Global Integrability Theorems for A-Harmonic Tensors

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Abstract. We extend global integrability theorems for the gradients of A-harmonic functions to the exterior derivative of differential forms satisfying rather general nonhomogeneous elliptic equations. These include the usual A-harmonic equations. Geometric conditions on the boundary of the domains of integration imply a corresponding exponent of integrability. In the process we generalize the weak reverse Hölder inequality to such differential forms.

1. Introduction

In 1955 Walter Rudin proved the following result. See [12].

Theorem 1.1. *There exists a Blaschke product on the unit disk $\mathbb{D}$ such that*

$$\iint_{\mathbb{D}} |B'(z)|\, dx\, dy = \infty.$$

Hence there are bounded harmonic functions on $\mathbb{D}$ whose gradients fail to be L^1-integrable.

If one carefully examines the proof of Theorem 1.1 one discovers in fact that

$$\iint_{\mathbb{D}} |B'(z)|(\log(2 + |B'(z)|))^{-1}\, dxdy = \infty.$$

In contrast, the following result, Theorem 1.2, appears in [6].

First, we call u an A-harmonic function in Ω if u is in the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ and satisfies

$$\mathrm{div} A(x, \bigtriangledown u) = 0$$

in the distributional sense. Here and throughout this paper $\Omega \subset \mathbb{R}^n$ is an open set and $A(x, \xi)$ satisfies boundedness and ellipticity conditions determined by $p > 1$. See Section 2.

Theorem 1.2. *If u is a bounded A-harmonic function in a ball $B \subset \mathbb{R}^n$, then*

$$\int_B |\bigtriangledown u| (\log(2 + |\bigtriangledown u|))^{-1-\varepsilon} < \infty$$

for all $\varepsilon > 0$.

On the other hand, for more general domains, we mention Theorem 1.3.

Theorem 1.3 follows from calculations on p. 216 of [1]. Here $\mathrm{loc}\,\mathrm{Lip}_k(\Omega)$, $0 < k \leq 1$, are those functions in the usual Lipschitz space $\mathrm{Lip}_k(B)$ for any ball $B \subset \Omega$. The Whitney cube number condition bounds the number of Whitney cubes in Ω of a given size. See Section 2. The geometry of the boundary of Ω determines the exponent λ in this condition.

Theorem 1.3. *Suppose that Ω satisfies a Whitney cube number condition with exponent $\lambda < n$. If u is A-harmonic in Ω and if $u \in \mathrm{loc}\,\mathrm{Lip}_k(\Omega)$, $0 < k \leq 1$, then*

$$\int_\Omega |\bigtriangledown u|^q < \infty$$

for $q < (n - \lambda)/(1 - k)$.

Our main result, Theorem 1.4, generalizes Theorems 1.2 and 1.3 simultaneously to differential forms in $\mathbb{R}^n$. The natural limit of $\mathrm{loc}\,\mathrm{Lip}_k(\Omega)$ as $k \to 0$ is the space BMO. In

Theorem 1.2, boundedness of u can be replaced by the BMO condition. Moreover, with the logarithmic term in place we attain the endpoint $(n-\lambda)/(1-k)$ in Theorem 1.3. See Section 2 for the notation below. We show by examples that Theorem 1.4 gives the correct integrability exponent in the regular case when $\lambda = n-1$.

Theorem 1.4. *Suppose that the differential form $\omega \in W^1_{p,\mathrm{loc}}(\Omega, \Lambda)$ satisfies (2.5) and Ω satisfies a Whitney cube number condition with exponent λ, $0 \le \lambda < n$.*

If $\omega \in \mathsf{BMO}(\Omega, \Lambda)$ and if $n-\lambda \le p$, then

$$\int_\Omega |d\omega|^{n-\lambda}(\log(2+|d\omega|)^{-1-\varepsilon} < \infty$$

for all $\varepsilon > 0$.

