LINEAR ORBITS OF ARBITRARY PLANE CURVES

Paolo Aluffi and Carel Faber

Dedicated to William Fulton on the occasion of his 60th birthday

Abstract. The 'linear orbit' of a plane curve of degree d is its orbit in $\mathbb{P}^{d(d+3)/2}$ under the natural action of PGL(3). In this paper we obtain an algorithm computing the degree of the closure of the linear orbit of an arbitrary plane curve, and give explicit formulas for plane curves with irreducible singularities. The main tool is an intersection-theoretic study of the projective normal cone of a scheme determined by the curve in the projective space $\mathbb{P}^8$ of 3×3 matrices; this expresses the degree of the orbit closure in terms of the degrees of suitable loci related to the limits of the curve. These limits, and the degrees of the corresponding loci, have been established in previous work.

§0. Introduction

The Gromov-Witten invariants of $\mathbb{P}^2$ compute, roughly speaking, the number of plane curves of given degree d and genus g containing the appropriate number of general points. In recent years it has been discovered that these invariants are coherently linked together by the apparatus of quantum cohomology, which exposes their structure as d and g are allowed to vary.

For *nonsingular* plane curves, however, these invariants do not carry much information: the set of nonsingular curves of a given degree d is an open set of a projective space $\mathbb{P}^{d(d+3)/2}$, so the corresponding invariant is simply 1. We can consider a more refined question by fixing, as well as the degree d (and therefore the genus $g = \frac{(d-1)(d-2)}{2}$), the moduli class in $\mathcal{M}_g$ of the curve. What data determines then the corresponding invariant? Can this invariant be effectively computed? Can other enumerative invariants be computed for the set of nonsingular curves of given degree and moduli class, such as the number of curves tangent to the appropriate number of general lines?

In this article we fully answer these questions, and a natural generalization of these questions to *arbitrary* (i.e., possibly singular, reducible, nonreduced) plane curves of any degree. The group PGL(3) of projective linear transformations of $\mathbb{P}^2$ acts naturally on the space $\mathbb{P}^{d(d+3)/2}$ parameterizing plane curves of degree d. Our main result is the computation of the degree of the closure in this space of the orbit of an arbitrary plane curve (in char. 0). Somewhat surprisingly, the

1991 *Mathematics Subject Classification.* Primary 14N10; Secondary 14L30.

Both authors gratefully acknowledge partial NSF support, under grants DMS-9500843 and DMS-9801257.

enumerative geometers and the invariant theorists of the 19th century do not seem to have worked on this question. The orbit closure of a curve is a natural object of study, and its degree has a simple enumerative meaning: for a reduced curve with finite stabilizer, it counts the number of translates of the curve which contain 8 given general points. For a nonsingular curve, this is the invariant mentioned above. In this sense, therefore, this problem is an isotrivial version of the problem of computing Gromov-Witten invariants.

The computation in this paper relies on our previous work on the subject, where we have dealt with special curves: nonsingular curves were in fact already treated in [A-F2]; plane curves whose orbit has dimension less than $\dim \mathrm{PGL}(3) = 8$ are classified and studied in [A-F3], [A-F4]. We have also determined in [A-F5] the *limits* of an arbitrary plane curve; these are the curves appearing in the boundary of the orbit, that is, the complement of the orbit in its closure. In the terminology of [H-M] (p. 138) this solves the 'isotrivial flat completion problem' for plane curves.

Our previous enumerative computations relied on the explicit construction (by means of a sequence of blow-ups over the $\mathbb{P}^8$ of 3×3 matrices) of smooth varieties dominating the orbit closures. The case of an arbitrary curve appears to be too complex for that approach, and we turn in this paper to a more direct study of the projective normal cone of the base locus (scheme) of the rational map

$$\mathbb{P}^8 \dashrightarrow \mathbb{P}^{d(d+3)/2}$$

extending the map $\mathrm{PGL}(3) \to \mathbb{P}^{d(d+3)/2}$ which surjects onto the orbit of a given curve. Our study of limits of curves in [A-F5] allows us to express the degree of the orbit closure of a curve in terms of enumerative information concerning curves in the boundary of the orbit, also available from our previous computations.

For an arbitrary curve, this provides us implicitly with an algorithm computing the degree of the orbit closure. We illustrate this algorithm in §4 and §5 on specific classes of curves. For example, a surprisingly simple formula can be obtained to compute the effect on the degree due to an *irreducible* singularity p of a curve (see Theorem 5.1) in terms of the multiplicity of the curve at p, the order of contact with the tangent line to the branch at p, and the Puiseux pairs describing the singularity.

Of course many questions remain concerning orbit closures, for example regarding their singularities (which curves have smooth orbit closure? smooth orbit closures of *configurations of points in* $\mathbb{P}^1$ are classified in [A-F1]), or other invariants such as Euler characteristic, Poincaré polynomials, behavior in positive characteristic, etc.

Acknowledgement. It is a pleasure to dedicate this paper to Bill Fulton. His encouragement over the years for our collaboration was vital to its success. Much of the work on this project was done during several joint visits at the University of Chicago, at his invitation.

§1. The problem, and the approach

Let C be a curve of degree d in the projective plane $\mathbb{P}^2$ over an algebraically closed field of characteristic 0; we may think of C as a point in the projective space $\mathbb{P}^N = \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d)))$, where $N = d(d+3)/2$. The standard action of PGL(3) on $\mathbb{P}^2$ induces a right action on $\mathbb{P}^N$; specifically, for $\varphi \in \mathrm{PGL}(3)$ we can consider the *translate* of C by φ: if C has equation $F(x_0 : x_1 : x_2) = 0$, then its translate $C \circ \varphi$ has equation

$$F(\varphi(x_0 : x_1 : x_2)) = 0 \quad .$$

The action $\varphi \mapsto C \circ \varphi$ defines a map

$$c : \mathrm{PGL}(3) \to \mathbb{P}^N$$

whose image is what we call the *linear orbit* of C. Our aim is the computation of the degree of the closure of this orbit, for an arbitrary plane curve C, in terms of a description of the irreducible components and the singularities of C.

Our general approach is based on compactifying $\mathrm{PGL}(3)$ to the space $\mathbb{P}^8$ of 3×3 matrices, and considering the rational map

$$\mathbb{P}^8 \dashrightarrow \mathbb{P}^N$$

determined by c. If $\tilde{c} : \widetilde{V} \to \mathbb{P}^N$ is a map resolving the indeterminacies of this rational map, so that the diagram

$$\begin{tikzcd} \widetilde{V} \arrow[d, "\pi"'] \arrow[dr, "\tilde{c}"] & \\ \mathbb{P}^8 \arrow[r, dashed, "c"] & \mathbb{P}^N \end{tikzcd}$$

commutes, then the orbit closure of C is the image of $\tilde{c}$. In special but important cases one can in fact construct and study a *nonsingular* such variety $\widetilde{V}$, by a suitable sequence of blow-ups along smooth centers over $\mathbb{P}^N$; this is carried out in [A-F2], [A-F3], [A-F4]. The work involved in the construction of an explicit resolution of the orbit closure pays off in terms of a simpler intersection-theoretic set-up, and opens the door to a more thorough study of the orbit closure.

