,c in the arithmetic identities can reduce to 0 only modulo those few primes.

1 Transformations of Fuchsian equations

Fuchsian differential equations are linear homogeneous differential equations with only regular singularities. That typically[1)] means that around each singularity there is a basis of local solutions of the form $x^{\lambda}\,(1+a_1x+a_2x^2+\ldots)$, with $\lambda\in\mathbb{C}$ called *a local exponent* of the differential equation at that point. Canonical Fuchsian equations with 3 or 4 singularities are, respectively, the hypergeometric equation

$$\frac{d^2y(z)}{dz^2}+\left(\frac{\gamma}{z}+\frac{\alpha+\beta-\gamma+1}{z-1}\right)\frac{dy(x)}{dz}+\frac{\alpha\beta}{z(z-1)}\,y(z)=0 \tag{1}$$

and the Heun equation

$$\frac{d^2Y(x)}{dx^2}+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\alpha+\beta-\gamma-\delta+1}{x-t}\right)\frac{dY(x)}{dx}+\frac{\alpha\beta x-q}{x(x-1)(x-t)}\,Y(x)=0. \tag{2}$$

The parameters $\alpha,\beta,\gamma,\delta$ determine the local exponents[2)] at the singularities $x=0$, $x=1$, $x=\infty$ (and $x=t$), while q is an *accessory parameter* for Heun's equation. The local solutions at the singularities for the hypergeometric equation are the well known *Gauss hypergeometric series* ${}_2\mathrm{F}_1\left(\left.{\alpha,\beta \atop \gamma}\right|z\right)$. The local solutions for Heun's equation are more complicates series [Mai07]. Let $E(a,b,c)$ denote a hypergeometric equation with the local exponent differences a,b,c, with the parameter order unimportant, and let $H(a,b,c,d)$ denote Heun's equation with the local exponent differences a,b,c,d.

Of particular interest are *pull-back transformations*

$$z\longmapsto\varphi(x),\qquad y(z)\longmapsto Y(x)=\theta(x)\,y(\varphi(x)), \tag{3}$$

between Fuchsian equations with a low number of singularities. Here $\varphi(x)$ is a rational function, and $\theta(x)$ is a product of powers of rational functions. Geometrically, the transformation *pull-backs* the starting differential equation on the projective line $\mathbb{P}^1_z$ to a differential equation on the projective line $\mathbb{P}^1_x$, with

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1) Logarithmic local solutions are possible as well.

2) The local exponents of the hypergeometric equation are: $0,1-\gamma$ at $z=0$; $0,\gamma-\alpha-\beta$ at $z=1$; and α,β at $z=\infty$. Hence the local exponent differences are $1-\gamma$, $\gamma-\alpha-\beta$, $\alpha-\beta$. Similarly, the local exponent differences of the Heun equation are $1-\gamma$, $1-\delta$, $\gamma+\delta-\alpha-\beta$, $\alpha-\beta$.

respect to the covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ determined by the rational function $\varphi(x)$. The factor $\theta(x)$ shifts the local exponents of the pull-backed equation, but it does not change the local exponent differences.

Classical transformations between hypergeometric functions with $\varphi(x)$ of degree 2, 3, 4 or 6 are known since Gauss, Kummer, Goursat [Gou81]. All transformations between hypergeometric functions are classified in [Vid09]. Recently systematic work has been done to classify all transformations between Heun and hypergeometric functions. In particular, Heun-to-hypergeometric transformations with a free parameter are classified in [FRV10], and the transformations from the so called *hyperbolic* hypergeometric equations are classified in [vHV11]. The hyperbolic hypergeometric equations are $E(1/k, 1/\ell, 1/m)$ with k, ℓ, m positive integers satisfying $1/k+1/\ell+1/m < 1$. There are 61 different parametric transformations (of degree up to 12), and 366 transformations from the hyperbolic equations (of degree up to 60). The transformations not completely considered yet are between the equations with a reducible, dihedral or a finite monodromy group.

Possible pullback coverings for Heun-to-hypergeometric transformations can be worked out by considering transformation of singularities and the local exponent differences. Typically, the points above the singularities of the original hypergeometric equation, and the branching points of the covering are singularities of the pullbacked equation. However, a branching point of order k above a singularity with the local exponent difference $1/k$ can be made non-singular. To keep the number of singularities of the transformed equation low, we typically[3)] need the covering to branch only above the 3 singularities of the hypergeometric equation, hence the covering should be a *Belyi map*. From Hurwitz theorem one gets [Vid09, Lemma 2.5] that Belyi coverings of degree d have exactly $d+2$ distinct points in the 3 branching fibers (while other coverings have more disctinct points in any 3 fibers). If we restrict the local exponents to the inverse integers $1/k, 1/\ell, 1/m$, we have at least

$$d+2-\left\lfloor \frac{d}{k} \right\rfloor - \left\lfloor \frac{d}{\ell} \right\rfloor - \left\lfloor \frac{d}{m} \right\rfloor \tag{4}$$

singularities of the transformed equation.

A suitable pullback covering for a Heun-to-hypergeometric transformation is defined by the rational function

$$F_{\mathrm{B31}}(x) = \frac{27\,(3\,x+13)^5\,(5\,x-21)^3\,(x-5)^2\,(8\,x+35)}{16\,(4\,x^6-210\,x^4-35\,x^3+3465\,x^2+903\,x-16415)^3}. \tag{5}$$

The degree is 18. One can check that the numerator of $F_{\mathrm{B31}}(x)-1$ is a square of a degree 9 polynomial. Together with $x=\infty$ we have 20 distinct points in the 3 fibers $F_{\mathrm{B31}}(x) \in \{0, 1, \infty\}$, exactly enough for a Belyi map. If we apply a pullback transformation with respect to this covering to a hypergeometric equation with the local exponent differences $1/2, 1/3, a \in \mathbb{C}$ at, respectively, $z=1$, $z=\infty$, $z=0$, we get a Fuchsian equation with 5 essential singularities at $x=5$, $x=21/5$, $x=-13/3$, $x=-35/8$ and $x=\infty$. The local exponent differences there are $2a$, $3a$, $5a$, a and $7a$, respectively. We can adjust the parameter a so that one of these local exponent differences is 1 and the respective point can be made non-singular (unless $x=-35/8$). Then a normalization by a Möbius transformation and a factor $\theta(x)$ in (3) gives Heun's equation. For example, with $a=1/7$ we conclude that

$$\left(1-\tfrac{3126x}{7^4}+\tfrac{5625x^2}{7^5}+\tfrac{39500x^3}{7^6}+\tfrac{2265x^4}{7^7}-\tfrac{54x^5}{7^8}+\tfrac{x^6}{7^{10}}\right)^{-1/28} {}_2\mathrm{F}_1\left(\begin{array}{c} 1/84, 29/84 \\ 6/7 \end{array} \middle| \, F_{\mathrm{B31}}\left(\frac{13x+147}{35-3x}\right)\right)$$

satisfies Heun's equation with[4)]

$$\alpha = \frac{3}{14}, \quad \beta = \frac{13}{14}, \quad \gamma = \frac{4}{7}, \quad \delta = \frac{5}{7}, \quad t = 2401, \quad q = \frac{3126}{49}. \tag{6}$$

[3)] Non-Belyi coverings can apply to Heun-to-hypergeometric transformations only if the monodromy group of the equations is reducible, dihedral or finite. In particular, Klein's theorem [Kle77] implies that any second order Fuchsian equation with a finite monodromy group (equivalently, with a basis of algebraic solutions) is a pullback of a hypergeometric equation with the local exponent differences $1/k, 1/\ell, 1/m$ with k, ℓ, m positive integers satisfying $1/k+1/\ell+1/m > 1$.

[4)] The degree 6 polynomial in $\theta(x)$ is just the respective Möbius transformation of the numerator in (5), and the accessory parameter q is found after a substitution of a power series into Heun's equation (2).

The parameter t of the transformed Heun equation is obtained after putting 3 of the 4 singularities to the locations $x = 0$, $x = 1$, $x = \infty$ by a Möbius transformation. It is equal to the *cross ratio* of the 4 singularities:

$$CrossRatio(a, b, c, d) = \frac{(a-c)(b-d)}{(b-c)(b-d)}, \qquad CrossRatio(\infty, a, b, c) = \frac{a-c}{b-c}.$$

We get the following t values from the 5 points in the fiber $F_{\mathrm{B31}}(x) = 0$, and thransformations between hypergeometric and Heun equations:

$$\begin{aligned}
CrossRatio\left(5, \frac{21}{5}, -\frac{13}{3}, -\frac{35}{8}\right) &= \frac{2401}{2400}, & E\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}\right) &\mapsto H\left(\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{5}{7}\right),\\
CrossRatio\left(\infty, -\frac{35}{8}, 5, \frac{21}{5}\right) &= -\frac{32}{343}, & E\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) &\mapsto H\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{7}{5}\right),\\
CrossRatio\left(\infty, -\frac{35}{8}, -\frac{13}{3}, 5\right) &= \frac{224}{225}, & E\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right) &\mapsto H\left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{7}{3}\right),\\
CrossRatio\left(\infty, -\frac{35}{8}, -\frac{13}{3}, \frac{21}{5}\right) &= \frac{1024}{1029}, & E\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}\right) &\mapsto H\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\right),\\
CrossRatio\left(\infty, -\frac{13}{3}, 5, \frac{21}{5}\right) &= -\frac{3}{32}, & E\left(\frac{1}{2}, \frac{1}{3}, 1\right) &\text{ has a logarithmic singularity.}
\end{aligned}$$

The subject of this paper is the observation that the t-values appear to factorize nicely as rational numbers. This is not strange, because Belyi maps are expected to degenerate only modulo a few small primes [Bec89]. This makes the t-values reduce to 0 or degenerate to ∞ only modulo those few primes, and often forces their factorization with high powers.

