Hypergeometric Solutions of Linear Differential Equations with Rational Function Coefficients

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  4. $\partial = \frac{d}{dx}$.

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- We want to find ${}_2F_1$-type solutions (if they exist), i.e, solutions of the form:
  $y = \exp(\int r\, dx) \left(r_0 S(f) + r_1 S(f)' \right) \neq 0$ such that $L(y) = 0$, where $S(x) = {}_2F_1(a, b; c \,|\, x)$, $r, r_0, r_1, f \in \mathbb{C}(x)$.

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  where $S(x) = {}_2F_1(a, b; c \,|\, x)$, $r, r_0, r_1, f \in \mathbb{C}(x)$.
- Why this format?
  Conjecture: if L has a convergent solution in $\mathbb{Z}[[x]]$ then it has a solution of this format.

An Example

- Consider the differential operator:

$$L = \partial^2 + \frac{\left(16\,x^3+16\,x^2+23\,x-5\right)}{3\,x(2\,x-1)(x^2+2\,x+5)}\,\partial - \frac{875\,x^2}{9\,(2\,x-1)^2(x^2+2\,x+5)}$$

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- Our algorithm on 'five singularities' solves L:

$$Sol_1(L) = \frac{20\,(2x-1)^{\frac{1}{6}}}{9\,(x^2+2x+5)^{\frac{7}{6}}\,x^{\frac{14}{3}}} \cdot \left[(x^2+2\,x+5)x^4\ {}_2F_1\left(\frac{1}{6}, \frac{1}{3};\ 1 \mid \frac{4\,(2\,x-1)}{x^4\,(x^2+2\,x+5)}\right) - (x^3+x^2+2\,x-2)^2\ {}_2F_1\left(\frac{7}{6}, \frac{4}{3};\ 2 \mid \frac{4\,(2\,x-1)}{x^4\,(x^2+2\,x+5)}\right)\right]$$

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- Complete algorithms for second order differential operators are very useful to solve higher order differential operators.

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- Fuchsian differential operators correspond to hypergeometric solutions.
- M. van Hoeij and R. Vidunas developed the tables of rational functions for 4 singularities (Heun equation).
- T. Fang and M. van Hoeij developed algorithm for 2-descent, which finds ${}_2F_1$-type solutions whenever f has degree 2, and also reduces a differential operator to another with fewer singularities.

Our Contribution

Let $L_{inp} \in \mathbb{C}(x)[\partial]$ be a second order linear differential operator with rational function coefficients. Let L_{inp} be irreducible and has no Liouvillian solutions.

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1. L_{inp} has five regular singularities where at least one of them is *logarithmic*. This is the topic of today!
2. L_{inp} has hypergeometric solution of degree three, i.e, L_{inp} is solvable in terms of ${}_2F_1(a, b; c \mid f)$ where f is a rational function of degree three.

Formal Solutions, Example

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- Exponents at $x = 1$ are: 0, 0. The point $x = 1$ is a logarithmic singularity.
- Regular points have exponents 0, 1.
- A change of variables $x \mapsto x^2$ turns $x = 0$ into a regular point. It turns $x = 1$ into two logarithmic singularities $x = \pm 1$.

Gauss Hypergeometric Differential Operator

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- The Gauss hypergeometric function ${}_2F_1(a, b; c \,|\, x)$ is a solution of $H_{c,x}^{a,b}$ where:

  $${}_2F_1(a, b; c \,|\, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}$$

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- The Pochhammer symbol $(a)_n$ is defined as:

  $$(a)_n = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)\dots(a+n-1) & \text{otherwise} \end{cases}$$

Transformations

- We define the following transformations on a second order differential operator:
  - (i) Change of variables: $y(x) \mapsto y(f)$
  - (ii) Gauge transformation: $y \mapsto r_0 y + r_1 y'$
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- These transformations are denoted $\xrightarrow{f}_C$, $\xrightarrow{r_0, r_1}_G$ and $\xrightarrow{r}_E$.

- $\xrightarrow{r_0, r_1}_G$ and $\xrightarrow{r}_E$ are equivalence relations. They do not affect the true singularities of a differential operator. $\xrightarrow{f}_C$ can change everything.