Such a construction is however not available for an *arbitrary* plane curve C. This is an indication of the fact that singularities of a plane curve can be extremely complicated, and that the orbit closure is highly sensitive to the local features of a curve. To treat the general case, we resort then essentially to using the most simple-minded (but highly singular) variety $\widetilde{V}$ as above—we will let $\widetilde{V}$ be the blow-up of $\mathbb{P}^8$ along the base scheme S of the rational map c—and pay the price of a more complicated intersection-theoretic set-up and of a careful local study of degenerations of C. In the end we will be able to express the degree of the orbit closure of C in terms of enumerative information concerning its *limits*, that is, the curves obtained as limits of translates $C \circ \varphi$ as φ approaches the base locus of c. This enumerative information has been obtained in our previous work; it relies on the explicit resolution of the orbit closure of the limits.

In this section we describe our degeneration technique, and the intersection theory formula we will use in the main computation. The degree of the orbit closure is the intersection number

$$h^{\dim \tilde{c}(\widetilde{V})} \cdot [\tilde{c}(\widetilde{V})] \quad ,$$

where h denotes the hyperplane class in $\mathbb{P}^N$. Pulling back to $\widetilde{V}$, we are then led to consider the class

$$h^{\dim \tilde{c}(\widetilde{V})} \cap [\widetilde{V}]$$

(following common practice, we omit evident pull-back notations); in fact, in order not to fix from the start the dimension of the orbit of C, we consider the class

$$\frac{[\widetilde{V}]}{c(\mathcal{O}(-h))} = (1 + h + h^2 + h^3 + \dots) \cap [\widetilde{V}] \quad ,$$

and its push-forward to $\mathbb{P}^8$:

$$\pi_* \frac{[\widetilde{V}]}{c(\mathcal{O}(-h))} = (1 + a_1 H + a_2 H^2 + \dots) \cap [\mathbb{P}^8] \quad ,$$

where H is the hyperplane class in $\mathbb{P}^8$, and a_i is the degree of $\pi_*(h^i \cap [\widetilde{V}])$. It is clear that

$$a_i = 0 \quad \text{for } i > \dim \tilde{c}(\widetilde{V})$$

and that $a_{\dim \tilde{c}(\widetilde{V})}$ equals the degree of the orbit closure times the degree of the closure of the stabilizer of C in $\mathbb{P}^8$. We call this number the 'predegree' of the orbit closure of C, and the whole class written above, which we think of as a polynomial in H, the *predegree polynomial* of (the orbit closure of) C.

Note. The 'polynomials' appearing in this paper are therefore nothing but classes in the Chow ring of $\mathbb{P}^8$. It will in fact be convenient to take rational coefficients, so that our polynomials will live in the ring $\mathbb{Q}[H]/(H^9)$. When manipulating polynomials we will implicitly work in this ring; in particular, all operations are truncated to H^8. This allows us some convenient abuse of language; for example,

$$\exp(dH) = 1 + dH + \frac{(dH)^2}{2} + \frac{(dH)^3}{3!} + \dots + \frac{(dH)^8}{8!}$$

with our conventions.

Our objective then becomes the following: *compute the predegree polynomial of an arbitrary plane curve C.* The degree of the orbit closure of a curve C is recovered from its predegree polynomial by dividing the top nonzero coefficient by the degree of the closure of the stabilizer of C. Predegree polynomials are a more natural object of study, since they carry enumerative information independently of the dimension of the orbit closure. The information in the predegree polynomial is equivalent to the information in what we call the *adjusted predegree polynomial (a.p.p.)*

$$\pi_*(ch(\mathcal{O}(h)) \cap [\widetilde{V}]) = 1 + a_1 H + a_2 \frac{H^2}{2} + a_3 \frac{H^3}{3!} + \dots \quad .$$

Computing *adjusted* predegree polynomials often leads to simpler formulas, so we focus on them in this paper. Adjusted predegree polynomials for curves with small orbits (i.e., of dimension < 8) are computed in [A-F3], [A-F4].

We can analyze the situation in a more general context. Let V be any variety, $\mathcal{L}$ a line bundle on V, and $\mathcal{E} \subset H^0(V, \mathcal{L})$ a nonzero linear system. These choices determine a rational map

$$\alpha : V \dashrightarrow \mathbb{P}^N = \mathbb{P}(\mathcal{E}^\vee) \quad .$$

Let S be the scheme-theoretic intersection of the sections in $\mathcal{E}$, so that the base locus of α is the support of S, and (the closure of) the graph Γ of α can be identified with the blow-up $\widetilde{V}$ of V along S. We let E be the exceptional divisor of the blow-up, that is, the part of the graph over S:

$$\begin{tikzcd}
E=\pi^{-1}S \arrow[rr] \arrow[dr] & & \Gamma=\widetilde{V} \arrow[rr] \arrow[dr, "\pi"] & & V\times\mathbb{P}^N \arrow[dl] \arrow[dr] & \\
& S \arrow[rr] & & V \arrow[rr, dashed, "\alpha"] & & \mathbb{P}^N
\end{tikzcd}$$

In other words, E is a realization of the projective normal cone of S in V. Let now $\widetilde{\mathcal{L}}$ denote the pull-back to Γ of the hyperplane class in $\mathbb{P}^N$, and notice that if $\mathcal{E}$ is base-point-free to begin with (so $S=\emptyset$), then $\widetilde{\mathcal{L}}=\mathcal{L}$ and the quantity corresponding to the adjusted predegree polynomial is simply

$$(*) \qquad \pi_*(ch(\widetilde{\mathcal{L}})\cap[\widetilde{V}]) = ch(\mathcal{L})\cap[V] \quad \text{in } (A_*V)_{\mathbb{Q}}.$$

The following proposition shows how to modify the fundamental class of $[V]$ in this formula to account for the base locus S of α. The correction term will be obtained from the cycle of E:

$$[E]=m_1[E_1]+\cdots+m_r[E_r] \quad ,$$

as follows. We denote by h the hyperplane class in $\mathbb{P}^N$ and its pull-backs (for example, $h=c_1(\widetilde{\mathcal{L}})$ on Γ); write $\ell=c_1(\mathcal{L})$, and let

$$L_i=\sum_{k\geq 0}\frac{1}{k+1}\sum_{j=0}^{k}\frac{(-\ell)^{k-j}}{j!(k-j)!}\pi_*(h^j\cap[E_i])$$

(so *a priori* the L_i might have nonzero terms in all dimensions from 0 to $\dim V-1$). Here is the main observation in this section:

Proposition 1.1.

$$\pi_*(ch(\widetilde{\mathcal{L}})\cap[\widetilde{V}]) = ch(\mathcal{L})\cap\big([V]-(m_1L_1+\cdots+m_rL_r)\big) \quad \textit{in } (A_*V)_{\mathbb{Q}}.$$

