$L^s(\mu)$-Averaging Domains

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Abstract. In this paper we first introduce $L^s(\mu)$-averaging domains which are generalizations of L^s-averaging domains introduced by Susan G. Staples. We characterize $L^s(\mu)$-averaging domains using the quasihyperbolic metric. As applications, we prove norm inequalities for conjugate A-harmonic tensors in $L^s(\mu)$-averaging domains which can be considered as generalizations of the Hardy and Littlewood theorem for conjugate harmonic functions. Finally, we give applications to quasiconformal and quasiregular mappings.

1. Introduction

Susan G. Staples introduces L^s-averaging domains in [S]: a proper subdomain $\Omega \subset \mathbb{R}^n$ is called an L^s-averaging domain, $s \geq 1$, if there exists a constant C such that

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_\Omega|^s dm\right)^{1/s} \leq C \sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B |u-u_B|^s dm\right)^{1/s}$$

for all $u \in L^s_{loc}(\Omega)$. Here $|\Omega|$ is the n-dimensional Lebesgue measure of Ω. In [S] these domains are characterized in terms of the quasihyperbolic metric. In this paper, we generalize L^s-averaging domains to weighted averaging domains and obtain a similar characterization. Here and throughout $d\mu = wdx$ where w is a weight satisfying certain properties. We also obtain a global norm inequality for conjugate A-harmonic tensors in $L^s(\mu)$-averaging domains.

Throughout this paper, we always assume Ω is a connected open subset of $\mathbb{R}^n$. We write $\mathbb{R} = \mathbb{R}^1$. Balls are denoted by B and σB is the ball with the same center as B and with $\mathrm{diam}(\sigma B) = \sigma\mathrm{diam}(B)$. We write $B(x, R)$ for the ball centered at x of radius R. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ a.e. For a function u we denote the μ-average over B by

$$u_{B,\mu} = \frac{1}{\mu(B)}\int_B u d\mu.$$

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Definition 1.1. We call w a doubling weight and write $w \in D(\Omega)$ if there exists a constant C such that

$$\mu(2B) \le C\mu(B)$$

for all balls B with $2B \subset \Omega$. If this condition holds only for all balls B with $4B \subset \Omega$, then w is weak doubling and we write $w \in WD(\Omega)$.

Definition 1.2. Let $\sigma > 1$. We say that w satisfies a weak reverse Hölder inequality and write $w \in WRH(\Omega)$ when there exist constants $\beta > 1$ and $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^\beta dx\right)^{1/\beta} \le C\frac{1}{|B|}\int_{\sigma B} w dx \tag{1.3}$$

for all balls $B \subset \Omega$ with $\sigma B \subset \Omega$. We say that w satisfies a reverse Hölder inequality when (1.3) holds with $\sigma = 1$ and we write $w \in RH(\Omega)$.

In fact the space $WRH(\Omega)$ is independent of $\sigma > 1$, see [Bu] and [IN]. We next define Muckenhoupt weights.

Definition 1.4. We say that a weight w satisfies the A_r-condition, where $r > 1$, and write $w \in A_r(\Omega)$ when

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)} dx\right)^{r-1} < \infty,$$

where the supremum is over all balls $B \subset \Omega$. If w satisfies the A_r-condition for all balls B with $2B \subset \Omega$, we write $w \in A_r^{loc}(\Omega)$. Also we write $A_\infty(\Omega) = \cup_{r>1} A_r(\Omega)$ and $A_\infty^{loc}(\Omega) = \cup_{r>1} A_r^{loc}(\Omega)$.

It is well known that $w \in A_\infty(\Omega)$ if and only if $w \in RH(\Omega)$. See [Bu]. This is also true for $A_\infty^{loc}(\Omega)$ and $WRH(\Omega)$. Moreover $A_\infty^{loc}(\Omega) \subset WD(\Omega)$.

We establish some notations and definitions. Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbb{R}^n)$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by

$$< \alpha, \beta >= \sum \alpha^I \beta^I$$

with summation over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $\star : \wedge \to \wedge$ by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \text{ and } \alpha \wedge \star\beta = \beta \wedge \star\alpha = < \alpha, \beta > (\star 1)$$

for all α, $\beta \in \wedge$.