Effect of $\xrightarrow{f}_C$

$H^{\frac{1}{8},\frac{3}{8}}_{1,x}$:

p	0	1	∞
Δ_p	0	$\frac{1}{2}$	$\frac{1}{4}$

p: singularity, Δ_p: exponent difference

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- There are algorithms to compute $\xrightarrow{r_0,r_1}_G \xrightarrow{r}_E$. The crucial part is to compute f and a, b, c.

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- There are many integer sequences in `oeis.org` whose generating functions are convergent and holonomic. Such generating functions satisfy linear differential operators. Such differential operators of order 2 and 3 tested so far have logarithmic singularities and have ${}_2F_1$- type solutions.

Motivation Contd.

- Moreover, such differential operators lie in the same class (minimal network of differential operators in terms of solvability), namely, $Class\left(H_{1,x}^{\frac{1}{12},\frac{5}{12}}\right)$;
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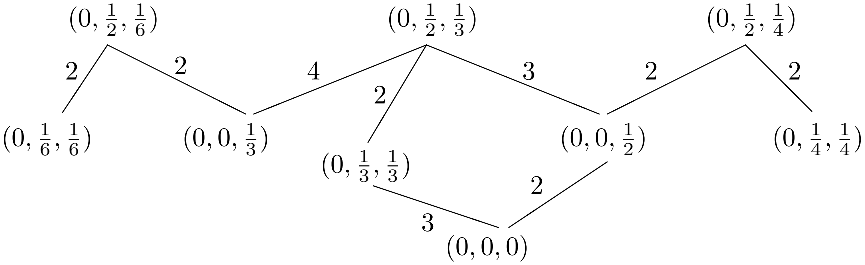
- K. Takeuchi classified commensurable classes of arithmetic triangle groups. The first class gives (e_0, e_1, e_∞) of Gauss hypergeometric differential operators that lie in $Class\left(H_{1,x}^{\frac{1}{12},\frac{5}{12}}\right)$:



Degree Bounds and Types of f

- For a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n, total amount of ramification is given by:

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- Riemann-Hurwitz's formula gives the following for our project:

1. Belyi maps: zero-dimensional families $f(x)$, ramify only above $\{0, 1, \infty\}$, degree bound 18.
2. Belyi-1 maps: one-dimensional families $f(x, t)$, ramify above one point outside $\{0, 1, \infty\}$, degree bound 12.
3. Belyi-2 maps: two-dimensional families $f(x, s, t)$, ramify above two points outside $\{0, 1, \infty\}$, degree bound 6.

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- Smaller cases are easy to find. For larger cases we use Elimination, Resultants, Parametrization etc. There are no maps of degree 17 for our project. We use special techniques given by F. Beukers and H. Montanus to compute degree 18 Belyi maps.

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- Smaller cases are easy to find. For larger cases we use Elimination, Resultants, Parametrization etc. There are no maps of degree 17 for our project. We use special techniques given by F. Beukers and H. Montanus to compute degree 18 Belyi maps.
- The major task is to prove that we have computed ALL Belyi and near Belyi maps relevant to our project.

The Major Task

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- **We have to develop a complete table T of relevant rational functions $f(x), f(x,t)$ and $f(x,s,t)$ such that there exists at least one $f \in T$ and a suitable Möbius transformation m for which**

$$H_{c,x}^{a,b} \xrightarrow{f(m)}_C H_{c,f(m)}^{a,b} \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_{inp}$$

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- Once we have a complete table, we can develop a differential solver from it.

