Zero-Dimensionality and Products of Commutative Rings*

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In March of 1988, Bill Heinzer visited Florida State University to give us a couple of talks. During his visit, he and I discussed a problem that he had assigned in a recent graduate course. In essence, the problem asked the following question (Q):

(Q) *If R is a Noetherian ring and n is a positive integer, is the set*

$$\mathcal{I}_n = \{I \ : \ I \text{ is an ideal of } R \text{ and } |R/I| = n\}$$

finite?

I found our proof that (Q) has an affirmative answer to be interesting, in that it involved concepts that I did not initially anticipate being a part of a proof. Among these unanticipated factors were products of commutative rings and questions of integrality concerning such products. Our proof led us into a series of questions that we considered to be worthy of investigation. We have been able to answer

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* This paper contains results presented in the first 1994 Barrett Lecture, given on April 7. Most of its results come from the paper [GH4] (see the list of references for the paper *Background and Preliminaries* ... in this volume), and represent work done jointly with W. Heinzer.

some, but far from all, of these questions. Our successes, and partial successes, have led to five papers [GH 3–7] that focus on questions involving zero-dimensional commutative rings. The 1994 Barrett Lectures were based on this group of papers. But back to question (Q). It turns out that P. Samuel had already published a solution to (Q) in [S, Prop. 13]. While both Bill and I were familiar with [S], we did not make the association between its theme, Euclidean rings, and (Q) when we were discussing the latter in 1988. Dan Anderson pointed out the connection to us in March 1993, after [GH4] had already been published sans any reference to [S]. In light of our work in [GH 3–9], I'm glad that an association between (Q) and [S] did not occur to us in 1988, for I have wondered whether we would have worked on zero-dimensional commutative rings at all had we remembered that Samuel had already answered (Q).

We give a proof below (Result 3) that (Q) has an affirmative answer. As already indicated, other proofs appear in [S, Prop. 13] and in [GH4, Thm. 2.2]. While no two of these proofs are identical, all three involve products of zero-dimensional rings. The proof of Result 3 uses two lemmas. In the statement of Lemma 1, a ring R is said to be *hereditarily zero-dimensional* if each subring of R is zero-dimensional.

Lemma 1. *If $\{R_a\}_{a\in A}$ is a family of finite rings such that $\{|R_a|\}$ is bounded, then $R = \prod_{a\in A} R_a$ is hereditarily zero-dimensional.*

Proof. If $n \in \mathbf{Z}^+$ is an upper bound of $\{|R_a|\}$, then the characteristic of R divides $n!$, and hence the prime subring π of R is finite. Therefore π is zero-dimensional, and to show that R is hereditarily zero-dimensional, it suffices to show that R is integral over π. We show, in fact, that there exists a single monic polynomial $h(X)$ in $\mathbf{Z}[X]$ that vanishes on R. First we consider the case of a finite ring R_a. If $r \in R$, not all powers of r are distinct, so there exist $m_r, n_r \in \mathbf{Z}^+$ with $m_r < n_r$ such that $r^{n_r} = r^{m_r}$. Then $h_r = X^{n_r} - X^{m_r} \in \mathbf{Z}[X]$ is monic and vanishes on r. If $R_a = \{h_{r_1}, \ldots, h_{r_k}\}$, $h_a = \prod_{i=1}^{k} h_{r_i}$ is a monic polynomial over $\mathbf{Z}$ that vanishes on R_a. For any positive integer N, there exist only finitely many isomorphism classes of rings of order N; since $\{|R_a|\}$ is bounded, it follows that a set of isomorphism class representatives of the family $\{R_a\}$ is finite — say $\{R_{a_i}\}_{i=1}^{v}$ is such a set. Then $h = \prod_{i=1}^{v} h_{a_i}$ is a monic polynomial over $\mathbf{Z}$ that vanishes identically on R. ∎

Lemma 2. *Suppose $\{J_b\}_{b\in B}$ is a nonempty family of ideals of the Artinian ring T. Then $J = \cap_{b\in B} J_b$ is a finite subintersection of the family $\{J_b\}$ — that is, there exist elements $b_1, \ldots, b_k \in B$ such that $J = \cap_{i=1}^{k} J_{b_i}$.*

Proof. Choose $b_1 \in B$. If $J = J_{b_1}$, take $k = 1$. Otherwise, there exists $b_2 \in B$ such that $J_{b_1} > J_{b_1} \cap J_{b_2}$. If $J = J_{b_1} \cap J_{b_2}$, we take $k = 2$. Otherwise we continue the process, noting that the Artinian condition in T implies that the process terminates after finitely many steps. ∎

If I is an ideal of a ring S, the *norm* of I is defined to be cardinality of the residue class ring S/I.