Proof. Note that $h=c_1(\widetilde{\mathcal{L}})=\ell-e$, where e is the class of E and as usual we omit obvious pull-back notations. Therefore

$$\begin{aligned}\pi_*(ch(\widetilde{\mathcal{L}})\cap[\widetilde{V}]) &= \pi_*(\exp(\ell-e)\cap[\widetilde{V}]) = \exp(\ell)\cap\pi_*(\exp(-e)\cap[\widetilde{V}])\\ &= \exp(\ell)\cap([V]-\pi_*(1-\exp(-e))\cap[\widetilde{V}])\end{aligned}$$

giving the correction term to the fundamental class as

$$-\pi_*(1-\exp(-e))\cap[\widetilde{V}] = -\pi_*\sum_{i\geq 0}\frac{(-e)^i}{(i+1)!}\cap[E] \quad , \text{ that is}$$

$$-\pi_*\sum_{i\geq 0}\frac{(h-\ell)^i}{(i+1)!}\cap(m_1[E_1]+\cdots+m_r[E_r]) \quad .$$

The statement follows by expanding this expression. $\square$

In our situation $V = \mathbb{P}^8$, $\mathcal{L} = \mathcal{O}(dH)$ (where d is the degree of the curve C), and $\mathcal{E}$ is the linear system corresponding to the rational map $c = \alpha$. We note that the support $|E|$ of $E \hookrightarrow \mathbb{P}^8 \times \mathbb{P}^N$ is described set-theoretically by

$$\begin{aligned}|E| = \{(\sigma, X) \in \mathbb{P}^8 \times \mathbb{P}^N :\\ X \text{ is a limit of } \alpha(\sigma(t)) \text{ for some curve germ } \sigma(t) \subset \mathbb{P}^8 \text{ centered at } \sigma \in S\}\end{aligned} \quad ,$$

so that it records the behavior of α as one approaches its base locus S. Since E is identified with the projective normal cone of S in $\mathbb{P}^8$, it is a scheme of pure dimension 7; invariably this will turn out to be reducible and nonreduced. Often challenging is the computation of the multiplicities m_i of the various components E_i of E; for our specific problem all this information can be found in [A-F5], and it will be recalled in the next section. In §3 we will compute explicit expressions

$$E_i = (\epsilon_1 h^N H + \cdots + \epsilon_8 h^{N-7} H^8) \cap [\mathbb{P}^8 \times \mathbb{P}^N] \quad ,$$

yielding

$$L_i = \sum_{k \geq 0} \left(\sum_{j=0}^{k} \frac{(-d)^{k-j} \epsilon_{j+1}}{j!(k-j)!} \right) \frac{H^{k+1}}{k+1} \quad .$$

According to Proposition 1.1, the a.p.p. can be computed by expanding

$$\exp(dH) \cdot \big(1 - (m_1 L_1 + \cdots + m_k L_k)\big) \quad .$$

This will be our main tool in §4 and §5.

Example 1.1. As an illustration, we describe the components of E for C a smooth curve of degree $d \geq 2$, with only ordinary flexes. Recall ([A-F2]) that in this case the base locus S consists of the set of rank-1 matrices whose image is a point of C. We will see (§2) that E consists of one component dominating S, and components dominating the set of matrices whose image is an inflection point of C.

More precisely, the first component is supported on the locus $G \subset \mathbb{P}^8 \times \mathbb{P}^N$:

$$G = \overline{\{(\sigma, C_\sigma) \,|\, \operatorname{im}\sigma \in C, \text{ and } C_\sigma \text{ is the union } \ell \cup c \text{ of a } (d-2)\text{-fold line } \ell \text{ supported on } \ker\sigma \text{ and a nonsingular conic } c \text{ tangent to } \ell\}}.$$

Computing the class of this locus is a standard exercise in the enumerative geometry of conics, and we obtain

$$[G] = 6d\, H^5 h^{N-4} + 4d(5d-9)\, H^6 h^{N-5} + 6d(d-2)(5d-8)\, H^7 h^{N-6} \quad ,$$

and the corresponding class

$$L_G = \frac{dH^5}{20} - \frac{d(5d+18)H^6}{360} + \frac{d(9d+8)H^7}{420} - \frac{d^2 H^8}{60}$$

in $\mathbb{P}^8$. The multiplicity of this component in the projective normal cone turns out to be 2 (Fact 2(ii) in §2).

For each flex p on C we will also find a component of E supported on $F \subset \mathbb{P}^8 \times \mathbb{P}^N$:

$$F = \overline{\{(\sigma, C_\sigma) | \operatorname{im}\sigma = p, \text{ and } C_\sigma \text{ is the union of a } (d-3)\text{-fold line } \ell \text{ supported on } \ker\sigma \text{ and a cuspidal cubic } c \text{ with cuspidal tangent } \ell\}}.$$

Again the computation of the class of this locus in $\mathbb{P}^8 \times \mathbb{P}^N$ is not hard, and yields

$$L_F = \frac{H^6}{144} - \frac{H^7}{70} + \frac{197H^8}{13440}$$

in $\mathbb{P}^8$; the multiplicity of F in the projective normal cone will be found to be 3 (Fact 4(ii) in §2). Since a smooth curve of degree $d \geq 2$ (and only ordinary flexes) has $3d(d-2)$ flexes, the adjusted predegree polynomial of such a curve is, according to Proposition 1.1,

$$\exp(dH) \cdot (1 - 2 \cdot L_G - 3d(d-2)\,3 \cdot L_F)$$

$$= 1 + d\,H + d^2\,\frac{H^2}{2} + d^3\,\frac{H^3}{3!} + d^4\,\frac{H^4}{4!} + (d^5 - 12d)\,\frac{H^5}{5!} + (d^6 - 97d^2 + 162d)\,\frac{H^6}{6!}$$
$$+ (d^7 - 427d^3 + 1566d^2 - 1488d)\,\frac{H^7}{7!} + (d^8 - 1372d^4 + 7992d^3 - 15879d^2 + 10638d)\,\frac{H^8}{8!} \quad .$$

The coefficient of $\frac{H^8}{8!}$ reproduces the result of the computation in [A-F2] for $d \geq 3$. Also note that, for $d = 2$, this expression reduces to

$$1 + 2H + \frac{4H^2}{2} + \frac{8H^3}{3!} + \frac{16H^4}{4!} + \frac{8H^5}{5!} \quad ,$$

the adjusted predegree polynomial for a smooth conic, in agreement with [A-F3], §4.2. We note in passing that the expression does *not* yield the a.p.p. of a line for $d = 1$; this is not surprising, since a line is not a curve with ordinary flexes.

§2. Limits of plane curves—summary of results

In this section we recall the results from [A-F5] which we need for the enumerative computations in this paper.

As we saw in §1, we are interested in the structure of the projective normal cone E of the base scheme S of the rational map

$$c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$$

extending the action of PGL(3) on a given plane curve C of degree d. Now $S \subset \mathbb{P}^8$ consists of all matrices whose image is contained in C; in particular, S has exactly one component for each component of C. More precisely, if no component of C is a line, then

$$|S| \cong \mathbb{P}^2 \times |C| \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \quad :$$

S consists of rank-1 matrices with arbitrary kernel, and image a point of C. Every linear component ℓ of C contributes a 5-dimensional component to S, consisting of the $\mathbb{P}^5$ of rank-≤ 2 matrices whose image is contained in ℓ.