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = < \alpha, \alpha > = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star (-1)^{l(n-l)} : \wedge^l \to \wedge^l$. Let $0 < p < \infty$, we denote the weighted L^p-norm of a measurable function f over E by

$$||f||_{p,E,w} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p}.$$

A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbb{R}^n)$. We denote the space of differential l-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$||\omega||_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega (\sum_I |\omega_I(x)|^2)^{p/2} dx \right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathbb{R})$. The notations $W_{p,loc}^1(\Omega, \mathbb{R})$ and $W_{p,loc}^1(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$. From the results in [IL], if $\omega \in W_p^1(\Omega, \Lambda^\ell)$, then there exists a form $\omega_Q \in D'(Q, \wedge^l)$ giving the following Poincaré inequality

$$\|\omega - \omega_Q\|_{p,Q} \le C(n,p) \text{ diam } Q \|d\omega\|_{p,Q}.$$

It is this form appearing in Theorem B below. Here Q is a cube or a ball in Ω.

We consider solutions to equations of the form

$$d^\star A(x, d\omega) = 0, \tag{1.5}$$

where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1} \text{ and } < A(x,\xi), \xi > \ge |\xi|^p$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(R^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.5). A solution to (1.5) is an element of the Sobolev space $W^1_{p,loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega < A(x, d\omega), d\varphi >= 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

Suppose that u is a solution to (1.5) in Ω. Then by Poincaré lemma, at least locally in a ball B, there exists a form $v \in W^1_q(B, \wedge^{l+1})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$A(x, du) = d^\star v. \tag{1.6}$$

Definition 1.7. When u and v satisfy (1.6) in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic tensors in Ω. Below p and q will represent the structure constants above.

Example 1.8. We call u a p-harmonic function if u satisfies the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

with $p > 1$. Its conjugate in the plane is a q-harmonic function w, $p^{-1} + q^{-1} = 1$, which satisfies

$$|\nabla u|^{p-2}\nabla u = (\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}).$$

We see that u and $v = w\,dx\,dy$ are conjugate A-harmonic tensors in plane domains where $A(x, \nabla u) = |\nabla u|^{p-2}\nabla u$.

2. $L^s(\mu)$-averaging domains and their properties

We denote the space of functions that are locally L^s-integrable in Ω with respect to the measure μ by $L^s_{loc}(\Omega, \mu)$. Now we introduce $L^s(\mu)$-averaging domains.

Definition 2.1. We call a proper subdomain $\Omega \subset \mathbb{R}^n$ an $L^s(\mu)$-averaging domain, $s \geq 1$, if there exists a constant C such that

$$\left(\frac{1}{\mu(B_0)}\int_\Omega |u - u_{B_0,\mu}|^s d\mu\right)^{1/s} \leq C \sup_{4B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u - u_{B,\mu}|^s d\mu\right)^{1/s} \tag{2.2}$$

for some ball $B_0 \subset \Omega$ and all $u \in L^s_{loc}(\Omega, \mu)$. Here the supremum is over all balls $B \subset \Omega$ with $4B \subset \Omega$. The factor 4 here is for convenience and in fact these domains are independent of this expansion factor, see [S] and [SS].

Definition 2.3. The quasihyperbolic distance between x and y in Ω is given by

$$k(x,y) = k(x,y;\Omega) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial\Omega)} ds, \tag{2.4}$$

where γ is any rectifiable curve in Ω joining x to y. Here $d(z,\partial\Omega)$ is the Euclidean distance between z and $\partial\Omega$.

We remark that for any two points x and y in Ω there is a quasihyperbolic geodesic arc joining them [GO].

Theorem 2.5. *Assume that $w \in WRH(\Omega)$. If Ω is an $L^s(\mu)$-averaging domain, then*

$$\left(\frac{1}{\mu(\Omega)} \int_{\Omega} k(x,x_0)^s d\mu \right)^{1/s} \leq a \tag{2.6}$$

for each x_0 in Ω, where a depends only on n, s, $\mu(\Omega)$, $\mu(B(x_0, d(x_0,\partial\Omega)/2))$ and the constant C in (2.2).

