Maple exercise
The square root of a symmetric matrix

This exercise is related to finding the coordinates of vectors from the gram matrix of the vectors.

If M is any matrix then the we call the symmetric matrix $G = M'M$ the *square* of M. We can think of M as a *square root* of G.

A matrix is called *diagonal* if all the entries not on the diagonal are zero. If all the entries of a diagonal matrix W are positive, then it is easy to find the square root, just take the square root of each entry on the diagonal. Call the resulting matrix $W^{1/2}$.

In what follows we assume that the eigenvalues of G are positive. We say that G is *positive definite*. Here are the steps for finding the square root of a positive definite symmetric matrix G. This will work if the eigenvalues are distinct (which they almost always are).

(1) Find the eigenvectors and eigenvalues of G
(2) Put the eigenvalues into a diagonal matrix E
(3) Put the corresponding eigenvectors as columns of a matrix V
(4) Check that $GV = VE$ and $W = V'V$ is diagonal.
(5) Let $M = (W^{1/2})^{-1}E^{1/2}V'$, then M is a square root of G, i.e., $M'M = G$.

Exercises:

(1) Write a function or procedure `sqm` to find the square of a matrix. Check that the square of

$$\begin{bmatrix} 1 & 1.0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

is

$$\begin{bmatrix} 3.0 & 0.0 & 1.0 \\ 0.0 & 2.0 & 0.0 \\ 1.0 & 0.0 & 1.0 \end{bmatrix}$$

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(2) Write a function or procedure `sqrtm` to find the square root of a diagonal matrix with positive entries. Check that the square root of

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & .25 \end{bmatrix}$$

is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & .5 \end{bmatrix}$$

(3) Look in the help menu for the `LinearAlgebra` package and read about the command `DiagonalMatrix`.

(4) Use the method described above to find a square root of

$$\begin{bmatrix} 3. & 0 & 1. \\ 0 & 2. & 0 \\ 1. & 0 & 1. \end{bmatrix}$$

Check your answer by squaring it and verifying that you get the above matrix.

(5) Use the method described above to find a square root of

$$\begin{bmatrix} 3. & 0 & 1. & 1. \\ 0 & 2. & 0 & -1 \\ 1. & 0 & 2. & 1. \\ 1. & -1. & 1. & 2. \end{bmatrix}$$

Check your answer by squaring it and verifying that you get the above matrix.

(6) Use the commands you used above to write a procedure to take the square root of any positive definite symmetric matrix. Check that it works on the two matrices above.