A BLOW-UP CONSTRUCTION AND GRAPH COLORING

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§0. Introduction

In this note we construct a non-singular algebraic variety V_G encoding the incidence information of a simple graph G, by a sequence of blow-ups of a projective space along suitable linear subspaces. The aim is to translate into the geometry of V_G the combinatorial information about G; we find that this can be done with surprising ease and efficiency.

For example, we prove that the *chromatic polynomial* of the graph—that is, the polynomial giving for each $m > 0$ the number of ways in which G can be colored using m colors, so that no two adjacent vertices are assigned the same color—is (up to a power of the variable) the intersection product of a fixed class γ in $A_1 V_G$ with a polynomial $S(t)$ in $\mathrm{Pic} V_G[t]$: the class is defined as the Poincaré dual of the pull-back of the hyperplane class, with respect to a natural basis of $\mathrm{Pic} V_G$, and $S(t)$ is also easily defined as a combination of the exceptional divisors arising in the blow-up construction. In §1 we describe the construction for graphs and state the above result precisely (but with no proofs), as a sales pitch for the rest of the paper, which examines the construction more carefully and gives deeper—but necessarily more technical—results.

In fact the right level of generality to perform our construction is that of 'combinatorial geometries which are projectively coordinatizable over some field'; for short (and a little improperly) we will refer to these as *matroids*. Our construction can be performed starting from any (loopless) matroid embedded in a projective space, and specializes to the one in §1 for the *cycle matroid* of a graph. We give this more general construction in §2: roughly, the variety of a matroid is obtained by blowing-up the ambient projective space along the flats of the matroid, in order of increasing dimension. We prove the above result concerning the chromatic polynomial of a graph by showing that the *characteristic polynomial* of a matroid equals the intersection product of a fixed 1-class by a suitable polynomial $S(t)$ in the Pic of its variety. A question that then arises naturally regards the positivity of $S(m)$ for a given m and a given class of matroids: we determine a large class (including cycle matroids of graphs) for which a close relative $\overline{S}(m)$ of $S(m)$ is generated by global sections for all positive m.

To support the point that our construction may offer a new angle on the theory of characteristic polynomials of matroids, in §3 we give 'geometric proofs' of a few basic results on these (our source of examples here is Zaslavsky's contribution to [W2]). The deletion-contraction rule and Stanley's 'modular factorization theorem' for example follow easily from the functoriality of the construction. Most likely these proofs could be translated word by word into standard combinatorial proofs; our point here is that our arguments are suggested by 'algebro-geometric intuition', and the hope is that this could lead to a fresh approach to the combinatorics. Also, we hope §3 will help to advertise this beautiful branch of combinatorics among the geometers.

Our favorite example of the interplay between the two fields is the following: if we were to hand our construction to a random algebraic geometer, and asked to provide us with an interesting numerical invariant of these objects, she would likely propose the intersection product of the canonical divisor (which is the first place where to look for an invariant) with the above class γ (dual to the pull-back of the hyperplane class, thus a priori defined for all varieties produced by the construction). The result would essentially be, as we show in §3, Crapo's *Beta invariant* of the matroid; the basic properties of this latter (like additivity, or vanishing for disconnected matroids) all follow from the adjunction formula for the canonical divisor.

One feature of our construction is that it produces an infinite tower of varieties, rather than a single one: the construction depends on a starting $\mathbb{P}^n$ in which the matroid is embedded, and we get a non-singular variety V^n of dimension n for each n strictly larger than the rank of the matroid. In addition, each V^n is naturally embedded as a divisor in V^{n+1}, in a way compatible with the construction: for example, the divisor $S(m)$ on V^n is the restriction of the corresponding divisor on V^{n+1}, etc. The facts discussed in the first three sections hold uniformly for each variety in the sequence, so we may choose one arbitrarily if we wish. We think however that interesting information can be extracted from the whole tower: one such facts is observed in §4. For simplicity, assume the matroid to be regular (for example, graphical) and consider the rational maps $V^n \dashrightarrow \mathbb{P}^N$ defined by $S(m)$. Define $d(m,n)$ to be the degree of the (closure of the) image of this map as a cycle of dimension n. These numbers are invariants of the starting matroid which, we argue, encode interesting information. $d(m,n)$ is hard to compute in general (this is almost always the case for the degree of the image of a rational map!); specific examples can however be worked out. Here is a table of $d(m,n)$ for a few small values of m,n, for the varieties constructed starting from the complete graph on three vertices (these entries and the table in §4 were checked with Schubert [K-S]):

$d(m,n)$	$m=2$	3	4	5
$n=3$	42	644	3888	15216
4	210	6312	64746	388704
5	930	58312	1045476	9756192
6	3906	529244	16764894	244093680
7	16002	4776396	268386264	6103281168

And here is the general result given in §4:

Theorem. *If n is prime and greater than the rank of the matroid, then*

$$d(m,n) \equiv p(m) \pmod n \quad ,$$

where $p(m)$ is the characteristic polynomial of the matroid.

For example, $p(m) = (m-1)(m-2)$ for the complete graph on three vertices, and e.g., $6103281168 \equiv 4 \cdot 3 \pmod 7$.

We note that, by this result, the statement of the celebrated four-color-theorem tranlates into: *For a planar graph with N vertices, there exists a prime $n \geq N$ such that $d(4,n)$ is not a multiple of n.*

The above table will immediately convince the reader that it is *not* true that $d(m,n) \equiv p(m) \pmod{n}$ for all n.

The numbers $d(m,n)$ above can also be defined without ever leaving the original projective space from which the construction starts: they can be written in terms of the Segre classes of specific schemes supported on a linear subspace of the projective space. A congruence formula similar to the above can then be written for the zero-dimensional term of these Segre classes; see §4 for a precise statement.

Translating coloring problems in terms of projective geometry is not a new idea: the 'critical problem' ([C-R], Chapter 16) is the foremost such construction. We also know of a different and more algebro-geometric interpretation of these problems due to R. Miranda ([M]; see also [C-M]). A feature common to the critical problem and Miranda's approach is that both work by coordinatizing the relevant combinatorial geometry over a *finite* field, which in a sense keeps track of the number of colors used. Our construction has a different flavor in that it is performed in any characteristic over which the relevant matroid can be embedded (for example over $\mathbb{C}$; graphical matroids can be embedded in any characteristic); different colorings correspond to different divisors within this one construction. Of course we would be very interested in learning about relations between our construction and Miranda's or the critical problem.

Granted, we offer no new coloring theorem here. One missing ingredient is an algebro-geometric tool to tell a priori when a variety V_G as above does in fact arise from a planar graph as per our construction: the next natural step to take in the program is a suitable translation of Kuratowski's theorem in this language.

A note about our references: we draw most of our combinatorics know-how from Crapo and Rota's 'Combinatorial geometries' ([C-R]) and from the excellent contributions of Brylawski and Zaslavsky to [W1], [W2]. We found these references extremely helpful for their thoroughness and accessibility to the complete outsider, of which this writer is a perfect specimen.

Finally, a note for the hasty reader: the reader who feels confortable with matroids can safely skip §1, which simply specializes the construction to graphs. Also, §4 can be read independently of §3.

§1. The chromatic polynomial as an intersection product

Let G be a graph (loopless and with no parallel edges). Place the vertices of G at linearly independent points of a projective space $\mathbb{P}^n$ (over any algebraically closed field), and draw for each edge the line joining the corresponding vertices. Intersecting the resulting reducible curve with a general hyperplane gives a configuration of points e_k (ordered in any fashion), each corresponding to an edge of the graph, which is the starting point of our construction: in §2 we will study more generally the construction obtained by starting with any finite collection of points in a projective space. Our goal is to extract information from the linear dependence of the points e_k; the above is the standard way to embed in a projective space the 'cycle matroid' corresponding to the graph. The (point corresponding to an) edge e is in the subspace spanned by edges $e_1, \dots, e_d$ if and only if e joins vertices in one connected component of the subgraph of G determined by $e_1, \dots, e_d$ (cf. e.g. [W1], p.19, or [C-R], chapter 6). For example, three of the e_k's are collinear in $\mathbb{P}^n$ precisely if the corresponding edges form a circuit in G.

Now for the construction. Consider all dimension-d subspaces x_r^d spanned by the e_k in $\mathbb{P}^n$, listed by dimension and otherwise in any order: so in particular the x_r^0's are simply the e_k's. Also, consider the subspaces y_r^d obtained by intersecting collections of the x's, provided these do not appear already in the list of the x's. Observe that the family of subspaces of $\mathbb{P}^n$ thus obtained is closed with respect to intersection.