Result 3. If R is Noetherian ring and n is a positive integer, then only finitely many ideals of R have norm n.

Proof. Let $\mathcal{I}_n$ be the set of ideals of R of norm n. We assume that $\mathcal{I}_n = \{I_a\}_{a \in A}$ is nonempty. Let $I = \cap_{a \in A} I_a$. Then R/I is imbedded in $S = \prod_{a \in A}(R/I_a)$, and S is hereditarily zero-dimensional by Lemma 1. Therefore R/I is a zero-dimensional Noetherian ring, hence an Artinian ring. By Lemma 2, $I = \cap_{i=1}^{k} I_{a_i}$ is a finite subintersection of the family $\{I_a\}$. But a well-known theorem of Pontryagin shows that a finite intersection of subgroups of finite index is of finite index, so R/I is finite. (Alternately, R/I is finite because it is imbedded in $\prod_{i=1}^{k}(R/I_{a_i})$, a ring of cardinality n^k.) Consequently, the set $\{I_a/I\}_{a \in A}$ is finite, and since the map $I_a \to I_a/I$ is injective, $\{I_a\}$ is also finite, as asserted. ∎

We make several remarks in connection with the three preceding results. First, if the notation and hypothesis are as in the statement of Lemma 1, the proof of that result shows that there exists a fixed monic polynomial $h \in \mathbf{Z}[X]$ that vanishes identically on R. A ring T with this latter property is hereditarily zero-dimensional (the characteristic of T is nonzero since no monic polynomial over $\mathbf{Z}$ vanishes identically on $\mathbf{Z}$), but the converse fails. For example, if $F_1 < F_2 < \ldots$ is a strictly increasing sequence of finite fields of characteristic p, the subring T of $\prod_{i=1}^{\infty} F_i$ consisting of all eventually constant sequences $\{a_i\}_1^{\infty}$ is integral over its prime subring $\pi \simeq \mathbf{Z}_p$, and hence T is hereditarily zero-dimensional. However, no monic polynomial over $\mathbf{Z}$ vanishes identically on T. We show in Corollary 9 that the converse is valid for a ring of the form $\prod R_a$, where each R_a is finite.

Clearly the condition considered in Lemma 2 characterizes Artinian rings. We remark that a Noetherian ring R has the property that the intersection of an arbitrary family of ideals of R is a countable subintersection of that family (R. Hartshorne, personal communication). This property does not characterize

Noetherian rings; for example, there exist non-Noetherian integral domains with only countable many ideals.

The Noetherian hypothesis on R enters the proof of Result 3 at only one point, and yet, the hypothesis is crucial for the conclusion. For example, if $\{X_a\}_{a\in A}$ is an infinite set of indeterminates over the finite field F, if $D = F[\{X_a\}]$, if $M = (\{X_a\})$, and if $T = D/M^2$, then T is a zero-dimensional quasilocal ring with infinitely many ideals of norm $|F|^k$ for each integer $k > 1$.

Suppose $\{R_a\}_{a\in A}$ is a family of finite rings. The proof of Result 3 prompted us to ask the following questions.

(Q1) *Is $R = \prod R_a$ zero-dimensional?*

(Q2) *Under what conditions is $R = \prod R_a$ hereditarily zero-dimensional?*

In relation to (Q2), some conditions on the family is necessary. For example, the ring $\prod\{\mathbf{Z}_p\ :\ p \text{ is a prime integer}\}$ has characteristic 0, and hence its prime subring is one-dimensional.

It is fairly easy to see that R itself need not be zero-dimensional. We use $J(\cdot)$ and $N(\cdot)$, respectively, to denote the Jacobson radical and nilradical of a ring. If $\{S_b\}_{b\in B}$ is an arbitrary family of rings, it is routine to show that $J(\prod S_b) = \prod J(S_b)$ and $N(\prod S_b) \subseteq \prod N(S_b)$. Thus, if $R = \prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$, then $J(R) = \prod_{n=1}^{\infty}(p\mathbf{Z}/p^n\mathbf{Z})$, so $p \in J(R)$. However, $p \notin N(R)$ so $N(R) < J(R)$, and this implies that R is not zero-dimensional. Incidentally, the ring $\prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$ has proved to be a good test case for several questions Heinzer and I have considered concerning products of zero-dimensional rings. Our general experience has been that if we are able to answer a question for $\prod_1^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$, then we can extend the answer to an arbitrary product of zero-dimensional PIR's.

