4. Matrices

4.1. Definitions.

Definition 4.1.1. *A* **matrix** *is a rectangular array of numbers. A matrix with m rows and n columns is said to have* **dimension** $m \times n$ *and may be represented as follows:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Definition 4.1.2. *Matrices A and B are* **equal**, *$A = B$, if A and B have the same dimensions and each entry of A is equal to the corresponding entry of B.*

Discussion

Matrices have many applications in discrete mathematics. You have probably encountered them in a precalculus course. We present the basic definitions associated with matrices and matrix operations here as well as a few additional operations with which you might not be familiar.

We often use capital letters to represent matrices and enclose the array of numbers with brackets or parenthesis; e.g., $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We do not use simply straight lines in place of brackets when writing matrices because the notation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ has a special meaning in linear algebra. $A = [a_{ij}]$ is a shorthand notation often used when one wishes to specify how the elements are to be represented, where the first subscript i denotes the row number and the subscript j denotes the column number of the entry a_{ij}. Thus, if one writes a_{34}, one is referring to the element in the 3rd row and 4th column. This notation, however, does not indicate the dimensions of the matrix. Using this notation, we can say that two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if $a_{ij} = b_{ij}$ for all i and j.

Example 4.1.1. *The following matrix is a 1×3 matrix with $a_{11} = 2$, $a_{12} = 3$, and $a_{13} = -2$.*

$$\begin{bmatrix} 2 & 3 & -2 \end{bmatrix}$$

EXAMPLE 4.1.2. *The following matrix is a* 2×3 *matrix.*

$$\begin{bmatrix} 0 & \pi & -2 \\ 2 & 5 & 0 \end{bmatrix}$$

4.2. Matrix Arithmetic. Let α be a scalar, $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, and $C = [c_{ij}]$ a $n \times p$ matrix.

(1) Addition: $A + B = [a_{ij} + b_{ij}]$
(2) Subtraction: $A - B = [a_{ij} - b_{ij}]$
(3) Scalar Multiplication: $\alpha A = [\alpha a_{ij}]$
(4) Matrix Multiplication: $AC = \left[\sum_{k=1}^{n} a_{ik} c_{kj} \right]$

Discussion

Matrices may be added, subtracted, and multiplied, provided their dimensions satisfy certain restrictions. To add or subtract two matrices, the matrices must have the same dimensions.

Notice there are two types of multiplication. Scalar multiplication refers to the product of a matrix times a scalar (real number). A scalar may be multiplied by a matrix of any size. On the other hand, matrix multiplication refers to taking the product of two matrices. The definition of matrix multiplication may not seem very natural at first. It has a great many applications, however, some of which we shall see. Notice that in order for the product AC to be defined, the number of columns in A must equal the number of rows of C. Thus, it is possible for the product AC may be defined, while CA is not. When multiplying two matrices, the order is important. In general, AC is not necessarily the same as CA, even if both products AC and CA are defined. In other words, matrix multiplication is *not commutative.*

4.3. Example 4.3.1.

EXAMPLE 4.3.1. *Suppose*

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \end{bmatrix}, \; \textit{and } C = \begin{bmatrix} 3 & 4 & -6 & 0 \\ 0 & -1 & 2 & 2 \\ 1 & -2 & 3 & 4 \end{bmatrix}$$

Then

$$A+B=\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 9 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 1 & -3 & 5 \\ -3 & 7 & -1 \end{bmatrix}$$

$$3A=\begin{bmatrix} 3 & -6 & 9 \\ 0 & 9 & 12 \end{bmatrix}$$

$$AC=\begin{bmatrix} 6 & 0 & -1 & 8 \\ 4 & -11 & 18 & 22 \end{bmatrix}$$

Let us break down the multiplication of A and C in Example 4.3.1 down to smaller pieces.

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+0+3 \end{bmatrix} = [6]$$

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3+0+3 & 4+2-6 \end{bmatrix} = \begin{bmatrix} 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & -6 & 0 \\ 0 & -1 & 2 & 2 \\ 1 & -2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 3+0+3 & 4+2-6 & -6-4+9 & 0-4+12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & -1 & 8 \end{bmatrix}$$

Now compute the second row to get

$$AC=\begin{bmatrix} 6 & 0 & -1 & 8 \\ 4 & -11 & 18 & 22 \end{bmatrix}.$$

4.4. Special Matrices.

1. A **square matrix** is a matrix with the same number of rows as columns.
2. A **diagonal matrix** is a square matrix whose entries off the main diagonal are zero.
3. An **upper triangular matrix** is a matrix having all the entries below the main diagonal equal to zero.
4. A **lower triangular matrix** is a matrix having the entries above the main diagonal equal to zero.
5. The $n \times n$ **identity matrix**, I, is the $n \times n$ matrix with ones down the diagonal and zeros elsewhere.
6. The **inverse** of a square matrix, A, is the matrix A^{-1}, if it exists, such that $AA^{-1} = A^{-1}A = I$.
7. The **transpose** of a matrix $A = [a_{ij}]$ is $A^t = [a_{ji}]$.
8. A **symmetric matrix** is one that is equal to its transpose.

Discussion

Many matrices have special forms and special properties. Notice that, although a diagonal matrix must be square, no such condition is put on upper and lower triangular matrices.