Proof. Let B be any ball in Ω with center x_1 and radius r. We first show

$$\left(\frac{1}{\mu(B)} \int_{B} k(x,x_1)^s d\mu \right)^{1/s} \leq \alpha, \tag{2.7}$$

where α is a constant independent of B. We may assume that $x_1 = 0$. Then for each $x \in B$, $d(x,\partial\Omega) \geq r - |x|$ and using Hölder's inequality we obtain

$$\begin{aligned} \int_B k(x,0)^s d\mu &\leq \int_B \left(\log \frac{r}{r-|x|} \right)^s d\mu \\ &\leq \left(\int_B \left(\log \frac{r}{r-|x|} \right)^{sp} dx \right)^{1/p} \left(\int_B w^q dx \right)^{1/q} \end{aligned} \tag{2.8}$$

for $1/p + 1/q = 1$.

Next

$$\int_B \left(\log \frac{r}{r-|x|}\right)^{sp} dx = \int_0^r \left(\log \frac{r}{r-\rho}\right)^{ps} \rho^{n-1} d\rho \int_{S^{n-1}} d\sigma \le C_1 r^n. \tag{2.9}$$

As such by (2.8) and (2.9)

$$\int_B k(x,0)^s d\mu \le C_2 |B|^{1/p} \left(\int_B w^q dx\right)^{1/q}$$

$$= C_2 |B|^{1/p} |B|^{1/q} \left(\frac{1}{|B|}\int_B w^q dx\right)^{1/q}.$$

We now use the fact that $w \in WRH(\Omega)$ and with $q = \beta$, (1.3) gives

$$\int_B k(x,0)^s d\mu \le C_3 \int_{2B} w dx = C_3 \mu(2B) \le C_4 \mu(B)$$

for all balls B with $4B \subset \Omega$. The last inequality holds since $w \in WD(\Omega)$. Now (2.7) follows. Finally set $u(x) = k(x, x_0)$. By the result in [N1], p.53, we have

$$\left(\frac{1}{\mu(B)}\int_B |u(x) - u_{B,\mu}|^s d\mu\right)^{1/s} \le 2\left(\frac{1}{\mu(B)}\int_B |u(x) - u(x_1)|^s d\mu\right)^{1/s}$$

$$\le 2\left(\frac{1}{\mu(B)}\int_B k(x,x_1)^s d\mu\right)^{1/s}$$

$$\le 2\alpha.$$

Since Ω is an $L^s(\mu)$-averaging domain, we have

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u(x) - u_{B,\mu}|^s d\mu\right)^{1/s} \le 2\alpha C.$$

We have completed the proof of Theorem 2.5. □

Theorem 2.10. *Let $w \in WD(\Omega)$. If*

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega k(x,x_0)^s d\mu\right)^{1/s} \le a \tag{2.11}$$

for some fixed point x_0 in Ω, then Ω is an $L^s(\mu)$-averaging domain and the constant C in (2.2) depends only on n, s and a.

In order to prove Theorem 2.10, we need the following lemmas.

Lemma 2.12. *Let $w \in WD(\Omega)$ and $s \geq 1$. If*

$$\sup_{4B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u-u_{B,\mu}|^s d\mu\right)^{1/s} \leq C,$$

then there exist constants c_1 and c_2 such that

$$\mu(\{x\in B : |u(x)-u_{B,\mu}|>t\}) \leq c_1 e^{-c_2 t}\mu(B)$$

for all balls B with $4B\subset\Omega$.