Maroscia [M, Prop. 2.6] has determined equivalent conditions for a product of zero-dimensional rings to be zero-dimensional.

Theorem 4. ([M, Prop. 2.6]) *Suppose $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings and let $R = \prod R_a$. The following conditions are equivalent.*

(1) R is zero-dimensional.

(2) $J(R) = N(R)$.

(3) $N(R) = \prod_{a\in A} N(R_a)$.

Proof. It is clear that (1) implies (2), and the implication (2)⇒(3) follows since $J(R) = \prod J(R_a) = \prod N(R_a)$. If (3) is satisfied, then $R/N(R) = \prod R_a/\prod N(R_a) \simeq$

$\prod_a (R_a/N(R_a))$. Each $R_a/N(R_a)$ is a von Neumann regular ring, and hence $\prod_a (R_a/N(R_a))$ is also von Neumann regular (see [BP, Result 3.2]). Therefore $0 = \dim(R/N(R)) = \dim(R)$. ∎

It is useful to have a reformulation of condition (3) of Theorem 4 that is expressed as an internal property of the family $\{R_a\}$. Such a reformulation is straightforward by using the concept of index of nilpotency, which we proceed to define.

Definition 5. If S is a ring and $x \in N(S)$, we define the *index of nilpotency* of x, denoted $\eta(x)$, to be the smallest positive integer k such that $x^k = 0$. The *index of nilpotency* of S, denoted $\eta(S)$, is defined to be $\sup\{\eta(x) \,:\, x \in N(S)\}$; we write $\eta(x) = \infty$ if $\{\eta(x) \,:\, x \in N(S)\}$ is unbounded.

Theorem 6. **([GH3, Thm. 3])** *If $R = \prod_{a \in A} R_a$, where each R_a is zero-dimensional, then R is zero-dimensional if and only if there exists $k \in \mathbf{Z}^+$ such that the set $\{a \in A \,:\, \eta(R_a) > k\}$ is finite.*

Proof. We show that the condition of Theorem 6 is equivalent to the condition $N(R) = \prod N(R_a)$ in Theorem 4. In the proof we represent elements of R as functions $f \,:\, A \to \cup R_a$ such that $f(a) \in R_a$ for each $a \in A$.

Assume that the condition in the statement of Theorem 6 is satisfied for the integer k, and let $\{a \in A \,:\, \eta(R_a) > k\} = \{a_1, \ldots, a_v\}$. If $f \in \prod_{a \in A} N(R_a)$ and if $n = \max\{k, \eta(f(a_1)), \ldots, \eta(f(a_v))\}$, then $f^n(a) = 0$ for each $a \in A$. Therefore $f^n = 0$, $f \in N(R)$, and $N(R) \supseteq \prod N(R_a)$, as we wished to show.

On the other hand, if the condition of Theorem 6 fails, we can choose a sequence $\{a_i\}_1^\infty$ of distinct elements of A such that $\eta(R_{a_i}) > i$ for each i. Choose $r_i \in N(R_{a_i})$ such that $r_i^i \neq 0$ for each i. If $f \in R$ is such that $f(a_i) = r_i$ for $i \in \mathbf{Z}^+$ while $f(a) = 0$ for each $a \in A - \{a_i \,:\, i \in \mathbf{Z}\}$, then $f \in \prod N(R_a)$, but $f^i \neq 0$ for each $i \in \mathbf{Z}^+$. Consequently, condition (3) of Theorem 4 fails, and this completes the proof. ∎

Index of nilpotency is also involved in our characterization (Theorem 8) of hereditarily zero-dimensional products, but before proving Theorem 8, we establish a lemma.