The Differential Solver	Table
$L = \partial^2 + \frac{(x+7)(x-39)}{(x-16)(x^2+18x-15)}\partial - \frac{25x^3-1006x^2-5523x-894}{36(x^2+18x-15)(x-16)(x^2-3)}$ $Sol = e^{\int r dx}\left(r_0 S(f) + r_1 S(f)'\right)$, $r, r_0, r_1 \in \mathbb{C}(x)$ and $S(f) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \mid \frac{4(x^2+18x-15)^2(x^2-3)^3}{9(4x^3-29x^2+42x-21)^3}\right)$	**Belyi maps:** $F_1(x) = \frac{4(2x-5)(7x+20)^4}{x^5(5x+28)^2(5x+12)}$ $\vdots$ $F_{383}(x) = \ldots$ **Belyi-1 maps:** $G_1(x,s) = \frac{-64(x^2+sx-s)^3 x^2}{s^3(x-1)^3(8x^2+9sx-9s)}$ $\vdots$ $G_{100}(x,s) = \ldots$ **Belyi-2 maps:**
$\exists\, m = \frac{ax+b}{cx+d}$ such that $f = \begin{cases} F_i(x) \circ m & \text{or} \\ G_j(x,s)\|_{s=?} \circ m & \text{or} \\ \text{Belyi-2 map} \end{cases}$	

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- A dessin is a connected, oriented and bi-colored graph where any two vertices of different color are joined by an edge.
- Dessin of a Belyi map f is the graph of $f^{-1}([0,1])$.
- A sequence $[g_1, g_2, \cdots, g_k]$ of permutations in S_n is called a constellation (or a k-constellation) of degree n if:
    1. the group $\langle g_1, g_2, \cdots, g_k \rangle$ is transitive,
    2. $g_1 g_2 \cdots g_k = 1$.

The Correspondence

- A dessin is a connected, oriented and bi-colored graph where any two vertices of different color are joined by an edge.
- Dessin of a Belyi map f is the graph of $f^{-1}([0,1])$.
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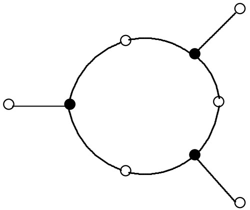
- The braid group B_k generated by the braids $\sigma_1, \ldots, \sigma_{k-1}$ acts on a k-constellation in the following way:
  $\sigma_i : [g_1, \ldots, g_i, g_{i+1}, \ldots, g_k] \mapsto [g_1, \ldots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \ldots, g_k]$

The Correspondence Contd.

- Here is an example of a dessin of degree 9:



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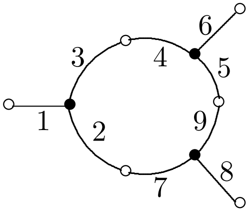

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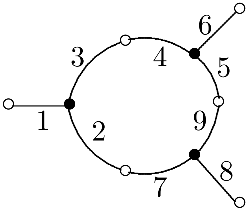

- This dessin has 3 black vertices (points above 0), 6 white vertices (points above 1) and 2 faces (correspond to poles).

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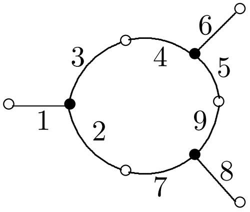

- Dessins do not have labels. The above 'labelled dessin' is useful to read the correspondence.

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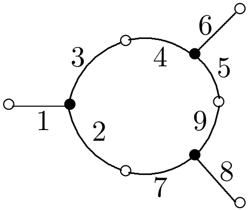

- This dessin corresponds to the following 3-constellation of degree 9 (unique up to conjugation):

$$g_0 = (1\ 2\ 3)\ (4\ 5\ 6)\ (7\ 8\ 9)$$
$$g_1 = (1)\ (6)\ (8)\ (2\ 7)\ (3\ 4)\ (5\ 9)$$
$$g_\infty = (g_0 g_1)^{-1} = (1\ 3\ 6\ 5\ 8\ 7)\ (2\ 9\ 4).$$

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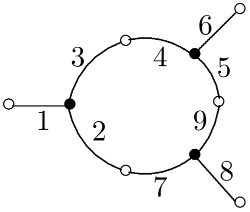

- and the following Belyi map (up to Möbius transformation):

$$f = \frac{4}{27}\,\frac{\left(x^3+1\right)^3}{x^3}$$

$$1 - f = -\frac{1}{27}\,\frac{\left(x^3+4\right)\left(2\,x^3-1\right)^2}{x^3}$$

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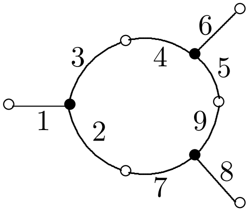


- A dessin is the equivalence class of 3-constellations mod conjugation. Conjugated 3-constellations give the same dessin (with different labelling).