Proof. It follows from Hölder's inequality and the hypotheses that

$$\sup_{4B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u-u_{B,\mu}| d\mu\right) \leq C.$$

Thus $u\in BMO(\Omega,\mu)$. So that by the John-Nirenberg Lemma for doubling weights [HKM], p. 313, we have

$$\mu(\{x\in B : |u(x)-u_{B,\mu}|>t\}) \leq c_1 e^{-c_2 t}\mu(B)$$

for $4B\subset\Omega$. □

Lemma 2.13. *Let $w \in WD(\Omega)$. Suppose that s and q are positive constants and that*

$$\mu(\{x\in B : |u(x)-u_{B,\mu}|>t\}) \leq s e^{-qt}\mu(B) \tag{2.14}$$

for each $t>0$ and each B with $4B\subset\Omega$. Then there exists a constant $C=C(s,q,n)$ such that

$$|u_{B(x),\mu}-u_{B(y),\mu}| \leq C(k(x,y)+1) \tag{2.15}$$

for all x and y in Ω. Here $B(x)$ is the ball $B(x,d(x,\partial\Omega)/4)$.

Proof. For each $z \in \Omega$, notice $4B(z) \subset \Omega$. Fix x and $y \in \Omega$ and choose a quasihyperbolic geodesic arc γ joining x and y in Ω. We define an ordered sequence of points $\{z_j\}$ on γ by induction as follows. First let $z_1 = x$. Next suppose that $z_1, \cdots, z_j$ have been defined and let $\beta_j = \gamma(z_j, y)$ denote that part of γ from z_j to y and γ_j the component of $\beta_j \cap B(z_j)$ which contains z_j. Define z_{j+1} as the other endpoint of γ_j. We simplify notation as follows: $B_j = B(z_j)$, $d_j = d(z_j, \partial\Omega)$, $r_j = d_j/4 =$ radius of B_j and $y = z_{m+1}$. From the definition of $\{z_j\}$, we have

$$|z_{j+1} - z_j| = r_j, j = 1, \cdots, m-1 \text{ and } |z_{m+1} - z_m| \leq r_m. \tag{2.16}$$

For $j = 1, \cdots, m-1$, pick $z_j' \in \partial\Omega$ so that $d(z_j, \partial\Omega) = |z_j - z_j'|$. If $z \in \gamma_j \subset B_j$, then

$$d(z, \partial\Omega) \leq |z - z_j'| \leq |z - z_j| + |z_j - z_j'| \leq 4d_j/3.$$

Hence

$$\int_{\gamma_j} \frac{1}{d(z, \partial\Omega)} ds \geq \frac{3}{4d_j} \int_{\gamma_j} ds \geq \frac{3|z_{j+1} - z_j|}{4d_j} = \frac{3}{16}.$$

Summing the above over j gives

$$\frac{3(m-1)}{16} \leq \sum_1^{m-1} \int_{\gamma_j} \frac{1}{d(z, \partial\Omega)} ds \leq \int_\gamma \frac{1}{d(z, \partial\Omega)} ds = k(x, y), \tag{2.17}$$

and we have $m < \infty$.

Consider now the relative size of neighboring balls. Fix j and choose $z, z' \in \partial\Omega$ so that $d_j = |z - z_j|$ and $d_{j+1} = |z' - z_{j+1}|$. Then $d_j \leq d_{j+1} + r_j$ and $d_{j+1} \leq d_j + r_j$, with the first inequality yielding

$$r_{j+1} = d_{j+1}/4 \geq (d_j - r_j)/4 \geq 3r_j/4,$$

and the second yielding $r_j \geq 3r_{j+1}/4$ if $r_{j+1} \geq r_j$. Hence

$$3r_j/4 \leq r_{j+1} \leq 4r_j/3. \tag{2.18}$$

Next we show that

$$\mu(B_j \cap B_{j+1}) > c_1(\mu(B_j) + \mu(B_{j+1})) \tag{2.19}$$

for a constant c_1. Fix j and let $v = (z_j + z_{j+1})/2$, $s = (\max(r_j, r_{j+1}))/8$, and B be the ball of radius s centered at v. In the case $r_j \geq r_{j+1}$ we have $s = r_j/8 \leq r_{j+1}/6$; similarly if $r_{j+1} \geq r_j$ we have $s \leq r_j/6$. Thus for $z \in B$ we obtain

$$|z - z_j| \leq |z - v| + |v - z_j| \leq s + \frac{1}{2}|z_j - z_{j+1}| \leq r_j,$$

a similar argument shows $|z - z_{j+1}| \le r_{j+1}$. Hence we conclude $B \subset B_j \cap B_{j+1}$. Since μ is doubling, we have

$$\mu(B_j \cap B_{j+1}) \ge \mu(B) \ge \mu(\frac{1}{8}B_j) \ge N_1(w)\mu(B_j),$$

and

$$\mu(B_j \cap B_{j+1}) \ge \mu(B) \ge \mu(\frac{1}{8}B_{j+1}) \ge N_2(w)\mu(B_{j+1}).$$

