From the Martingale Zoo

If you play a succession of fair games, your fortune is a martingale. If the number of games played has a fixed upper bound, the intuition that "a martingale is fair" works just fine. However, if it is merely the case that play eventually stops, the expected value of your final fortune may not be your initial fortune, even though the expected value of your fortune after game n is the same for all n (Indeed, your final fortune may not even have an expected value.) Thus, to refine one's intuition, it is helpful to look at some examples of how martingales can misbehave.

Let X_0 be your initial fortune, which in our examples will be a constant. You are allowed to place bets (of possibly varying amounts) on independent fair games of chance at times $1, 2, 3, ...$; your fortune after game n is X_n . The amount that you bet on game $n+1$ is allowed to depend on the outcomes of the previous games. The martingale property is that, after the game at time $n \geq 1$ has taken place (and at time $n = 0$, when there is no game), the expected value of your fortune after game $n+1$ is your fortune's current value.

Most of our examples involve coin flipping. For convenience, we allow negative bets: a bet of -1 on heads is the same as a bet of 1 on tails. The amount that we can bet is not restricted by our fortune: borrowing the amount staked is allowed, and our fortune may be negative.

Our first example is simple yet striking. Start with an initial fortune of 1, and bet your entire fortune on successive flips of a coin. With probability 1, your final fortune is 0. Note that this apparent pathology occurs even though your fortune is always nonnegative and there is no borrowing.

Closely related to the above is the so-called "doubling strategy". Start with 0, bet 2^{n-1} on heads until heads come up, and then stop. With probability 1, your final fortune is 1. Thus, this strategy seems to be a sure-fire way to make money.

We can get more dramatic examples using "lottery tickets". We consider the old-fashioned kind of lottery, where one ticket always wins the entire pot. We buy one ticket at price 1 for each lottery, and assume that the lottery at time n has a pot of 2^n. Using the first Borel-Cantelli lemma, we see that, with probability 1, every ticket from some point on will be a loser. Thus, in this martingale our losses will undoubtedly go to infinity. If we sell a "lottery ticket" at each n (that is, play a game which pays 1 with probability $1 - \frac{1}{2^n}$ and $1 - 2^n$ with probability $\frac{1}{2^n}$), then our earnings (winnings) will undoubtedly grow without bound.

We now consider a general setup, which we shall then specialize to illustrate various types of martingale behavior, most of which are rather technical. Let $a_0, a_1, a_2, ...$ be any sequence of real numbers. Our initial fortune is $X_0 = a_0$. Our plan is to bet on heads until heads comes up and then stop, with the amounts of the bets selected so that if heads comes up for the first time in game n then our fortune upon winning that game is a_n. Thus, we bet $a_1 - a_0$ (which may be negative) on heads in game 1; if tails comes up, our fortune is $x_1 = 2a_0 - a_1$. If tails come up on games $1, ..., n$, leaving us with a fortune of x_n, then we bet $a_{n+1} - x_n$ on heads in game $n+1$, and have $x_{n+1} = 2x_n - a_{n+1}$ if we get tails. We see that $x_n = 2^n a_0 - \sum_{k=1}^{n} 2^{n-k} a_k$ for $n \geq 1$; for later use, we extend the notation by setting $x_0 = a_0$. For $n \geq 1$, our fortune X_n is a_k with probability $\frac{1}{2^k}$ for $k = 1, ..., n$, and x_n with probability $\frac{1}{2^n}$. This martingale converges with probability 1 to our final fortune X_∞, which is a_n with probability $\frac{1}{2^n}$ for each $n \geq 1$.

1. Our first example was the case $a_0 = 1$, $a_n = 0$ for $n \geq 1$. The "doubling strategy" is the case $a_0 = 0$, $a_n = 1$ for $n \geq 1$.

2. According to the Submartingale Convergence Theorem (SCT), a martingale X will converge if $\{\mathbb{E}[|X_n|]\}$ is bounded (X is "bounded in L^1 "). Taking, for example, $a_n = 3^n$, we see that this boundedness condition is by no means necessary for convergence. However, in this case X_∞ is not integrable, and when the SCT hypotheses apply the limit is integrable. In our setup, we can show that $\mathbb{E}[|X_n|] \leq 2\,\mathbb{E}\,[|X_\infty|] + |a_0|$, so if X_∞ is integrable then $\{\mathbb{E}[|X_n|]\}$ is bounded. (After our discussion of this class of examples, we give an example to show that this implication is not always true.)