Computing Relevant Dessins

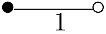
We have developed the table of Belyi maps. To prove the completeness we first enumerate all '5 singularity' dessins using combinatorial search including various techniques to prevent computational explosion. Then we compare the table of dessins with our table of Belyi maps. Steps:

Computing Relevant Dessins

We have developed the table of Belyi maps. To prove the completeness we first enumerate all ‘5 singularity’ dessins using combinatorial search including various techniques to prevent computational explosion. Then we compare the table of dessins with our table of Belyi maps. Steps:

- Computing 3-constellations
- Computing dessins, i.e, discarding conjugates
- Discarding non-planar dessins, as well as dessins whose Weighted Singularity Count is too high
- Choosing only relevant dessins

Computing ‘Labelled Dessins’ or 3-constellations



$$g_0 = (1)$$
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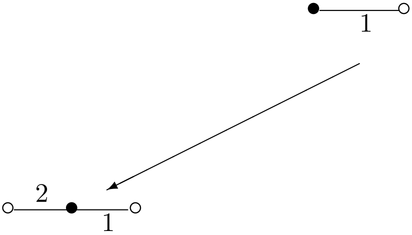

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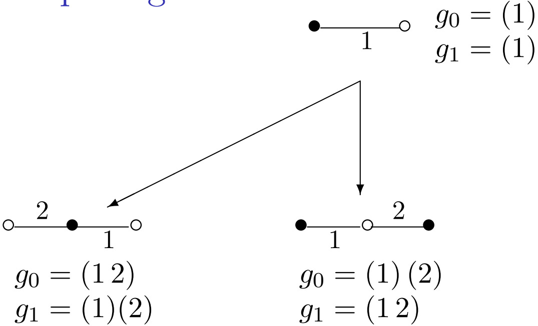

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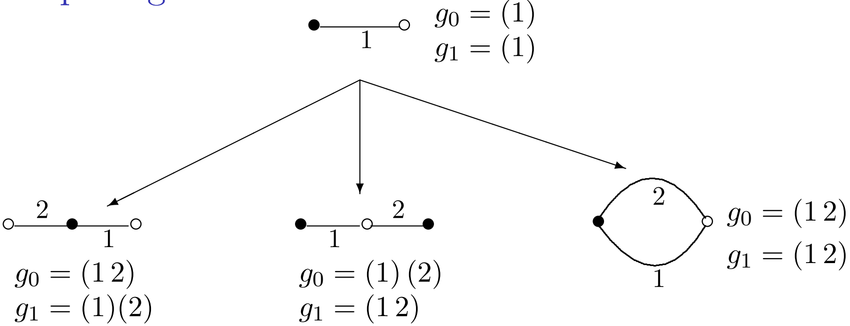

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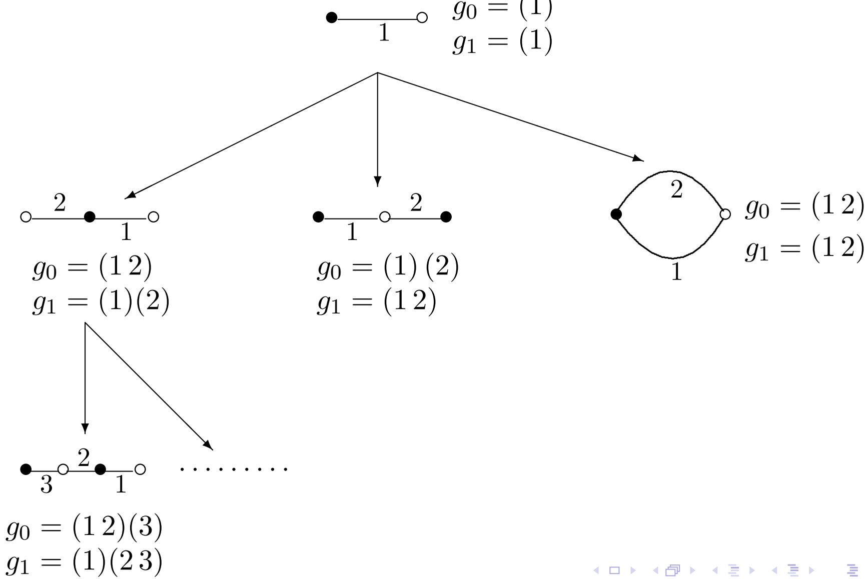