2 Relation to the ABC conjecture

The cross ratios are determined up to the Möbius transformations

$$z \mapsto \left\{ z,\ 1-z,\ \frac{z}{z-1},\ \frac{1}{z},\ 1-\frac{1}{z},\ \frac{1}{1-z} \right\}. \tag{7}$$

Correspondingly, the t-values of Heun's equation go through an orbit of (generally) 6 values under its fractional-linear transformations [Mai07] permuting the 4 singularities. An orbit of 6 values in $\mathbb{Q}$ can be encoded by an arithmetic identity $a + b = c$ with coprime integers a, b, c. The orbit is recovered as $\{a/c,\ b/c,\ c/a,\ c/b,\ -a/b,\ -b/a\}$. The orbit of t-values is also characterized by the j-invariant

$$j(t) = \frac{256\,(t^2 - t + 1)^3}{t^2\,(t-1)^2}. \tag{8}$$

With the possible exception for 2, the primes dividing a, b, c appear in the denominator of $j(t)$.

Arithmetic identities $a + b = c$ with large $a, b, c \in \mathbb{Z}$ reducing to 0 only modulo a few primes have notable interest. In particular, existence of solutions of Fermat's equation $x^n + y^n = z^n$ fascinated people for centuries. With this equation finally settled by Wiles, a weaker general supposition is the ABC conjecture formulated by Masser and Oesterlé. It implies that for any real $\eta > 1$ there are only finitely many identities $a+b = c$ with coprime a, b, c such that $\max(|a|, |b|, |c|)$ exceeds $\mathrm{rad}(a, b, c)^\eta$, where $\mathrm{rad}(a, b, c)$ is the product of all primes dividing the product abc, the *radical* of the $a+b = c$ identity. The identities with a high *quality ratio*

$$Q(a, b, c) = \frac{\log \max(|a|,\ |b|,\ |c|)}{\log \mathrm{rad}(a, b, c)} \tag{9}$$

are actively searched for and tabulated [Nit]. So far the identity with the highest known quality ratio 1.629911... is $2 + 3^{10}\,109 = 23^5$, due to E. Reyssat.

For this paper we looked through the list of computed Heun-to-hypergeometric transformations and checked the quality ratio of the *abc* triples characterizing the t-values of involved Heun's equations. We also checked other cross ratios of found Belyi coverings. The results are reported in the next section.

Many coverings and t-values are not defined over $\mathbb{Q}$ but over an algebraic number field K. An orbit of 6 algebraic t-values can still be represented by an identity $a + b = c$ with a, b, c algebraic integers. There is an extension of the ABC conjecture to algebraic number fields as well. In the algebraic generalization, the radical is the product of the field discriminant and of the residue field sizes of the primes ideals where a, b or c reduce to 0, but excluding the prime ideals which "divide" all three a, b, c in the same power. The absolute value in $\max(|a|,\, |b|,\, |c|)$ is replaced by the $\mathbb{Q}$-norm of an algebraic integer, divided by the residue ring size of the ideal generated by a, b, c. More elegantly, one considers the respective point $(a : b : c)$ in the projective plane over K; then $\max(|a|,\, |b|,\, |c|)$ is replaced by the *height* of this point, and the radical is the product of the field discriminant and of the residue field sizes of the primes ideals where the point reduces to $(0 : 1 : 1)$, $(1 : 0 : 1)$ or $(1 : -1 : 0)$. The best known algebraic example is

$$\frac{\sqrt{13}+1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^5 + \frac{\sqrt{13}-1}{2}\left(\frac{\sqrt{13}+3}{2}\right)^5 = 2^9, \qquad (10)$$

with the quality ratio $\log(4^9)/\log(13\cdot 3\cdot 3\cdot 4) = 2.029228...$, due to T. Dokchitzer [Dok03].

Nitaj's tables [Nit] lists all found integer ABC triples with the quality ration greater than 1.4, and all algebraic ABC triples with the quality ration greater than 1.5. The t values of the found Heun equations give several of the well known ABC identities with a high quality ratio; the best arising ABC triple is defined over $\mathbb{Q}(\sqrt{-7})$ and has the quality ratio 1.707222... We found a new example with the quality ratio 1.581910..., defined over $\mathbb{Q}(\sqrt{-14})$. This number field has the class number 4 however, and the high powers of involved prime ideals have to distributed into principal ideals. Hence the factorized ABC expressions for the new example do not look spectacular.

3 The coverings and ABC triples

As mentioned, there are 61 parametric Heun-to-hypergeometric transformations and 366 transformations from hyperbolic hypergeometric functions. The parametric transformations are realized by 48 different Belyi coverings; 38 of them occur in Herfurtner's list [Her91] of elliptic surfaces over $\mathbb{P}^1$ with 4 singular fibers. The coverings for transformations of hyperbolic equations are all different. The website with the computed data is indicated in the reference [vHV11].

A priori, there were 89 possible branching patterns for the parametric transformations and 376 branching patterns for the hyperbolic transformations. But Belyi coverings exist not for every prescribed branching pattern above three points, neither they have to be unique. The Belyi coverings $\mathbb{P}^1 \to \mathbb{P}^1$ with a prescribed branching pattern are generally defined over an algebraic number field, and the Galois action on them is the subject of Grothendieck's theory of *dessins d'enfant*. Computation of Belyi coverings of degree higher than 20 is still considered hard. We developed 2 computational methods, one of them with modular lifting, both employing presumed existence of transformations between hypergeometric and Heun equations. Roughly speaking, we get extra equations for undetermined coefficients of the rational Belyi function by comparing the power series of presumed solutions of the two differential equations. With our methods, even the coverings of the maximal degree 60 were computed in comfortable time. The degree 60 transformations are between $E(1/2, 1/3, 1/7)$ and $H(1/7, 1/7, 1/7, 1/7)$. There are two cubic Galois orbits of them, defined by the polynomials $x^3 - x^2 - 2x + 1$ and $x^3 + 2x^2 + 6x - 8$. The highest degree algebraic number field for defining a found Belyi covering is 15, for a degree 37 transformation from $E(1/2, 1/3, 1/7)$ to $H(1/2, 1/3, 1/7, 1/7)$.

The parametric transformations are denoted by P1 to P61. The transformations from hyperbolic hypergeometric equations are denoted as follows:

A1–A24:	the transformations with $j(t)=1728$, that is $t \in \{-1,2,1/2\}$;
B1–B34:	the other transformations with $t \in \mathbb{Q}$;
C1–C42:	the transformations with $j \in \mathbb{Q}$, and t in a real quadratic extension of $\mathbb{Q}$;
D1–D50:	the transformations with $j \in \mathbb{Q}$, and t in an imaginary quadratic extension of $\mathbb{Q}$;
E1–E25:	the transformations with $j \in \mathbb{Q}$, and t in the splitting field (of degree 6) of a cubic polynomial in $\mathbb{Q}[\xi]$;
F1–F25:	the transformations with j in a real quadratic extension of $\mathbb{Q}$;
G1–G52:	the transformations with j in an imaginary quadratic extension of $\mathbb{Q}$;
H1–H53:	the transformations with j in a cubic extension of $\mathbb{Q}$;
I1–I33:	the transformations with j in an extension of $\mathbb{Q}$ of degree 4 or 5;
J1–J28:	the transformations with j in an extension of $\mathbb{Q}$ of degree 6 and higher.

The transformations are ordered so that the coverings defined over the same number field or with the same j-invariant are next to each other. However, the coverings with the same number field for t might be in different classes, as the extension degree from the j-field to the t-field varies. In our examples, this extension degree is $1, 2$ or 6, never 3.

In total, we have 30 different $j(t)$ values for the parametric transformations and 266 different $j(t)$ values for hyperbolic transformations; all together we have 277 different $j(t)$ values. Together with the alternative cross ratios (not extending the field of definition of our coverings) we got 1799 different $j(t)$ values.

3.1 The ABC identities over integers

By far the most frequent orbit of t-values for the encountered Heun equations is $t \in \{-1,2\}$, with the j-invariant 1728. It gives the most simple ABC identity $1+1=2$. This t-orbit occurs for the Heun equations of 14 parametric and 24 hyperbolic transformations. We also counted 566 instances when this t-orbit occurs as an alternative cross ratio of 4 points in the 3 singular fibers of our Belyi coverings (but a small portion of cases is doubly counted.)

In total there are 19 different orbits of t-values in $\mathbb{Q}$ from the encountered Heun equations. They are listed in the upper part of Table 1. Among any cross ratios of 4 points in our Belyi coverings, we found 68 different orbits of t-values in $\mathbb{Q}$ in total. The alternative most notable ABC identities are listed in the lower part of Table 1; we skipped

$$1+2=3, \quad 2+3=5, \quad 1+5=6, \quad 1+6=7, \quad 3+4=7, \quad 1+7=8, \quad 3+5=8,$$
$$2+7=9, \quad 4+5=9, \quad 1+9=10, \quad 3+7=10, \quad 1+10=11, \quad 7+8=15,$$
$$1+15=16, \quad 7+9=16, \quad 4+21=25, \quad 7+18=25, \quad 5+28=33.$$

As we can see, the same coverings like B31, B30, B33 tend to produce the best alternative ABC triples as well. We already considered B31. Here are other two prominent coverings:

$$F_{\rm B30}(x) = -4\,\frac{(x+8)\left(9\,x^5-300\,x^3+400\,x^2+1945\,x-3512\right)^3}{(9\,x-28)\,(5\,x+22)^2\,(13\,x-40)^5},$$
$$F_{\rm B33}(x) = \frac{16\left(108\,x^5-6930\,x^3-23485\,x^2+7700\,x+98252\right)^3}{6561\,(8\,x-77)\,(15\,x+22)^3\,(11\,x+46)^4}.$$

Sometimes an alternative cross ratio is a t-value of a transformation between Heun and hypergeometric functions with a finite monodromy group rather than of the hyperbolic type (as we saw with B31). We marked these cases with *dih, tet, oct, ico* to indicate dihedral or the tetrahedral, octahedral, icosahedral monodromy groups. But the cross ratios that do not arise from a Heun-to-hypergeometric transformations can give just as good ABC triples. For example, the best cross ratios from B30 and B33 are $CrossRatio(\infty, -22/5, 28/9, 40/13)$ and $CrossRatio(\infty, 77/8, -46/11, -15/22)$; they have an unramified point in the same fiber with several points of the same branching order 3 or 2.

