Hybrid Method for Stokes Flow with Interfaces with Several Applications

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Overview

Interfaces may:

- drive the flow
- be driven by the flow
- complicate the flow
- surround dynamic regions

Overall goal is to develop a method that is flexible enough to address all of these for zero Reynolds number flows

Outline

- Background of the approach
- Hybrid Method: Interface Motion (BIM) + Flow Field (Regularized Stokeslets)
- Example 1: Liver cell/β-cell interaction
- Example 2: Biofilm Disinfection
- Example 3: Mucociliary interaction + biofilm colonization

The beginning...

- Interested in biofilm disinfection → advection/diffusion/reaction
- Small length scales → Stokes flow
- How do you handle the fluid dynamics? (Lots and lots of methods)

Regularized Stokeslets (RS)-Ricardo Cortez

- For stationary interfaces $\rightarrow$ flow around irregular obstacles
- Fundamental solution to Stokes flow $\rightarrow$ Recover the velocity from point force
- Stokes is linear so sum forces along discrete interface
- Cortez uses 'blobs' - smooth approximations of δ-functions $\rightarrow$ *regularized* Stokeslets.
- Invert the force/velocity relationship to find forces to enforce velocities/boundary conditions

Far Field (RS)

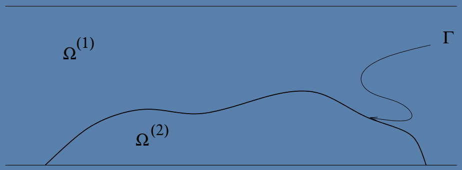
$$\mu \Delta \vec{U} = \nabla p - \sum_i \vec{f}_i \delta_\epsilon(x_i, x)$$
$$\nabla \cdot \vec{U} = 0$$

Then

$$\vec{U} = \vec{U_0} + \frac{1}{\mu} \sum_i \left[(\vec{f}_i \cdot \nabla) \nabla B_\epsilon(x_i, x) - \vec{f}_i G_\epsilon(x_i, x) \right]$$

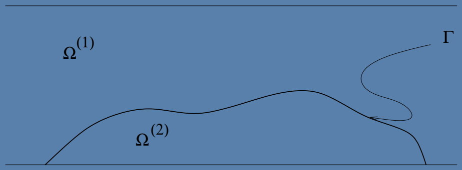
- Determine $\vec{f}_i$ to match flow at a point (invert force/velocity relationship from Stokeslet)
- Use $\vec{f}_i$ to determine the flow everywhere (force/velocity)

From Stokeslets to BIM



- Discretize an interface
- Sum of forces is like discrete integral
- Kernel is the Stokeslet
- BIM

Interface Motion (BIM)



- Interface separates two Stokes fluids (with different viscosities)
- Transform Stokes equations to integral equations (reciprocal relation)
- Solution gives velocity of the interface

Comparison Flows

External Flow:

$$\nabla \cdot \sigma^{(1)} = 0$$
$$\nabla \cdot \mathbf{U}^{(1)} = 0$$

Internal Flow:

$$\nabla \cdot \sigma^{(2)} = 0$$
$$\nabla \cdot \mathbf{U}^{(2)} = 0$$

Fundamental:

$$\nabla \cdot \sigma' = \mathbf{f}\delta(\mathbf{x} - \mathbf{x_0})$$
$$\nabla \cdot \mathbf{U}' = 0$$

$$\sigma^* = \mu^*(\nabla \mathbf{U}^* + \nabla {\mathbf{U}^*}^T) - P^*\mathbf{I}$$

Reciprocal Relations

$$\nabla \cdot (\mathbf{U}^{(*)}\sigma') - \nabla \cdot (\mathbf{U}'\sigma^{(*)}) \;=\; \mathbf{f}\delta(\mathbf{x} - \mathbf{x_0})\mathbf{U}$$

- Analogous to Green's Theorem
- $\mathbf{U}'$ - Single layer potential (Stokeslet)
- σ' - Double layer potential

Boundary Integral Formulation

- Flow in $\Omega^{(1)}$ due to singular force in $\Omega^{(1)}$

$$U_j^{(1)}(\mathbf{x_0}) = -\frac{1}{4\pi\mu^{(1)}}\int_\Gamma \sigma_{ik}^{(1)}\eta_k(\mathbf{x})\mathbf{G}_{ij}(\mathbf{x},\mathbf{x_0})\ dl(\mathbf{x}) + \frac{1}{4\pi}\int_\Gamma U_i(\mathbf{x})\mathbf{T}_{ijk}(\mathbf{x},\mathbf{x_0})\eta_k(\mathbf{x})\ dl(\mathbf{x})$$

- Flow in $\Omega^{(2)}$ due to singular force in $\Omega^{(1)}$

$$0 = \int_\Gamma \sigma_{ik}^{(2)}\eta_k(\mathbf{x})\mathbf{G}_{ij}(\mathbf{x},\mathbf{x_0})dl(\mathbf{x}) - \mu^{(2)}\int_\Gamma U_i(\mathbf{x})\mathbf{T}_{ijk}(\mathbf{x},\mathbf{x_0})\eta_k(\mathbf{x})dl(\mathbf{x})$$

Integral Equations

Combine these:

$$\begin{aligned} U_j^{(1)}(\mathbf{x_0}) &= -\frac{1}{4\pi\mu^{(1)}} \int_\Gamma \Delta\sigma_{ik}\eta_k \mathbf{G}_{ij}(\mathbf{x}, \mathbf{x_0})\ dl(\mathbf{x}) \\ &\quad + \frac{1-\lambda}{4\pi} \int_\Gamma U_i(\mathbf{x}) \mathbf{T}_{ijk}(\mathbf{x}, \mathbf{x_0}) \eta_k(\mathbf{x})\ dl(\mathbf{x}) \end{aligned}$$

$\lambda = \frac{\mu^{(1)}}{\mu^{(2)}}$

Constitutive assumption that $\Delta\sigma_{ik}$ is proportional to curvature

Hybrid combination

- Use BIM to find motion of interface
- Regularized Stokeslet finds forces to match velocities (or enforce boundary conditions)
- Sum of forces gives velocity field
- If boundary is fixed $\rightarrow$ Regularized Stokeslets
- If only interested in boundary motion $\rightarrow$ BIM

Example 1: Liver cell/β-cell interaction

with R. Bertram and M. Roper

- β-cell responds to glucose and produces insulin
- Liver cell responds to insulin and produces/consumes glucose
- Body synchronizes this process
- Design microfluidic experiments to test synchronization models

β-cell

Example 2: Biofilm Disinfection

- Biofilms affected by the external flow
- Antibiotic advects and diffuses
- Track surviving bacteria
- Predict optimal dosing and spatial distribution of surviving bacteria

biofilm

Example 3: Type II biofilm/mucociliary

with A. Dixon

- Mucus in the lung pumped by cilia
- Use envelope model (don't track individual cilia)
- Biofilm infections alter the clearance rates
- Explore the role of the alterations

type II

Conclusions

- Flexible with lots of applications
- BIM on the interface, where RS is less accurate
- RS away from the interface, where BIM is slow
- R. Bertram, M. Roper (Liver/β-cell)
  R. Cortez, L. Fauci (Biofilm/RS)
  A. Dixon (Type II Biofilm) and NSF for support.