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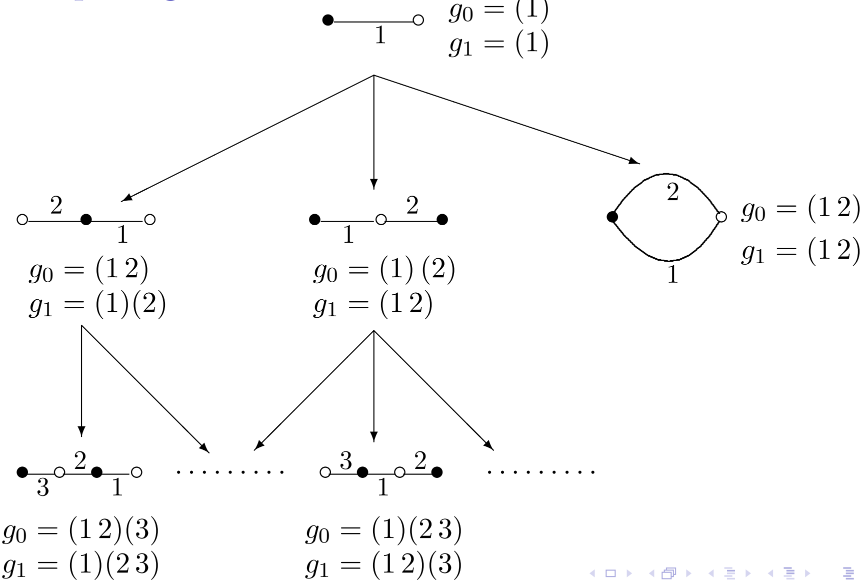

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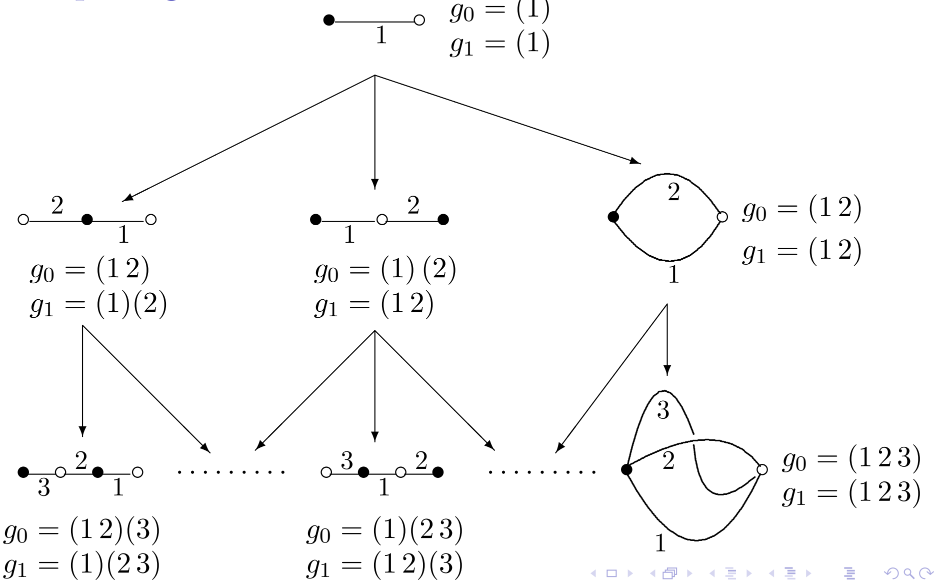

Computing ‘Labelled Dessins’ or 3-constellations Contd.