3.2 ABC identities in the quadratic fields

Among the Heun equations occurring in the parametric or hyperbolic Heun-to-hypergeometric transormations, there are 64 rational values of $j(t) \in \mathbb{Q}$ with the t-values defined over a quadratic extension of $\mathbb{Q}$. The square roots are taken of these 22 numbers:

$$-39, -35, -15, -14, -7, -6, -5, -3, -2, -1, 2, 3, 5, 6, 7, 10, 13, 21, 105, 273, 385.$$

The number fields and the coverings are identified in Tables 2 and 3. The tables give also: factorization of the $\mathbb{Q}$-norms of a, b, c; the radical; the quality ratio; and the identities themselves. In the identities, u denotes a primitive unit, and v_k an algebraic integer of the norm k. The bar in $\overline{u}$ or $\overline{v}_k$ denote the conjugation $a + b\sqrt{d} \mapsto a - b\sqrt{d}$, and the tilede in $\tilde{u}$ or $\tilde{v}_k$ denote the conjugation composed with the multiplication by -1: $a + b\sqrt{d} \mapsto b\sqrt{d} - a$. The later notation is illustrative in real quadratic fields (to keep the numbers positive), and more generally, with units (to denote the multiplicative inverse). The σ denotes the current square root, as reminded in most entries of Tables 2 and 3.

The ABC identities for D37/D39 and C18 are well-known algebraic identities with a high *abc* quality ratio; they are currently No 11 and 12 in Nitaj's list [Nit]. The identity D42 is a new identity with the quality ratio > 1.5.

Some of the quadratic number fields are not principal ideal domains, and factorization of their integers is not unique. The fields $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{105})$, $\mathbb{Q}(\sqrt{273})$, $\mathbb{Q}(\sqrt{285})$, $\mathbb{Q}(\sqrt{-5})$, $\mathbb{Q}(\sqrt{-6})$, $\mathbb{Q}(\sqrt{-15})$, $\mathbb{Q}(\sqrt{-35})$ have the class number 2, while $\mathbb{Q}(\sqrt{-14})$, $\mathbb{Q}(\sqrt{-39})$ have the class number 4. Distributing the primes into principal ideals reduces the arithmetic appeal of ABC identites in these fields.

Quadratic extensions from the j-field to the t-field usually arise when the rational function realizing the covering in a minimal field has two conjugate points with the same ramification order in the same fiber, and those 2 points are singularities of the pullbacked Heun equation. For example, the covering

$$F_{\text{D37}}(x) = 64\,\frac{7x^2 + 7x + 8}{x^5\,(x-7)^2}, \tag{11}$$

pullbacks $E(1/2, 1/5, 1/5)$ to $H(1/2, 1/5, 1/5, 2/5)$, with the t value in $\mathbb{Q}(\sqrt{-7})$ equal to the cross ratio of 0, 7 and the roots of $7x^2 + 7x + 8$. In the ABC identities of these cases, two of the numbers have equal $\mathbb{Q}$-norms. But those two numbers are not necessarily conjugate as units might break the symmetry; consider C29, for example. The conjugation symmetry can be restored if the difference is by a unit that is a square (multiplied by -1, possibly) in the number field..

There are 14 cases with both $j(t)$ and t defined in a quadratic number field, of type F or G and also P60, P61. They are represented in Tables 2 and 3 as well. In their ABC identities, the three numbers have different $\mathbb{Q}$-norms.

Alternative cross ratios from our Belyi coverings give 207 other rational $j(t) \in \mathbb{Q}$ with the t-values in a quadratic extension, and 72 other quadratic $j(t)$ with with the t-values in the same field. New square roots appear in these identities, of $-38, -21, 14, 15, 17, 19, 30, 39, 42, 70, 190, 357$. The best alternative ABC triples are:

$$(\sqrt{2}-1)^4 + 3\,(\sqrt{2})^7 = (\sqrt{2}+1)^4, \qquad (8-3\sqrt{7})^2\,(2+\sqrt{7})^7 + (3+\sqrt{7}) = (8+3\sqrt{7})^3(\sqrt{7}-2)^7,$$

$$\left(\frac{\sqrt{5}-1}{2}\right)^6 + 2^3\sqrt{5} = \left(\frac{\sqrt{5}+1}{2}\right)^6, \qquad (2-\sqrt{-1})^4 + \sqrt{-1}\,(1+\sqrt{-1})^7\,(2+\sqrt{-1}) = 1.$$

They have they quality ratios 1.418414, 1.235869, 1.252575, 1.219532 and come from C5/P7/P39, C29, F13, C29, A4/B21/B22, respectively.

Transformations		The j-invariant	$a+b=c$		Quality
Hyperb.	Param.		in integers	factorized	ratio
B31		$73^3 193^3 409^3/2^2 3^2 5^4 7^8$	$1+2400=2401$	$1+2^5\,3\cdot 5^2=7^4$	1.455673
B20/B24	P33/P34	$7^3 2287^3/2^6 3^2 5^6$	$3+125=128$	$3+5^3=2^7$	1.426565
B25		$2^6 7^3 31^3 271^3/3^{10} 11^4$	$1+242=243$	$1+2\cdot 11^2=3^5$	1.311101
B15	P27/P28	$6481^3/3^8 5^2$	$1+80=81$	$1+2^4\,5=3^4$	1.292030
B5/B14	P19/P24	$2^2 73^3/3^4$	$1+8=9$	$1+2^3=3^2$	1.226294
B33		$2^4 106791301^3/3^{14} 5^2 7^8 11^6$	$1331+9604=10935$	$11^3+2^2 7^4=3^7\,5$	1.200739
B26/B28	P35	$4993^3/2^2 3^8 7^4$	$32+49=81$	$2^5+7^2=3^4$	1.175719
B30		$2^4 181^3 2521^3/3^6 5^4 13^4$	$1+675=676$	$1+3^3 5^2=2^2 13^2$	1.092195
B34		$829^3 30469^3/3^6 5^6 7^8 19^4$	$2401+3375=5776$	$7^4+3^3 5^3=2^4 19^2$	1.044614
B29		$2^4 3^3 7^6 103^3/5^6 11^4$	$4+121=125$	$2^2+11^2=5^3$	1.027196
B18/B19	P31/P32	$7^3 127^3/2^2 3^6 5^2$	$5+27=32$	$5+3^3=2^5$	1.018975
A1/A24	P1/P14	$2^6 3^3$	$1+1=2$	$1+1=2$	1.
	P36	$19^3 1459^3/2^4 3^6 5^6 7^2$	$64+125=189$	$2^6+5^3=3^3\,7$	0.980296
B16/B17	P29	$2^6 7^3 97^3/3^6 5^4$	$2+25=27$	$2+5^2=3^3$	0.969023
B32		$49201^3/2^8 3^6 5^2 11^4$	$121+135=256$	$11^2+3^3\,5=2^8$	0.956215
	P26	$13^3 37^3/3^4 5^4$	$9+16=25$	$3^2+4^2=5^2$	0.946395
	P30	$2^4 757^3/3^6 7^2$	$1+27=28$	$1+3^3=2^2\,7$	0.891519
B1/B4	P15/P18	$2^4 13^3/3^2$	$1+3=4$	$1+3=2^2$	0.773706
	P25	$2^4 3^3 7^3/5^2$	$1+4=5$	$1+2^2=5$	0.698970
B31-*dih3*		$19^3\,31^3\,1789^3/2^{12}\,3^2\,5^2\,7^6$	$5+1024=1029$	$5+2^{10}=3\,7^3$	1.297214
B30		$2^6\,4804939^3/3^{14}\,5^2\,13^6$	$10+2187=2197$	$2\cdot 5+3^7=13^3$	1.289752
B33		$13^3 21513637^3/2^{10} 3^{12} 5^4 7^2 11^8$	$3584+14641=18225$	$2^9\,7+11^4=3^6 5^2$	1.266694
B29		$2^6\,31^3\,53881^3/3^8\,5^8\,11^6$	$81+1250=1331$	$3^4+2\,5^4=11^3$	1.240485
B20		$13^3\,61^3\,457^3/2^4\,3^4\,5^8\,7^4$	$49+576=625$	$7^2+2^6\,3^2=5^4$	1.203969
B25		$2^2\,19^3\,92683^3/3^6\,7^4\,11^6$	$8+1323=1331$	$2^3+3^3\,7^2=11^3$	1.172457
A14		$37^3\,109^3/2^4\,3^4\,7^2$	$1+63=64$	$1+3^2\,7=2^6$	1.112694
B26, B31-*ico*		$13^3\,9973^3/2^2\,3^2\,5^6\,7^6$	$32+343=375$	$2^5+7^3=3\,5^3$	1.108436
B26		$2^4\,277^3\,337^3/3^{10}\,5^4\,7^6$	$100+243=343$	$2^2\,5^2+3^5=7^3$	1.091755
B33-*tet*		$67^3 205507^3/2^6 3^{12} 5^2 7^6 11^2$	$128+3645=3773$	$2^7+3^6\,5=7^3\,11$	1.063347
C29-*dih*		$13^3\,181^3/3^2\,7^4$	$1+48=49$	$1+2^4\,3=7^2$	1.041242
B31-*tet*		$13^3\,3877^3/2^2\,3^4\,5^4\,7^2$	$1+224=225$	$1+2^5\,7=3^2\,5^2$	1.012903
B30-*dih3*		$2^4\,7^9\,307^3/3^8\,5^4\,13^2$	$1+324=325$	$1+2^2\,3^4=5^2\,13$	0.969441
B33-*oct*		$31^3 277^3 283^3/2^6 3^2 5^2 7^6 11^6$	$384+1331=1715$	$2^7\,3+11^3=5\,7^3$	0.961545
B8, B23-*dih4*, C19, ...		$2^2\,601^3/3^2\,5^4$	$1+24=25$	$1+2^3\,3=5^2$	0.946395
B30-*ico*		$2^6\,7^3\,19^3\,367^3/3^8\,5^6\,13^4$	$81+169=250$	$3^4+13^2=2\,5^3$	0.925465
B17-*tet*		$2^6\,3^3\,19^3\,43^3/5^4\,7^4$	$1+49=50$	$1+7^2=2\,5^2$	0.920802
B17-*ico*, B20-*tet*, B26-*ico*		$2^6\,12979^3/3^6\,5^6\,7^4$	$27+98=125$	$3^3+2\,7^2=5^3$	0.902977
B29-*dih3*		$2^4\,7^3\,1723^3/3^8\,5^6\,11^2$	$44+81=125$	$2^2\,11+3^4=5^3$	0.832598
B29-*tet*		$2^2\,13^3\,877^3/3^8\,5^2\,11^4$	$40+81=121$	$2^3\,5+3^4=11^2$	0.826990
B26		$2^2\,13^3\,397^3/3^8\,5^4\,7^2$	$25+56=81$	$5^2+2^3\,7=3^4$	0.821837
B25		$7^3\,79^3/3^6\,11^2$	$11+16=27$	$11+2^4=3^3$	0.786661
B25-*oct*		$2^2\,11113^3/3^4\,7^4\,11^4$	$49+72=121$	$7^2+2^3\,3^2=11^2$	0.781638
B20-*dih3*		$3361^3/2^4\,3^2\,5^2\,7^4$	$15+49=64$	$3\,5+7^2=2^6$	0.777782
B25		$2^6\,31^3\,313^3/3^4\,7^4\,11^2$	$1+98=99$	$1+2\,7^2=3^2\,11$	0.748932
B17		$2^6\,2671^3/3^6\,5^2\,7^4$	$5+49=54$	$5+7^2=2\,3^3$	0.746008
B26, B26-*dih4*		$2^2\,1801^3/3^2\,5^4\,7^4$	$24+25=49$	$2^3\,3+5^2=7^2$	0.727837
B26-*dih3*		$2^4\,2221^3/3^4\,5^2\,7^4$	$4+45=49$	$2^2+3^2\,5=7^2$	0.727837