We have realized E set-theoretically as a subset of pure dimension 7 of $\mathbb{P}^8 \times \mathbb{P}^N$:

$$|E| = \{(\sigma, X) \in \mathbb{P}^8 \times \mathbb{P}^N :$$
$$X \text{ is a limit of } c(\sigma(t)) \text{ for some curve germ } \sigma(t) \subset \mathbb{P}^8 \text{ centered at } \sigma \in S\} \quad .$$

We are interested in a description of the components of this locus, as well as the multiplicities with which they appear in E. A given component may arise in several ways according to the procedure described in this section; its multiplicity in E will be understood to be the sum of all multiplicities listed in each case.

A first rough description of the components of E can be given in terms of the locus on S they dominate:

Fact 1. *There is one component of E dominating each component of S (hence, one for each component of C), and components dominating loci $\cong \mathbb{P}^2$:*

$$\{\sigma \in \mathbb{P}^8 \,|\, \sigma \textit{ is a rank-1 matrix with image } p \in C\}$$

where p is either a flex or a singular point of C.

We call the first kind of components 'global', and the second kind 'local'.

Components are usually best described as orbit closures of specific elements (σ, C_σ) of $\mathbb{P}^8 \times \mathbb{P}^N$ under the induced (right) action of PGL(3). In each case C_σ will be the limit obtained along a germ centered at σ; thus it will be clear a priori that the given locus is a component of E. The content of the results listed below is that they provide an exhaustive list of all components of E for a given curve, and compute the multiplicity with which each component appears. Also, of course in each case C_σ will be a curve with small linear orbit; these curves have been studied in [A-F3] and [A-F4], and we use the terminology employed there.

Global components are easy to describe precisely:

Fact 2. *(i) Let ℓ be a line appearing with multiplicity m in C, and let λ be the $(d-m)$-tuple of points cut out on ℓ by the other components of C. Then the component of E corresponding to ℓ is the orbit closure of*

$$\begin{array}{l}(\sigma, C_\sigma), \text{where } \sigma \text{ is a rank-2 matrix with image } \ell, \text{ and } C_\sigma \text{ is a fan consisting of a star} \\ \text{centered at } \ker\sigma \text{ and reproducing projectively the tuple } \lambda, \text{ and of a residual } m\text{-fold line}\end{array} ,$$

with multiplicity m.

(ii) Let C' be a non-linear component appearing with multiplicity m in C. Then the component of E corresponding to C' is the closure of the locus

$$\begin{array}{l}\{(\sigma, C_\sigma) \in \mathbb{P}^8 \times \mathbb{P}^N \,|\, \sigma \text{ is a rank-1 matrix with image a point of } C', \text{ and } C_\sigma \text{ consists of a} \\ (d-2m)\text{-fold line supported on } \ker\sigma, \text{ and of an } m\text{-fold smooth conic tangent to } \ker\sigma\}\end{array} ,$$

with multiplicity $2m$.

We call components as in part (i) *components of type I,* and components as in part (ii) *components of type II.*

Local components of E are substantially harder to describe, since the germs of curves $\sigma(t)$ in $\mathbb{P}^8$ giving rise to such components have to be carefully tailored to the local features of C. As shown in [A-F5], only two kinds of germs must be considered, requiring separate discussions: one kind (1-parameter subgroups, or 1-PS for short) accounts for limits with multiplicative stabilizer; the other will be responsible for limits with additive stabilizer.

We start with the (simpler) case of 1-PS limits. Again, we first give a rough description of the situation.

Fact 3. *Let p be either a flex or a singular point of C. For each line in the tangent cone to C at p, there is a corresponding Newton polygon. The possible components of E due to 1-PS centered at p are indexed by sides of these Newton polygons; further, an additional component is present if the tangent cone is supported on at least three distinct lines.*

To be more precise, suppose p has multiplicity m, and denote by λ the tangent cone to C at p (hence λ determines an m-tuple in the pencil of lines through p).

Fact 4(i). *The component present exactly when λ is supported on three or more distinct lines is the orbit closure of*

$$(\sigma, C_\sigma), \text{where } \sigma \text{ is a rank-1 matrix whose image is } p, \text{ and } C_\sigma \text{ is a fan consisting of a} \atop \text{star projectively equivalent to } \lambda, \text{ and of a residual } (d-m)\text{-fold line supported on } \ker\sigma \quad ,$$

with multiplicity mA, where A is the number of automorphisms of λ as a tuple in the pencil of lines through p.

(The reason why this locus is not a component of E if λ is supported on ≤ 2 lines is simply that it is not big enough to be one: it is immediately checked that this locus has dimension 7 if and only if λ is supported on ≥ 3 lines.) We call such components *components of type III.*

To determine the components corresponding to a line ℓ in the tangent cone, choose coordinates $(x : y : z)$ in $\mathbb{P}^2$ so that $p = (1 : 0 : 0)$ and ℓ is the line $z = 0$; then consider the Newton polygon for the curve, that is, the boundary of the convex hull of the union of the positive quadrants with origin at the points (j, k) for which the coefficient of $x^i y^j z^k$ in the equation for C is nonzero (see [B-K], p. 380). Note that the part of the Newton polygon consisting of line segments with slope strictly between -1 and 0 does not depend on the choice of coordinates. Consider the 1-PS

$$\sigma(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix} \quad ,$$

with $1 \leq b < c$ relatively prime integers, and $-b/c$ a slope of a side of the Newton polygon for C.

Fact 4(ii). *For each line ℓ in the tangent cone of C, and for each 1-PS selected by the above procedure, there is a component E' of E supported on the orbit closure of*

$$(\sigma, C_\sigma), \text{where } C_\sigma \text{ is the limit as } t \to 0 \text{ of } C \text{ along the selected 1-PS } \sigma(t), \text{ and } \sigma = \sigma(0) \quad ,$$

provided this locus has dimension 7. If $x^{\overline{q}} y^r z^q \prod_{j=1}^{S} \left(y^c + \alpha_j x^{c-b} z^b\right)$ is the limit obtained along the 1-PS $\sigma(t)$, then the contribution to the multiplicity of E' is

$$(Sbc + rb + qc)\,\frac{A}{\delta} \quad ,$$

where A is the number of components of the stabilizer of the limit, and δ is the degree of the map from E' to its image in $\mathbb{P}^N$.

The limits appearing in this statement are among the curves with small orbit studied in [A-F3]. The number δ is 1 unless $c = 2$ and $q = \overline{q}$, in which case it is 2 (see [A-F5]). The number A/δ can be computed directly in terms of the tuple $\{\alpha_j\}$ (see [A-F3], Lemma 3.1). We will see in §3 that this factor is absorbed by other terms in the computation of the contribution of such components.

We call components arising as in Fact 4(ii) *components of type IV.*

In order to visualize part of this somewhat complicated recipe, note that if (j_0, k_0), (j_i, k_1), $j_0 < j_1$, are vertices of a side of the Newton polygon of C of slope strictly between -1 and 0, then the corresponding multiplicity (provided the locus specified in the statement has dimension 7) is

$$\frac{j_1 k_0 - j_0 k_1}{S} \frac{A}{\delta} \quad ,$$

where $S+1$ is the number of lattice points on the selected side. Also, note that $\overline{q} = d - j_1 - k_1$, $r = j_0$, and $q = k_1$ with these notations; and $\delta = 2$ exactly when (j_0, k_0), (j_i, k_1), and $(d, 0)$ lie on a line with slope $-1/2$. The tuple $\{\alpha_j\}$ is determined by the specific coefficients appearing along the side.