Let $V_0 = \mathbb{P}^n$, and inductively let V_{d+1}, $d \geq 0$, be the blow-up of V_d along the proper transforms of the x_r^d's and y_r^d's. Blowing up along the subspaces of dimension d separates the proper transforms of the subspaces of dimension $d+1$ containing them, so at each stage the centers of the blow-ups are necessarily disjoint, and the blow-ups can be performed in any order: in other words, these varieties do not depend on the specific ordering given to the x's and y's (in each dimension). Since G is finite, this construction stops at some stage, and we let V_G be the resulting variety. Of course V_G depends on the dimension n of the initial projective space $\mathbb{P}^n$; however, in most of the paper this will not play a rôle.

In V_G we single out several natural divisor classes: the pull-back H_0 of the hyperplane class from $\mathbb{P}^n$; the pull-backs E_r^d of the exceptional divisors arising by blowing up along x_r^d; the pull-backs F_r^d of the exceptional divisors arising by blowing up along y_r^d; and the classes H_r^d of the proper transforms of the general hyperplanes containing x_r^d. We define a divisor class $S(t)$ as follows: let R be the dimension of the subspace x^R spanned by *all* the x_r^0 ($R+1$ equals then the number of edges in a spanning forest of G; equivalently, the number of vertices of the graph minus the number of its connected components—cf. [W1], 6.1.2); then set

$$S(t) = t^{R+1} H_0 - \sum_{d,r} t^{R-d} E_r^d \quad .$$

Remark. Notice that the $F's$ are not used in this definition: in fact, most computations in the following can be performed 'modulo F' (that is, modulo combinations of F_r^d's). A construction could be concocted without introducing the auxiliary subspaces y_r^d and the corresponding F's, and still obtaining many of the results of the paper. We have chosen this alternative path because the construction as presented here is more natural in that it is independent of the ordering of the subspaces, and moreover blowing-up along the y's makes the H_r^d's generated by global sections (in fact, this amounts to resolving at one time all maps defined in terms of line bundles corresponding to nonnegative combinations of the H_r^d's). Is there an equally natural construction that does not invoke the use of these 'auxiliary' subspaces and divisors?

The following is the prototype of the results in the paper. We defer more general statements (and proofs) to later sections. Observe that

$$H_0, \text{ the } H_r^d, \text{ and the } F_r^d$$

give a basis of the Pic of V_G. Now by Poincaré duality we can find a class $\gamma \in A_1(V_G)$ dual to H_0 with respect to this basis: that is, such that

$$H_0 \cdot \gamma = 1, \quad H_r^d \cdot \gamma = 0, \quad F_r^d \cdot \gamma = 0 \quad \text{for all } d, r.$$

In other words, given a divisor D in V_G, $D \cdot \gamma$ picks the coefficient of H_0 in the (unique) expression of D in terms of H_0, H_r^d's, and the F_r^d's.

Theorem 1.1. *Let c be the number of connected components of G. Then the number of ways in which G can be colored properly with m colors (that is, so that no two adjacent vertices are given the same color) is given by the intersection product*

$$m^c\, S(m) \cdot \gamma \quad .$$

Corollary 1.2. *G can be colored properly with m colors if and only if $S(m) \cdot \gamma \neq 0$.*

Examples. (1) If G has at least 1 edge, then $S(1) = H_0 - \sum_{d,r} E_r^d$ is, modulo F, the class of the proper transform of the hyperplane containing all the x_r^d's; so (by definition of γ) $S(1) \cdot \gamma = 0$. If G has no edges, then $V_G = \mathbb{P}^n$, $S(1) = H_0$, and thus $S(1) \cdot \gamma = 1$. The corresponding facts about proper colorings are of course trivial.

(2) Let G be the complete graph on 4 vertices. The six x_r^0 are placed at the points of intersection of four general lines of a plane; on each of these four lines $x_1^1, \dots, x_4^1$ lie three of the x_k^0. There are three pairs of x_k^0's not lying on the same one line in this configuration; these pairs determine three more lines x_5^1, x_6^1, x_7^1. Finally, there is one plane x^2 containing the whole configuration. By using the definition of γ, we find

$$\begin{gathered} E_k^0 \cdot \gamma = 1, \quad k = 1, \dots, 6; \\ E_r^1 \cdot \gamma = -2, \quad r = 1, \dots, 4, \quad \text{and } E_r^1 \cdot \gamma = -1, \quad r = 5, 6, 7; \\ E^2 \cdot \gamma = 6 \quad , \end{gathered}$$

so

$$\begin{aligned} m\, S(m) &= m\,(m^3 - 6 \cdot 1\, m^2 - (-2 \cdot 4 - 1 \cdot 3)\, m - 1 \cdot 6) \\ &= m^4 - 6m^3 + 11m^2 - 6m = m\,(m-1)\,(m-2)\,(m-3) \end{aligned}$$

as it should be: each vertex must be assigned a different color from the palette.

We can prove a stronger statement than Theorem 1.1, which exploits one of the basic features of the construction: V_G encodes at once the combinatorial information of G and of all its *contractions*. Each x_r^d corresponds to a choice of edges of the original graph; let G_r^d be the graph obtained from G by contracting each edge in this collection, and removing parallel edges that might be created in the process (note: no loops arise by this operation). Also, let γ_r^d be the dual of H_r^d in the above basis. Up to a power of m, then, $S(m) \cdot \gamma_r^d$ *counts the proper m-colorings of the contraction G_r^d.* (This will follow from Theorem 2.3 in the more general setting of §2).

In other words, denote by $\overline{S}(m)$ the divisor equivalent to $S(m)$ modulo F and in the span of H_0, H_r^d: then the above says that
G and all its contractions can be colored properly with m colors if and only if $\overline{S}(m)$ is in the interior of the cone generated by H_0, H_r^d in $PicV_G$.
For example, the four-color-theorem ([A-K]) says that if G is a planar graph, then $\overline{S}(4)$ is in the interior of the cone generated by H_0, H_r^d (since all contractions of a planar graph are planar).

We end the section by remarking that in the case we have considered here (that is, varieties arising from graphs), the $\overline{S}(m)$, $m > 0$, turn out to be all generated by global sections (see Proposition 2.4): indeed, the H_r^d's are, and, by the above results, $\overline{S}(m)$ is a nonnegative combination of the H_0 and the H_r^d's in the graph case. This does not seem obvious a priori, for it is *not* true for the analogous construction for matroids examined in the next section (we will find there a class of matroids for which this holds, cf. Proposition 2.5). In the graph case, it follows that for positive m there always is a hypersurface in $\mathbb{P}^n$ generically smooth along the maximal x^R, with multiplicity m along the x_r^{R-1}'s, multiplicity m^2 along the x_r^{R-2}'s, ... , multiplicity m^R at the x_r^0's and degree m^{R+1}: simply take general hyperplanes containing the x_r^d's as dictated by the expression of $\overline{S}(m)$ in terms of H_0 and the H_r^d's. The class $\overline{S}(m)$ is then the class of the proper transform of such a hypersurface.

Conversely, we may view the above as a recipe to compute the chromatic polynomial of a graph: given the collection of x_r^d's obtained as above, construct a hypersurface by taking enough general hyperplanes containing each x_r^d to satisfy the above multiplicity prescription (multiplicity 1 along the maximal subspace x^R, t along codimension 1 subspaces, t^2 along codimension 2, etc.). By the above, this will always be possible: the number needed at x_r^d is $S(t) \cdot \gamma_r^d \geq 0$; and the number of hyperplanes not containing any of the x_r^d's, needed to get a hypersurface of degree t^{R+1}, multiplied by t to a power equal to the number of connected components of G, will give the value at t of the chromatic polynomial of G (this is of course nothing but "Möbius inversion" at work).

§2. Matroid varieties

In section 1 we gave the standard embedding in a projective space of the 'cycle matroid' associated with the graph G, and constructed a variety V_G from this data. The construction can be performed for the lattice $\mathcal{L} = \mathcal{L}(\mathcal{C})$ of subspaces spanned by any finite collection $\mathcal{C}$ of points in $\mathbb{P}^n$. $\mathcal{L}$ is (partially) ordered by inclusion; 0 will be the empty set (the minimum of the lattice), 1 the maximal subspace, spanned by all points; we require this to have codimension at least 2 in $\mathbb{P}^n$. We denote elements of $\mathcal{L}$ by letters $x, y, z, \dots$, by $\leq$ the ordering in $\mathcal{L}$, and by $\vee, \wedge$ resp. the join and meet in the lattice. The 'rank' $r(x)$ of $x \in \mathcal{L}, x \neq 0$, is one plus its dimension as a subspace of $\mathbb{P}^n$: so the points of $\mathcal{C}$ are the rank-1 elements of $\mathcal{L}$. The rank of $0 = \emptyset$ is 0; the 'rank of $\mathcal{L}$' is $r(\mathcal{L}) = r(1)$.