Table 1: The ABC triples for rational cross ratios

Id	a	b	c	rad	quality	$a+b=c$
$\mathbb{Q}(\sqrt{2})$						with $\sigma=\sqrt{2}$, $u=\sqrt{2}+1$, $v_7=2\sqrt{2}+1$
C1, P37	1	1	2^5	2	1.25	$\tilde{u}^2+\sigma^5=u^2$
C2	7^3	7^3	2^{11}	$2\,7^2$	1.144080	$\tilde{u}^2\,v_7^3+u^2\,\tilde{v}_7^3=\sigma^{11}$
C3/C4	7^4	7^4	$2^9 3^4$	$2\,3^2 7^2$	1.199866	$\tilde{v}_7^4+\sigma^9\,3^2=v_7^4$
F1/F3	2	3^6	7^4	$2\,3^2\,7$	1.125499	$\tilde{u}\,v_7^4+3^3=u^5\,\sigma$
$\mathbb{Q}(\sqrt{3})$						with $\sigma=\sqrt{3}$, $u=2+\sqrt{3}$, $v_2=\sqrt{3}+1$, $v_{11}=2\sqrt{3}+1$, $v_{13}=4+\sqrt{3}$
P38	1	1	2^4	2	0.872417	$1+u^2=\overline{u}\,v_2^4$
C5, P39	1	1	$2^6 3$	$2\,3$	1.229344	$1+\overline{u}\,v_2^6\,\sigma=u^4$
C6/C7, P40/P41	1	1	$2\,3^3$	$2\,3$	0.932732	$u^2+\overline{u}=v_2\,\sigma^3$
C8	2^{10}	11^3	11^3	$2\,11^2$	0.902160	$v_{11}^3+\overline{u}^4\,v_2^{10}=u^2\,\tilde{v}_{11}^3$
C9	11^2	11^2	$2^8 3^3$	$2\,3\,11^2$	0.974490	$v_{11}^2+\overline{u}^2\,v_2^8\,\sigma^3=u^4\,\tilde{v}_{11}^2$
C10/C11	13^2	13^2	$2^{12} 3^5$	$2\,3\,13^2$	1.468212	$v_{13}^2+\overline{u}^2\,v_2^{12}\,\sigma^5=u^8\,\overline{v}_{13}^2$
C12	11^3	11^3	$2^8 17^4$	$2\,11^2 17^2$	1.237368	$\tilde{v}_{11}^3+\overline{u}\,v_2^8\,17^2=u^6 v_{11}^3$
C13	$11^5 13^2$	$11^5 13^2$	$2\,3^{17}$	$2\,3\,11^2 13^2$	1.363827	$\tilde{v}_{11}^5\,v_{13}^2+v_2\,\sigma^{17}=u\,v_{11}^5\,\overline{v}_{13}^2$
F9	2^7	13^2	3^6	$2\,3\,13$	0.963467	$\sigma^6+\overline{u}^4\,v_2^7=v_{13}^2$
$\mathbb{Q}(\sqrt{5})$						with $\sigma=\sqrt{5}$, $u=\frac{\sqrt{5}+1}{2}$, $v_{11}=2\sqrt{5}+3$
C14	1	1	1	1	0.	$1+\tilde{u}=u$
C15/C17, P42	1	1	2^4	2^2	0.925513	$\tilde{u}^3+2^2=u^3$
C18	1	1	$2^8 3^4 5$	$2^2 3^2 5$	1.697794	$\tilde{u}^{12}+2^4\,3^2\,\sigma=u^{12}$
P43	1	1	5^3	5	1.5	$u^5+\tilde{u}^5=\sigma^3$
F10	5^3	2^8	11^3	$2^2\,5\,11$	1.027220	$\sigma^3+\tilde{u}^3\,v_{11}^3=2^4\,u^4$
F11/F12	2^2	11^2	5^5	$2^2\,5\,11$	1.149095	$\sigma^5+2\,\overline{u}^7=v_{11}^2$
$\mathbb{Q}(\sqrt{6})$						with $u=5+2\sqrt{6}$, $v_2=\sqrt{6}+2$, $v_3=3+\sqrt{6}$, $v_5=\sqrt{6}+1$, $\tilde{v}_2\,v_3=\sqrt{6}$
C19	1	1	$2^2 3^5$	$2\,3$	1.384228	$1+u^3=\tilde{v}_2^2\,v_3^5$
C20/C21	5^3	5^3	$2^2 3^7$	$2\,3\,5^2$	1.108429	$v_5^3+\overline{u}^3\,v_2^2\,v_3^7=u^3\,\tilde{v}_5^3$
C22/C23	5^2	5^2	$2^9 3^3$	$2\,3\,5^2$	1.164309	$\tilde{v}_5^2+\overline{u}^2\,v_2^9\,v_3^3=u^2\,v_5^2$
C24/C25	$2^{11} 3$	5^6	5^6	$2\,3\,5^2$	1.179264	$u^2\,\tilde{v}_5^6+\overline{u}^4\,v_2^{11}\,\overline{v}_3=v_5^6$
$\mathbb{Q}(\sqrt{7})$						with $\sigma=\sqrt{7}$, $u=8+3\sqrt{7}$, $v_2=3+\sqrt{7}$, $v_3=\sqrt{7}+2$, $2=\overline{u}\,v_2^2$, $100=u^2\,\overline{v}_2^4\,5^2$
C26	1	1	2^8	2	1.377563	$1+u^2=\overline{u}^3\,v_2^8$
C27	3^3	3^3	$2^4 5^4$	$2\,3^2 5^2$	0.975522	$\tilde{v}_3^3+100=v_3^3$
C28	$2^6 7$	3^6	3^6	$2\,3^2 7$	0.806964	$u^4\,\tilde{v}_3^6+\overline{u}\,v_2^6\,\sigma=v_3^6$
C29	3^8	3^8	$2\,7^5$	$2\,3^2 7$	1.275964	$u^3\,\tilde{v}_3^8+v_2\,\sigma^5=\overline{u}^2\,v_3^8$
$\mathbb{Q}(\sqrt{10})$						with $\sigma=\sqrt{10}$, $u=\sqrt{10}+3$, $v_6=4+\sqrt{10}$, $v_9=\sqrt{10}+1$
C30/C31	3^5	3^5	2^{13}	$2\,3^2$	1.369596	$v_6\,v_9^2+\overline{v}_6\,\tilde{v}_9^2=2^7$
C32	3^6	3^6	$2^9 5$	$2\,3^2 5$	0.958366	$\tilde{u}\,v_9^3+2^4\,\sigma=u\,\tilde{v}_9^3$
$\mathbb{Q}(\sqrt{13})$						with $\sigma=\sqrt{13}$, $u=\frac{\sqrt{13}+3}{2}$, $v_3=\frac{\sqrt{13}+1}{2}$
C33	$2^{12} 13$	3^{10}	3^{10}	$2^2 3^2 13$	1.260828	$u^2\,\tilde{v}_3^{10}+2^6\,\sigma=\tilde{u}^2\,v_3^{10}$
$\mathbb{Q}(\sqrt{21})$						with $u=\frac{5+\sqrt{21}}{2}$, $v_3=\frac{\sqrt{21}+3}{2}$, $v_5=\frac{\sqrt{21}+1}{2}$, $v_{17}=\sqrt{21}+2$, $96\sigma=u\,2^5\,\tilde{v}_3^3\,\frac{7+\sqrt{21}}{2}$
C34	5^3	5^3	2^8	$2^2 5^2$	0.724889	$\tilde{v}_5^3+2^4=v_5^3$
C35	5	5	2^{12}	$2^2 5^2$	1.087333	$\overline{u}^2\,\tilde{v}_5+2^6=u^2\,v_5$
C36	17^3	17^3	$2^{12} 3^7$	$2^2 3\,17^2$	1.429819	$\tilde{v}_{17}^3+u^3\,v_{17}^3=2^6\,\overline{u}^2\,v_3^7$
C37/C38	5^6	5^6	$2^{10} 3^3 7$	$2^2 3\,5^2 7$	1.138299	$\tilde{v}_5^6+96\sigma=v_5^6$
F23	5	2^6	$5^3\,17^2$	$2^2\,5^2\,17$	1.001129	$\overline{u}\,2^3+v_5^3\,v_{17}^2=u^4\,\tilde{v}_5$
$\mathbb{Q}(\sqrt{105})$						with $u=41+4\sqrt{105}$, $v_4=\frac{11+\sqrt{105}}{2}$, $v_5=\sqrt{105}+10$, $v_6=\frac{\sqrt{105}+9}{2}$, $49\sigma=7^2\,v_5\,(21-2\sigma)$
C39	2^2	2^2	5^7	$2^2 5$	1.472747	$v_4+\overline{u}^2\,v_5^7=u^3\,\overline{v}_4$
C40	2^{11}	2^{11}	$5\,7^5$	$2^2 5\,7$	1.181686	$\overline{u}^2 v_4^5\,\tilde{v}_6+u^2\,\overline{v}_4^5\,v_6=49\sigma$
$\mathbb{Q}(\sqrt{273})$						with $u=727+44\sqrt{273}$, $v_4=\frac{17+\sqrt{273}}{2}$, $v_{22}=\frac{19+\sqrt{273}}{2}$, $7203\sigma=u\,7^4(2\sigma-33)^3(182+11\sigma)$
C41	$2^{19} 11^3$	$2^{19} 11^3$	$3^3 7^9 13$	$2^2 3\,7\,11^2 13$	1.343253	$\overline{u}^2 v_4^8\overline{v}_{22}^3+7203\sigma=u^2\tilde{v}_4^8 v_{22}^3$
$\mathbb{Q}(\sqrt{385})$:						$u=95831+4884\sqrt{385}$, $v_4=\frac{59+3\sqrt{385}}{2}$, $v_{14}=\frac{21+\sqrt{385}}{2}$, $125\sigma=5^3(1295+66\sigma)(726-37\sigma)$
C42	2^{13}	2^{13}	$5^7 11$	$2^2 5\,11$	1.204205	$\overline{u}^2 v_4^6 v_{14}+125\sigma=u^2\overline{v}_4^6\overline{v}_{14}$