Example 2.1. Suppose that C has a general multiple point at p, by which we mean an ordinary multiple point such that the tangent line to each branch intersects that branch with multiplicity 2 at p. Let m be the multiplicity of C at p. For each line in the tangent cone, the Newton polygon contains exactly one side as in the prescription given above, from $(m-1,1)$ to $(m+1,0)$; each line then contributes a multiplicity of $(m+1)A/\delta$ to the component consisting of the orbit closure of

$$(\sigma, C_\sigma), \text{where } \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } C_\sigma \text{ is the curve } x^{d-m-1}y^{m-1}(y^2+xz)=0 \quad .$$

This component therefore appears in E with multiplicity $m(m+1)A/\delta$. Note that here $\delta = 2$ exactly when the curve has degree $m+1$. Also, if $m \geq 3$ we find one component supported on the orbit closure of

$$(\sigma, C_\sigma), \text{where } \sigma \text{ is a rank-1 matrix whose image is } p, \text{ and } C_\sigma \text{ is a fan consisting of a translate of the tangent cone at } p, \text{ and of a residual } (d-m)\text{-fold line supported on } \ker\sigma \quad ,$$

with multiplicity m.

The real subtleties in the discussion occur in the next and last case, dealing with limits with additive stabilizer. The components of E detect an interaction between different (formal) branches of C sharing a tangent at a singular point. This phenomenon does not occur for e.g., ordinary multiple points.

Consider a line in the tangent cone to C at p, and as above choose coordinates so that $p = (1:0:0)$, and the line is $z = 0$. Let m be the multiplicity of C at p. It is well-known (cf. [B-K]) that there are m formal branches of C at p, where nonreduced branches are counted according to their multiplicity. For a general choice of y, these can be written

$$z = f(y) = \sum_i \gamma_{\lambda_i} y^{\lambda_i} \quad ,$$

where $f(y)$ is a power series with fractional exponents $\lambda_i \in \mathbb{Q}$, $\lambda_0 < \lambda_1 < \dots$.

Let B be the collection of all m branches of the curve at p. We then have a finite sequence of rational numbers $c > 1$, determined as those numbers c for which at least two of the branches tangent to $z = 0$ agree modulo y^c, differ at y^c, and satisfy $\lambda_0 < c$. Call B_c the collection of those branches.

Each c determines a finite number of truncations $\underline{f(y)}$: these are the truncations at y^c (excluding y^c) of the branches in B_c. These truncations determine germs

$$\sigma(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^a & t^{ab} & 0 \\ \underline{f(t^a)} & \underline{f'(t^a)t^{ab}} & t^{ac} \end{pmatrix} \quad ,$$

where $b = \frac{c-\lambda_0}{2}+1$, and a is the least positive integer clearing all denominators in the exponents. We identify truncations if the corresponding germs are equivalent after reparameterization, that is, after multiplication on the right by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^{ab} & 0 \\ 0 & 0 & \eta^{ac} \end{pmatrix}$, with η a primitive a-th root of unity.

To each such germ we associate two numbers ℓ and W. The number ℓ is defined as the least positive integer μ such that $\underline{f(y^\mu)}$ has integer exponents. The weight W is defined as follows. For each branch β in B, let v_β be the first exponent at which β and $\underline{f(y)}$ differ, and let w_β be the minimum of c and v_β. Then W is the sum $\sum w_\beta$.

Fact 5. *Each germ $\sigma(t)$ contributes a component to E: the orbit closure of*

$$(\sigma, C_\sigma), \text{where } C_\sigma \text{ is the limit of } C \text{ along the germ } \sigma(t), \text{ and } \sigma = \sigma(0) \quad ,$$

with multiplicity ℓWA, where A is the number of components of the stabilizer of C_σ.

The limits C_σ appearing in this statement consist of unions of quadritangent conics, plus possibly a multiple of the distinguished tangent; these curves have been studied in [A-F3], §4.1. For enumerative purposes, they can be described in terms of the multiplicities s_i of the different conics, and of the number A of components of their stabilizer. As in the case of 1-PS limits, this number A will be absorbed by other terms in the computation of the contribution to the predegree of C.

We call the components identified in Fact 5 *components of type V.*

An example will clarify the procedure described above.

Example 2.2. Consider the quartic given in affine coordinates by

$$(y^2 - xz)^2 = y^3 z \quad .$$

Expanding at the origin gives two formal branches

$$z = y^2 \pm y^{5/2} + \dots \quad ;$$

with the notations used above: $c = \frac{5}{2}$, $b = \frac{5/2-2}{2} + 1 = \frac{5}{4}$, and $\underline{f(y)} = y^2$; hence the weight W is $\frac{5}{2} + \frac{5}{2} = 5$, $\ell = 1$, and the germ determined by the truncation is

$$\sigma(t) = \begin{pmatrix} 1 & 0 & 0 \\ t^4 & t^5 & 0 \\ t^8 & 2t^9 & t^{10} \end{pmatrix} \quad .$$

The corresponding component of E is the orbit closure of

$$(\sigma, C_\sigma), \text{where } \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } C_\sigma \text{ is the curve } (y^2 - xz + x^2)(y^2 - xz - x^2) \quad ;$$

one checks $A = 4$, and concludes that the multiplicity of this component in E is $1 \cdot 5 \cdot 4 = 20$.

To close the section, we remark that *not all* singular points of (the support of) a curve contribute components to the projective normal cone:

Example 2.3. If ℓ_1, ℓ_2 are lines contained in C (with any multiplicity), and $p = \ell_1 \cap \ell_2$ is not a point of the remainder of the curve, then p does *not* contribute a component to E.

Indeed, the tangent cone to C at p consists of only two lines, so there are no components of type III; next, the Newton polygon at p with respect to either line has no sides of slope between -1 and 0, so there are no components of type IV; finally, the branches of C at p only consist of lines, so they do not interact in the sense of providing a 'truncation' as in Fact 5.

§3. Contributions to the adjusted predegree polynomial

The task in this section is to apply the results of [A-F3], [A-F4] and obtain explicit expressions for the contributions to the adjusted predegree polynomials of a curve C due to the various possible components of the corresponding projective normal cone E. Together with the description of the projective normal cone recalled in §2, the results of this section yield a procedure computing the predegree polynomial of any given plane curve, in terms of the multiplicities of its components and a description of its flexes and singular points.

Recall from §1 that we have expressed the adjusted predegree polynomial (a.p.p.) of a curve as

$$\exp(dH) \cdot (1 - (m_1 L_1 + \cdots + m_k L_k)) \quad ;$$

our objective here is to obtain explicit expressions for the different 'correction' terms

$$-m_i L_i$$

due to the various components of the projective normal cone described in §2. The results will be used in §4 and §5 to obtain explicit expressions for contributions to the a.p.p. due to various features of a plane curve. A correction term $-m_i L_i$ yields an *additive contribution*

$$\exp(dH) \cdot (-m_i L_i)$$

to the a.p.p. of a curve of degree d. All expressions $-m_i L_i$ will only have terms of degree 3 or higher in H; those corresponding to *local* components will only have terms of degree 6 or higher. Hence, the effect of a local correction term on the a.p.p. of a curve can also be expressed as a *multiplicative contribution* by

$$(1 - m_i L_i) \quad ;$$

we will often prefer this alternative, since it does not involve the degree of the curve. Also, sometimes we may list the effect of a component as a correction term to the predegree of a curve, taking account of other effects such as the number of flexes absorbed by a given singularity.