Now $V_{\mathcal{L}}$ is constructed as in section 1. First we close the family $\mathcal{L}$ of subspaces of $\mathbb{P}^n$ with respect to intersection: let $\mathcal{M}$ be the family of subspaces $\notin \mathcal{L}$ obtained by intersecting collections of elements of $\mathcal{L}$; we extend rank and ordering to elements of $\mathcal{M}$. Next, $V_{\mathcal{L}}$ is obtained from $\mathbb{P}^n$ by blowing up the (proper transforms of the) $x \neq 0$ in $\mathcal{L}$ and $\mathcal{M}$ in order of increasing dimension; again we observe that since $\mathcal{L} \cup \mathcal{M}$ is closed under intersections, blowing-up all x of rank r separates the proper transforms of the subspaces of rank $r+1$, hence the construction is independent of the specific order in which the blow-ups are executed (within each rank).

We note that $V_G = V_{\mathcal{L}}$ if $\mathcal{L}$ corresponds to G as in section 1. Keeping the same style of notations as in §1, we let H_x be the class of the proper transform of the general hyperplane containing x (so the pull-back of the hyperplane class is H_0), we let E_x be the pull-back of the exceptional divisor over $x \in \mathcal{L}, x \neq 0$, and F_x be the pull-back of the exceptional divisor over $x \in \mathcal{M}$. For $x \in \mathcal{L}$, γ_x is a 1-class such that $\gamma_x \cdot H_x = 1, \gamma_x \cdot H_y = 0$ for all $y \in \mathcal{L}, y \neq x$, and $\gamma_x \cdot F_z = 0$ for all $z \in \mathcal{M}$. $S(t)$ is the class

$$S(t) = t^{r(1)} H_0 - \sum_{x \in \mathcal{L}, x \neq 0} t^{r(1)-r(x)} E_x$$

(as in §1, we will soon introduce a class $\overline{S}(t)$ equivalent to $S(t)$ 'modulo F' but somewhat better behaved.)

§2.1. Compatibilities with contractions, deletions, etc. We will now show how the construction behaves with respect to three basic matroid operations. All the results in §3 will essentially follow from a closer look at the compatibilities sketched below; a detailed analysis of the functorial properties of the construction is well beyond the scope of this note. For the hasty reader: only contractions will be used in the rest of this section.

Contractions. The variety $V_{\mathcal{L}}$ contains a 'compatible' copy of $V_{\mathcal{L}/x} = V_{[x,1]}$ for each $x \in L$. More precisely: the fiber of the exceptional divisor obtained when blowing-up along $x \in \mathcal{L}$ is a projective space $\mathbb{P}^{n-r(x)}$, met by all and only the $z \geq x$ in $\mathcal{L}$. The lattice of subspaces these form in this projective space is the interval $[x, 1]$, isomorphic to the 'geometric contraction' $\mathcal{L}/x$ of $\mathcal{L}$ by x ([W1], p. 141). In terms of graphs, this is the contraction determined by a choice of a collection of edges, as described in §1. Now the blow-up process is compatible with restriction to this $\mathbb{P}^{n-r(x)}$: the general fiber of E_x (that is, the proper transform of $\mathbb{P}^{n-r(x)}$ in $V_{\mathcal{L}}$) is the blow-up of $\mathbb{P}^{n-r(x)}$ along its intersection with the $z \in \mathcal{L} \cup \mathcal{M}$, $z \geq x$, that is nothing but a copy of $V_{\mathcal{L}/x}$. Further, all expected compatibilities among the definitions of the relevant classes hold; for example, the class γ_x in $V_{\mathcal{L}}$ is the push-forward of the class γ_0 in $V_{\mathcal{L}/x}$, etc. Typically, anything proved about $\mathcal{L}$ by means of $V_{\mathcal{L}}$ will automatically restrict to a statement about all its contractions.

Modular elements. At the same time, $V_{\mathcal{L}}$ also contains a copy of $V_{[0,x]}$ (where $[0, x]$ denotes the lattice of elements $z \in \mathcal{L}$ such that $0 \leq z \leq x$), provided that x be *modular*. An element $x \in \mathcal{L}$ is 'modular' if $x \wedge z = x \cap z$ for all z in $\mathcal{L}$ (where $\wedge$ denotes the meet in the lattice, while $\cap$ denotes intersection in $\mathbb{P}^n$); for example, all rank-1 elements of $\mathcal{L}$ are modular. Now consider any subspace $\mathbb{P}_x$ of $\mathbb{P}^n$, of dimension $> r(x)$ and intersecting $1 \in \mathcal{L}$ precisely along x; then

Claim 2.1. *If x is modular, then the proper transform of $\mathbb{P}_x$ in $V_{\mathcal{L}}$ is isomorphic to a variety $V_{[0,x]}$.*

Proof. $\mathbb{P}_x$ contains a copy of $[0, x]$. Let $\mathcal{M}_x$ denote for a moment the set of subspaces defined when constructing $V_{[0,x]}$ (that is, all $y \cap z \notin [0, x]$, where $y, z \in$

$[0,x]$). Then it is easily checked that modularity implies $[0,x] = \{z \cap \mathbb{P}_x, z \in \mathcal{L}\}$ and $\mathcal{M}_x = \{z \cap \mathbb{P}_x, z \in \mathcal{M}\}$. Taking the proper transform of $\mathbb{P}_x$ amounts then to performing precisely the same sequence of blow-ups producing $V_{[0,x]}$ as dictated by the construction. □

Deletions. The construction is also compatible with substructures. Let $\mathcal{C}'$ be a subset of the set of rank-1 elements of $\mathcal{L}$ (that is, of the original set $\mathcal{C}$ of points in $\mathbb{P}^n$ generating $\mathcal{L}$); these generate a sublattice $\mathcal{L}(\mathcal{C}')$ of $\mathcal{L}$, a 'deletion' of $\mathcal{L}$. Then there is a map $V_{\mathcal{L}} \to V_{\mathcal{L}(\mathcal{C}')}$: this follows from the universal property of blow-ups, once we observe that the inverse image of all subspaces generated by elements of $\mathcal{C}'$ (and all their intersections) are Cartier divisors in $V_{\mathcal{L}}$. For example, for $\mathcal{C}' = \emptyset$, the resulting map $V_{\mathcal{L}} \to V_{\mathcal{L}(\emptyset)} = \mathbb{P}^n$ is simply the sequence of blow-ups defining $V_{\mathcal{L}}$.

Nesting. Finally, we observe that we get a variety $V^n = V_{\mathcal{L}}$ by blowing up $\mathbb{P}^n$ as above, for each $n > r(1)$; most results of the paper do not depend on the specific choice of n. These different varieties are nested into each others like Russian dolls: for all $n > r(1)$, V^n can be embedded as a divisor of class H_1 in V^{n+1}. Indeed, the proper transform of any $\mathbb{P}^n$ containing $1 \in \mathcal{L}$ in $\mathbb{P}^{n+1}$ is a copy of V^n: this is Claim 2.1 for $x = 1$ (1 is always modular!).

§2.2. The characteristic polynomial. Now for a bit of well known and beautiful combinatorics, and its translation into the intersection ring of $V_{\mathcal{L}}$. Recall ([W2], Chapter 7) that the 'Möbius function' of a lattice $\mathcal{L}$ is the function $\mu_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ satisfying

$$\sum_{x \le y \le z} \mu_{\mathcal{L}}(x,y) = \begin{cases} 0 \text{ if } x \neq z \\ 1 \text{ if } x = z \end{cases} \quad \text{if } x \le z, \quad \mu_{\mathcal{L}}(x,z) = 0 \text{ if } x \not\le z$$

We will write μ for $\mu_{\mathcal{L}}$ if no ambiguity is feared. The 'characteristic polynomial' of $\mathcal{L}$ is the polynomial

$$p(\mathcal{L}, t) = \sum_{x \in \mathcal{L}} \mu(0,x)\, t^{r(1)-r(x)}$$

Now the key observation is the following (see for example [W2], §7.5): the number of proper colorings of a graph G with t colors (that is, the 'chromatic polynomial' of G) is given by

$$t^c\, p(\mathcal{L}, t) \quad ,$$

where c is the number of connected components of G and $\mathcal{L}$ is the lattice determined by G. So Theorem 1.1 will be proved once we show that for any matroid in $\mathbb{P}^n$ as above:

Theorem. $p(\mathcal{L}, t) = S(t) \cdot \gamma_0 \quad .$

In turn, given the definition of $S(t)$, this is proved once we observe that $\mu(0,0) = 1 = H_0 \cdot \gamma_0$, and show that $\mu(0,z) = -E_z \cdot \gamma_0$ for $z \in \mathcal{L}, z \neq 0$. In fact:

Lemma 2.2. *$E_z \cdot \gamma_x = -\mu(x,z)$ for all $z \in \mathcal{L}, z \neq 0$.*

Proof. First we observe that by restricting to the general fiber of E_x we may assume $x = 0$ (by compatibility with contractions). So we just have to show

$\mu(0,z) = -E_z \cdot \gamma_0$ for $z \neq 0$. By definition of the Möbius function, this amounts to showing

$$\mu(0,0) + \sum_{0<y\leq z} (-E_y \cdot \gamma_0) = 0$$

whenever $z \neq 0$. But observe that the construction gives

$$H_z = H_0 - \sum_{\substack{y\in\mathcal{L} \\ 0<y\leq z}} E_y - \sum_{\substack{x\in\mathcal{M} \\ x<z}} F_x \quad ,$$

so that

$$\begin{aligned} \mu(0,0) + \sum_{0<y\leq z} (-E_y \cdot \gamma_0) &= 1 + \sum_{0<y\leq z} (-E_y \cdot \gamma_0) \\ &= (H_0 - \sum_{0<y\leq z} E_y - \sum_{x<z} F_x) \cdot \gamma_0 \\ &= H_z \cdot \gamma_0 = 0 \end{aligned}$$

by definition of γ_0. $\square$

As pointed out, this lemma implies the theorem above, and this in turn implies Theorem 1.1. There is a substantial advantage, however, in giving a more comprehensive statement dealing with all contractions of $\mathcal{L}$ at once. For this, let $\overline{S}(t)$ denote the divisor equivalent to $S(t)$ modulo F and in the span of the H_x's. Note that $S(t)$ and $\overline{S}(t)$ have the same intersection numbers against any combination of the γ_x, $x \in \mathcal{L}$.