If $\omega \in \mathrm{loc}\,\mathrm{Lip}_k(\Omega, \Lambda)$, and if $(n-\lambda)/(1-k) \le p$, then

$$\int_\Omega |d\omega|^{(n-\lambda)/(1-k)}(\log(2+|d\omega|))^{-1-\varepsilon} < \infty$$

for all $\varepsilon > 0$.

The definitions of $\mathsf{BMO}(\Omega, \Lambda)$ and $\mathrm{locLip}_k(\Omega, \Lambda)$ are in Definition 2.2.

Example 1.5. Let Ω be the Cartesian product of the unit disk $\mathbb{D}$ with $n-2$ copies of $(0,1)$, $\Omega = \mathbb{D} \times (0,1) \times \cdots \times (0,1)$. Define $U(x_1, x_2, \ldots, x_n) = u(x_1 + ix_2)$ where $u = \mathrm{Re}\, B(z)$ and $B(z)$ is the Blaschke product in Theorem 1.1. Then $U : \Omega \to \mathbb{R}$ is harmonic. Also Ω satisfies a Whitney cube number condition with $\lambda = n-1$. In view of the remark following Theorem 1.1,

$$\int_\Omega |\bigtriangledown U|(\log(2+|\bigtriangledown U|))^{-1} = \infty.$$

Furthermore since U is bounded, $U \in \mathsf{BMO}(\Omega, \Lambda^0)$. Hence for $\lambda = n-1$ we must have $\varepsilon > 0$ in Theorem 1.4 in the BMO case.

Example 1.6. For each $\frac{1}{2} < k < 1$ there is a quasiconformal mapping f of the complex plane $\mathbb{C}$ such that f is uniformly Hölder continuous with exponent k and

$$\begin{aligned} \frac{|f'(x+iy)|}{c} &\le |y|^{k-1} \\ &\le c|f'(x+iy)| \end{aligned}$$

for some constant c and for all $x + iy \in \mathbb{C}\backslash\mathbb{R}$. As such

$$\iint_{\mathbb{D}} |f'(x+iy)|^{1/(1-k)}(\log(2+|f'(x+iy)|))^{-1}\,dx\,dy = \infty.$$

An example is due to P. Koskela, [7], where the quasiconformal mapping is an extension of a quasisymmetric mapping of the real line onto a van Koch snowflake-type curve. Now since f is a quasiconformal mapping, the components of $f = (f_1, f_2)$ satisfy an equation of the form (2.5). Furthermore $f_i \in \operatorname{loc}\operatorname{Lip}_k(\mathbb{D}, \Lambda^0)$ for $i = 1, 2$. As such Theorem 1.4 states that

$$\int_{\mathbb{D}} |f'|^{1/(1-k)}(\log(2+|f'|)))^{-1-\varepsilon} < \infty$$

for all $\varepsilon > 0$.

2. Preliminaries

Let $e_1, e_2, \ldots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $\ell = 0, 1, \ldots, n$, the linear space of ℓ-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_\ell}$ corresponding to all ordered ℓ-tuples $I = (i_1, i_2, \ldots, i_\ell)$, $1 \le i_1 < i_2 < \ldots < i_\ell \le n$, is denoted by $\Lambda^\ell = \Lambda^\ell(\mathbb{R}^n)$. Thus, $\Lambda^0 = \mathbb{R}$ and $\Lambda^1 = \mathbb{R}^n$. The Grassman algebra $\Lambda = \oplus\Lambda^\ell$ is a graded algebra with respect to the exterior product. For $\alpha = \sum \alpha^I e_I \in \Lambda$ and $\beta = \sum \beta^I e_I \in \Lambda$, the inner product in Λ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

with summation over all ℓ-tuples $I = (i_1, \ldots, i_\ell)$ and all integers $\ell = 0, 1, \ldots, n$. We define the Hodge star operator $\star : \Lambda \to \Lambda$ by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

and

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$$

for all $\alpha, \beta \in \Lambda^\ell$, $\ell = 1, 2, \ldots, n$. The norm of $\alpha \in \Lambda$ is then

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbb{R}.$$

The Hodge star is an isometric isomorphism on Λ with $\star : \Lambda^\ell \to \Lambda^{n-\ell}$ and $\star\star(-1)^{\ell(n-\ell)} : \Lambda^\ell \to \Lambda^\ell$.