The following matrix is a diagonal matrix (it is also upper and lower triangular).

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The following matrix is upper triangular.

$$\begin{bmatrix} -1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

The next matrix is the transpose of the previous matrix. Notice that it is lower triangular.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \\ -2 & 5 & 3 \end{bmatrix}$$

The identity matrix is a special matrix that is the multiplicative identity for any matrix multiplication. Another way to define the identity matrix is the square matrix $I = [a_{ij}]$ where $a_{ij} = 0$ if $i \neq j$ and $a_{ii} = 1$. The $n \times n$ identity I has the property that $IA = A$ and $AI = A$, whenever either is defined. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -4 & -2 \\ 2 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -2 \\ 2 & 7 & 0 \end{bmatrix}$$

The inverse of a matrix A is a special matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. A matrix must be square to define the inverse. Moreover, the inverse of a matrix does not always exist.

Example 4.4.1.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

The transpose of a matrix is the matrix obtained by interchanging the rows for the columns. For example, the transpose of

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 5 & 6 \end{bmatrix} \qquad \text{is} \qquad A^t = \begin{bmatrix} 2 & -2 \\ 3 & 5 \\ -1 & 6 \end{bmatrix}$$

If the transpose is the same as the original matrix, then the matrix is called symmetric. Notice a matrix must be square in order to be symmetric.

We will show here that matrix multiplication is distributive over matrix addition.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices and let $C = [c_{ij}]$ be an $n \times p$ matrix. We use the definitions of addition and matrix multiplication and the distributive properties of the real numbers to show the distributive property of matrix multiplication. Let i and j be integers with $1 \le i \le m$ and $1 \le j \le p$. Then the element in the i-th row and the j-th column in $(A+B)C$ would be given by

$$\begin{aligned}\sum_{k=1}^{n}(a_{ik}+b_{ik})(c_{kj}) &= \sum_{k=1}^{n}(a_{ik}c_{kj}+b_{ik}c_{kj}) \\ &= \sum_{k=1}^{n}a_{ik}c_{kj}+\sum_{k=1}^{n}b_{ik}c_{kj} \\ &= \sum_{k=1}^{n}a_{ik}c_{kj}+\sum_{k=1}^{n}b_{ik}c_{kj}\end{aligned}$$

This last part corresponds to the form the element in the i-th row and j-th column of $AC+BC$. Thus the element in the i-th row and j-th column of $(A+B)C$ is the same as the corresponding element of $AC+BC$. Since i and j were arbitrary this shows $(A+B)C = AC+BC$.

The proof that $C(A+B) = CA+CB$ is similar. Notice that we must be careful, though, of the order of the multiplication. Matrix multiplication is *not* commutative.

4.5. Boolean Arithmetic. If a and b are binary digits (0 or 1), then

$$a \wedge b = \begin{cases} 1, & \text{if } a = b = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$a \vee b = \begin{cases} 0, & \text{if } a = b = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Definitions 4.5.1. *Let A and B be $n \times m$ matrices.*

1. *The* **meet** *of A and B: $A \wedge B = [a_{ij} \wedge b_{ij}]$*
2. *The* **join** *of A and B: $A \vee B = [a_{ij} \vee b_{ij}]$*

Definition 4.5.1. *Let $A = [a_{ij}]$ be $m \times k$ and $B = [b_{ij}]$ be $k \times n$. The* **Boolean product** *of A and B, $A \odot B$, is the $m \times n$ matrix $C = [c_{ij}]$ defined by*

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee (a_{i3} \wedge b_{3j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Discussion

Boolean operations on zero-one matrices is completely analogous to the standard operations, except we use the Boolean operators $\wedge$ and $\vee$ on the binary digits instead of ordinary multiplication and addition, respectively.

4.6. Example 4.6.1.

EXAMPLE 4.6.1. *Let* $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, *and* $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Then

1. $A \wedge B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

2. $A \vee B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

3. $A \odot C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Here are more details of the Boolean product in Example 4.6.1:

$$A \odot C = \begin{bmatrix} (1\wedge1)\vee(1\wedge0)\vee(0\wedge0)\vee(1\wedge1) & (1\wedge1)\vee(1\wedge1)\vee(0\wedge1)\vee(1\wedge0) & (1\wedge0)\vee(1\wedge0)\vee(0\wedge1)\vee(1\wedge0) \\ (0\wedge1)\vee(1\wedge0)\vee(1\wedge0)\vee(0\wedge1) & (0\wedge1)\vee(1\wedge1)\vee(1\wedge1)\vee(0\wedge0) & (0\wedge0)\vee(1\wedge0)\vee(1\wedge1)\vee(0\wedge0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 1 \vee 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 0 & 0 \vee 1 \vee 1 \vee 0 & 0 \vee 0 \vee 1 \vee 0 \end{bmatrix}$$

EXERCISE 4.6.1.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Find

(1) $A \vee B$
(2) $A \wedge B$

EXERCISE 4.6.2.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find

(1) $A \odot A$
(2) $A \odot A \odot A$
(3) $A \odot A \odot A \odot A$

EXERCISE 4.6.3.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find $\odot_{k=1}^{n} A$*, the Boolean product of* A *with itself* n *times. Hint: Do exercise 4.6.2 first.*