In Propositions 3.1–3.5 below we will compute the correction terms $-m_i L_i$. As in §2, we start with the global components.

§3.1. Type I contributions.

Proposition 3.1. *Let ℓ be a line appearing with multiplicity m in C, and let r_i denote the multiplicities of the intersections of ℓ with the rest of C. Then the correction term due to ℓ is the antiderivative (w.r.t. H) with 0 constant term of*

$$-\frac{m^3}{2} \exp(-dH)\, H^2 \prod_i \left(1 + r_i H + \frac{r_i^2 H^2}{2}\right) \quad .$$

Explicitly:

$$-\left(\frac{m^3H^3}{6}-\frac{m^4H^4}{8}+\frac{m^5H^5}{20}-\frac{m^3(m^3+\sum r_i^3)H^6}{72}+\frac{m^3(m^4+4m\sum r_i^3+3\sum r_i^4)H^7}{336}\right.$$
$$\left.-\frac{m^3(m^5+10m^2\sum r_i^3+15m\sum r_i^4+6\sum r_i^5)H^8}{1920}\right)\quad .$$

Proof. According to Fact 2(i) in §2, the component E_ℓ of E corresponding to ℓ is the orbit closure in $\mathbb{P}^8\times\mathbb{P}^N$ of (σ,C_σ), where σ has image ℓ and C_σ is a fan consisting of an m-fold line and a star C'_σ of lines with multiplicities r_1, r_2, etc., centered at $\ker\sigma$. Denote by

$$[E_\ell]=(\epsilon_1 Hh^N+\cdots+\epsilon_8 H^8h^{N-7})\cap[\mathbb{P}^8\times\mathbb{P}^N]$$

the class of this component, so that $\epsilon_i=H^{8-i}h^{i-1}\cdot[E_\ell]$.

Claim. *Let $\beta_0+\beta_1H+\cdots+\beta_5H^5$ be the adjusted predegree polynomial of C'_σ. Then*

$$\epsilon_i=\begin{cases}0 & i<3\\ \dfrac{m^2}{2}(i-1)!\,\beta_{i-3} & i\ge 3\end{cases}$$

To see this, consider the embedding

$$\mathbb{P}^{N'}\times\mathbb{P}^2\to\mathbb{P}^N\quad ,$$

where $\mathbb{P}^{N'}$ parameterizes plane curves of degree $d-m$, $\mathbb{P}^2$ parameterizes lines, and the embedding attaches to a given curve of degree $d-m$ an m-fold line. We get an embedding

$$(\mathbb{P}^8\times\mathbb{P}^{N'})\times\mathbb{P}^2\xrightarrow{\iota}\mathbb{P}^8\times\mathbb{P}^N\quad ,$$

and it is readily understood that $E_\ell=\iota(E'_\ell\times\mathbb{P}^2)$, where E'_ℓ is the orbit closure of (σ,C'_σ). Pulling back to $(\mathbb{P}^8\times\mathbb{P}^{N'})\times\mathbb{P}^2$, we see that $\epsilon_i=0$ for $i<3$, and

$$\epsilon_i=m^2\binom{i-1}{2}H^{8-i}{h'}^{i-3}\cdot[E'_\ell]$$

for $i\ge 3$, where h' is the hyperplane in $\mathbb{P}^{N'}$. Now note that E'_ℓ is the part of the closure of the graph of the map

$$\mathbb{P}^8\dashrightarrow\mathbb{P}^{N'}$$

(extending the action of PGL(3) on the star C'_σ) over the $\mathbb{P}^5$ of matrices whose image is $\subset\ell$. By Remark 2.4 in [A-F4]

$$H^{8-i}{h'}^{i-3}\cdot[E'_\ell]=(i-3)!\,\beta_{i-3}\quad ,$$

and the claim follows.

The a.p.p. for a star is computed in Theorem 2.5 in [A-F4]:

$$\beta_0+\beta_1H+\cdots+\beta_5H^5=\left\{\prod_i\left(1+r_iH+\frac{r_i^2H^2}{2}\right)\right\}_5$$

($\{\}_5$ denotes truncation to H^5). Also, the multiplicity of this component of E is m, according to Fact 2(i) in §2. By the claim and Proposition 1.1, the correction term is therefore

$$-m\sum_{k\geq 0}\left(\sum_{j=0}^{k}\frac{(-d)^{k-j}\epsilon_{j+1}}{j!(k-j)!}\right)\frac{H^{k+1}}{k+1}=-\frac{m^3}{2}\sum_{k\geq 0}\left(\sum_{j=2}^{k}\frac{(-d)^{k-j}}{(k-j)!}\beta_{j-2}\right)\frac{H^{k+1}}{k+1}\quad ,$$

yielding the expressions given in the statement. □

Example 3.1. The a.p.p. of a curve consisting of a union of lines, with multiplicity m_i and no three meeting at a point, is

$$\prod_i\left(1+m_iH+\frac{m_i^2H^2}{2}\right)$$

(by our notational convention, this expression stands for its truncation at H^8).

Indeed, by Example 2.3 there are no components of E due to the points of intersection of such a configuration of lines; the only components are therefore those corresponding to the lines themselves. Using Proposition 3.1, the total correction term evaluates to

$$-\Bigg(\frac{\sum m_i^3H^3}{6}-\frac{\sum m_i^4H^4}{8}+\frac{\sum m_i^5H^5}{20}-\frac{(\sum m_i^3)^2H^6}{72}+\frac{(7(\sum m_i^3)(\sum m_i^4)-6\sum m_i^7)H^7}{336}$$
$$-\frac{(15(\sum m_i^4)^2+16(\sum m_i^3)(\sum m_i^5)-30\sum m_i^8)H^8}{1920}\Bigg)\quad .$$

Applying Proposition 1.1 yields the expression given in the statement.

This computation reproduces results from §2 of [A-F4], where a more general 'multiplicativity' of adjusted predegree polynomials for configurations of lines meeting transversally is discussed.

