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ABOUT THE COVER: ON THE DISTRIBUTION OF PRIMES—GAUSS' TABLES

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In 1791, Carl Wilhelm Ferdinand, the Duke of Brunswick, gave fourteen-year-old Gauss a collection of mathematics books which included logarithm tables by Schulze.[1] These were the first logarithm tables that Gauss possessed. The tables listed decimal logarithms up to 7 digits as well as natural logarithms of all natural numbers up to 2200 and prime numbers up to 10,009 [3]. Gauss worked on extending the tables: a computation of ln(10037) can be found in his papers from that time.

It is conceivable that seeing and using the tables of logarithms of primes, Gauss was led to discover the law governing the distribution of primes, the Prime Number Theorem. Throughout his life Gauss returned again and again to this issue, matching data from published tables of prime numbers with his prediction.

Here are the first lines of a four-page letter from Gauss to his student Johann Franz Encke, lieutenant of artillery and later astronomer in Berlin, dated December 24, 1849:[2]

My distinguished friend,

Your remarks concerning the frequency of primes were of interest to me in more ways than one. You have reminded me of my own endeavors in this field which began in the very distant past, in 1792 or 1793, after I had acquired the Lambert supplements to the logarithmic tables. Even before I had begun my more detailed investigations into higher arithmetic, one of my first projects was to turn my attention to the decreasing frequency of primes, to which end I counted the primes in several chiliads and recorded the results on the attached white pages. I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm, so that the number of primes below a given bound n is approximately equal to

$$\int \frac{dn}{\log(n)},$$

where the logarithm is understood to be hyperbolic. Later on, when I became acquainted with the list in Vega's tables (1796) going up to 400031, I extended my computation further, confirming that estimate. In 1811, the appearance of Chernau's cribrum gave me much pleasure and I have frequently (since I lack the patience for a continuous count) spent an idle quarter of an hour to count another chiliad here and there; although I eventually gave it up without quite getting through a million. Only some time later did I make use of the diligence of Goldschmidt to fill some of the remaining gaps in the first million and to continue the computation

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[1] The cover page of this book is at www.math.princeton.edu/~ytschink/.gauss/tafel2.pdf.

[2] The scan of the original is at www.math.princeton.edu/~ytschink/.gauss/Briefe-B.pdf. This translation is from Appendix B to [2].

according to Burkhardt's tables. Thus (for many years now) the first three million have been counted and checked against the integral. A small excerpt follows: ....

The continuation of the letter (p. 2) is reproduced on the cover of this issue of the *Bulletin*. The scan on p. 14 of [1] shows one of Gauss' tables summarizing information about the distribution of primes between 1.9 and 2 million. There are slight discrepancies with actual numbers. For example, there are 896 primes between 1910000 and 1920000 and not 897 as in the table. The correct total number is 6904 and not 6902. We have

$$\int_{1900000}^{2000000} \frac{\mathrm{d}x}{\ln(x)} = 6904.54423628\ldots.$$

Gauss computed 6904.54424; the scan on p. 29 of [1] shows his computation of $\mathrm{Li}(x)$, for $x = 10^5, \ldots, 2 \cdot 10^6$, in steps of 100,000. The table on p. 20 of [1] shows numbers of primes in intervals between 2000000 and 3000000; the actual number of primes is 67883 and not 67862.

Clearly, questions about the distribution of primes were much more than of "idle" interest for Gauss—he had a multitude of obligations.[3] It is perhaps worth noticing that Gauss never plotted the graph of the function $\pi(x)$ or $\mathrm{Li}(x)$, the way one would introduce the problem nowadays [4].

Acknowledgments

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References

[1] C. F. Gauss, *Tables*, `http://www.math.princeton.edu/~ytschink/.gauss/Gauss-Nachlass-Math-18.pdf`.

[2] L. J. Goldstein, *A history of the prime number theorem*, Amer. Math. Monthly 80 (1973), 599–615. MR0313171 (47:1726)

[3] F. Klein, *Materialien für eine wissenschaftliche Biographie von Gauss*, Leipzig, Teubner Verl. (1911).

[4] D. Zagier, *The first* 50 000 000 *prime numbers*, Math. Intelligencer 0 (1977), 7–19. MR0643810 (83e:10003)

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[3] From a letter to Dirichlet, September 9, 1826: *"I wish you from my heart a situation in which you can, as much as possible, remain the master of your time and choice of your tasks."*

Unter	gibt es Primzahlen	Integral $\int \frac{dn}{\log n}$	Differ.	Ihre Formel	Abweich.
500 000	41 556	41606,4	+ 50,4	41596,9	+ 40,9
1000 000	78 501	78627,5	+ 126,5	78672,7	+ 171,7
1500 000	114 112	114263,1	+ 151,1	114374,0	+ 264,0
2000 000	148883	149054,8	+ 171,8	149233,0	+ 350,0
2500 000	183016	183245,0	+ 229,0	183495,1	+ 479,1
3000 000	216745	216970,6	+ 225,6	217308,5	+ 563,6

Dass Legendre sich auch mit diesem Gegenstande beschäftigt hat, war mir nicht bekannt; auf Veranlassung Ihres Briefes habe ich in seiner Theorie des Nombres nachgesehen, und in der zweiten Ausgabe einige darauf bezügliche Seiten gefunden, die ich früher übersehen (oder seitdem vergessen) haben muss. Legendre gebraucht die Formel

$$\frac{n}{\log n - A}$$

wo A eine Constante sein soll, für welche er 1,08366 setzt. Nach einer flüchtigen Rechnung finde ich danach in obigen Fällen die Abweichungen

− 23,3
+ 42,2
+ 68,1
+ 92,8
+ 159,1
+ 167,6

Diese Differenzen sind noch kleiner als die ~~nach~~ mit dem Integral, sie scheinen aber bei zunehmendem n ~~stärker~~ schneller zu wachsen als diese, so dass leicht möglich wäre, dass bei viel weiterer Fortsetzung jene die letztern überträfen. Um Zählung und Formel in Übereinstimmung zu bringen, müsste man respective anstatt $A = 1,08366$ setzen

1,09040
1,07682
1,07582
1,07529
1,07179
1,07297