Lemma 7. *The ring T is hereditarily zero-dimensional if and only if $T/N(T)$ is hereditarily zero-dimensional.*

Proof. It is clear that $T/N(T)$ is hereditarily zero-dimensional if T is [BP, (3.7)]. Conversely, if $T/N(T)$ is hereditarily zero-dimensional and if S is a subring of T, then $(S+N(T))/N(T) \simeq S/(N(T)\cap S) = S/N(S)$ is zero-dimensional, and hence S is also zero-dimensional. ∎

Theorem 8. **([GH4, Thm. 4.9])** *If $R = \prod_{a\in A} R_a$, where each R_a is zero-dimensional, then R is hereditarily zero-dimensional if and only if each R_a is hereditarily zero-dimensional and there exists $k \in \mathbf{Z}^+$ such that the set*

$$B = \{a \in A \ : \ \eta(R_a) > k \text{ or there exists } P \in \operatorname{Spec}(R_a) \text{ such that } |R_a/P| > k\}$$

is finite.

Proof. Assume first that R is hereditarily zero-dimensional. Each R_a has the same property since each R_a is a homomorphic image of R. Suppose that no $k \in \mathbf{Z}^+$ exists such that the corresponding set B is finite. Since R is hereditarily zero-dimensional, Theorem 6 then implies that there exists a sequence $a_1, a_2, \ldots$ of distinct elements of A such that $\operatorname{Spec}(R_{a_i})$ contains an element P_i such that $|R_{a_i}/P_i| > i$. Let $K_i = R_{a_i}/P_i$. The ring $\prod_{i=1}^{\infty} K_i$ is a homomorphic image of R, so without loss of generality we assume that $A = \mathbf{Z}^+$ and $R = \prod_1^{\infty} K_i$. Since $\operatorname{char}(R) \neq 0$, the set $\{\operatorname{char}(K_i)\}_{i=1}^{\infty}$ is finite. Thus, by passage to a homomorphic image of R, we may assume that $\operatorname{char}(K_i) = p \neq 0$ and $|K_i| \geq p^i$ for each i. Then the prime subring π of R is isomorphic to $\mathbf{Z}_p$, and for each i, K_i contains an element c_i of degree at least i over $\mathbf{Z}_p$. Thus, the element $c = \{c_i\}_1^{\infty}$ of R is transcendental over π, $\dim \pi[c] = 1$, and this contradicts the assumption that R is hereditarily zero-dimensional. This establishes necessity of the stated condition on k and B.

Conversely, assume that the condition on k and B is satisfied. Theorem 6 shows that R is zero-dimensional, and hence $N(R) = \prod_{a\in A} N(R_a)$ by Theorem 4. To show R is hereditarily zero-dimensional, it suffices, by Lemma 7, to show that $R/N(R) \simeq \prod_{a\in A}(R_a/N(R_a))$ is hereditarily zero-dimensional. Thus, we assume without loss of generality that each R_a is reduced. Let $B = \{a_1, \ldots, a_m\}$. Then $R \simeq R_{a_1} \oplus \ldots \oplus R_{a_m} \oplus T$, where $T = \prod_{a\in A\setminus B} R_a$. Since each R_{a_i} is hereditarily zero-dimensional, it suffices to show that T also has this property [BP, (3.7)]. Thus we assume that $R = T$ so that each R_a is reduced with residue fields of cardinality at most k. Each R_a is canonically imbedded in the ring $S_a = \prod\{(R_a)_P \ : \ P \in \operatorname{Spec}(R_a)\}$, where each $(R_a)_P$ is a field of cardinality at most k. Hence R is imbedded in $S = \prod_{a \in A} S_a$, a product of fields of cardinality at most k. It follows from Lemma 1 that S, and hence R, is hereditarily zero-dimensional. ∎

Corollary 9. *If $\{R_a\}_{a \in A}$ is a family of finite rings such that $R = \prod_{a \in A} R_a$ is hereditarily zero-dimensional, then there exists a monic polynomial $h \in \mathbf{Z}[X]$ that vanishes identically on R.*

Proof. Choose $k \in \mathbf{Z}^+$ such that the set

$$B = \{a \in A \,:\, \eta(R_a) > k \text{ or there exists } P \in \operatorname{Spec}(R_a) \text{ such that } |R_a/P| > k\}$$

is finite, say $B = \{a_1, \ldots, a_m\}$. Then $R \simeq S \oplus T$, where $S = R_{a_1} \oplus \ldots \oplus R_{a_m}$ is finite and $T = \prod_{a \in A \setminus B} R_a$. The proof of Lemma 1 shows that there exists a monic polynomial f over $\mathbf{Z}$ that vanishes on S, so Corollary 9 reduces to the case where $R = T$. We have $\eta(T) \leq k$, and as in the proof of Theorem 8, $T/N(T)$ is imbedded in a product of fields of cardinality at most k. It follows from the proof of Lemma 1 that there exists a monic polynomial g over $\mathbf{Z}$ such that $g(T) \subseteq N(T)$, and hence $g^k \in \mathbf{Z}[X]$ is monic and vanishes identically on T. ∎