Table 2: The ABC triples in real quadratic fields

Id	a	b	c	cond	quality	
$\mathbb{Q}(\sqrt{-1})$						with $\sigma=\sqrt{-1}$, $v_2=\sigma+1$, $v_5=2+\sigma$, $v_{13}=2\,\sigma+3$
D1, P44	5^2	5^2	2^6	$2\,5^2$	0.784944	$v_5^2+v_2^6=\overline{v}_5^2$
D2/D6, P45/P46	2^4	5^3	5^3	$2\,5^2$	0.911292	$v_5^3+\overline{v}_5^3=-v_2^4$
G1	1	2^2	5	$2\,5$	0.436295	$1+\sigma\,v_5=v_2^2$
G2	5^3	$3^6\,13^2$	$2^3\,5^6$	$2\,3^2\,5^2\,13$	1.166550	$\overline{v}_2^3\,\overline{v}_5^6+\sigma\,v_5^3=3^3\,v_{13}^2$
P60	1	2^2	5	$2\,5$	0.436295	$1+v_2^2=v_5$
$\mathbb{Q}(\sqrt{-2})$						with $\sigma=\sqrt{-2}$, $v_3=\sigma+1$, $v_{11}=3+\sigma$, $v_{19}=1+3\,\sigma$
D7/D8	2^3	3^3	3^3	$2\,3^2$	0.663171	$v_3^3+\sigma^3=\overline{v}_3^3$
D9/D11, P48/P49	3^4	3^4	2^7	$2\,3^2$	0.976300	$\overline{v}_3^4+\sigma^7=v_3^4$
D12/D14, P50	2^2	3^5	3^5	$2\,3^2$	1.105285	$v_3^5+\overline{v}_3^5=-\sigma^2$
D15	2^9	$3\,11^5$	$3\,11^5$	$2\,3^2\,11^2$	1.340223	$v_3\,v_{11}^5+\sigma^9=\overline{v}_3\,\overline{v}_{11}^5$
D16	2^{13}	$3^6\,19^2$	$3^6\,19^2$	$2\,3^2\,19^2$	1.149361	$\overline{v}_3^6\,v_{19}^2+\sigma^{13}=v_3^6\,\overline{v}_{19}^2$
D17	$3^{10}\,11^2$	$3^{10}\,11^2$	$2^{15}\,5^4$	$2\,3^2\,5^2\,11^2$	1.296545	$\overline{v}_3^{10}\,\overline{v}_{11}^2+5^2\,\sigma^{15}=v_3^{10}\,v_{11}^2$
G10	11^2	3^6	$2^8\,3$	$2\,3^2\,11$	0.901744	$v_{11}^2+\sigma^8\,\overline{v}_3=v_3^6$
$\mathbb{Q}(\sqrt{-3})$						with $\sigma=\sqrt{-3}$, $u=\frac{1+\sigma}{2}$, $v_7=2+\sigma$, $v_{13}=2\,\sigma+1$
D18/D24, P51/P55	1	1	1	1	0.	$1+\tilde{u}=u$
D25	1	1	3	3	0.500000	$1+u\,\sigma=\tilde{u}$
D26	1	13^2	13^2	13^2	0.823616	$u\,v_{13}^2+\overline{u}=\overline{v}_{13}^2$
D27/D29	$2^8\,3$	7^4	7^4	$2^2\,3\,7^2$	1.041242	$\overline{v}_7^4+2^4\,\sigma=v_7^4$
P56	3^3	7^2	7^2	$3\,7^2$	0.639151	$v_7^2+u\,\sigma^3=\overline{v}_7^2$
P61	7	3^3	2^4	$2^2\,3\,7$	0.596054	$v_7+\sigma^3=\tilde{u}\,2^2$
G19	13^2	$2^6\,3^3$	7^4	$2^2\,3\,7\,13$	0.961611	$v_7^4+2^3\,u\,\sigma^3=\overline{v}_{13}^2$
$\mathbb{Q}(\sqrt{-5})$						with $\sigma=\sqrt{-5}$, $v_9=2+\sigma$, $v_{14}=3+\sigma$, $v_{21}=4+\sigma$, $v_{49}=2+3\,\sigma$, $v_{129}=2+5\,\sigma$
D30/D32	$2^8\,5$	3^8	3^8	$2\,3^2\,5$	1.172550	$v_9^4+2^4\,\sigma=\overline{v}_9^4$
D33	$3^3\,7^4$	$3^3\,7^4$	$2\,5^7$	$2\,3^2\,5\,7^2$	1.050218	$\overline{v}_{14}\,v_{21}^3+2\,\sigma^7=v_{14}\,\overline{v}_{21}^3$
D34	$2^{14}\,5$	$3^2\,7^6$	$3^2\,7^6$	$2\,3^2\,5\,7^2$	1.218253	$v_9\,\overline{v}_{49}^3+2^7\sigma=\overline{v}_9\,v_{49}^3$
D35	$2^{16}\,5^3$	$3^{10}7^4 43^2$	$3^{10}7^4 43^2$	$2\,3^2\,5\,7^2 43^2$	1.390402	$v_9^2\overline{v}_{21}^4\overline{v}_{129}^2+2^8\sigma^3=\overline{v}_9^2v_{21}^4v_{129}^2$
$\mathbb{Q}(\sqrt{-6})$						with $\sigma=\sqrt{-6}$, $v_7=\sigma+1$, $v_{25}=2\,\sigma+1$
D36	$2^5\,3^7$	$5^8\,7^2$	$5^8\,7^2$	$2\,3\,5^2\,7^2$	1.387965	$v_7^2\,\overline{v}_{25}^4+3\,\sigma^5=\overline{v}_7^2\,v_{25}^4$
$\mathbb{Q}(\sqrt{-7})$						with $\sigma=\sqrt{-7}$, $v_2=\frac{1+\sigma}{2}$, $v_{11}=2+\sigma$, $v_{23}=4+\sigma$, $v_{29}=2\,\sigma+1$
P57	2^7	2^7	7^3	$2^2 7$	1.106026	$v_2^7+\sigma^3=\overline{v}_2^7$
D37/D39	7	2^{13}	2^{13}	$2^2\,7$	1.707222	$v_2^{13}+\sigma=\overline{v}_2^{13}$
D40	3^4	$2^8\,11^2$	$2^8\,11^2$	$2^2\,3^2\,11^2$	1.001525	$v_2^8\,\overline{v}_{11}^2+\overline{v}_2^8\,v_{11}^2=3^2$
D41	$3^6\,7^3$	$2\,11^5$	$2\,11^5$	$2^2\,3^2\,7\,11^2$	1.033533	$\overline{v}_2\,\overline{v}_{11}^5+3^3\,\sigma^3=v_2\,v_{11}^5$
G30	2^5	2^{10}	11^3	$2^2\,11$	1.255421	$v_2^{10}+\overline{v}_2^5=-v_{11}^3$
G31	3^6	$2^3\,11^2$	2^{11}	$2^2\,3^2\,11$	0.961815	$v_2^{11}+\overline{v}_2^3\,v_{11}^2=3^3$
G32	$2^7 23^2$	$2^8\,11^3$	$3^6\,29^2$	$2^2 3^2 11\,23\,29$	0.923504	$3^3\,v_{29}^2+\overline{v}_2^8\,v_{11}^3=v_2^7\,v_{23}^2$
$\mathbb{Q}(\sqrt{-14})$						with $\sigma=\sqrt{-14}$, $v_{81}=5+2\,\sigma$, $v_{135}=11+\sigma$
D42	$2^3\,7^3$	$3^{13}\,5^3$	$3^{13}\,5^3$	$2\,3^2\,5^2\,7$	1.581910	$\overline{v}_{81}\,v_{135}^3+\sigma^3=v_{81}\,\overline{v}_{135}^3$
$\mathbb{Q}(\sqrt{-15})$						with $\sigma=\sqrt{-15}$, $v_4=\frac{1+\sigma}{2}$, $v_6=\frac{3+\sigma}{2}$, $v_{46}=\frac{7+3\sigma}{2}$
D43/D45, P58	3^5	2^9	2^9	$2^2\,3$	1.201305	$\overline{v}_4^4\,v_6+v_4^4\,\overline{v}_6=-3^3$
P59	2^6	2^6	$3^3 5$	$2^2\,3\,5$	0.721110	$v_4^3+3\,\sigma=\overline{v}_4^3$
D46	$3\,5^3$	2^{10}	2^{10}	$2^2\,3\,5$	1.018975	$\overline{v}_4^5+5\,\sigma=v_4^5$
D47	5^7	$2^{11}\,23^2$	$2^{11}\,23^2$	$2^2\,5\,23^2$	1.160407	$\overline{v}_4^4v_6\overline{v}_{46}^2+5^3\sigma=v_4^4\overline{v}_6v_{46}^2$
$\mathbb{Q}(\sqrt{-35})$						with $\sigma=\sqrt{-35}$, $v_9=\frac{1+\sigma}{2}$, $v_{39}=\frac{11+\sigma}{2}$
D48	$3^{16} 13^2$	$3^{16}\,13^2$	$2^{14}\,5^7\,7$	$2^2 3^2 5\,7\,13^2$	1.448172	$\overline{v}_9^7\,v_{39}^2+2^7 5^3\sigma=v_9^7\,\overline{v}_{39}^2$
$\mathbb{Q}(\sqrt{-39})$						with $\sigma=\sqrt{-39}$, $v_{10}=\frac{1+\sigma}{2}$, $v_{16}=\frac{5+\sigma}{2}$, $v_{22}=\frac{7+\sigma}{2}$, $v_{40}=\frac{11+\sigma}{2}$
D49	$3^3\,13^3$	$2^{19}\,5$	$2^{19}\,5$	$2^2\,3\,5^2\,13$	1.238591	$v_{16}^4\,\overline{v}_{40}+\sigma^3=\overline{v}_{16}^4\,v_{40}$
D50	$3^5\,13^3$	$2^9\,5^6\,11^3$	$2^9\,5^6\,11^3$	$2^2 3\,5^2 11^2 13$	1.380232	$v_{10}^6\,\overline{v}_{22}^3+3\,\sigma^3=\overline{v}_{10}^6\,v_{22}^3$