§**3.2. Type II contributions.** Next, we consider nonlinear components of C:

Proposition 3.2. *Let C' be a component of C of degree $e>1$, appearing with multiplicity m in C. Then the correction term due to C' is*

$$-2em^5\left(\frac{H^5}{20}-\frac{(5d+18m)H^6}{360}+\frac{(9d+8m)mH^7}{420}-\frac{dm^2H^8}{60}\right)\quad .$$

Proof. According to Fact 2(ii), the corresponding component of E is the locus $E_{C'}$ of (σ, C_σ), where the image of σ is a point of C' and C_σ consists of a $(d-2m)$-fold line supported on $\ker\sigma$, and of an m-fold conic tangent to $\ker\sigma$. Let

$$[E_{C'}]=(\epsilon_1Hh^N+\cdots+\epsilon_8H^8h^{N-7})\cap[\mathbb{P}^8\times\mathbb{P}^N];$$

then $\epsilon_i=H^{8-i}h^{i-1}\cdot[E_{C'}]$. To evaluate this, note that $E_{C'}$ is contained in $B\times\mathbb{P}^N\subset\mathbb{P}^8\times\mathbb{P}^N$, where $B=\mathbb{P}^2\times C'$ is the set of rank-1 matrices σ with image on C'. Denoting by k the pull-back to B of the hyperplane class from the $\mathbb{P}^2$ factor, and by ℓ the pull-back of the restriction of the hyperplane class from the other factor, we have

$$\epsilon_i=(k+\ell)^{8-i}h^{i-1}\cdot[E_{C'}]=(8-i)k^{7-i}\ell h^{i-1}\cdot[E_{C'}]\quad ;$$

in particular $\epsilon_i = 0$ unless $i = 5$, 6, or 7. The class ℓ splits $E_{C'}$ into e components, each of which consists of points (σ, C_σ) with σ constrained to have a fixed image. Also note that intersecting by k amounts to imposing a linear condition on the distinguished tangent line in C_σ; therefore, $\epsilon_i = (8-i)e$ times the number (counted with multiplicity) of curves C_σ through $i-1$ general points, with tangent line constrained to contain $7-i$ general points, where $i = 5$, 6, or 7.

For these values of i, the corresponding number of configurations (in case $d > 2m$) is computed by arguing as in [A-F3], Proposition 4.1:

$$\epsilon_i = (8-i)\, e\, \frac{(i-1)!}{6!} \frac{\partial^{7-i}}{\partial \overline{q}^{7-i}} P(\overline{q})|_{\overline{q}=d-2m} \quad ,$$

where $P(\overline{q})$ is the polynomial giving the degree for a curve such as C_σ, with distinguished tangent taken with multiplicity $\overline{q}$. This is the coefficient of $t^6/6!$ in the a.p.p. for C_σ (computed in §4.2 of [A-F3]: set $n = 2$; $m = \overline{m} = 1$; $S = s_1 = m$; $r = q = 0$ in the formulas given there), divided by 4, the degree of the stabilizer:

$$P(\overline{q}) = 12m^5\overline{q} + 30m^4\overline{q}^2 \quad .$$

The same formula holds in the case $d = 2m$. This yields

$$[E_{C'}] = em^4 \left(6H^5h^{N-4} + 4(5d-9m)H^6h^{N-3} + 6(5d-8m)(d-2m)H^7h^{N-2}\right) \quad .$$

According to Fact 2(ii) in §2 this locus appears in E with multiplicity $2m$. From this we obtain the stated correction term. $\square$

Example 3.2. If C is reduced and irreducible, then the only component of type II considered in Proposition 3.2 is the one dominating the whole curve. Setting $e = d$, $m = 1$ we get a correction term of

$$-2d\left(\frac{H^5}{20} - \frac{(5d+18)H^6}{360} + \frac{(9d+8)H^7}{420} - \frac{dH^8}{60}\right)$$

agreeing with the class $-2\,L_G$ used in Example 1.1

§**3.3. Type III contributions.** Moving on to the correction terms due to local features of the curve, we first establish a technical lemma, which will be used in the proofs of the statements that follow, and which explains a recurrent feature of the correction terms we will compute.

The components of type III, IV, and V, arising from local features of the curve, consist of orbit closures of points $(\sigma, C_\sigma) \in \mathbb{P}^8 \times \mathbb{P}^N$, where σ is a rank-one matrix with a given image point and C_σ is a curve with a distinguished line, that is supported on $\ker \sigma$ and has multiplicity $\overline{q} = d - \rho$ (where ρ changes from case to case). Let $P(\overline{q})$ denote the coefficient of H^7 in the predegree polynomial for such a curve; this is always a polynomial of degree at most two in $\overline{q}$. Also, let δ be the degree of the map from the component to its image in $\mathbb{P}^N$. As pointed out already in §2, this number is 1 in almost all cases.

Lemma 3.3.1. *The corresponding contribution to the correction term is*

$$-\delta\left(\frac{P''(-\rho)H^6}{42\cdot 6!} + \frac{P'(-\rho)H^7}{7\cdot 7!} + \frac{P(-\rho)H^8}{8!}\right) \quad .$$

Proof. Let E' denote a component of E arising from a point p of the curve, and let

$$[E'] = (\epsilon_1 H h^N + \cdots + \epsilon_8 H^8 h^{N-7}) \cap [\mathbb{P}^8 \times \mathbb{P}^N]$$

be its class. Since E' is the orbit closure of a point $(\sigma, C_\sigma) \in \mathbb{P}^8 \times \mathbb{P}^N$, with σ a rank-1 matrix with image p, E' is in fact contained in $\mathbb{P}^2 \times \mathbb{P}^N \subset \mathbb{P}^8 \times \mathbb{P}^N$, where $\mathbb{P}^2$ consists of all rank-1 matrices with image p. If k denotes the hyperplane class in $\mathbb{P}^2$, pulling back to $\mathbb{P}^2 \times \mathbb{P}^N$ shows that

$$\epsilon_i = k^{8-i} h^{i-1} \cdot [E'] \quad ;$$

this gives immediately $\epsilon_i = 0$ unless $i = 6$, 7, or 8. Also, note that under the identification of $\mathbb{P}^2$ with rank-1 matrices σ with fixed image, the class k imposes a linear condition on the line $\ker \sigma$. Now, C_σ consists in each case of a curve with a distinguished line supported on $\ker \sigma$, appearing with multiplicity $\overline{q} = d - \rho$ in our notations. Let $P(\overline{q}) = \alpha \overline{q}^2 + \beta \overline{q} + \gamma$ be the polynomial in $\overline{q}$ giving the coefficient of H^7 in the predegree polynomial for such a curve. Using Proposition 4.1 in [A-F3] we get

$$\frac{\epsilon_i}{\delta} = \begin{cases} \frac{P''(d-\rho)}{42} & i = 6 \\ \frac{P'(d-\rho)}{7} & i = 7 \\ P(d-\rho) & i = 8 \end{cases} \quad \text{and therefore}$$

$$[E'] = \frac{2\alpha}{42} H^6 h^{N-5} + \frac{2\alpha(d-\rho) + \beta}{7} H^7 h^{N-6} + (\alpha(d-\rho)^2 + \beta(d-\rho) + \gamma) H^8 h^{N-7}.$$

Computing the corresponding correction term as prescribed in §1 gives the stated expression. □

This observation explains why the degree d of C does *not* appear explicitly in the correction terms we will list. Note that a similar phenomenon also occurs in the second formula in Proposition 3.1.

Let p be a singular point of C. As recalled in §2, Fact 4(i), a component of type III of the projective normal cone is present if the tangent cone to C at p is supported on ≥ 3 distinct lines.

Proposition 3.3. *Let e_i denote the elementary symmetric functions in the multiplicities of the distinct lines in the tangent cone to C at p (so $e_1 =$ the multiplicity of C at p). Then the correction term corresponding to this component is*

$$-e_1(e_2 e_3 - e_1 e_4 - e_5) \left(\frac{H^6}{24} - \frac{e_1 H^7}{28} + \frac{e_1^2 H^8}{64} \right) \quad .$$

Note that the expression given in this statement vanishes automatically if the tangent cone is supported on ≤ 2 lines.