Theorem 4 provides an avenue for showing that question (Q1) has a negative answer, and Theorem 8 yields a satisfactory answer to (Q2). But more probing questions lie close at hand; in particular, we considered the following pair of questions.

(Q3) *If $\{R_a\}_{a \in A}$ is any family of rings* (not necessarily zero-dimensional), *what is* $\dim(R)$, *where $R = \prod_{a \in A} R_a$?*

(Q4) *How are general hereditarily zero-dimensional rings characterized?*

In considering (Q3), Bill and I initially thought of two items that were relevant: (1) J. Keisler in 1963 showed me a proof that a product of countably many copies of $\mathbf{Z}$ is infinite-dimensional, and (2) we recalled the result of Maroscia recorded in Theorem 4. Bill later remembered a 1983 conversation with M. Hochster and R. Wiegand in which they outlined a proof that our test-case product, $\prod_{n=1}^{\infty}(\mathbf{Z}/p^n\mathbf{Z})$, is infinite-dimensional. And in 1993, at the same time he pointed out the connection between Samuel's paper [S] and question (Q), Dan Anderson also mentioned that Aoyama in [A] had considered properties of $\mathrm{Spec}(\prod V_\lambda)$, where each V_λ is a rank-one discrete valuation domain. Indeed, all the necessary tools for showing that $\prod V_\lambda$ may be infinite-dimensional can be found in [A], but Aoyama did not provide such an example in [A], probably because the focus of his paper was the ring of entire functions, rather than the dimension of an infinite product of rings.

We provide an answer to (Q3) in Theorem 11 below. Our next result is a special case of Theorem 11. Proofs of Lemma 10 and Theorem 11 use the following statements, which are well-known (cf. [BP, (2.1)]).

(#) If x is a regular element of the ring R, then $\dim R > \dim(R/(x))$.

(##) $\dim(R_a \oplus \ldots \oplus R_n) = \sup\{\dim(R_i) \,:\, 1 \leq i \leq n\}$.

Statement (#) follows from the fact that a minimal prime of a ring consists of zero divisors, so P is not minimal in R for each prime ideal $P/(x)$ of $R/(x)$. And (##) is a consequence of the fact that each ideal of $R_1 \oplus \ldots \oplus R_n$ is of the form $I_1 \oplus \ldots \oplus I_n$, where I_j is an ideal of R_j.

Lemma 10. ([GH4, Prop. 3.2]) *Let $k \in \mathbf{Z}^+$ and let $\{R_a\}_{a\in A}$ be an infinite family of rings, each of dimension at least k. If $R = \prod_{a\in A} R_a$, then $\dim R \geq k+1$.*

Proof. By passage to an appropriate homomorphic image of R, we assume without loss of generality that $A = \mathbf{Z}^+$ and each R_a is an integral domain.

The proof is by induction on k. If $k = 1$, choose a nonzero nonunit $r_a \in R_a$ for each $a \in A$ and let $r = \{r_a^a\}_{a=1}^\infty \in R$. Then r is a regular element of R, so by (#) above, $\dim R > \dim(R/rR)$. However, $rR = \prod_{a=1}^\infty r_a^a R_a$, so $R/rR \simeq \prod_{a=1}^\infty (R_a/r_a^a R_a)$. Each $R_a/r_a^a R_a$ is zero-dimensional, and $\{r_a + r_a^a R_a\}_1^\infty$ is an element of $\prod_1^\infty N(R_a/r_a^a R_a)$ that is not nilpotent. By Theorem 4, $\dim(\prod_1^\infty (R_a/r_a^a R_a)) > 0$, and hence $\dim R > 1$.