Throughout this paper Q is a cube in Ω with diameter $\operatorname{diam} Q$. We write $L^q(\Omega, \mathbb{R})$, $0 < q \leq \infty$, for the usual L^q space of real-valued functions with respect to Lebesgue measure. The norm of $f \in L^q(\Omega, \mathbb{R})$ is denoted

$$|f|_{q,\Omega} = \left(\int_\Omega |f(x)|^q \, dx \right)^{1/q}$$

for $0 < q < \infty$. We also write

$$L^q_{\text{loc}}(\Omega, \mathbb{R}) = \cap L^q(\Omega', \mathbb{R})$$

where the intersection is over all Ω' compactly contained in Ω. The Sobolev space $W^1_q(\Omega, \mathbb{R})$ is the subspace of $L^q(\Omega, \mathbb{R})$ whose distributional first derivatives are also in $L^q(\Omega, \mathbb{R})$. Similarly we have the local space $W^1_{q,\text{loc}}(\Omega, \mathbb{R})$.

A differential ℓ-form ω on Ω is a Schwartz distribution on Ω with values in $\Lambda^\ell(\mathbb{R}^n)$. We denote the space of differential ℓ-forms by $\mathcal{D}'(\Omega, \Lambda^\ell)$. We write $L^q(\Omega, \Lambda^\ell)$ for the ℓ-forms $\omega(x) = \sum_I \omega_I(x) \, dx_I = \sum \omega_{i_1 i_2 \cdots i_\ell}(x) \, dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_\ell}$ with $\omega_I \in L^q(\Omega, \mathbb{R})$ for all ordered ℓ-tuples I. Thus $L^q(\Omega, \Lambda^\ell)$ is a Banach space with norm

$$
\begin{aligned}
|\omega|_{q,\Omega} &= \Big(\int_\Omega |\omega(x)|^q dx \Big)^{1/q} \\
&= \Big(\int_\Omega \big(\sum_I |\omega_I(x)|^2 \big)^{q/2} dx \Big)^{1/q}.
\end{aligned}
$$

Similarly $W^1_q(\Omega, \Lambda^\ell)$ are those differential ℓ-forms on Ω whose coefficients are in $W^1_q(\Omega, \mathbb{R})$. The notations $W^1_{q,\mathrm{loc}}(\Omega, \mathbb{R})$ and $W^1_{q,\mathrm{loc}}(\Omega, \Lambda^\ell)$ are self-explanatory. We denote the exterior derivative by

$$d : \mathcal{D}'(\Omega, \Lambda^\ell) \to \mathcal{D}'(\Omega, \Lambda^{\ell+1})$$

for $\ell = 0, 1, 2, \ldots, n$. Its formal adjoint (the Hodge codifferential) is the operator

$$d^\star : \mathcal{D}'(\Omega, \Lambda^{\ell+1}) \to \mathcal{D}'(\Omega, \Lambda^\ell)$$

given by

$$d^\star = (-1)^{n\ell+1} \star d \star$$

on $\mathcal{D}'(\Omega, \Lambda^{\ell+1})$, $\ell = 0, 1, \ldots, n$. We require a version of the Poincaré inequality for differential forms. The following result appears in [5].

Theorem 2.1. *Let $\omega \in \mathcal{D}'(Q, \Lambda^\ell)$ be such that $d\omega \in L^q(Q, \Lambda^{\ell+1})$, then $\omega - \omega_Q$ is in $W^1_q(Q, \Lambda^\ell)$ and with $1 < q < \infty$,*

$$|\omega - \omega_Q|_{q,Q} \le C(n,q) \mathrm{diam}\, Q |d\omega|_{q,Q}.$$

Here ω_Q is an appropriate closed ℓ-form defined in [5]. When $\ell = 0$, ω_Q is the average value of ω over Q.

Next, using the above form ω_Q, we define the following spaces of forms. See [11].