Theorem 2.3. *Denoting by $\mathcal{L}/x \cong [x,1]$ the sublattice of $\mathcal{L}$ consisting of all $z \in \mathcal{L}$ such that $x \leq z$:*

$$S(t) \cdot \gamma_x = p(\mathcal{L}/x, t)$$

for all $x \in \mathcal{L}$. In other words,

$$\overline{S}(t) = \sum_{x\in\mathcal{L}} p(\mathcal{L}/x, t)\, H_x \quad .$$

Proof.

$$\begin{aligned} S(t) \cdot \gamma_x &= t^{r(1)} H_0 \cdot \gamma_x - \sum_{y\in\mathcal{L}, x\neq 0} t^{r(1)-r(y)} E_y \cdot \gamma_x \\ &= \sum_{y\in\mathcal{L}} t^{r(1)-r(y)} \mu(x,y) \qquad \text{by Lemma 2.2} \\ &= \sum_{y\geq x} t^{r(1)-r(y)} \mu(x,y) \\ &= p([x,1],t) = p(\mathcal{L}/x, t) \quad . \quad \square \end{aligned}$$

Theorem 2.3 implies the extension of Theorem 1.1 discussed in section 1.

$S(t)$ is easier to define, while $\overline{S}(t)$ is better behaved in some respects. For example, $\overline{S}(m)$ is automatically globally generated for $m > 0$ in the graph case (as mentioned in §1), because:

Proposition 2.4. *Non-negative linear combinations of the H_x's are generated by global sections.*

Proof. We only need to show that each H_x is generated by global sections. Now H_0 clearly is, since it 'already' is in $\mathbb{P}^n$; for $x \neq 0$, observe that any $x \in \mathcal{L}$ is the intersection of $n+1-r(x)$ general hyperplanes containing it. In the construction, every center of blow-up is either included in the proper transform of x, or it is disjoint from it (note: this would not necessarily be the case if we didn't blow-up along the elements of $\mathcal{M}$ as well!). It follows that the proper transforms of the hyperplanes still intersect exactly along the proper transform of x after each blow-up, and get separated when x itself is blown up. They give then $n+1-r(x)$ sections of H_x generating it globally. $\square$

Remark. What was shown in this proof was in fact that $n+1-r(x)$ general representatives of H_x have empty intersection in $V_{\mathcal{L}}$.

In the graph case, the coefficients of H_x in $\overline{S}(t)$ are (up to powers of t) chromatic polynomials, thus nonnegative at positive integers: so $\overline{S}(m)$ is in the cone generated by the H_* in $\mathrm{Pic} V_G$ for all positive m, and is globally generated.

This does not seem at all obvious a priori, say from the definition of $S(t)$; in fact, it is *not* true for arbitrary matroids! For example, consider the matroid L_4 generated by four collinear points: if $\overline{S}(2)$ were generated by global sections, then (at least in char. 0) by Bertini there would be a nonsingular irreducible hypersurface of class $\overline{S}(2)$ in V_{L_4}; this would map down to $\mathbb{P}^n$ to a hypersurface of degree 4, generically smooth along a line, and having multiplicity 2 at (at least) 4 points on this line. This cannot be: the general plane section (through the line) of this hypersurface would be a plane quartic curve containing a line, whose residual cubic meets the line at four distinct isolated points. Thus $\overline{S}(2)$ is not generated by global sections in general.

It would be interesting to find a characterization of planar graphical matroids in terms of properties of the divisors $\overline{S}(m)$. A more ambitious goal would be to find for each given matroid M an algebro-geometric property of $V_{\mathcal{L}}$ that can signal whether $\mathcal{L}$ is the lattice of a matroid *none of whose minors is isomorphic to M.* Such a tool would allow us to mirror the characterization of classes of matroids in terms of 'excluded minors' (see pp. 146–7 in [W1]); in particular a characterization of varieties arising from planar graphical matroids would follow.

The only result of this sort that we know is the following. Following the common terminology, we denote by L_4 the 'four point line' of the above example, and by F_7 the 'seven point plane' (that is, the matroid defined by the projective plane over the 2-element field).

Proposition 2.5. *Let $\mathcal{L}$ be the lattice corresponding to a given matroid M, and $\overline{S}(m)$ the divisor on $V_{\mathcal{L}}$ defined as above. Then the following are equivalent:*

1. *$\overline{S}(2)$ and $\overline{S}(3)$ are in the cone generated by the H_x, $x \in \mathcal{L}$;*
2. *M has no minor isomorphic to L_4 or F_7;*
3. *All $\overline{S}(m)$, $m > 0$, are in the cone generated by the H_x, $x \in \mathcal{L}$.*

Remark. This amounts to saying that the class defined in (2) is precisely the class of matroids whose contractions all have characteristic polynomials which are non-negative at each positive integer. This must be a well-known characterization in combinatorics, but we could not trace it in the literature; we apologize for the missing reference and provide the following straightforward (and hopefully correct) argument.

Proof. (3) $\implies$ (1) is trivial.

(1) $\implies$ (2): if M has a minor isomorphic to L_4, then by the 'scum theorem' (Prop. 7.4.11 in [W1]) L_4 is obtained from M by a contraction M/I followed by a sequence of deletions: $L_4 = M/I - e_1 - \cdots - e_r$. Now $p(L_4, m) = m^2 - 4m + 3$, so $p(\mathcal{M}/I - e_1 - \cdots - e_r, 2) = p(L_4, 2) = -1$; we claim that this implies some contraction of M has negative characteristic polynomial at 2. Indeed, by [W2], Theorem 7.2.4,

$$p(M/I - e_1 - \cdots - e_r, 2) = p(M/I - e_1 - \cdots - e_{r-1}, 2) \\ + p(M/(I \vee e_r) - e_1 - \cdots - e_{r-1}, 2)$$

if e_r is not an isthmus in $M/I - e_1 - \cdots - e_{r-1}$, and

$$p(M/I - e_1 - \cdots - e_r, 2) = p(M/I - e_1 - \cdots - e_{r-1}, 2)$$

if e_r is an isthmus in $M/I - e_1 - \cdots - e_{r-1}$. In either case, the polynomial is necessarily negative at 2 for a contraction of M followed by fewer deletions: the claim follows. Finally, the coefficients in the expression of $\overline{S}(2)$ in terms of the H_* are precisely the values of the characteristic polynomials of the (geometric) contractions of M (by Theorem 2.3), so we can conclude that $\overline{S}(2)$ is not in the cone generated by the H_*. The argument for F_7 is entirely similar, given that $p(F_7, m) = m^3 - 7m^2 + 14m - 8$ is negative for $m = 3$.

