MAP 5485 Fall 2005
Homework 3

(1) Consider the orientation preserving euclidean transformation

$$T : X \to AX + B$$

where A is a rotation and B is a vector. This problem indicates how to find the screw axis of T.

Suppose U is an axis direction for A and let $\mathcal{P}$ be the plane through the origin perpendicular to U, that is, the set of vectors X such that $X \cdot U = 0$. Let B_1 be the orthogonal projection of B onto $\mathcal{P}$,

$$B = B_1 + sU \qquad B_1 \cdot U = 0$$

and s a scalar.

(a) Show that $\mathcal{P}$ is left fixed by A.
(b) Show that $\mathcal{P}$ is left fixed by $T_1 : X \to AX + B_1$.
(c) Let P be the fixed point of T_1. Show that the line through P parallel to U, $\{P + tU | t \in \mathbb{R}\}$, is left fixed by T. It is the screw axis for T.

Solution:
a)
If $X \cdot U = 0$ then since $AU = U$,

$$(AX) \cdot U = AX \cdot AU = X \cdot U = 0.$$

b)
Since If X is in $\mathcal{P}$ and B_1 is in $\mathcal{P}$, AX is in $\mathcal{P}$ and $AX + B_1$ is in $\mathcal{P}$.
c)

$$T(P + tU) = AP + B_1 + A(tU) + sU = P + (s + t)U.$$

(2) Find the fixed line (screw axis) of the transformation

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Solution: Use the method from the first problem. The axis of A is the z axis. The resulting 2D transformation is

$$(x, y) \to (-y + 1, x)$$

and its fixed point is $(1/2, 1/2)$. So the screw axis is given by the line $x = 1/2$, $y = 1/2$, or in parametric equations

$$(x, y, z) = (1/2, 1/2, 0) + t(0, 0, 1)$$

If P $=(\frac{1}{2}, \frac{1}{2}, 0)'$ then the transformation can be written

$$X \to A(X - P) + P + (0, 0, 1)'$$

in the form of a screw translation.

(3) Let T be the transformation in problem 1. Show that T^4 is a translation. What is the translation vector?

Solution: Write transformation as $AX + B$, then

$$T^4 X = X + A^3 B + A^2 B + AB + B$$

$$= X + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= X + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix},$$

so the translation vector is

$$\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

(4) Consider the euclidean transformations below:

$$T_1 : (x, y, z) \to \left(x + \frac{1}{2}, \frac{1}{2} - y, -z\right)$$
$$T_2 : (x, y, z) \to \left(-x, \frac{1}{2} + y, \frac{1}{2} - z\right)$$
$$T_3 : (x, y, z) \to \left(\frac{1}{2} - x, -y, \frac{1}{2} + z\right).$$

Show that the product $T_1 T_2 T_3$ is the identity transformation.

Solution: Here is a solution using representations of Euclidean motions as 4×4 matrices. Let

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} -1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$T_1 T_2 T_3 = I.$$

The problem can also be done by direct substitution.

(5) Make a copy of the pattern *pmm* above. Draw a coordinate system on the pattern and indicate where the origin and x and y axes are. Write in the form $AX + B$ several different transformations that leave the pattern fixed. Include

(a) reflections in at least 3 different horizontal axes
(b) reflections in at least 3 different vertical axes
(c) rotations about at least 3 different fixed points.

Solution: a) Let $X = (x, y)'$
Reflection in y axis:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X$$

Reflection in line $y = 1/2$:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Reflection in line $y = 1$.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

b)Reflection in y axis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X$$

Reflection in line $x = 1/2$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Reflection in line $x = 1$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

c)
Rotation $180°$ about origin:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} X$$

Rotation $180°$ about $(1/2, 0)$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Rotation $180°$ about $(0, 1/2)$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(6) In this problem you show that every rotation can be written as

$$R_z(\alpha)R_y(\beta)R_z(\gamma)$$

in terms of three Euler angles.

Suppose A is a rotation. A vector of length 1 can be written in spherical coordinates as

$$(\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta).$$

The angles β and α are the spherical coordinates. The angle β is called the azimuthal angle and α the polar angle. Suppose α and β are the spherical coordinate of Ae_3.

(a) Show that

$$Ae_3 = R_z(\alpha)R_y(\beta)e_3$$

(b) Show that

$$R_y(-\beta)R_z(-\alpha)A$$

is rotation about the z axis.

(c) Show that

$$A = R_z(\alpha)R_y(\beta)R_z(\gamma)$$

for some γ.

Solution: a)

Check that

$$R_z(\alpha)R_y(\beta)R_z(\gamma) =$$

$$\begin{bmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{bmatrix}$$

so that

$$R_z(\alpha)R_y(\beta)R_z(\gamma)e_3 = R_z(\alpha)R_y(\beta)e_3 = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)'$$

b) by part a),

$$R_y(-\beta)R_z(-\alpha)Ae_3 = e_3.$$

Since e_3 is left fixed, the rotation is about the z axis.

c) Write the rotation in b) as $R_z(\gamma)$.