A VERY BRIEF LINEAR ALGEBRA REVIEW
for MAP 5485
Introduction to Mathematical Biophysics
Fall 2005

Matrices. Example

$$\mathrm{A} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

A is a matrix with 2 rows and 2 columns i.e a 2×2 matrix.

A matrix with m rows and n columns is called an $m \times n$ matrix.

A matrix with the same number of rows and columns is called a square matrix.

3×3 square matrix:

$$\mathrm{B} = \begin{bmatrix} 3 & 1 & 7 \\ -1 & 2 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

3×2 matrix:

$$\mathrm{C} = \begin{bmatrix} 2 & 0 \\ -9 & 10 \\ 1 & 14 \end{bmatrix}$$

A 1×1 matrix is the same as a number or scalar,

$$3 = [3].$$

Vectors. Matrices with 1 row are called row vectors and matrices with 1 column are called column vectors.

$$\mathrm{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

are column vectors.

$$\mathrm{C} = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad \mathrm{D} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

are row vectors. Usually we will assume vectors are column vectors. A row vector can be converted into a column vector (or vice versa) by the transpose operation, which changes rows to columns.

Example.

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathrm{t}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Complex Numbers. Complex numbers can be used in matrices.

A number $z = a + bi$ where a and b are real numbers and $i^2 = -1$ is called a complex number. The number a is called the *real part*, $a = \Re z$ and b the imaginary part, $b = \Im z$.

If $z = a + bi$ where a and b are real numbers, then the *complex conjugate* of z is $\overline{z} = a - bi$. The number z is *real* if $b = 0$, or equivalently, $z = \bar{z}$.

If Z is a matrix then the matrix conjugate is formed by taking the complex conjugate of each entry.

Example Let

$$\mathrm{Z} = \begin{bmatrix} 1+i & 2 \\ 3-i & \frac{1}{2}+2i \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} i$$
$$= A + Bi$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & \frac{1}{2} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

.

The matrix A is the real part of Z and the matrix B is the imaginary part of Z.

The conjugate of Z is

$$\overline{\mathrm{Z}} = \begin{bmatrix} 1-i & 2 \\ 3+i & \frac{1}{2}-2i \end{bmatrix}$$

or using the real and imaginary parts of the matrix,

$$\overline{\mathrm{Z}} = A - Bi$$

Properties of the matrix congugate.

$$\overline{\mathrm{AB}} = \overline{\mathrm{A}}\,\overline{\mathrm{B}}$$

$$\overline{(\mathrm{A}+\mathrm{B})} = \overline{\mathrm{A}} + \overline{\mathrm{B}}$$

Transpose. The *transpose* of a matrix A, A^{t}, is obtained by changing rows to columns (or equivalently, changing columns to rows).

Sometimes the transpose is denoted A' rather than A^{t}.

$$\mathrm{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathrm{A}^{\mathrm{t}} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Properties of transpose.

$$(\mathrm{AB})^{\mathrm{t}} = \mathrm{B}^{\mathrm{t}}\mathrm{A}^{\mathrm{t}}$$
$$(\mathrm{A}+\mathrm{B})^{\mathrm{t}} = \mathrm{A}^{\mathrm{t}} + \mathrm{B}^{\mathrm{t}}$$

Adjoint. The *Hermitian transpose* or *adjoint* is the conjugate transpose given by

$$\mathrm{A}^* = \bar{\mathrm{A}}^{\mathrm{t}}.$$

Example.

$$\begin{bmatrix} 1, 2-i, 3 \end{bmatrix}^* = \begin{bmatrix} 1 \\ 2+i \\ 3 \end{bmatrix}$$

Example. For the matrix Z given above,

$$\mathrm{Z}^* = \overline{\mathrm{Z}^t} = \begin{bmatrix} 1-i & 3+i \\ 2 & \frac{1}{2}-2i \end{bmatrix}$$

Properties of Adjoint.

$$(\mathrm{AB})^* = \mathrm{B}^*\mathrm{A}^*$$
$$(\mathrm{A}+\mathrm{B})^* = \mathrm{A}^* + \mathrm{B}^*$$

Dot product. Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be two vectors, then the *dot product* is given by

$$v \cdot w = v^t w = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$$

Let $v = \begin{bmatrix} 1+i \\ 1 \\ i \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1-i \\ 3 \end{bmatrix}$ then the *Hermitian dot product* is given by

$$\langle v, w\rangle = v^* w = \overline{v} \cdot w = (1-i)2 + 1(1-i) + (-i)3 = 3 - 6\,i.$$

The length of a vector $|v|$ is given by

$$|v|^2 = \langle v, v\rangle.$$

Two vectors v and w are *orthogonal* or *perpendicular* if $\langle v, w\rangle = 0$. In general for real vectors

$$\langle v, w\rangle = |v||w|\cos\theta$$

where θ is the angle between the vectors.

Matrix Addition. Add matrices by adding corresponding entries.

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+i & 1 \\ 0 & 5 \end{bmatrix}$$

Scalar Multiplication. Every matrix entry is multiplied by the scalar.

$$2i \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4i & 2i \\ -2i & 6i \end{bmatrix}$$

Matrix Multiplication. Multiplication AB can be done only if the number of columns of A is the same as the number of rows of B. Each entry of the product is the dot product of a row of the first matrix with a column of the second.