Assume Lemma 10 is valid for $k = m-1$, where $m \geq 2$, and consider the case where $k = m$. For each $a \in A$, choose $P_a \in \mathrm{Spec}(R_a)$ such that $\dim(R_a/P_a) \geq m-1$, and choose a nonzero element $x_a \in P_a$. If $x = \{x_a\}_{a=1}^\infty \in R$, then x is regular in R and $R/xR \simeq \prod_1^\infty (R_a/x_a R)$. Since $\prod_1^\infty (R_a/P_a)$ is a homomorphic image of $\prod_1^\infty (R_a/x_a R_a)$, the induction hypothesis implies that $\dim(\prod_1^\infty (R_a/x_a R_a)) \geq m$. It then follows from (#) that $\dim R > m$. ∎

Theorem 11. **([GH4, Thms. 3.3 and 3.4])** *Suppose $\{R_a\}_{a\in A}$ is an infinite family of nonzero rings. Let $R = \prod_{a\in A} R_a$.*

(1) If $\dim R_a > 0$ for infinitely many elements $a \in A$, then $\dim R = \infty$.

(2) If $\dim R_a > 0$ for only finitely many elements $a_1, \ldots, a_t \in A$, then $\dim R = \sup\{\dim(R_{a_1}), \ldots, \dim(R_{a_t}), \dim(\prod_{b\in B} R_b)\}$, where $B = A - \{a_1, \ldots, a_t\}$.

(3) If each R_a is zero-dimensional, then $\dim R \in \{0, \infty\}$, and Theorem 4 gives equivalent conditions for R to be zero-dimensional.

Proof. (1): As in the proof of Lemma 10, by passage to a homomorphic image of R we may assume that $A = \mathbf{Z}^+$ and each R_a is an integral domain. To prove R is infinite- dimensional, we prove by induction on k that there exists a homomorphic image of R of the form $S^{(k)} = \prod_{a=1}^{\infty} S_a^{(k)}$, where each $S_a^{(k)}$ is an integral domain of dimension at least k. For $k = 1$ we can take $S_a^{(1)} = R_a$ for each $a \in \mathbf{Z}^+$. Assume the result is true for $k = t$, where $t \in \mathbf{Z}^+$. To extend to the case $k = t+1$, partition A into an infinite sequence $A_1, A_2, \ldots$ of infinite subsets A_i and let $T_a = \prod_{i\in A_a} R_i$ for each $a \in A$. Then $R \simeq \prod_{a\in A} T_a$. The induction hypothesis and Lemma 10 show that each T_a admits a homomorphic image $S_a^{(t+1)}$, where $S_a^{(t+1)}$ is an integral domain of dimension at least $t+1$. Then $S^{(t+1)} = \prod_{a=1}^{\infty} S_a^{(t+1)}$ is a homomorphic image of R, and this completes the argument by the principle of mathematical induction.

(2) follows immediately from (##) above.

(3): Assume that each R_a is zero-dimensional. We show that if $\dim R \neq 0$, then R is infinite-dimensional. Thus, Theorem 6 shows that the set $\{a \in A : \eta(R_a) > k\}$ is infinite for each $k \in \mathbf{Z}^+$. Using a traditional diagonal process, we can partition A into an infinite sequence $A_1, A_2, \ldots$ of infinite subsets A_i such that the set $\{a \in A_i : \eta(R_a) > k\}$ is infinite for all $i, k \in \mathbf{Z}^+$. Let $S_i = \prod_{a\in A_i} R_a$ for each $i \in \mathbf{Z}^+$. Then $R \simeq \prod_{i=1}^{\infty} S_i$, and Theorem 4 shows that $\dim(S_i) > 0$ for each i. It then follows from (1) that R is infinite-dimensional. ∎

A characterization of general hereditarily zero-dimensional rings, the subject of (Q4), is fairly easy to obtain. Recall that if R is a subring of S, then S is integral over R if and only if S/P is integral over $R/(P \cap R)$ for each $P \in \mathrm{Spec}(S)$ (cf. [GH4, Prop. 4.1]). Also recall that a field F is *absolutely algebraic* if it is algebraic over its prime subfield.

Theorem 12. **([GH4, Cor. 4.2 and 4.3])** *The following conditions on a ring S are equivalent.*

(1) *S is hereditarily zero-dimensional.*
(2) *S/P is an absolutely algebraic field of nonzero characteristic for each $P \in$ $\mathrm{Spec}(S)$.*
(3) char $S \neq 0$ *and S is integral over its prime subring.*

Proof. Assume that S is hereditarily zero-dimensional. If $P \in \mathrm{Spec}(S)$, the field S/P is hereditarily zero-dimensional, and this clearly implies that S/P is absolutely algebraic of nonzero characteristic. Therefore (1) implies (2).