Table 3: The ABC triples in imaginary quadratic fields

Id	discrim	a	b	c	cond	quality	$a+b=c$
	$\mathbb{Q}(\sqrt{2+\sqrt{2}})$	with $\sigma=\sqrt{2+\sqrt{2}}$, $\rho=\sqrt{2}$, $u_0=\rho-1$, $u_1=\sigma+1$, $u_2=\rho\,\sigma+\rho+1$, $v_{17}=2\,\sigma+1$					
F4	2^{11}	17^3	17^3	2^{25}	$2\,17^2$	1.239162	$u_1^{-2}\tilde{u}_2^2 v_{17}^3+\tilde{u}_1^{-2}u_2^2\tilde{v}_{17}^3=\sigma^{25}u_0^{10}$
	$\mathbb{Q}(\sqrt{6+3\sqrt{2}})$	with $u_0=\rho+1, u_1=3+\rho+\rho\,\sigma, u_2=3-\rho+(2-\rho)\sigma, v_2=2+\sigma+\rho, v_7=\sigma+1, v_9=3+\sigma$					
F5	$2^{11}\,3$	$2^9\,3^6$	7^8	7^8	$2\,3^2\,7^2$	1.003989	$\tilde{v}_7^8\,\tilde{u}_1\,u_2^3+3\,\rho^4\,u_0^7\,\sigma=v_7^8\,u_1\,\tilde{u}_2^3$
	$\mathbb{Q}(\sqrt{-5+2\sqrt{2}})$	with $u=\rho-1$, $v_2=\frac{\rho-1+\sigma}{2}$					
F6	$2^6\,17$	17^3	2^{17}	2^{17}	$2^2\,17$	1.051009	$\overline{v}_2^{17}+u^9\,\sigma^3=v_2^{17}$
	$\mathbb{Q}(\sqrt{3+6\sqrt{2}})$	with $u_0=\rho-1$, $u_1=2+\sigma+\rho$, $v_2=\frac{\sigma+\rho+1}{2}$, $v_7=2+\frac{\sigma+1}{2}\rho$, $v_9=\frac{3+\sigma}{2}\rho$					
F7/F8	$-2^6\,3^2\,7$	2^6	2^6	$3^6\,7$	$2^2\,3^2\,7$	0.617259	$v_2^6\,\overline{u}_1^3+u_1^{-1}\,\overline{v}_7\,v_9^3=\tilde{v}_2^6\,u_1^3$
	$\mathbb{Q}\left(\sqrt{\frac{1+\sqrt{5}}{2}}\right)$	with $u=\sigma+1$, $u_2=\sigma$, $v_4=\sigma^3+1$					
F13	$-2^4\,5^2$	1	1	2^{12}	2^2	1.127411	$\tilde{u}_1^4+\sigma^{-4}\tilde{u}_1^3\,v_4^6=u_1^4$
	$\mathbb{Q}(\sqrt{\sqrt{5}})$	with $u_0=\frac{\rho+1}{2}$, $u_1=\frac{3+\rho+\sigma+\rho\,\sigma}{2}$, $v_4=\sigma+1, v_9=1+\frac{\rho+1}{2}\,\sigma, v_{11}=2+\sigma$					
F14	$-2^4\,5^3$	2^8	$3^6\,11^2$	$3^6\,11^2$	$2^2\,3^4\,11^2$	0.626462	$\overline{u}_1\,\tilde{v}_9^3\,v_{11}^2+2^2\,u_0^5=u_1\,v_9^3\,\overline{v}_{11}^2$
F15	$-2^4\,5^3$	$3^8\,11^2$	$3^8\,11^2$	$2^{12}\,5^5$	$2^2\,3^4\,5\,11^2$	0.827061	$\overline{u}_1^4\,v_9^4\,v_{11}^2+40\,u_0^5\,\sigma=u_1^4\,\tilde{v}_9^4\,\overline{v}_{11}^2$
	$\mathbb{Q}(\sqrt{5+\sqrt{5}})$	with $u_0=\frac{\rho+1}{2}$, $u_1=1+2\rho+2\sigma$, $u_2=\sigma+\frac{3+\rho}{2}$, $v_4=2+\sigma$, $v_5=3\sigma+5$, $v_{11}=\sigma+1$					
F16	$2^6\,5^3$	11^4	11^4	$2^{10}\,5^5$	$2^2\,5\,11^2$	0.892718	$u_1\,\tilde{u}_2^4\,v_{11}^4+20\,u_0^5\,\sigma=\overline{u}_1\,u_2^4\,\tilde{v}_{11}^4$
	$\mathbb{Q}(\sqrt{4+\sqrt{5}})$	with $u_0=\frac{\rho-1}{2}$, $u_1=5+2\,\sigma$, $u_2=\frac{5+\rho}{2}+\sigma$, $u_3=\sigma+\rho+1$, $v_4=\sigma+1$, $v_5=2+\sigma$					
F17	$2^4\,5^2\,11$	5^6	5^6	$2^{16}\,11^3$	$2^2\,5^2\,11$	1.187860	$\overline{u}_1^2\,v_5^6+u_0^{11}u_2^{-4}v_4^8\,\sigma^3=u_1^2\,\overline{v}_5^6$
	$\mathbb{Q}(\sqrt{9+4\sqrt{6}})$	with $u_0=5+2\,\rho$, $u_1=2+\sigma+\rho$, $v_2=\frac{\sigma+1}{2}, v_5=2+\sigma-\rho\,\frac{\sigma+1}{2}$					
F18/F19	$-2^6\,3^3\,5$	5	2^{10}	2^{10}	$2^2\,5$	0.574754	$\overline{u}_1\,v_2^{10}+u_0^4\,v_5=\tilde{v}_2^{10}$
	$\mathbb{Q}(\sqrt{3},\sqrt{7})$	with $\sigma=\sqrt{3}$, $\rho=\sqrt{7}$, $u_1=\frac{\rho+\sigma}{2}$, $u_2=\sigma+\frac{1+\rho\,\sigma}{2}$, $u_3=2+\sigma$, $v_3=1+\frac{\rho+\sigma}{2}$					
F20	$2^4\,3^2\,7^2$	1	1	3^7	3	0.772098	$\overline{u}_3^2+u_1^7\,u_3^2=v_3^7$
	$\mathbb{Q}(\sqrt{5+2\sqrt{7}})$	with $u_0=8-3\rho$, $u_1=\frac{3+\rho-\sigma+\rho\,\sigma}{2}$, $v_2=\frac{\rho-1+3\sigma-\rho\sigma}{2}$, $v_3=1+3\sigma-\rho\sigma$, $v_{31}=2\sigma+\rho+2$					
F21	$-2^6\,3\,7^2$	2	$3^{13}\,31^2$	$3^{13}\,31^2$	$2\,3^2\,31^2$	1.118591	$u_0v_2+\overline{u}_1^{10}v_3^{13}\tilde{v}_{31}^2=u_1^{11}\tilde{v}_3^{13}v_{31}^2$
	$\mathbb{Q}(\sqrt{2+\sqrt{7}})$	with $u_0=8-3\,\rho$, $u_1=18+7\,\rho+9\,\sigma+3\,\rho\,\sigma$, $v_2=5+2\,\rho+2\,\sigma+\rho\,\sigma$, $v_3=2+\sigma$					
F22	$-2^8\,3\,7^2$	2^8	3^7	3^7	$2\,3^2$	0.572791	$u_1^4\,\tilde{v}_3^7+\overline{u}_1^2\,u_0^2\,v_2^8=v_3^7$
	$\mathbb{Q}(\sqrt{\frac{3+\sqrt{21}}{2}})$	with $u=\frac{1+\rho+2\sigma}{4}$, $2+\sigma, v_5=\sigma+1, v_{25}=\frac{\rho-1}{2}$					
F24	$-3^3\,7^2$	5^3	5^3	$3^7\,5^4$	$3\,5^4$	0.959523	$\tilde{u}^3\,\tilde{v}_5^3+\sigma^7\,v_{25}^2=u^3\,v_5^3$
	$\mathbb{Q}(\sqrt{2},\sqrt{-1})$	with $u=\sigma+1$, $v_2=1+\frac{\sigma+\rho\,\sigma}{2}$, $v_{25}=2+\rho$, $v_9=1+\rho\,\sigma$, $v_{17}=1+\sigma+\rho\,\sigma$					
G3	$2^4\,3^2\,7^2$	$3^6 17^2$	$3^6 17^2$	$2^4\,5^6$	$2\,3^4\,5^2\,17^2$	0.636805	$\overline{v}_{17}^2\,v_9^3\,u^2+v_2^4\,\tilde{u}\,v_{25}^3=v_{17}^2\,\overline{v}_9^3$
	$\mathbb{Q}(\sqrt{1+2\sqrt{-1}})$	with $u=\frac{1-\rho+\sigma-\rho\,\sigma}{2}$, $v_2=\frac{1+\rho+\sigma-\rho\,\sigma}{2}$, $v_{17}=\rho\,\sigma+\rho+1$					
G4/G5	$2^6\,5$	1	1	$2^4\,5$	$2\,5$	0.542941	$u^4+v_2^4\,\sigma=1$
G6	$2^6\,5$	17^3	17^3	$2^8\,5^3$	$2\,5\,17^2$	0.755131	$v_{17}^3+\tilde{u}^2\,v_2^8\,\sigma^3=u^4\,\overline{v}_{17}^3$
	$\mathbb{Q}(\sqrt{2+\sqrt{-1}})$	with $u=2-\rho+\sigma-\rho\,\sigma$, $v_2=\sigma+1$, $v_5=2+\sigma$					
G7	$2^8\,5$	5^6	5^6	$2^{12}\,5^3$	$2\,5^3$	1.037078	$v_5^6+\rho\,\overline{u}^2\,v_2^{12}\,\sigma^3=u^8\,\overline{v}_5^6$
	$\mathbb{Q}(\sqrt{3+2\sqrt{-1}})$	with $u=\frac{3-\rho+\sigma-\rho\,\sigma}{2}$, $v_2=\frac{1-\rho+\sigma-\rho\,\sigma}{2}$, $v_5=2\,\rho+\rho\,\sigma$, $v_9=\sigma+\rho+1$					
G8	$2^6\,13$	$3^6\,5^2$	$3^6\,5^2$	$2^4\,13^3$	$2\,3^4\,5^2\,13$	0.594901	$\overline{u}^2\,v_9^3\,v_5^2+I\,v_2^4\,\sigma^3=u^6\,\overline{v}_5^2\,\overline{v}_9^3$
	$\mathbb{Q}(\sqrt{1+8\sqrt{-1}})$	with $u=2+2\,\rho+\sigma$, $v_2=\rho+\frac{1+\sigma}{2}$, $v_5=\frac{3-\rho+\sigma-\rho\,\sigma}{2}$, $v_{13}=\frac{5+\rho-\sigma+\rho\,\sigma}{2}, v_{25}=2-\rho$					
G9	$2^4 5\,13$	2^{10}	2^{10}	$5^7\,13^3$	$2^2\,5^3\,13$	1.205664	$\tilde{u}^7\,v_2^{10}+\rho\,v_{13}^3\,v_5\,v_{25}^3=u^5\,\overline{v}_2^{10}$