3. Our martingale $X = \{X_n\}$ is uniformly integrable (equivalently, $\{X_n\}$ converges to X_∞ in L^1) if and only if X_∞ is integrable and $\mathbb{E}[X_\infty] = \mathbb{E}[X_0] = a_0$. The conditions are necessary by,

for example, Billingsley, *Probability and Measure*, Theorem 16.14(i). If the conditions hold, one can compute that $x_n = \sum_{k=1}^{\infty} \frac{a_{n+k}}{2^k}$ and from this that the expected value of X_∞, given the results of games $1, ..., n$, is X_n. This shows uniform integrability. More concretely, one can show $\mathbb{E}[|X_n - X_\infty|] \leq 2 \sum_{k=n+1}^{\infty} \frac{|a_k|}{2^k}$. (In our later example, $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$ but $\{X_n\}$ is not uniformly integrable: we emphasize that the equivalence of uniform integrability with $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$ holds for this class of examples but not in general. The equivalence of uniform integrability with convergence in L^1 holds for any martingale, via SCT and general convergence results.)

4. Let $a_n = 2^{\alpha n}$ for $n \geq 1$, where α is a fixed real number. If $\alpha \geq 1$ then X_∞ is not integrable. We suppose $\alpha < 1$. We set $a_0 = \mathbb{E}[X_\infty]$, so that the martingale X is uniformly integrable. We calculate $x_n = \frac{2^{\alpha n}}{2^{1-\alpha} - 1}$ for all $n \geq 0$. If $\alpha \leq 0$ then 2^α is a uniform bound for X, and so X is perfectly well behaved. Let us consider the case $0 < \alpha < 1$. For $p > 0$, we find by explicit calculation that X is bounded in L^p if and only if $p < 1/\alpha$.

5. Let $X^* = \sup_n |X_n|$. Let Y have value $|x_n|$ if heads occurs for the first time in game $n+1$ (in which case $Y = |X_n|$), and $Y = 0$ if heads never occurs; here we are using our extended notation $x_0 = a_0$. Then $0 \leq Y \leq X^*$, so if X^* is integrable then so is Y. Suppose $a_n \geq 0$ for $n \geq 1$ and that the martingale X is uniformly integrable, so that $a_0 = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$ and $x_n = \sum_{k=1}^{\infty} \frac{a_{n+k}}{2^k}$. Then $\mathbb{E}[Y] = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{n a_n}{2^n}$. If we set $a_n = \frac{2^n}{n^2}$ for $n \geq 1$, we thus have an example in which X is uniformly integrable but X^* is not integrable. This shows that the uniform integrability is not the result of $\{X_n\}$ being dominated by an integrable random variable; more technically, the martingale $M_n = X_n - X_0$ is uniformly integrable but not in the Hardy space $\mathcal{H}_0^1$.

We can easily give an example (outside our general setup) of a martingale which is not bounded in L^1 but converges to an integrable limit. Starting with 0, bet 3^n on heads until heads comes up, and then bet your entire fortune on heads until you lose. This martingale converges to 0, its initial value, but it is not bounded in L^1, and thus the convergence does not take place in L^1. Note that $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$; for *nonnegative* martingales, this condition implies convergence in L^1.

Let us now briefly move on to the next level: we use a martingale to define a game sequence. If Z is a martingale (say, one of the ones we have constructed), then we consider $Z_n - Z_{n-1}$ to be the payoff per unit stake in game n, $n \geq 1$; there is no game at $n = 0$. The amount that we decide to stake on game n must be "previsible": that is, dependent only on the information accumulated up until time $n-1$. We note that our games may no longer be independent: in our examples, it was usually the case that if $Z_n - Z_{n-1} = 0$ for some n, then the same was true for all subsequent n. We shall consider a single example, using the martingale M defined above to define our game sequence. We start with a fortune of $X_0 = 0$. We introduce a second (independent) coin, and flip it at time $n-1$ to decide whether to stake 1 or -1 on game n. Our resulting fortune X_n is a martingale. Since $M_n - M_{n-1}$ is almost surely eventually zero, X converges with probability 1. We claim that X_∞ is not integrable. Consider the event in which (for the coin defining M) heads occurs for the first time at time n, so that $X_\infty = X_n$. Then $M_n - M_{n-1} = a_n - x_{n-1}$ is approximately $\frac{-2^{n-1}}{n+1}$, and we can show $|M_n - M_{n-1}| > \frac{2^{n-2}}{n+1}$ if $n \geq 3$. Half of the time $X_n - X_{n-1}$ will have the same sign as X_{n-1}, in which case $|X_\infty| > \frac{2^{n-2}}{n+1}$. We thus see that $\mathbb{E}[|X_\infty|] \geq \sum_{n=3}^{\infty} \frac{1}{2} \frac{2^{n-2}}{n+1} \frac{1}{2^n} = \infty$. On a technical level, we have illustrated the fact that the discrete stochastic integral of a bounded previsible process with respect to a uniformly integrable martingale need not be uniformly integrable (or even bounded in L^1); see the discussion in Williams, *Probability with Martingales*, pages 151 - 152, which tells where some of these creatures may be found.