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 0 + 3\cdot 2 \\ (-1)\cdot 1 + 1\cdot 0 + 2\cdot 2 \\ 3\cdot 1 + 1\cdot 0 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & \frac{1}{2} \\ 3 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2\cdot 1 + 1\cdot 1 + 3\cdot 3 & 2\cdot(-1) + 1\cdot 2 + 3\cdot 0 & 2\cdot 0 + 1\cdot\frac{1}{2} + 3\cdot 1 \\ 1\cdot 1 + 0\cdot 1 + 1\cdot 3 & 1\cdot(-1) + 0\cdot 2 + 1\cdot 0 & 1\cdot 0 + 0\cdot\frac{1}{2} + 1\cdot 1 \\ 2\cdot 1 + 1\cdot 1 + 0\cdot 3 & 2\cdot(-1) + 1\cdot 2 + 0\cdot 0 & 2\cdot 0 + 1\cdot\frac{1}{2} + 0\cdot 1 \end{bmatrix} =$$

$$\begin{bmatrix} 12 & 0 & 3.5 \\ 4 & -1 & 1 \\ 3 & 0 & .5 \end{bmatrix}$$

An important thing to remember about matrix multiplication is that it is *not commutative*; in general, $\mathrm{AB} \neq \mathrm{BA}$. For example

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Matrix multiplication and dot product. An $m \times n$ matrix can be considered as a list of $n \times 1$ column vectors,

$$\mathrm{A} = [v_1, \ldots, v_n]$$

and the transpose as a list of row vectors

$$\mathrm{A}^{\mathrm{t}} = \begin{bmatrix} v_1^{\mathrm{t}} \\ \vdots \\ v_n^{\mathrm{t}} \end{bmatrix}$$

If

$$\mathrm{B} = \begin{bmatrix} w_1^{\mathrm{t}} \\ \vdots \\ w_k^{\mathrm{t}} \end{bmatrix}$$

is a $k \times m$ matrix given as row vectors, then

$$\mathrm{BA} = \begin{pmatrix} w_1 \cdot v_1 & \ldots & w_1 \cdot v_n \\ \vdots & & \vdots \\ w_k \cdot v_1 & \ldots & w_k \cdot v_n \end{pmatrix}$$

is a matrix of dot products.

Symmetric Matrices. A matrix B is *symmetric* if $\mathrm{B} = \mathrm{B}^{\mathrm{t}}$

Example:

$$\begin{bmatrix} 3 & 1+i & 2 \\ 1+i & 0 & -5 \\ 2 & -5 & 2 \end{bmatrix}$$

is symmetric.

A matrix B is *self adjoint* (or *Hermitian symmetric*) if $\mathrm{B}^* = \mathrm{B}$

Example:

$$\begin{bmatrix} 2 & 1+i & \frac{1}{2}+2i \\ 1-i & 3 & 5 \\ \frac{1}{2}-2i & 5 & 4 \end{bmatrix}$$

is self adjoint.

Determinant. The determinant of a 2×2 matrix is given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix,

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

This is called *expansion by minors.* Likewise the determinant is defined for any square matrix.

Properties of determinant:

$$\det \mathrm{AB} = \det \mathrm{A} \det \mathrm{B}$$

$$\det \mathrm{A} \neq 0 \text{ implies } \mathrm{A}^{-1} \text{ exists}$$

Vector Cross Product.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} bz - cy \\ -az + cx \\ ay - bx \end{bmatrix}.$$

$$= (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

where

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The formula for cross product is often remembered by pretending that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are numbers and writing

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{bmatrix}$$

Identity Matrix. The matrix I denotes a square matrix whose entries are a_{ij} where

$$a_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

The matrix I is called the unit or *identity matrix.* Identity matrices come in different sizes:

$$\mathrm{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathrm{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix inverse. Let A be a square matrix. If $\det A \neq 0$ there is an inverse A^{-1} such that $A^{-1}A = AA^{-1} = I$.

The inverse of a 2×2 matrix is easy to find. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{\alpha} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

where $\alpha = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of A.

Eigenvalue and Eigenvectors. The scalar λ, a real or complex number, is an *eigenvalue* of a matrix A corresponding to an *eigenvector* $v \neq 0$ if $Av = \lambda v$.

Example 1

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvalue is 3, and an eigenvector is $[1,1]^{\mathrm{t}}$. Note that $[2,2]^{\mathrm{t}}$ is also an eigenvector.

Example 2

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The eigenvalue is i, and an eigenvector is $[1,-i]^t$.

Example 3

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The eigenvalue is $-i$, and an eigenvector is $[1,i]^t$.

Note that the equation in example 3 is the conjugate on the one in example 2.

Also note that a matrix with real entries can have complex eigenvalues and eigenvectors.

Eigenvalues of self adjoint matrices are real. This fact is essential in many areas of mathematics and is also a key fact in the mathematical formulation of quantum mechanics. Here is a proof:

If A is self adjoint and

$$Av = \lambda v$$

then taking the adjoint of both sides gives

$$v^*A = \overline{\lambda} v^*.$$

Multiplying the first equation on the left by v^* and the second on the right by v gives

$$v^*Av = \lambda |v|^2 = \overline{\lambda} |v|^2.$$

Since $|v| \neq 0$ we have $\overline{\lambda} = \lambda$ and so λ is real.

A consequence is that the eigenvalues of a symmetric real matrix are real.

Rotation matrices. In dimension 2, rotation of a column vector by an angle θ counterclockwise is given by multiplying on the left by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In dimension 3 rotations about the three axes are given by

- Rotation an angle θ about the x axis
$$\mathrm{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
- Rotation an angle θ about the y axis
$$\mathrm{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
- Rotation an angle θ about the z axis
$$\mathrm{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrices can be found for rotation of any angle about axis, where an axis is given by a direction (vector of length one).