Let $c_1 = \frac{1}{2}\min\{N_1(w), N_2(w), 4s\}$, then we have

$$\mu(B_j \cap B_{j+1}) \ge \frac{1}{2}(N_1(w)\mu(B_j) + N_2(w)\mu(B_{j+1}))$$

$$\ge c_1(\mu(B_j) + \mu(B_{j+1})).$$

For $j = 1, 2, \cdots, m+1$, let

$$E_j = \{x \in B_j : |u - u_{B_j,\mu}| > t\},$$

where $t = (\log(2s/c_1))/q$. By (2.14) we obtain

$$\mu(E_j) \le c_1\mu(B_j)/2. \tag{2.20}$$

Combining (2.19) and (2.20) yields

$$\mu((B_j \cap B_{j+1})\backslash(E_j \cup E_{j+1})) > 0.$$

Therefore there exists $x \in (B_j \cap B_{j+1})\backslash(E_j \cup E_{j+1})$ and hence

$$|u_{B_j,\mu} - u_{B_{j+1},\mu}| \le |u - u_{B_j,\mu}| + |u - u_{B_{j+1},\mu}| \le 2t.$$

Summing and using (2.17) we obtain

$$|u_{B(x),\mu} - u_{B(y),\mu}| \le \sum_1^m |u_{B_j,\mu} - u_{B_{j+1},\mu}| \le 2mt$$

$$\le 32tk(x,y)/3 + 2t$$

$$\le \frac{32t}{3}(k(x,y)+1)$$

$$= \frac{32(\log(2s/c_1))}{3q}(k(x,y)+1)$$

$$= C(k(x,y)+1).$$

We have completed the proof of Lemma 2.13.

Lemma 2.21. *Let $w \in WD(\Omega)$. Suppose that $A, q > 0$ and that*

$$\mu(\{x \in B : |u(x) - u_{B,\mu}| > t\}) \le Ae^{-qt}\mu(B) \tag{2.22}$$

for each $t > 0$ and each ball B satisfying $4B \subset \Omega$. If

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega k(x, x_0)^s d\mu\right)^{1/s} \tag{2.23}$$

is finite for some point x_0 in Ω with $s \ge 1$, then

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u - u_{B(x_0),\mu}|^s d\mu\right)^{1/s} \tag{2.24}$$

is finite and depends only on A, q, n, s and the quantity (2.23).

Proof. Fix $x_0 \in \Omega$. Let $W = \{Q_i\}$ be a Whitney decomposition of Ω into closed cubes. As such $\Omega = \bigcup\limits_{i=1}^{\infty} Q_i$, $Q_i^0 \cap Q_j^0 = \emptyset$ when $i \ne j$ and there are constants c_1 and c_2 so that

$$c_1 d(Q_i, \partial\Omega) \le \text{diam } Q_i \le c_2 d(Q_i, \partial\Omega)$$

for all i. See [S]. Here $d(E, \partial\Omega)$ is the Euclidean distance between E and $\partial\Omega$. Let B_i be the ball with the same center as Q_i and with diam $B_i =$ diam Q_i. One can arrange the Whitney decomposition so that $4B_i \subset \Omega$ for all i. As such

$$\int\limits_\Omega |u - u_{B_0,\mu}|^s \, d\mu \le c_1 \left[\sum_i \int\limits_{B_i} |u - u_{B_i,\mu}|^s \, d\mu + \sum_i \int\limits_{B_i} |u_{B_0,\mu} - u_{B_i,\mu}|^s \, d\mu\right].$$