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$$i.e,\ T_n = 1, 3, 24, 360, 8640, 302400, 14515200, 914457600, \ldots$$

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- **Problem:** The program computes every 3-constellation. A dessin is a conjugacy class of 3-constellations. To prevent computing the same dessin many times, we should compute only one element from each conjugacy class (otherwise the output will be"error, out of memory" long before we reach $n = 18$). The next step is to identify conjugated 3-constellations and discard all but one of them.

Computing Dessins

- Any two 3-constellations $[g_0, g_1, g_\infty]$ and $[\tilde{g}_0, \tilde{g}_1, \tilde{g}_\infty]$ represent the same dessin iff $\exists\, \sigma \in S_n$ such that $\tilde{g}_i = \sigma g_i \sigma^{-1}$, $i \in \{0, 1, \infty\}$.

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- We will obtain $\sigma\pi = [\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n)]$ after applying the repeated action of $\tilde{g}_0$ and $\tilde{g}_1$ on $\sigma(b)$. Moreover,

$$(\sigma\pi)^{-1}\tilde{g}_i(\sigma\pi) = \pi^{-1}\sigma^{-1}\sigma g_i \sigma^{-1}\sigma\pi = \pi^{-1} g_i \pi,\ \ i \in \{0, 1\}$$

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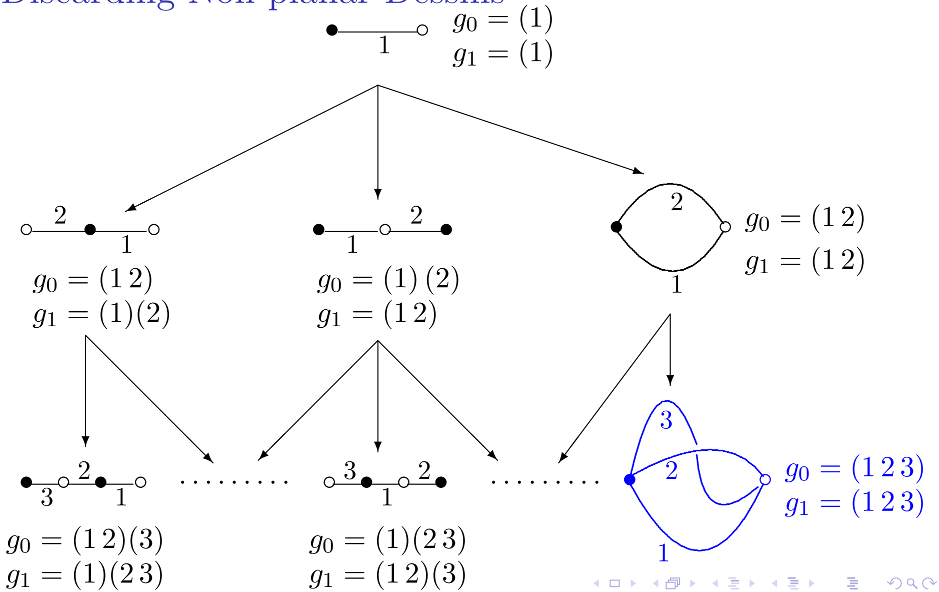
- Including this procedure discards conjugated 3-constellations and gives the following growth:

$$T_n = 1, 3, 7, 26, 97, 624, 4163, 34470, 314493, 3202839, \ldots$$

Discarding Non-planar Dessins



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- The growth is much smaller, but still too large to reach $n = 18$ (we still get "error! out of memory").

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- We can discard a 3-constellation D as soon as $W(D)$ exceeds the desired number of singularities. This tool is very useful as each 3-constellation contributes $n^2 - 1$ new 3-constellations in the next level.

Putting it All Together

By discarding

1. all but one member from each conjugacy class
2. non-planar dessins
3. dessins whose Weighted Singularity Count is too high

the table grows much more slowly. Not only are we able to compute all relevant dessins for $d = 5\,(n \leq 18)$ we can also do the same for $d = 6\,(n \leq 24)$.