(2) $\implies$ (3): the class defined in (2) is closed under contractions, so we just need to show that the characteristic polynomial of any matroid in it is nonnegative at positive integers. By a result of Seymour (cf. [W1], p. 147), the class is in fact the class of 'direct sums and 2-sums of regular matroids and copies of F_7^*'. Now observe that $p(F_7^*, m) = m^4 - 7m^3 + 21m^2 - 28m + 13$ is ≥ 0 for all integer $m > 0$; also, regular matroids have nonnegative characteristic polynomial because of a result of Crapo (Theorem III in [C]: the value of the polynomial at m is the number of 'H-coboundaries with kernel 0', for H a group of order m). Next, nonnegativity is preserved by direct sums by Theorem 7.2.4 (ii) in [W2]; so we just have to show it is preserved under 2-sums. Now the 2-sum of two matroids M_1, M_2 is obtained from their parallel connection by deletion of the base point: in the notation of [W1], p. 180

$$S_2(M_1, M_2) = P(M_1, M_2) - p \quad ,$$

where p is not an isthmus of either M_1 or M_2. It follows that p is not an isthmus of $P(M_1, M_2)$, so applying 7.2.4 (i) from [W2], $7.6.7_P$ from [W1], 7.2.9 and 7.2.4 (ii) from [W2] we get

$$\begin{aligned} p(S_2(M_1, M_2), m) &= p(P(M_1, M_2), m) + p(P(M_1, M_2)/p, m) \\ &= p(P(M_1, M_2), m) + p(M_1/p \oplus M_2/p, m) \\ &= \frac{p(M_1, m)\, p(M_2, m)}{m-1} + p(M_1/p, m)\, p(M_2/p, m) \quad : \end{aligned}$$

each summand on the right is non-negative, so we are done. $\square$

All matroids representable over any field, and in particular all graphical matroids, belong to the class defined in this proposition; however, for such matroids one can prove (3) more directly, cf. the discussion following Proposition 2.4. For all matroids satisfying (3), the line bundles corresponding to $\overline{S}(m)$ are globally generated, so they define maps from the variety of the matroid to a projective space. We feel that studying these maps would be quite fruitful; we will obtain a simple result about the degree of the image of such maps in §4.

Of course a characterization of planar graphs in a fashion similar to Proposition 2.5 would be desirable.

§3. Characteristic polynomial basics, Crapo's invariant: a geometric viewpoint

In this section we run through basic material concerning characteristic polynomials, illustrating it in the context of the construction introduced in §2. The reader is encouraged to compare the 'geometric' proofs given here with more standard combinatorial arguments, as presented for example in Chapter 7 of [W2].

The general strategy is the following: in a given situation, write the most fundamental relation suggested by the geometry; then applying the results in §2 will yield an equally fundamental combinatorial statement. As an appetizer, the following is the simplest possible example of such an argument:

Proposition 3.1. *With notations as in §2, $\sum_{x\in\mathcal{L}}\gamma_x$ equals the class of the pull-back ℓ of a line from $\mathbb{P}^n$.*

Proof. Dot both classes against all divisors. $\square$

And here is the translation into combinatorics:

Corollary 3.2. $\sum_{x\in\mathcal{L}} p(\mathcal{L}/x,t) = t^{r(1)}$

Proof. By Theorem 2.3 and Proposition 3.1, the left-hand-side is $S(t)\cdot\sum_{x\in\mathcal{L}}\gamma_x = S(t)\cdot\ell$. But the pull-back of a line vanishes against all exceptional divisors, so $S(t)\cdot\ell = t^{r(1)}H_0\cdot\ell = t^{r(1)}$. $\square$

The other examples in this section are a little more complex, but motivated by the same simple geometric intuition.

§3.1. Deletion-contraction rule. Let $e\in\mathcal{C}$ be a rank-1 element in $\mathcal{L}$—that is, one of the points in the set used to generate the subspaces in $\mathcal{L}$. Denote by $\mathcal{L}-e$ the lattice of subspaces spanned by the *other* points ($\mathcal{L}-e$ is a 'deletion' of $\mathcal{L}$). We observed in §2.1 that the universal property of blow-ups gives then a map

$$\alpha : V_{\mathcal{L}} \to V_{\mathcal{L}-e}$$

compatible with the blow-up maps from the matroid varieties to $\mathbb{P}^n$. In particular, this map is proper, birational and onto. We use notations as in §2, and append a $'$ to denote objects in $V_{\mathcal{L}-e}$: so e.g., H_0' is the pull-back of the hyperplane class to $V_{\mathcal{L}-e}$ (and it follows $\alpha^*(H_0')=H_0$), etc.

Proposition 3.3. $\alpha^*(\gamma_0') = \gamma_0 + \gamma_e$

Proof. It is clear that the class vanishes against 'F divisors'; we have to show $H_x \cdot \alpha^*(\gamma_0) = 0$ if $x \neq 0, e$, and $= 1$ otherwise. Now any $x \in \mathcal{L}$, $x \neq 0, e$, contains a maximal $x' \in \mathcal{L} - e$, $x' \neq 0$; the reader will then check that $\alpha_*(H_x) = H'_{x'}$. Since α is birational, and using the projection formula, $H_x \cdot \alpha^*(\gamma_0') = H'_{x'} \cdot \gamma_0' = 0$ since $x' \neq 0$. By the same token, $\alpha_*(H_e) = \alpha_*(H_0) = H_0'$, from which $H_e \cdot \alpha^*(\gamma_0') = H_0 \cdot \alpha^*(\gamma_0') = 1$. $\square$

Proposition 3.3 'stands behind' the deletion-contraction rule for the characteristic polynomial (Theorem 7.2.4(i) in [W2]), curiously regardless of e being or not an *isthmus* of $\mathcal{L}$ (a rank-1 element e of $\mathcal{L}$ is an 'isthmus' if the rank of $\mathcal{L}$ is strictly larger than the rank of $\mathcal{L} - e$). More precisely:

Corollary 3.4(a). *If e is not an isthmus, then $p(\mathcal{L}, t) = p(\mathcal{L} - e, t) - p(\mathcal{L}/e, t)$.*

Proof. If e is not an isthmus, then $r(\mathcal{L}) = r(\mathcal{L} - e)$, and it follows that $\alpha_*(S(t)) = S(t)'$ by definition. By Theorem 2.3 and using the projection formula:

$$\begin{aligned} p(\mathcal{L} - e, t) &= S(t)' \cdot \gamma_0' = \alpha_*(S(t)) \cdot \gamma_0' = S(t) \cdot \alpha^*(\gamma_0') \\ &= S(t) \cdot (\gamma_0 + \gamma_e) \qquad \text{by the proposition} \\ &= p(\mathcal{L}, t) + p(\mathcal{L}/e, t) \end{aligned}$$

again by Theorem 2.3. $\square$

Corollary 3.4(b). *If e is an isthmus, then $p(\mathcal{L}, t) = (t-1)\, p(\mathcal{L} - e, t)$.*

Proof. If e is an isthmus, then $r(\mathcal{L}) = r(\mathcal{L} - e) + 1$. From this it follows that $\alpha_*(S(t)) = t\, S(t)'$, so

$$\begin{aligned} t\, p(\mathcal{L} - e, t) &= t\, S(t)' \cdot \gamma_0' = \alpha_*(S(t)) \cdot \gamma_0' \\ &= S(t) \cdot (\gamma_0 + \gamma_e) \qquad \text{arguing as above} \\ &= p(\mathcal{L}, t) + p(\mathcal{L}/e, t) \\ &= p(\mathcal{L}, t) + p(\mathcal{L} - e, t) \end{aligned}$$

since $\mathcal{L}/e = \mathcal{L} - e$ if e is an isthmus. The statement follows. $\square$

§3.2. Stanley's modular factorization theorem. If $\mathcal{L}$ is the product $\mathcal{L}_1 \times \mathcal{L}_2$ of two lattices, we could argue as above and prove the multiplicativity of the characteristic polynomial under direct sums, by studying the map $V_{\mathcal{L}} \longrightarrow V_{\mathcal{L}_1}$. However, as pointed out in [W2], p. 122, this is a particular case of a more general factorization result ([S], Theorem 2); so we present the latter.

Recall from §2 that we have an injection $i : V_{[0,x]} \hookrightarrow V_{\mathcal{L}}$ whenever x is a modular element of $\mathcal{L}$. Again we use notations as in §2, appending a $''$ to denote objects of $V_{[0,x]}$.

Proposition 3.5. *If x is modular, and with notations as above:*

(1) $i^*(H_z) = H''_{x \wedge z}$;
(2) $i^*(E_z) = E''_z$ *if* $z \leq x$, *0 otherwise;*
(3) $i^*(F_z) = F''_z$ *if* $z \leq x$, *0 otherwise;*
(4) $i^*(S(t)) = t^{r(1)-r(x)}\, S(t)''$ *and* $i^*(\overline{S}(t)) = t^{r(1)-r(x)}\, \overline{S}(t)''$

Proof. (2) and (3) follow from a chase of the diagram of blow-ups producing the two varieties. For example, if $z \not\leq x$ then x and z are separated when blowing-up along $x \cap z$; the proper-transform of $\mathbb{P}_x$ (that is, $V_{[0,x]}$ by Claim 2.1) is then disjoint from the exceptional divisor above z, and the corresponding pull-back must vanish.