Definition 2.2. Assume that $\omega \in L^1_{\text{loc}}(\Omega, \Lambda^\ell)$, $\ell = 0, 1, \ldots, n$. We write $\omega \in \mathsf{BMO}(\Omega, \Lambda^\ell)$ if

$$\sup_{\sigma Q \subset \Omega} |Q|^{-1} |\omega - \omega_Q|_{1,Q} < \infty \text{ for some } \sigma > 1. \tag{2.3}$$

Similarly, we write $\omega \in \operatorname{loc} \operatorname{Lip}_k(\Omega, \Lambda^\ell)$, $0 < k \leq 1$, if

$$\sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} |\omega - \omega_Q|_{1,Q} < \infty \text{ for some } \sigma > 1. \tag{2.4}$$

When ω is a 0-form, (2.3) is the classical definition of $\mathsf{BMO}(\Omega)$ which turns out to be independent of σ. The space $\mathsf{BMO}(\Omega, \Lambda) = \bigoplus_{\ell=0}^{n} \mathsf{BMO}(\Omega, \Lambda^\ell)$ has many similar properties to those of $\mathsf{BMO}(\Omega)$.

A continuous 0-form which satisfies (2.4) is in the usual space $\operatorname{loc} \operatorname{Lip}_k(\Omega)$. This result is in [9]. The space $\operatorname{loc} \operatorname{Lip}_k(\Omega)$ was studied in [2]. Again the condition (2.4) is independent of σ.

We consider solutions to equations of the form

$$d^* A(x, d\omega) = B(x, d\omega). \tag{2.5}$$

Here $A : \Omega \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^\ell(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^{\ell-1}(\mathbb{R}^n)$ are measurable mappings which satisfy the following assumptions:

$$\begin{aligned} |A(x,\xi)| &\leq a|\xi|^{p-1} \\ |B(x,\xi)| &\leq b|\xi|^{p-1} \\ A(x,\xi) \cdot \xi &\geq |\xi|^p \end{aligned} \tag{2.6}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^\ell(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $p, 1 < p < \infty$, is a fixed exponent associated with (2.5). The exponent p is used for this purpose throughout this paper. A solution to (2.5) is an element of the Sobolev space $W^1_{p,\text{loc}}(\Omega, \Lambda^{\ell-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle + \langle B(x, d\omega), \varphi\rangle = 0$$

for all $\varphi \in W^1_{p,\mathrm{loc}}(\Omega, \Lambda^{\ell-1})$ with compact support. When $B(x,\xi) \equiv 0$, such differential forms ω are called A-harmonic tensors, see [4], [5] and [15].

We write, for $u \in \Lambda^0(\mathbb{R})$,

$$u^+ = \max\{u, 0\} \quad \text{and}$$

$$u^- = \min\{u, 0\}.$$

Also

$$\omega^+ = \sum_I \omega_I^+ \, dx_I \quad \text{and}$$

$$\omega^- = \sum_I \omega_I^- \, dx_I.$$

Theorem 2.7. *Let ω be a solution to (2.5) and $\eta \in C_0^\infty(\Omega)$ with $\eta \geq 0$. There exists a constant C, depending only on a, b and p, such that*

$$\int_\Omega \left|d\omega^+\right|^p \eta^p \leq C \left(\int_\Omega \left|\omega^+\right|^p |\nabla \eta|^p + \int_\Omega \left|\omega^+\right|^p \eta^p \right). \tag{2.8}$$

The same is true for ω^-.

Proof. Using the test form $\varphi = -\omega^+ \eta^p$ and using the inequalities (2.6) we get

$$\int_\Omega \left|d\omega^+\right|^p \eta^p \leq C_1 \int_\Omega \left|\omega^+\right| |\nabla \eta| \left|d\omega^+\right|^{p-1} \eta^{p-1} + C_2 \int_\Omega \left|\omega^+\right| \left|d\omega^+\right|^{p-1} \eta^p.$$

Using Hölder's inequality, the above is

$$\leq C_1 \left(\int_\Omega \left|\omega^+\right|^p |\nabla \eta|^p \right)^{1/p} \left(\int_\Omega \left|d\omega^+\right|^p \eta^p \right)^{(p-1)/p}$$

$$+ C_2 \left(\int_\Omega \left|\omega^+\right|^p \eta^p \right)^{1/p} \left(\int_\Omega \left|d\omega^+\right|^p \eta^p \right)^{(p-1)/p}.$$

The calculations for ω^- are similar.