Choosing Relevant Dessins

- Finally, we consider only those dessins which produce 5 non removable singularities from $0, 1, \infty$ with $(e_0, e_1, e_\infty) = (0, \frac{1}{2}, \frac{1}{k})$ where $k \in \{3, 4, 6\}$.

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- We computed all such dessins which produce up to 6 singularities (degree ≤ 24). The details for $(0, \frac{1}{2}, \frac{1}{3})$ up to 5 singularities are as follows:

d	n	dessin count for $(0, \frac{1}{2}, \frac{1}{3})$
3	≤ 6	1, 2, 1, 1, 0, 2
4	≤ 12	0, 1, 3, 4, 3, 6, 4, 6, 4, 4, 0, 6
5	≤ 18	0,0,2,6,12,19,22,26,32,39,36,50,40,42,32,32,0,26

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- COMPLETENESS: Once each member from our table of Belyi maps corresponds to a member from the table of dessins and vice versa, the table of Belyi maps is complete.

Computing Relevant Near Dessins

- There is a correspondence between Belyi-1 maps (up to Möbius transformation) and 4-constellations $[g_0, g_1, g_t, g_\infty]$ (up to conjugation and braid action) where g_t is a 2-cycle.

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- Computing relevant near dessins involves the following steps:
  1. Listing all branching patterns (up to degree 12) which produce 5 non removable singularities from $\{0, 1, \infty\}$.
  2. Computing near dessins (4-constellations mod conjugation) for each branching pattern.
  3. Grouping near dessins together by braid orbits.

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- The next slides will explain the procedure of computing relevant near dessins of degree 9 for $(e_0, e_1, e_\infty) = (\frac{1}{3}, \frac{1}{2}, 0)$.

Listing Branching Patterns

- Branching patterns above $0, 1$ are $[3, 3, 3], [1, 2, 2, 2, 2]$ respectively. Following is the list of branching patterns above ∞:
  $[1, 1, 1, 6], [1, 1, 2, 5], [1, 1, 3, 4], [1, 2, 2, 4], [1, 2, 3, 3], [2, 2, 2, 3]$

- 4 poles and a root above 1 produce 5 non removable singularities.

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- For each of the $945 \cdot 36 = 34020$ triples (g_0, g_1, g_t) we check the following:
  1. Is $\langle g_0, g_1, g_t \rangle$ transitive?
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- Computing near dessins (4-constellations mod conjugation) is similar to the procedure of computing dessins. Here we use the action of g_0, g_1 and g_t.

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- Our computation produces braid orbits with the following branching patterns above ∞:
  $[1, 1, 1, 6], [1, 1, 1, 6], [1, 1, 2, 5], [1, 1, 3, 4], [1, 2, 2, 4], [1, 2, 3, 3], [2, 2, 2, 3]$

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- Following are the Belyi-1 maps with branching pattern $[1,1,1,6]$:

$$f_1(x,s) = \frac{4}{27}\,\frac{\left(sx^3 - 2\,sx^2 + sx - 3\right)^3}{sx^3 - 2\,sx^2 + sx - 4}$$

$$f_2(x,s) = \frac{\left(sx^3 - 2\,sx^2 - 9\,x^2 + 18\,x + sx - 3\right)^3}{27\left(sx^3 - 2\,sx^2 - 9\,x^2 + 18\,x + sx - 1\right)}$$

Braid Orbits Contd.

- For f_1 the fourth branch point $t = \frac{1}{19683} \frac{(4\,s-81)^3}{s-27}$.

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- For each fixed value of $t \notin \{0, 1, \infty\}$, we get 3 distinct values of s which produce 3 distinct near dessins.
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Braid Orbits Contd.

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- Monodromy groups of f_1 and f_2 have different order. Hence $\{f_1, f_2\}$ completely cover the branching pattern $[1, 1, 1, 6]$.

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- Our program gives two branching patterns for Belyi-2 maps which occur only for $(0, \frac{1}{2}, \frac{1}{3})$;
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- I dedicate this achievement to my sweet family. It would not be possible without their love, care and company.