Now to estimate the first sum we use (2.22),

$$\begin{aligned}\int\limits_{B_i} |u - u_{B_i,\mu}|^s &= s\int\limits_0^\infty t^{s-1}\mu\left\{x \in B_i : |u - u_{B_i,\mu}| > t\right\} dt \\ &\le sA\int\limits_0^\infty t^{s-1}e^{-qt}\mu(B_i)dt \\ &\le c_2\mu(B_i).\end{aligned}$$

On the other hand, by using the estimate (2.15), we get

$$\int_{B_i} |u_{B_0,\mu} - u_{B_i,\mu}|^s \, d\mu \le c_3 \int_{B_i} (k(x_0,x)+1)^s \, d\mu$$

$$\le c_3 \int_{B_i} (k(x_0,x)+k(x_i,x)+1)^s \, d\mu$$

$$\le c_4 \int_{B_i} k(x_0,x)^s d\mu.$$

Hence, using the fact that $w \in WD(\Omega)$ and the properties of the Whitney cover,

$$\int_{\Omega} |u - u_{B_0,\mu}|^s \le c_5 \left[\sum_i \mu(B_i) + \sum_i \int_{B_i} k(x_0,x)^s d\mu \right]$$

$$\le c_6 \left[\mu(\Omega) + \int_{\Omega} k(x_0,x)^s d\mu \right].$$

Now, in view of (2.23), (2.24) follows.

We show below that John domains are $L^s(\mu)$-averaging domains when w satisfies the A_r-condition. For a reference on John domains see [M].

The proof of Theorem 2.31 is an application of Theorem 2.28. Theorem 2.28 appears as Theorem 3.15 in [N1].

Theorem 2.28. *Suppose that f and g are measurable functions in a John domain Ω, $q > 0$, and $w \in A_r(\Omega)$. If there exists a constant A such that*

$$\int_{Q} |f - f_{Q,\mu}|^q \, d\mu \le A \int_{\sigma Q} |g|^q \, d\mu \tag{2.29}$$

for all cubes Q with $\sigma Q \subset \Omega$, then there is a constant B, independent of f and g, such that

$$\int_{\Omega} |f - f_{Q_0,\mu}|^q \, d\mu \le B \int_{\Omega} |g|^q \, d\mu \tag{2.30}$$

for some cube $Q_0 \subset \Omega$.

Theorem 2.31. *Let Ω be a John domain and $w \in A_r$. Then Ω is an $L^s(\mu)$-averaging domain.*

Proof. Fix a point $x \in \Omega$ and let $k = k(x, x_0)$. Using again the result from [N1], p. 53, and the triangle inequality we get

$$\left(\frac{1}{\mu(Q)} \int\limits_Q |k - k_Q|^s \, d\mu\right)^{1/s} \leq 2 \left(\frac{1}{\mu(Q)} \int\limits_Q |k - k(x_0, x_Q)|^s \, d\mu\right)^{1/s}$$

$$\leq 2 \left(\frac{1}{\mu(Q)} \int\limits_Q k(x_0, x_Q)^s d\mu\right)^{1/s}$$

for each cube $Q \subset \Omega$ with x_Q the center of the cube Q. A similar calculation as the proof of (2.7) shows the above is bounded independent of the cube Q. As such (2.29) holds with $k = f$ and $g = 1$. It follows from Theorem 2.28 that

$$\left(\frac{1}{|Q|)} \int\limits_\Omega |k - k_{Q_0,\mu}|^s \, d\mu\right)^{1/s}$$

is finite and so, by Theorem 2.10, the proof of Theorem 2.31 is complete.

3. Applications of $L^s(\mu)$-averaging domains

We prove some norm inequalities for conjugate A-harmonic tensors in $L^s(\mu)$-averaging domains. Conjugate A-harmonic tensors are generalizations of conjugate harmonic functions and p-harmonic functions. See [B], [BM], [I], [IM] and [N2]. Theorem A follows from results in [HL].

Theorem A. *For each $p > 0$, there is a constant C such that*

$$\int_D |u - u(0)|^p dxdy \leq C \int_D |v - v(0)|^p dxdy$$

for all analytic functions $f = u + iv$ in the unit disk D.