(1) follows from (2) and (3). The first part of (4) follows from the definitions of $S(t)$, $S(t)''$ and from (1) and (2). The second part of (4) follows from the first, by killing F terms on both sides. □

Corollary 3.6. *For all modular $x \in \mathcal{L}$ and all $y \in [0, x]$*

$$\sum_{z\in\mathcal{L}, z\wedge x=y} p(\mathcal{L}/z, t) = t^{r(1)-r(x)}\, p([y, x], t)$$

Proof. Using Theorem 2.3 to write out the second part of (4) from the proposition:

$$\begin{aligned} t^{r(1)-r(x)} \sum_{0\leq y\leq x} p([0, x]/y, t)\, H''_y &= i^*\big(\sum_{z\in\mathcal{L}} p(\mathcal{L}/z, t)\, H_z\big) \\ &= \sum_{z\in\mathcal{L}} p(\mathcal{L}/z, t)\, H''_{z\wedge x} \qquad \text{by (1) above} \\ &= \sum_{0\leq y\leq x} \left(\sum_{z\in\mathcal{L}, z\wedge x=y} p(\mathcal{L}/z, t)\right) H''_y \quad . \end{aligned}$$

The statement follows by dotting with γ''_y and observing $[0, x]/y = [y, x]$. □

Setting $y = 0$ in the statement and isolating $p(\mathcal{L}/0, t) = p(\mathcal{L}, t)$ gives

$$p(\mathcal{L}, t) = t^{r(1)-r(x)}\, p([0, x], t) - \sum_{\substack{z\in\mathcal{L}, z\neq 0 \\ z\wedge x=0}} p(\mathcal{L}/z, t)$$

Corollary 3.7. *(Modular factorization theorem) If x is a modular element of $\mathcal{L}$, then*

$$p(\mathcal{L}, t) = p([0, x], t) \sum_{y\in\mathcal{L}, y\wedge x=0} \mu(0, y)\, t^{r(1)-r(x)-r(y)}$$

Proof. By induction on the rank of $\mathcal{L}$. The statement is clear if the rank of $\mathcal{L}$ equals $r(x)$ (because this forces $x = 1$). If x is modular in $\mathcal{L}$ and $z \wedge x = 0$, then $z \vee x$ is modular in $[z, 1] = \mathcal{L}/z$; and $r(\mathcal{L}/z) < r(\mathcal{L})$ if $z \neq 0$, so we may assume the statement for $\mathcal{L}/z$ in this case. Doing so in the formula preceding the statement of this corollary gives the induction step. □

§3.3. Crapo's beta invariant. Writing down an expression for the canonical divisor $\omega_{\mathcal{L}}$ of a matroid variety $V_{\mathcal{L}}$ is an elementary exercise:

$$\omega_{\mathcal{L}} = -(n+1)\, H_0 + \sum_{x\in\mathcal{L}, x\neq 0} (n - r(x))\, E_x + \sum_{y\in\mathcal{M}} (n - r(y))\, F_y \quad ,$$

where n denotes as usual the dimension of $V_{\mathcal{L}}$. On the other hand, an important invariant of a matroid is its *beta invariant*

$$\beta(\mathcal{L}) = (-1)^{r(1)-1} \frac{d}{dt}\, p(\mathcal{L}, 1)$$

(our source is §7.3 in [W2]). The beta invariant contains a surprising amount of information: for example, it vanishes precisely if the matroid is a direct sum (or it is trivial). Now it turns out that the beta invariant of $\mathcal{L}$ is intimately related to the canonical divisor of $V_{\mathcal{L}}$—thus its relevance is clear from an algebro-geometric perspective.

Proposition 3.8. *Assume $\mathcal{L} \neq 0$. Then*

$$\beta(\mathcal{L}) = (-1)^{r(1)}(1 + \omega_{\mathcal{L}} \cdot \gamma_0)$$

We will see in a moment (Proposition 3.10) that knowing the canonical divisor of $V_{\mathcal{L}}$ (modulo F) is in fact equivalent to knowing the beta invariant of all contractions of $\mathcal{L}$.

Proof. $p(\mathcal{L}, t) = S(t) \cdot \gamma_0$ (Theorem 2.3), so

$$\begin{aligned}
&(1 + \omega_{\mathcal{L}} \cdot \gamma_0) - (-1)^{r(1)}\beta(\mathcal{L}) = 1 + \left(\omega_{\mathcal{L}} + \frac{dS}{dt}(1)\right) \cdot \gamma_0 \\
&= 1 + \left(-(n+1)\,H_0 + \sum_{x \neq 0}(n - r(x))E_x + r(1)\,H_0 - \sum_{x \neq 0}(r(1) - r(x))\,E_x\right) \cdot \gamma_0 \\
&= 1 - \left(H_0 + (n - r(1))\left(H_0 - \sum_{x \neq 0} E_x\right)\right) \cdot \gamma_0 \\
&= 1 - (H_0 + (n - r(1))\,H_1) \cdot \gamma_0 \qquad\qquad (\text{modulo } F) \\
&= 0
\end{aligned}$$

as needed. □

Notice that the canonical divisor depends on the dimension n of $V_{\mathcal{L}}$; as the proposition shows, its intersection with γ_0 does not (if $\mathcal{L} \neq 0$). The reason is that each variety is embedded in the next as a divisor of class H_1: so their canonical divisors differ by multiples of H_1 by adjunction, and their difference is not detected by γ_0 by definition of the latter.

The excluded case ($\mathcal{L} \neq 0$) and the shape of the formula in Proposition 3.8 reflect a little white noise in the definitions. We can improve the situation by modifying the definition slightly in order to make it independent of n and fully compatible with contractions. To this effect, define the 'beta divisor' of $V_{\mathcal{L}}$ to be

$$\widetilde{\omega}_{\mathcal{L}} = H_0 + (\dim V_{\mathcal{L}} - r(\mathcal{L}))\,H_1 + \omega_{\mathcal{L}}$$

The beta divisor is more natural with respect to the construction, in the sense that it is compatible with the operations we have encountered so far. More precisely, let $\alpha : V_{\mathcal{L}} \longrightarrow V_{\mathcal{L}-e}$ and $i : V_{[0,x]} \hookrightarrow V_{\mathcal{L}}$ be as in §§3.1,2 (with the same $'$, $''$ notations), and view $V_{\mathcal{L}/x}$ as a subvariety of $V_{\mathcal{L}}$ as usual (§2.1); then

Proposition 3.9. *For $e \in \mathcal{L}, r(e) = 1$ and $x \in \mathcal{L}$:*

(1) $\widetilde{\omega}_{\mathcal{L}}|_{V_{\mathcal{L}/x}} = \widetilde{\omega}_{\mathcal{L}/x}$
(2) $\alpha_*(\widetilde{\omega}_{\mathcal{L}}) = \widetilde{\omega}_{\mathcal{L}-e} - (r(\mathcal{L}) - r(\mathcal{L} - e))H_1'$
(3) *if x is modular,* $i^*(\widetilde{\omega}_{\mathcal{L}}) = \widetilde{\omega}_{[0,x]} - (r(1) - r(x))H_1''$

Proof. These are all immediate from the definition and the adjunction formula. For example, let's check (3): $V_{[0,x]}$ is embedded in $V_{\mathcal{L}}$ as the proper transform of a space intersecting $1 \in \mathcal{L}$ precisely along x; it follows that $V_{[0,x]}$ is cut out by $\dim V_{\mathcal{L}} - \dim V_{[0,x]}$ representatives of H_x. Its normal bundle has then first Chern class $= (\dim V_{\mathcal{L}} - \dim V_{[0,x]})H_x$, so the adjunction formula and Proposition 3.5(1) give

$$i^*(\omega_{\mathcal{L}}) = \omega_{[0,x]} - i^*((\dim V_{\mathcal{L}} - \dim V_{[0,x]})H_x) = \omega_{[0,x]} - (\dim V_{\mathcal{L}} - \dim V_{[0,x]})H_x''$$

Plugging this into the definition of the beta divisor gives (3). □

These compatibility properties of the beta divisor are in our view the motor behind the basic properties of the beta invariant (e.g., 7.3.1, 7.3.2 in [W2]). To support this viewpoint, we derive a few of these in the remaining of this section. For a start, let's observe explicitly that knowing the beta divisor (modulo F) is equivalent to knowing the beta invariant of $\mathcal{L}$ and of all its contractions. Indeed:

Proposition 3.10. *For all $x \in \mathcal{L}$:*

$$\widetilde{\omega}_{\mathcal{L}} \cdot \gamma_x = (-1)^{r(1)-r(x)}\beta(\mathcal{L}/x)$$

Proof. For $x = 0$ this follows at once from Proposition 3.8, or by explicit computation if $\mathcal{L} = 0$; the general case reduces to $x = 0$ by compatibility with contractions (Proposition 3.9(1)). □

$(-1)^{r(1)-r(x)}\beta(\mathcal{L}/x)$ is called the 'signed beta function', $B(x)$, in [W2], §7.3. Proposition 3.10 simply says

$$\widetilde{\omega}_{\mathcal{L}} = \sum_{x\in\mathcal{L}} B(x)\, H_x \qquad \text{modulo } F.$$

Corollary 3.11. $\displaystyle\sum_{x\in\mathcal{L}, x\geq y} B(x) = r(y) - r(1)$

Proof. Reduce to $y = 0$ by replacing $\mathcal{L}$ by $\mathcal{L}/y$. Then by the last proposition

$$\begin{aligned}\sum_{x\in\mathcal{L}} B(x) &= \widetilde{\omega}_{\mathcal{L}} \cdot \sum_{x\in\mathcal{L}} \gamma_x \\ &= \widetilde{\omega}_{\mathcal{L}} \cdot \ell \qquad \text{by Proposition 3.1} \\ &= 1 + (n - r(1)) - (n+1) = -r(1)\end{aligned}$$

by the definition of $\widetilde{\omega}_{\mathcal{L}}$. □

The above formula corrects an oversight in [W2], p. 126 (the first formula on p. 126 holds only if $x \neq 1$, so inversion hides one term in the second).