(2)⇒(3): The hypothesis of (2) implies that S is zero-dimensional. If π denotes the prime subring of S, then S/P is integral over $\pi/(P \cap \pi)$ for each $P \in \mathrm{Spec}(S)$, and hence S is integral over π. Therefore, π is zero-dimensional and char $S \neq 0$.

The implication (3)⇒(1) is patent. ∎

It follows from Theorem 12 that a hereditarily zero-dimensional ring S is a directed union of Artinian subrings. This follows because, if π is the prime subring of S, then S is the directed union of the family $\{S_a\}$ of finitely generated ring extensions of π in S. Each S_a is Noetherian and zero-dimensional, hence Artinian. The converse fails — that is, a ring T that is a directed union of Artinian subrings need not be hereditarily zero-dimensional, although T itself is zero-dimensional. In the vein of Theorems 6 and 8, we state in Theorem 13 equivalent conditions in order that a product of rings should be a directed union of Artinian subrings. We omit the proof of Theorem 13 because of its similarity to the proof of Theorem 8 (see [GH4, Thm. 6.7]).

Theorem 13. *Let $\{R_a\}_{a \in A}$ be a family of rings and let $R = \prod_{a \in A} R_a$. Then R is a directed union of Artinian subrings if and only if each R_a has this property and there exists a positive integer k such that the set*

$$\{a \in A \ : \ \eta(R_a) > k \text{ or there exists } P \in \mathrm{Spec}(R_a) \text{ such that } |R_a/P| > k\}$$

is finite.

We conclude with two questions, labeled (P1) and (P2), that are related to material discussed in this paper.

(P1) *Suppose $\{R_a\}$ is a family of zero-dimensional rings such that the product $R = \prod R_a$ is infinite-dimensional.*

(a) Does R admit a homomorphic image S that is of positive finite dimension?

(b) If so, must S be one-dimensional?

This question is obviously probing more deeply into the nature of Spec(R). We have been able to show that (a) and (b) have affirmative answers in the test case $\prod(\mathbf{Z}/p^n\mathbf{Z})$ or, more generally, in the case where each R_a is a PIR [GH9, Cor. 1.7, Thm. 3.3].** We remark that the ring E of entire functions shares this property of $\prod(\mathbf{Z}/p^n\mathbf{Z})$ that each of its homomorphic images of finite positive dimension is of dimension one [H2]. Many of the basic properties of Spec(E) were established by Henriksen in [H2] using analytic methods, and a primary goal of Aoyama in his paper [A] mentioned earlier seems to have been to prove these properties algebraically, using infinite products of commutative rings.

(P2) *Generalize the result that in a Noetherian ring R, there exist only finitely many ideals of a fixed finite norm* (Result 3).

There are two directions in which a generalization might proceed. One direction would weaken the Noetherian hypothesis on R; another would consider infinite residue fields. In the first direction, a natural question asks whether Result 3 generalizes to the case of a coherent ring. This is easily seen to fail; a Prüfer domain is coherent, but may have uncountably many ideals of a fixed finite norm. At one point there were several specific questions in the second direction — for example, if R is Noetherian and c is an infinite cardinal, is it true that $|\{I : I \text{ is an ideal of } R \text{ such that } |R/I| \leq c\}| \leq c$? Most of these questions were settled by R. Heitmann (personal communication), who observed that, through use of a construction from his doctoral dissertation (see [He]), he could construct a principal ideal domain D with uncountably many maximal ideals, each with countable residue field. In such a domain D it is, in fact, true that (0) is the only ideal I of D such that $|D/I|$ is uncountable.

** Bill and I were able to answer (P1) during the summer of 1994. In relation to (a), we showed that R need not admit a homomorphic image of finite positive dimension. Concerning (b), we showed that for any $d \in \mathbf{Z}_0 \cup \{\infty\}$, there exists an R such that the set of integers realizable as the dimension of a nonzero homomorphic image of R is $\{k \in \mathbf{Z}_0 : 0 \leq k \leq d\}$. We expect our results to appear in a paper entitled *Infinite products of zero-dimensional commutative rings*; this paper is still under construction.