Theorem 2.9. *Let ω be a solution to (2.5) in Ω and $\eta \in C_0^\infty(\Omega)$ with $\eta \geq 0$. For each $q \geq 0$, there exists a constant C, depending only on a, b, p, q and n, such that*

$$\int_\Omega |\omega^+|^q |d\omega^+|^p \eta^p \leq C \int_\Omega |\omega^+|^{p+q} \left(|\nabla\eta|^p + \eta^p\right). \tag{2.10}$$

Inequality (2.10) holds also for ω^-.

Proof. The proof is the same as that of Theorem 2.11 in [11], but using instead (2.8).

Theorem 2.11. *Let ω be a solution to (2.5) in Ω, $\sigma > 1$ and $0 < s$, $t < \infty$. Then there exists a constant C, depending only on s, t, p, b, α, σ and n, such that*

$$|\omega|_{s,Q} \leq C|Q|^{(t-s)/ts} |\omega|_{t,\sigma Q}. \tag{2.12}$$

for all cubes Q with $\sigma Q \subset \Omega$.

Proof. This follows as the proof of Theorem 3.34 of [3] if one chooses the test functions η so that $\eta \leq C_0|\nabla\eta|$ over each cube, for some constant C_0. See [11]. This clearly can be done, for example with a "tent" over each cube, while maintaining the inequality $|\nabla\eta| \leq \frac{C}{(\sigma-1)\operatorname{diam} Q}$.

Theorem 2.13. *Let ω be a solution to (2.5). The following are equivalent.*

(a) $\omega \in \mathsf{BMO}(\Omega, \Lambda)$.

(b) $\sup\{|Q|^{(p-n)/pn}|d\omega|_{p,Q} \,|\, \sigma Q \subset \Omega\} < \infty$ *for some* $\sigma > 1$.

Similarly the following are equivalent.

(c) $\omega \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \Lambda)$.

(d) $\sup\{|Q|^{(p-pk-n)/pn}|d\omega|_{p,Q} \,|\, \sigma Q \subset \Omega\} < \infty$ *for some* $\sigma > 1$.

Proof. The proof of Theorem 2.13 is the same as the proof of Theorem 6.4 in [11]. One only needs the weak reverse Hölder inequality (2.12).

Also we make use of a Whitney decomposition W of Ω. See [14]. Our Whitney decomposition here is a family of closed dyadic cubes with disjoint interiors which satisfy

(a) $\Omega = \bigcup_{Q \in W} Q$

(b) $\operatorname{diam} Q \leq d(Q, \partial\Omega) \leq 4 \operatorname{diam} Q$

(c) $\frac{1}{4} \operatorname{diam} Q_2 \leq \operatorname{diam} Q_1 \leq 4 \operatorname{diam} Q_2$

when $Q_1 \bigcap Q_2 \neq \emptyset$. Here $d(Q, \partial\Omega)$ is the Euclidean distance between Q and the boundary of Ω, $\partial\Omega$.

Definition 2.14. Here $N(j)$ is the number of Whitney cubes with sidelength 2^{-j}.

We say that Ω satisfies a Whitney cube number condition with exponent λ if there is a constant M such that

$$N(j) \leq M 2^{\lambda j}.$$

Balls and cubes satisfy a Whitney cube number condition with $\lambda = n - 1$. To state a result in [10], we define the λ-dimensional Hausdorff type premeasure obtained by covering a set $E \subset \mathbb{R}^n$ with balls of equal radii r,

$$H_\lambda(E, r) = \inf\{kr^\lambda | E \subset \bigcup_{i=1}^{k} B(x_i, r)\}.$$

Here $B(x_i, r) = \{y \big| \, |x_i - y| < r\}$ and the infinum is over all possible such covers.