The compatibility of the beta divisor with deletions (that is, Proposition 3.9(2)) leads to the additivity property of the beta invariant:

Corollary 3.12. *If e is not an isthmus, then $\beta(\mathcal{L}) = \beta(\mathcal{L} - e) + \beta(\mathcal{L}/e)$*

Remark. Loops do not appear in this statement because our matroids are loopless by assumption, cf. the introduction.

Proof. If e is not an isthmus, then $r(\mathcal{L}) = r(\mathcal{L} - e)$ so $\alpha_*(\widetilde{\omega}_{\mathcal{L}}) = \widetilde{\omega}_{\mathcal{L}-e}$ by Proposition 3.9(2). Using Propositions 3.3 and 3.10:

$$\begin{aligned}(-1)^{r(\mathcal{L}-e)}\beta(\mathcal{L}-e) &= \gamma_0' \cdot \widetilde{\omega}_{\mathcal{L}-e} \\ &= (\gamma_0 + \gamma_e)\cdot\widetilde{\omega}_{\mathcal{L}} \qquad \text{by the projection formula} \\ &= (-1)^{r(\mathcal{L})}\beta(\mathcal{L}) + (-1)^{r(\mathcal{L}/e)}\beta(\mathcal{L}/e) \quad .\end{aligned}$$

Since $r(\mathcal{L} - e) = r(\mathcal{L}) = r(\mathcal{L}/e) + 1$, the statement follows. □

If e is an isthmus, an extra $-H_1'$ term appears in $\alpha_*(\widetilde{\omega}_{\mathcal{L}})$; if $\mathcal{L} \neq [0, e]$, the argument in this proof gives $\beta(\mathcal{L} - e) = -\beta(\mathcal{L}) + \beta(\mathcal{L}/e)$ (since in this case $r(\mathcal{L}-e) = r(\mathcal{L}/e) = r(\mathcal{L}) - 1$); and since $\mathcal{L} - e = \mathcal{L}/e$ if e is an isthmus, it follows that $\beta(\mathcal{L}) = 0$ in this case. If $\mathcal{L} = [0, e]$ itself is an isthmus, then $e = 1$ and the extra H_1' term kicks in, giving $\beta([0, e]) = 1$ as it should (cf. [W2], 7.3.1(b)).

The vanishing of the beta invariant in the presence of an isthmus is a particular case of the fact that the invariant vanishes on direct sums. This will follow in a moment from Corollary 3.14 below; it could also be checked easily by studying the deletion map $V_{\mathcal{L}_1\times\mathcal{L}_2} \to V_{\mathcal{L}_1}$. We leave this as a pleasant exercise to the reader (although the conventional proof, which simply takes the derivative of a product, is much easier!)

Proposition 3.9(3) translates into:

Corollary 3.13. *If $x \in \mathcal{L}$ is modular, and $y < x$, then*

$$(-1)^{r(x)-r(y)}\beta([y, x]) = \sum_{z\wedge x=y} B(z)$$

Proof. Writing $\widetilde{\omega}$ modulo F and using (3) from Proposition 3.9 yields

$$i^*(\sum_{z\in\mathcal{L}} B(z)\,H_z) = \sum_{0\le y\le x} B(y)''\,H_y'' - (r(1) - r(x))H_x'' \quad .$$

But Proposition 3.5 says

$$\begin{aligned}i^*(\sum_{z\in\mathcal{L}} B(z)\,H_z) &= \sum_{z\in\mathcal{L}} B(z)\,H''_{z\wedge x} \\ &= \sum_{0\le y\le x}\left(\sum_{z\wedge x=y} B(z)\right) H_y'' \quad ;\end{aligned}$$

comparing the two expressions and dotting with γ_y'' for $y < x$ gives the statement. □

Setting $y = 0$ in this corollary and isolating the term $B(0)$ gives:

$$(*) \qquad (-1)^{r(1)}\beta(\mathcal{L}) = (-1)^{r(x)}\beta([0, x]) - \sum_{z\neq 0,\, z\wedge x=0} B(z)$$

if $x \neq 0$ is modular. The following statement follows:

Corollary 3.14. *If $x \in \mathcal{L}, x \neq 0$ is modular, then*

$$\beta(\mathcal{L}) = (-1)^{r(1)-r(x)} \beta([0,x]) \sum_{y \wedge x = 0} \mu(0,y)$$

Proof. Induction: if $r(\mathcal{L}) = r(x)$ then $x = 1$ and there is nothing to prove; next, the terms in the summation in (*) are (up to sign) beta invariants of lattices of lower rank, so we may apply the statement to them (because $z \vee x$ is modular in $[z,1]$ and $[z, z \vee x] \cong [0,x]$ if $z \wedge x = 0$); doing so yields the induction step. □

The statement of the last corollary is a 'modular decomposition' expression for the beta invariant. It could also be derived easily from Stanley's modular factorization theorem; the above proof, however, seems more direct. For $x = e$ a rank-1 element of $\mathcal{L}$ (thus automatically modular), the corollary says

$$\beta(\mathcal{L}) = (-1)^{r(\mathcal{L})-1} \sum_{y \not\geq e} \mu(0,y) \quad ,$$

that is 7.3.1(d) in [W2]. For $x = (1,0)$ in $\mathcal{L}_1 \times \mathcal{L}_2$, Corollary 3.14 implies the vanishing of the beta invariant on direct sums: indeed in this case $y \wedge x = 0 \iff y \in \{0\} \times \mathcal{L}_2$, so $\sum_{y \wedge x = 0} \mu(0,y) = 0$ if $\mathcal{L}_2 \neq 0$.

One last observation: the last formula can also be written

$$B(0) - \sum_{y \geq e} \mu(0,y) = 0 \quad ,$$

which translates back into $(\widetilde{\omega}_{\mathcal{L}} + \sum_{y \geq e} E_y) \cdot \gamma_0 = 0$ by Proposition 3.10 and Lemma 2.2, or in fact into

$$(\widetilde{\omega}_{\mathcal{L}} + \sum_{y \geq e} E_y) \cdot \gamma_z = 0 \qquad \text{if } z \not\geq e$$

(once more by compatibility with contractions). In other words, writing

$$\widetilde{\omega}_{\mathcal{L}} + \sum_{y \geq e} E_y = \sum_{z \in \mathcal{L}} a_z H_z$$

modulo F, we find $a_z = 0$ necessarily for all $z \not\geq e$. This also has a pretty geometric explanation. By induction and compatibility with contractions, it is enough to show $\sum_{z \not\geq e} a_z = 0$. Now consider the hypersurface D obtained by taking the union of a general representative of H_0 and of all E_y with $y \geq e$. D is non-singular along H_0 and E_e away from certain divisors of these latter; the complements $\widetilde{H}_0$, $\widetilde{E}_e$ of these divisors in H_0, E_e are isomorphic to the complement of sets of codimension at least 2 in $\mathbb{P}^{n-1}$: so their Pic is $\mathbb{Z}$, generated by a hyperplane class h, and their canonical divisor is $-n\,h$. Now

$$\omega_{\mathcal{L}} + H_0 + \sum_{y \geq e} E_y = \omega_{\mathcal{L}} + D$$

restricts, by adjunction, to the canonical divisors of $\widetilde{H}_0$, $\widetilde{E}_e$ on each of these: so restricting $\widetilde{\omega}_{\mathcal{L}} + \sum_{y \geq e} E_y = \omega_{\mathcal{L}} + D + (n - r(1))H_1$ to $\widetilde{H}_0, \widetilde{E}_e$ and reading the coefficient of h gives respectively

$$\sum_{z \geq 0} a_z = -n + (n - r(1)) = -r(1) \quad ,$$
$$\sum_{z \geq e} a_z = -n + (n - r(1)) = -r(1)$$

(H_0 meets all H_z, while E_e only meets the H_z with $z \geq e$). Comparing the two expressions gives $\sum_{z \not\geq e} a_z = 0$, as needed.