Table 4: ABC triples in quartic fields

Id	discrim	a	b	c	cond	quality	$a+b=c$
	$\mathbb{Q}(\sqrt{\sqrt{-2}})$	with $\sigma=\sqrt{\sqrt{-2}}$, $\rho=\sqrt{2}$, $u=1-\rho+\rho\,\sigma$, $v_3=\sigma+1$, $v_9=\rho+1$					
G11	2^{11}	3^8	3^8	$2^{15}\,3^4$	$2\,3^4$	1.163578	$\tilde{u}^4\,\overline{v}_3^8+\sigma^{15}\,v_9^2=u^4\,v_3^8$
	$\mathbb{Q}(\sqrt{-1+\sqrt{-2}})$	with $u=3+\rho+2\sigma-\rho\sigma$, $v_2=\sigma+\rho+1$, $v_3=2+\sigma-\rho\sigma$, $v_{17}=1+\rho+2\,\sigma$					
G12	$2^{10}\,3$	$3^6 17^2$	$3^6 17^2$	$2^{18}\,3^7$	$2\,3^3\,17^2$	1.140309	$v_3^6\,\overline{v}_{17}^2+\overline{u}^5\,v_2^{18}\,\sigma^7=u^8\,\overline{v}_3^6\,v_{17}^2$
	$\mathbb{Q}(\sqrt{2+2\sqrt{-2}})$	$u=9+2\rho+6\sigma, v_2=2-\rho-\sigma+\rho\sigma, v_3=\sigma+1, \hat{v}_3=1+2\rho+\sigma+\rho\sigma, v_{11}=\sigma+\rho+1$					
G13	$2^{11}\,3$	3^6	3^6	$2^{11}\,3^3$	$2\,3^3$	0.859052	$\overline{u}^2\,v_3^6+u^3\,v_2^{11}\,\hat{v}_3^3=\overline{v}_3^6$
G14	$2^{11}\,3$	$3^4 11^3$	$3^4 11^3$	2^{31}	$2\,3^2\,11^2$	1.309467	$v_3^4\,\overline{v}_{11}^3+u^{15}\,v_2^{31}=\overline{u}\,\overline{v}_3^4\,v_{11}^3$
	$\mathbb{Q}(\sqrt{1+4\sqrt{-2}})$	$u=2+\sigma+\rho$, $v_2=\frac{1+\sigma}{2}$, $v_3=1+\rho\,\frac{\sigma-1}{2}$, $v_9=1-\rho$, $v_{11}=1+\rho\,\frac{\sigma-3}{2}$, $v_{19}=1+\rho\,\frac{\sigma+1}{2}$					
G15	$2^6\,3\,11$	$3^4 11^3$	2^{23}	2^{23}	$2^2\,3^2\,11$	1.169070	$\overline{u}^6\,v_2^{23}+u^3\,\overline{v}_2^{23}=v_9^2\,v_{11}^3$
I16	$2^6\,3\,11$	$2^4 3^3$	2^{25}	$3^6 11\,19^2$	$2^2 3^3 11\,19$	0.974608	$\overline{u}^5 v_2^2 5+u^4\overline{v}_2^4 v_3^3=v_9^3 v_{11} v_{19}^2$
	$\mathbb{Q}(\sqrt{7+2\sqrt{-2}})$:	$u=2+3\rho+\sigma+\rho\sigma, v_2=\frac{3+\rho+\sigma}{2}, v_3=\rho\frac{3+\sigma}{2}, v_9=\rho+1, v_{11}=\rho\frac{\sigma+1}{2}, v_{17}=2+\sigma, v_{19}=3+\sigma-\rho\frac{\sigma+1}{2}$					
G16	$2^6 3\,19$	$2^{29}17^3$	$2^{29}17^3$	$3^8 11^6 19^3$	$2^2 3^2 11^2 17^2 19$	1.270620	$u^{12}\overline{v}_2^{29}v_{17}^3+v_9^4 v_{11}^3\overline{v}_{11}^3 v_{19}^3=\overline{u}^9 v_2^{29}\overline{v}_{17}^3$
	$\mathbb{Q}(\sqrt{\sqrt{-3}})$	with $u_0=\frac{1+\rho}{2}$, $u_1=\frac{1+\rho}{2}+\sigma$, $v_4=\sigma+1$, $v_7=\sigma+\rho+1$					
G20	$2^4\,3^3$	7^2	7^2	$2^{12}\,3^3$	$2^2\,3\,7^2$	0.933183	$u_1^7\,\overline{v}_7^2+u_0\,\sigma^3\,v_4^6=\overline{u}_1\,v_7^2$
	$\mathbb{Q}(\sqrt{3+2\sqrt{-3}})$	with $u_0=\frac{1+\rho}{2}$, $u_1=\frac{\sigma+1}{2}$, $v_3=\frac{3+\sigma}{2}, v_7=1+\frac{\sigma-1}{2}\,\rho, v_{13}=2+\sigma$					
G21	$3^3\,7$	13^2	13^2	$2^{16}\,3^3\,7$	$2^4\,3\,7\,13^2$	1.008855	$u_0\,u_1^{15}\,\overline{v}_{13}^2+2^4\,v_3^3\,v_7=u_1^{-5}\,v_{13}^2$
	$\mathbb{Q}(\sqrt{-2+3\sqrt{-3}})$	with $u_0=\frac{1+\rho}{2}$, $u_1=\frac{5-\rho}{2}-\rho\,\sigma$, $v_3=\sigma+\rho+1$, $v_4=\rho+\sigma$, $v_{13}=\sigma+\frac{3+\rho}{2}$					
G22	$2^4\,3^2\,7$	3^3	3^3	$2^4\,7^3$	$2^2\,3^2\,7$	0.691861	$v_3^3+v_4^2\,\sigma^3=-u_0\,u_1^2\,\overline{v}_3^3$
G23	$2^4\,3^2\,7$	3^6	3^6	$2^{12}\,7$	$2^2\,3^2\,7$	0.824713	$u_1^7\,\overline{v}_3^6+v_4^6\,\sigma=\overline{u}_1\,v_3^6$
G24	$2^4\,3^2\,7$	$3^{10}13^2$	$3^{10}13^2$	$2^{20}7^3$	$2^2\,3^2\,7\,13^2$	1.120945	$\overline{u}_1 v_3^{10} v_{13}^2+u_0 v_4^{10}\sigma^3=u_0^{-1}u_1^{11}\overline{v}_3^{10}\overline{v}_{13}^2$
	$\mathbb{Q}\left(\sqrt{\frac{-1+\sqrt{-7}}{2}}\right)$	with $u=\frac{1-\rho+\sigma-\rho\,\sigma}{4}$, $v_4=\frac{1+\rho}{2}$, $v_{11}=1+\frac{\rho+1}{2}\,\sigma$					
G33	$2^3\,7^2$	11^2	11^2	2^{15}	$2^3\,11^2$	0.809342	$\tilde{u}^6\,\overline{v}_{11}^2+\sigma^7\,v_4^4=u^6\,v_{11}^2$
	$\mathbb{Q}(\sqrt{-1},\sqrt{-7})$	with $u=\frac{3+\rho+3\sigma-\rho\,\sigma}{2}$, $v_2=1+\frac{\rho+1}{2}\sigma, \hat{v}=1+\frac{1-\rho}{2}\,\sigma, v_9=2-\rho\,\sigma, v_{37}=1+\rho+3\,\sigma$					
G34	$2^4\,7^2$	2^{17}	$3^6\,37^2$	$3^6\,37^2$	$2^2\,3^4\,37^2$	0.702370	$\sigma\,u^{-7}v_9^3\,v_{37}^2+u^4\,\hat{v}_2\,v_2^{16}=\tilde{v}_9^3\,\overline{v}_{37}^2$
	$\mathbb{Q}(\sqrt{\sqrt{-7}})$	with $u=\frac{3-\rho+\sigma+\rho\sigma}{4}, v_2=\sigma+\frac{3+\rho}{2}, v_4=\frac{1-\rho}{2}, v_9=2\sigma+\rho+2, v_{11}=3-\rho+\rho\sigma, v_{23}=2+\sigma$					
G35/G36	$2^2\,7^3$	3^6	3^6	2^{16}	$2\,3^4$	0.900804	$v_9^3+u^{17}\,v_2^{16}=-\overline{u}^{14}\,\overline{v}_9^3$
I2	$2^2\,7^3$	$2^7 23^2$	$3^{12}11^3$	$3^{10}7^7$	$2^3 3^4 7\,11\,23$	1.161980	$u^8 v_9^6 v_{11}^3+\overline{u}^7\sigma^7\overline{v}_9^5=v_2 v_4^3 v_{23}^2$
	$\mathbb{Q}\left(\sqrt{\frac{1+\sqrt{-7}}{2}}\right)$	with $u=\frac{1-\rho+\sigma+\rho\,\sigma}{2}$, $v_2=\sigma+1$, $v_7=1+\sigma\,\frac{1-\rho}{2}$, $v_{11}=\frac{1-\rho}{2}+\sigma$					
G37	$2^5\,7^2$	7^5	7^5	2^{20}	$2^2\,7^2$	1.097128	$v_7^5+\overline{u}^6\,\overline{v}_7^5=u^6\,\sigma^2\,v_2^{18}$
G38	$2^5\,7^2$	11^3	11^3	2^{15}	$2^2\,11^2$	0.767909	$\overline{u}^2\,v_{11}^3+u^5\,\sigma^3\,v_2^{12}=\overline{v}_{11}^3$
	$\mathbb{Q}(\sqrt{-2+\sqrt{-7}})$	with $u=\frac{7+\rho+3\,\sigma-\rho\,\sigma}{2}$, $v_2=\frac{5-\rho+\sigma-\rho\,\sigma}{4}$, $\hat{v}_2=\frac{1+\rho}{2}+\sigma$, $v_{23}=\sigma+\rho+1$					
G40	$2^2 7^2 11$	2^8	2^8	$2\,11^3$	$2^3\,11$	0.648943	$u^4\,\overline{v}_2^8+\overline{u}^3\,v_2^8=-\hat{v}_2\,\sigma^3$
G41	$2^2 7^2 11$	$2^{20}11$	$2^{10}23^3$	$2^{10}23^3$	$2^3\,11\,23^2$	0.886760	$v_2^{10}\,\overline{v}_{23}^3+\overline{u}^7\,\hat{v}_2^{20}\,\sigma=u^6\,\overline{v}_2^{10}\,v_{23}^3$
	$\mathbb{Q}(\sqrt{3+2\sqrt{-7}})$	with $u=\frac{1-3\,\rho+\sigma+\rho\,\sigma}{4}$, $v_2=\frac{1+\sigma}{2}$, $v_4=\frac{1-\rho}{2}$, $v_{29}=2+\sigma$					
G43	$7^2\,37$	$2^{10}29^3$	$2^{10}29^3$	$2^{12}37^3$	$2^4\,29^2\,37$	0.928698	$u\,v_2^{10}\,\overline{v}_{29}^3+\tilde{u}\,\overline{v}_2^{10}\,v_{29}^3=-v_4^6\,\sigma^3$