Lemma 2.15. *Assume that Ω is bounded. If*

$$\limsup_{r \to 0} H_\lambda(\partial\Omega, r) < \infty,$$

then Ω satisfies a Whitney cube number condition with exponent λ.

Lemma 2.15 appears as Lemma 3.9 in [10].

It turns out that John domains (and hence bounded uniform domains) satisfy a Whitney cube number condition with an exponent $\lambda < n$ which depends only on the domain constants and the dimension, see [8] and [10]. Also we have Theorem 2.16, which appears on p. 123 of [13]. We call Ω, with $|\Omega| < \infty$ an L^q-averaging domain if there exists a constant c, independent of $u \in L^q_{\text{loc}}(\Omega)$, such that

$$|\Omega|^{-1/q}|u - u_\Omega|_{q,\Omega} \leq c \sup_{Q \subset \Omega} |Q|^{-1/q}|u - u_Q|_{q,Q}.$$

Here $u_\Omega = |\Omega|^{-1} \int\limits_\Omega u$. John domains are L^q-averaging domains for all $q \geq 1$. See [13].

Theorem 2.16. *If Ω is an L^q-averaging domain, $q \geq 1$, then*

$$\sum_j j^q 2^{-nj} N_j < \infty. \tag{2.17}$$

3. Proof of Theorem 1.4

To prove Theorem 1.4 we first fix a Whitney cube $Q_j \subset \Omega$ with side length 2^{-j} and define

$$U_j = \{x \in Q_j \big| \ |d\omega(x)| \geq 2^{(1-k)j/2}\}.$$

Then

$$\begin{aligned}\int\limits_{Q_j} |d\omega|^{(n-\lambda)/(1-k)}(\log(2 + |d\omega|))^{-1-\varepsilon} &\leq |Q_j| 2^{(n-\lambda)j/2} + \int\limits_{U_j} |d\omega|^{(n-\lambda)/(1-k)}(\log(2 + |d\omega|))^{-1-\varepsilon} \\ &\leq 2^{-nj}2^{(n-\lambda)j/2} + C(n, \lambda)j^{-1-\varepsilon} \int\limits_{Q_j} |d\omega|^{(n-\lambda)/(1-k)}.\end{aligned}$$

Combining Hölder's inequality with Theorem 2.13 we see that the above is

$$\begin{aligned}&\leq 2^{-(n+\lambda)/2j} + Cj^{-1-\varepsilon}|Q_j|^{1-n-\lambda/n} \\ &= 2^{-(n+\lambda)/2j} + Cj^{-1-\varepsilon}2^{-\lambda j}.\end{aligned}$$

Next let $R_j = \{\text{Whitney cubes } Q_j \subset \Omega \text{ with side length } 2^{-j}\}$. Then

$$\int\limits_{R_j} |d\omega|^{(n-\lambda)/(1-k)} (\log(2+|d\omega|))^{-1-\varepsilon} \le N(j)\{2^{-(n+\lambda)/2j} + Cj^{-1-\varepsilon}2^{-\lambda j}\}.$$

Now if Ω satisfies a Whitney cube number condition with exponent λ, then

$$N(j) \le M2^{\lambda j}.$$

Hence,

$$\int\limits_{R_j} |d\omega|^{(n-\lambda)/(1-k)} (\log(2+|d\omega|))^{-1-\varepsilon} \le C\{2^{-(n-\lambda)/2j} + j^{-1-\varepsilon}\}.$$

As such

$$\begin{aligned}\int\limits_{\Omega} |d\omega|^{(n-\lambda)/(1-k)} (\log(2+|d\omega|))^{-1-\varepsilon} &= \sum_{j=0}^{\infty} \int\limits_{R_j} |d\omega|^{(n-\lambda)/(1-k)} (\log(2+|d\omega|))^{-1-\varepsilon} \\ &< \infty.\end{aligned}$$

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