Recently, Theorem A was generalized as follows [N2].

Theorem B. *Let u and v be conjugate A-harmonic tensors in $\Omega \subset \mathbb{R}^n$ with $u \in \mathcal{D}'(\Omega, \Lambda^0)$ and $v \in \mathcal{D}'(\Omega, \Lambda^2)$. If Ω is an L^s-averaging domain, if $v - c \in L^t(\Omega, \Lambda^2)$ and if $q < p$ with*

$$t = \frac{nq}{p-q},$$

then $u - u_{Q_0} \in L^s(\Omega, \Lambda^2)$ and there exists a constant C, independent of u and v, such that

$$\|u - u_{Q_0}\|_{s,\Omega} \le C|Q_0|^{1/s}\|v - c\|_{t,\Omega}^{q/p}. \tag{3.1}$$

If $p = q$, then (3.1) holds with $t = \infty$. Here c is any form in $\mathcal{D}'(\Omega, \Lambda^2)$ with $d^\star c = 0$.

The following is Theorem 2.4 in [D]. We use it to obtain a weighted version of Theorem B.

Theorem 3.2. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$, $w \in A_r^{loc}(\Omega)$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u and v, such that*

$$\left(\int_B |u - u_B|^s w dx\right)^{1/s} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^{pt/qs} dx\right)^{q/pt} \tag{3.3}$$

for all balls B with $2\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $\gamma = 1/s + 1/n - (1/t + 1/n)q/p$ and c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^* c = 0$.

Theorem 3.4. *Assume the hypotheses of Theorem 3.2. Then there is a constant C, independent of u and v, such that*

$$\left(\frac{1}{\mu(B)}\int_B |u - u_B|^s w\, dx\right)^{1/s} \le C|B|^\gamma \left(\int_{\sigma B} |v - c|^{q\kappa} dx\right)^{1/(p\kappa)} \tag{3.5}$$

for all balls B with $2\sigma B \subset \Omega$. Here $\gamma = 1/(\beta s) + 1/n - (1/t + 1/n)q/p$, σ and c are as in Theorem 3.2, $\kappa = \beta ts/(\beta qs - pt)$, where β is the exponent in the weak reverse Hölder inequality (1.3) and $\beta qs - pt > 0$.

Proof. By Theorem 3.2 and Hölder's inequality

$$\mu(B)^{-1/s}\left(\int_B |u-u_B|^s w\,dx\right)^{1/s} \le C_1\mu(B)^{-1/s}\left(\int_{\sigma B} w^\beta dx\right)^{1/\beta s}$$

$$|B|^\gamma\left(\int_{\sigma B} |v-c|^{\beta qts/(\beta qs-pt)}dx\right)^{(\beta qs-pt)/\beta pts}.$$

Since $w \in A_r^{loc}(\Omega)$ it satisfies the weak reverse Hölder inequality (1.3) and the weak doubling condition and so

$$\mu(B)^{-1/s}\left(\int_{\sigma B} w^\beta\right)^{1/\beta s} \le C_2\mu(B)^{-1/s}|B|^{(1-\beta)/(\beta s)}\mu(\sigma B)^{1/s}$$

$$\le C_3|B|^{(1-\beta)/(\beta s)}\mu(B)^{-1/s}\mu(B)^{1/s} = C_3|B|^{(1-\beta)/(\beta s)}$$

and Theorem 3.4 follows. $\square$

We now prove a norm inequality for conjugate A-harmonic tensors in $L^s(\mu)$-averaging domains which can be considered as a generalization of the Hardy and Littlewood theorem for conjugate harmonic functions in the unit disk of the plane.

Theorem 3.6. *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate A-harmonic tensors in an $L^s(\mu)$-averaging domain $\Omega \subset \mathbb{R}^n$. If $v - c \in L^t(\Omega, \Lambda^2)$, $w \in A_r^{loc}(\Omega)$ and*

$$t = \Psi(s) = \frac{nqs\beta}{np + (p-q)s\beta},$$

then there exists a constant C, independent of u and v, such that

$$\left(\frac{1}{\mu(\Omega)}\int_\Omega |u-u_{B_0,\mu}|^s w\,dx\right)^{1/s} \le C\left(\int_\Omega |v-c|^{q\kappa}dx\right)^{1/(p\kappa)},$$

where κ is again as above and $d^*c = 0$. Here B_0 is the ball in Definition 2.1.