§4. Degrees of matroid varieties; Segre classes

The line bundles associated with the divisors

$$S(m) = m^{r(1)} H_0 - \sum_{x \in \mathcal{L}, x \neq 0} m^{r(1) - r(x)} E_x$$

introduced in §2 on the n-dimensional matroid variety $V^n = V_{\mathcal{L}}$ define for $m > 0$ rational maps

$$\sigma_{m,n} : V^n \dashrightarrow \mathbb{P}^{N(m,n)}$$

to a projective space. We will write σ_m for short (disregarding n) because these maps are compatible with the natural inclusions $V^n \subset V^{n+1} \subset \cdots$ discussed in §2.1 (since the $S(m)$ are).

Example. For $m = 1$ we have $S(1) = H_1$ modulo F, and it follows that σ_1 is the blow-up map $V^n \to \mathbb{P}^n$ followed by the projection with center $1 \in \mathcal{L}$.

Now define for $m > 0, n > r(\mathcal{L})$:

$$d(m,n) = (\deg \sigma_m)(\deg \overline{\sigma_m(V^n)})$$

So $d(m,n) = 0$ if $\dim \sigma_m(V^n) < n$, while $d(m,n)$ is just the degree of $\overline{\sigma_m(V^n)}$ if σ_m is generically injective; for example $d(1,n) = 0$ for all n.

At this stage we do not know a general formula for $d(m,n)$. In a sense that is not surprising because, as we will show in a moment, the characteristic polynomial of the original matroid can be recovered from a fraction of the information carried by the $d(m,n)$'s. More precisely, let $\{a\}_n$ denote the smallest nonnegative residue of a modulo n; then Theorem 4.4 will imply:

Let $d(m,n)$ be the numbers defined above for the cycle matroid of a simple graph G; and let c be the number of components of G. Then the value of the chromatic polynomial of G at $m > 0$ equals

$$m^c \{d(m,n)\}_n \quad ,$$

where n is an arbitrary sufficiently large prime.

Also, observe that the V^n's are birational to $\mathbb{P}^n$ (via the blow-up map), so that σ_m and the $d(m,n)$ could be defined starting from the original $\mathbb{P}^n$ in which

$\mathcal{L}$ is embedded, thus bypassing the blow-up construction. The right language to express this is that of *Segre classes:* we will show that the $d(m,n)$ are determined by the Segre classes of specific subschemes of $\mathbb{P}^n$ supported on $1 \in \mathcal{L}$. Advances in the theory of Segre classes could thus be relevant to problems of graph coloring!

In this section we say for short that a matroid is 'nice' if it belongs to the class defined in Proposition 2.5: that is, if all its geometric contractions have characteristic polynomials with nonnegative value at positive integers. In particular, for nice matroids the divisor

$$\overline{S}(m) = \sum_{x\in\mathcal{L}} p(\mathcal{L}/x, m)\, H_x$$

introduced in §2 is generated by global sections for all $m > 0$. Thus graphical matroids, for example, are nice in this sense.

Lemma 4.1. *For nice matroids*

(1) *the σ_m's are in fact regular maps;*
(2) *the pull-back of the hyperplane class via σ_m is $\overline{S}(m)$.*

Proof. $\overline{S}(m) = S(m)$ modulo F: thus the rational maps defined by $\overline{S}(m)$ and $S(m)$ agree on a non-empty open subset of V^n (the complement of the F divisors), hence they are the same. Now $\overline{S}(m)$ is globally generated for nice matroids and $m > 0$, so the map is regular and $\overline{S}(m)$ is the hyperplane section. □

(2) implies:

Corollary 4.2. *For nice matroids: $d(m,n) = \overline{S}(m)^n$ (the n-th self-intersection of $\overline{S}(m)$ in V^n).*

Now computing $\overline{S}(m)^n$ is a challenge. The following trivial observation is our only tool:

Lemma 4.3. $H_0^n = 1$; $H_x^n = 0$ *for* $x \neq 0$.

Proof. H_0 is the pull-back of the hyperplane from $\mathbb{P}^n$ via the blow-up maps, so the first formula follows from the projection formula. The second follows from the remark following Proposition 2.4: the intersection of $n+1-r(x) \le n$ general representatives of H_x is empty. □

Still, this is enough to obtain the result mentioned in the introduction:

Theorem 4.4. *If $\mathcal{L}$ is the lattice corresponding to a nice matroid (e.g., a graphical matroid) and $n \ge r(\mathcal{L})$ is a prime number, then*

$$p(\mathcal{L}, m) \equiv d(m,n) \pmod{n}$$

In particular, let $\{a\}_n$ denote the smallest nonnegative residue of a modulo n. Then:

Corollary 4.5. *If $\mathcal{L}$ corresponds to a nice matroid,*

$$p(\mathcal{L},m)=\{d(m,n)\}_n \qquad \text{for all primes } n\gg 0$$

For graphs, the corollary implies the statement in italics given earlier in this section, by the relation between the chromatic polynomial of a graph and the characteristic polynomial of its cycle matroid.

Proof of the Theorem. From Corollary 4.2 and Theorem 2.3

$$\begin{aligned} d(m,n)=\overline{S}(m)^n&=\Big(\sum_{x\in\mathcal{L}}p(\mathcal{L}/x,m)\,H_x\Big)^n\\ &\equiv\sum_{x\in\mathcal{L}}(p(\mathcal{L}/x,m)\,H_x)^n \pmod n &&\text{since } n \text{ is prime}\\ &\equiv p(\mathcal{L},m)^n \pmod n &&\text{by Lemma 4.3}\\ &\equiv p(\mathcal{L},m) \pmod n &&\text{by Fermat's little theorem.} \quad\square\end{aligned}$$

The $d(m,n)$ can alternatively be obtained in terms of the Segre classes of subschemes of $\mathbb{P}^n$ supported on $1\in\mathcal{L}$, whose definition can be given without reference to the rest of the construction. For each $m>0$, consider the subscheme $X(m,n)$ of $\mathbb{P}^n$ defined by the intersection of all degree-$m^{r(1)}$ hypersurfaces satisfying the multiplicity prescription mentioned in the end of §1—in short, multiplicity $m^{r(1)-r(x)}$ along x for all $x\neq 0$ in $\mathcal{L}$ (such hypersurfaces do exist for nice matroids: map the general representative of $\overline{S}(m)$ down to $\mathbb{P}^n$; and their intersection is clearly supported on the maximal subspace $1\in\mathcal{L}$). Next, let $s_0(m,n)$ be the *degree of the zero-dimensional component of the Segre class* $s(X(m,n),\mathbb{P}^n)$ (see [F], Chapter 4, for the notion and properties of Segre classes). The result is then

Theorem 4.6. *If $\mathcal{L}$ is the lattice corresponding to a nice matroid (e.g., a graphical matroid) and $n\geq r(\mathcal{L})$ is a prime number, then*

$$p(\mathcal{L},m)\equiv m^{r(\mathcal{L})}-s_0(m,n) \pmod n$$

Thus, the characteristic polynomial can be recovered in terms of these numbers as well. Also, we note explicitly that the statement of the four-color-theorem translates into:

for a planar graph G, there exists a prime n such that $s_0(4,n)\not\equiv 4^r \pmod n$, where r is the number of edges in a spanning forest of G.

Theorem 4.6 follows from the following relation between the $d(m,n)$ and the above Segre classes:

Lemma 4.7. *For $m>0$ and $n>r(1)$:*

$$d(m,n)=\left(m^{r(1)}\right)^n-\int_{X(m,n)}(1+m^{r(1)}H)^n\cap s(X(m,n),\mathbb{P}^n)$$

where H is the hyperplane class in $\mathbb{P}^n$, and $\int$ denotes degree in the sense of [F], §1.4.

Proof. The linear system defined by the hypersurfaces of $\mathbb{P}^n$ satisfying the multiplicity prescription defines a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^{N(m,n)}$. Now we claim that this map, composed with the blow-up sequence defining V^n, gives the map σ_m defined at the beginning of this section: this follows from Lemma 4.1, since the proper transform of the hypersurfaces has class $\overline{S}(m)$ in V^n. Then applying Proposition 4.4 in [F] gives the statement. □

To prove Theorem 4.6, just read the Lemma modulo n and apply Theorem 4.4.

Here is a table of $s_0(m,n)$ for the complete graph on three vertices:

$s_0(m,n)$	$m=2$	3	4	5
$n=3$	10	58	160	334
4	30	213	726	1821
5	74	692	3020	9308
6	166	2143	12226	46795
7	354	6510	49080	234282

We believe the $s_0(m,n)$ might in general be easier to control than the $d(m,n)$.

To conclude, we mention that yet another congruence result similar to Theorems 4.4, 4.6 can be stated in terms of Fulton's *canonical classes* (4.2.6(a) in [F]). For this, denote by $X_H(m,n)$ the general hyperplane section of $X(m,n)$, and by $c_0(m,n)$ the degree of $c_0(X(m,n)) - c_0(X_H(m,n))$ (notations as in [F], Example 4.2.6); then one can show

$$c_0(m,n) \equiv s_0(m,n) \pmod{n}$$

for n prime. Unfortunately, few properties of Fulton's canonical classes are known as yet.

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