Table 5: ABC triples in quartic fields (continued)

3.3 The ABC identities in higher degree fields

There are 15 rational $j(t)$ giving t-orbits defined over splitting fields of a cubic polynomial. All three numbers in their ABC identities have the same norms, actually $2^2, 2^9, 2^{11}, 2^{13}, 2^{15}, 2^{16}, 2^{23}, 3^5, 3^{13}, 2^4 3^6, 5^7, 2^{22}5^7, 5^{11}7^5, 2^9 13^7, 2^{28}3^{10}13^5$. The quality ratio of the last case might be 1.378086. Here is the ABC triple for the norm 2^{23}, for the coverings E10/E11. Let ζ denote a root of $z^6 + 4z^4 - 3z^2 + 2$. The field $\mathbb{Q}(\zeta)$ is the splitting field of $x^3 + 2x^2 + 3x + 4$, obtained by adjoining $\sqrt{-2}$ to the cubic field. The identity is

$$\zeta^{23} + \left(\frac{\zeta+\zeta^2}{2} - \frac{5\zeta^3+\zeta^5}{4}\right)^{23} \left(\frac{1-\zeta}{2} - \frac{3\zeta^2-3\zeta^3+\zeta^4-\zeta^5}{4}\right)^{-6}$$
$$= \left(\frac{-\zeta+\zeta^2}{2} + \frac{5\zeta^3+\zeta^5}{4}\right)^{23} \left(\frac{1+\zeta}{2} - \frac{3\zeta^2+3\zeta^3+\zeta^4+\zeta^5}{4}\right)^{-6}. \quad (12)$$

The numbers under the 23rd power have the norm 2, while the numbers in the (-6)th power are units. Among alternative cross-ratios, there are 152 $j(t) \in \mathbb{Q}$ giving ABC identities in a splitting field (of degree 6) of a cubic polynomial, and 7 $j(t) \in \mathbb{Q}$ with t in a cubic field. Of them, 11 and 4 (respectively) give unit equations. The j-values are: $2^9 3, 2^{10}3, -2^{10}3, 2^9 3^2, -2^8 3^3, 2^9 3^3, -2^9 3^3, 2^{10}3^3, -2^{11}3^5, 2^8 5, -2^{15}3\cdot 5^3$ and $2^8 3^2, 2^8 3^3, 2^8 3^3 7, 2^8 3^2 7^3$.

Tables 4 and 5 give some ABC identities from the encountered Heun equations with the t-values in a quartic field. We attempted to give all cases with the trivial class group where the quartic field is either a nested quadratic extension, or a composite field of two quadratic extensions. The towered or composite quartic fields certainly appear in the F and G cases. In the ABC identities, σ denotes the outer square root (or the first given square root for the composite fields) and ρ denotes the inner square root (or the second given square root for the composite fields). Some units are indexed. The encountered tower quartic fields with apparently a non-trivial class group are

$$\mathbb{Q}\left(\sqrt{-22+4\sqrt{22}}\right), \quad \mathbb{Q}\left(\sqrt{9+8\sqrt{-3}}\right), \quad \mathbb{Q}\left(\sqrt{5+4\sqrt{-5}}\right), \quad \mathbb{Q}\left(\sqrt{3\sqrt{-7}}\right),$$
$$\mathbb{Q}\left(\frac{\sqrt{-9+\sqrt{-7}}}{2}\right), \quad \mathbb{Q}\left(\sqrt{\sqrt{-15}}\right), \quad \mathbb{Q}\left(\frac{\sqrt{3+\sqrt{-15}}}{2}\right), \quad \mathbb{Q}\left(\frac{\sqrt{-3+\sqrt{-15}}}{2}\right).$$

References

[Bec89] S. Beckmann. Ramified primes in the field of moduli of branched coverings of curves. *J. Algebra*, 125:236–255, 1989.

[Dok03] T. Dokchitzer. LLL & ABC. Available at http://arxiv.org/math.NT/0307322, 2003.

[FRV10] G. Filipuk, R.Maier, and R. Vidunas. General transformations between the heun and gauss hypergeometric functions. Available ar arxiv:0910.3087[math/CA]., 2010.

[Gou81] E. Goursat. Sur l'équation différentielle linéaire qui adment pour intégrale la série hypergéométrique. *Ann. Sci. École Noprm. Sup.(2)*, 10:S3–S142, 1881.

[Her91] S. Herfurtner. Elliptic surfaces with four singular fibres. *Math. Ann.*, 291:319–342, 1991.

[Kle77] F. Klein. Uber lineare differentialgleichungen I. *Math. Annalen*, 11:115–118, 1877.

[Mai07] R. S. Maier. The 192 solutions of the Heun equation. *Math. Comp.*, 76, 2007.

[Nit] A. Nitaj. The abc conjecture home page. www.math.unicaen.fr/nitaj/abc.html.

[vHV11] M. van Hoeij and R. Vidunas. Transformations between the Heun and Gauss hypergeometric functions of the hyperbolic type. Paper in progress; the data is available at the website http://www.math.fsu.edu/~hoeij/files/Heun/TextFormat, 2011.

[Vid09] R. Vidūnas. Algebraic transformations of Gauss hypergeometric functions. *Funkcialaj Ekvacioj*, 52(2):139–180, 2009. Available at http://arxiv.org/math.CA/0408269.