Proof. By the result in [N1], p. 53, we can replace u_B by u_{B_0},μ in Theorem 3.4. Using the definition of an $L^s(\mu)$-averaging domain we obtain

$$\begin{aligned}\left(\frac{1}{\mu(B_0)}\int_\Omega |u-u_{B_0,\mu}|^s w dx\right)^{1/s} &\le C_1 \sup_{B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u-u_{B,\mu}|^s d\mu\right)^{1/s}\\ &\le C_1 \sup_{B\subset\Omega}\left(\int_{\sigma B} |v-c|^{q\kappa} dx\right)^{1/(p\kappa)}\\ &\le C_2 \left(\int_\Omega |v-c|^{q\kappa} dx\right)^{1/(p\kappa)}.\end{aligned}$$

We have completed the proof of Theorem 3.6.

We use the following theorem that is Theorem 3 in [GO].

Theorem 3.7. *Let Ω, Ω' be proper subdomains of $\mathbb{R}$, $n \ge 2$, and let $f : \Omega \to \Omega'$ be K-quasiconformal. There exists a constant c, depending only on n and K, such that*

$$k_{\Omega'}(f(x_1), f(x_2)) \le c(n,K)\max(k_\Omega(x_1,x_2), k_\Omega^\alpha(x_1,x_2)),$$

for all $x_1, x_2 \in \Omega$.

Example 3.8. Let Ω be a proper subdomain of $\mathbb{R}^n$, Ω' an $L^s(m)$-averaging domain where $s \ge 1$ and m is n-dimensional Lebesgue measure, and let $f : \Omega \to \Omega'$ a K-quasiconformal mapping. Then Ω is an $L^s(\mu)$-averaging domain where $d\mu = J_f dm$ and J_f is the Jacobian determinant of f.

Proof. Notice that $f^{-1} : \Omega' \to \Omega$ is K-quasiconformal. Fix $x_0 \in \Omega$ and let $y_0 = f(x_0)$.

Also let $A = \{y \in \Omega' | k_{\Omega'}(y, y_0) \le 1\}$. Using Theorem 3.7

$$\begin{aligned}
\int_\Omega k_\Omega(x, x_0)^s J_f \, dx &= \int_\Omega k_\Omega(f^{-1}(y), f^{-1}(y_0))^s J_f \, dx \\
&\le c(n, K)^s \int_{\Omega'} k_{\Omega'}(y, y_0)^s \, dy \\
&\le c(n, K)^s \left\{ |A| + \int_{\Omega'} k_{\Omega'}(y, y_0)^s \, dy \right\} \\
&< \infty.
\end{aligned}$$

Notice that any simply-connected proper subdomain Ω of $\mathbb{R}^2$ is $L^s(\mu)$-averaging where $d\mu = J_f \, dx$ and f is the conformal mapping of Ω onto the unit disk.

Example 3.9. Let $f = (f^1, f^2, \ldots, f^n) : \Omega \to \mathbb{R}^n$ be K-quasiregular. The tensors

$$\begin{aligned}
u &= f^1 \\
v &= \star f^2 df^3 \wedge \ldots \wedge df^n
\end{aligned}$$

are conjugate A-harmonic tensors with $p = n$ and $q = n/(n-1)$. See [IM]. We then have a special case of Theorem 3.6. If Ω is an $L^s(\mu)$-averaging domain, then, with notation consistent with above and (3.7) we obtain

$$\left(\frac{1}{\mu(\Omega)} \int_\Omega |f^1 - f^1(0)|^s w \, dx \right)^{1/s} \le C \left(\int_\Omega | \star f^2 df^3 \wedge \ldots \wedge df^n - c|^{n\kappa/(n-1)} \, dx \right)^{1/(p\kappa)}$$

where C is independent of f and $d^* c = 0$.

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