Proof. Using Fact 4(i) and Lemma 3.3.1, the main ingredient in the computation is the polynomial $P(\overline{q})$ expressing the degree for a fan C_σ with star projectively equivalent to the tangent cone to C at p, and residual $\overline{q}$-fold line. From [A-F4], Theorem 2.5(ii), this polynomial is

$$P(\overline{q}) = \frac{630\, \overline{q}^2}{A} (e_2 e_3 - e_1 e_4 - e_5) \quad ,$$

where A is the number of automorphisms of the tuple determined by the lines in the tangent cone as elements of the pencil of lines through p. By Lemma 3.3.1, with $\overline{q} = d - e_1$, the correction term is

$$-\frac{(e_2e_3 - e_1e_4 - e_5)}{A}\left(\frac{H^6}{24} - \frac{e_1H^7}{28} + \frac{e_1^2H^8}{64}\right)$$

times the multiplicity with which the component appears in the projective normal cone. By Fact 4(i) this multiplicity is e_1A, and the statement follows. $\square$

Example 3.3. If the tangent cone consists of m distinct reduced lines, then Proposition 3.3 evaluates its corresponding correction term as

$$-m\left(\binom{m}{2}\binom{m}{3} - m\binom{m}{4} - \binom{m}{5}\right)\left(\frac{H^6}{24} - \frac{mH^7}{28} + \frac{m^2H^8}{64}\right) \quad ,$$

that is

$$-m^2(m-1)(m-2)(m^2+3m-3)\left(\frac{H^6}{720} - \frac{mH^7}{840} + \frac{m^2H^8}{1920}\right) \quad .$$

As an illustration, consider a star of d reduced lines through a point. The point will contribute as above, with $m = d$; also, according to Proposition 3.1 each line contributes

$$-\left(\frac{H^3}{6} - \frac{H^4}{8} + \frac{H^5}{20} - \frac{(1+(d-1)^3)H^6}{72} + \frac{(1+4(d-1)^3+3(d-1)^4)H^7}{336} - \frac{(1+10(d-1)^3+15(d-1)^4+6(d-1)^5)H^8}{1920}\right) \quad .$$

From the discussion of §2, we know that there are no other correction terms. Putting everything together and using Proposition 1.1, the a.p.p. of this curve is

$$\exp(dH)\left(1 - \frac{dH^3}{6} + \frac{dH^4}{8} - \frac{dH^5}{20} - \frac{d^2(d-3)(d^3+3d^2-11d+12)H^6}{720} + \frac{d^3(2d^4-35d^2+70d-42)H^7}{1680} - \frac{d^4(d^4-16d^2+30d-16)H^8}{1920}\right)$$

that is

$$1 + dH + \frac{d^2H^2}{2} + \frac{d(d-1)(d+1)H^3}{6} + \frac{d(d-1)(d^2+d-3)H^4}{24} + \frac{d(d-1)(d-2)(d^2+3d-3)H^5}{120}$$

Note that the polynomial detects that the orbit closure of this curve has dimension ≤ 5; of course the stated expression is the truncation

$$\left\{\left(1 + H + \frac{H^2}{2}\right)^d\right\}_5$$

as prescribed by [A-F4], Theorem 2.5(i). In fact, Propositions 3.1 and 3.3 suffice to compute the a.p.p. for an arbitrary configuration of lines in the plane, recovering Theorem 2.8 in [A-F4].

§**3.4. Type IV contributions.** Next, let p be a singular or an inflection point of (the support of) C, and consider a line ℓ of the tangent cone to C at p. We have recalled in Fact 4(ii) that these choices determine a Newton polygon, and that there are components (of type IV) of the projective normal cone corresponding to the sides of this polygon of slope strictly between -1 and 0.

Consider then such a side Σ, from (j_0, k_0) to (j_1, k_1) $(j_0 < j_1)$; and let $S+1$ be the number of lattice points on Σ. Let $\gamma_0, \dots, \gamma_S$ be the coefficients on Σ of the equation for C, and consider the S-tuple in $\mathbb{P}^1$ determined by the polynomial

$$\gamma_0 \xi^S + \gamma_1 \xi^{S-1}\eta + \dots + \gamma_S \eta^S \quad ;$$

let s_i be the multiplicities of the points of this S-tuple (so for example $S = \sum s_i$).

The side Σ determines then the following expressions:

- $R(\Sigma) = (j_1 k_0 - j_0 k_1)$, that is, twice the area of the triangle with vertices at $(0,0)$, (j_0, k_0), and (j_1, k_1);
- a polynomial $G(\Sigma) = \frac{1}{S}\left(4\sum_i s_i^5 \frac{H^6}{6!} - 36\sum_i s_i^6 \frac{H^7}{7!} + 192\sum_i s_i^7 \frac{H^8}{8!}\right)$;
- and a polynomial $L(\Sigma)$ given by

$$\begin{gathered}
(6j_0^2k_0^2 + 3j_0j_1k_0^2 + j_1^2k_0^2 + 3j_0^2k_0k_1 + 4j_0j_1k_0k_1 + 3j_1^2k_0k_1 + j_0^2k_1^2 + 3j_0j_1k_1^2 + 6j_1^2k_1^2)\,\frac{H^6}{6!} \\
-\big(30j_0^3k_0^2 + 18j_0^2j_1k_0^2 + 9j_0j_1^2k_0^2 + 3j_1^3k_0^2 + 30j_0^2k_0^3 + 12j_0j_1k_0^3 + 3j_1^2k_0^3 + 12j_0^3k_0k_1 + 18j_0^2j_1k_0k_1 \\
+ 18j_0j_1^2k_0k_1 + 12j_1^3k_0k_1 + 18j_0^2k_0^2k_1 + 18j_0j_1k_0^2k_1 + 9j_1^2k_0^2k_1 + 3j_0^3k_1^2 + 9j_0^2j_1k_1^2 + 18j_0j_1^2k_1^2 \\
+30j_1^3k_1^2 + 9j_0^2k_0k_1^2 + 18j_0j_1k_0k_1^2 + 18j_1^2k_0k_1^2 + 3j_0^2k_1^3 + 12j_0j_1k_1^3 + 30j_1^2k_1^3\big)\,\frac{H^7}{7!} \\
+\big(90j_0^4k_0^2 + 60j_0^3j_1k_0^2 + 36j_0^2j_1^2k_0^2 + 18j_0j_1^3k_0^2 + 6j_1^4k_0^2 + 180j_0^3k_0^3 + 90j_0^2j_1k_0^3 + 36j_0j_1^2k_0^3 \\
+ 9j_1^3k_0^3 + 90j_0^2k_0^4 + 30j_0j_1k_0^4 + 6j_1^2k_0^4 + 30j_0^4k_0k_1 + 48j_0^3j_1k_0k_1 + 54j_0^2j_1^2k_0k_1 + 48j_0j_1^3k_0k_1 \\
+ 30j_1^4k_0k_1 + 90j_0^3k_0^2k_1 + 108j_0^2j_1k_0^2k_1 + 81j_0j_1^2k_0^2k_1 + 36j_1^3k_0^2k_1 + 60j_0^2k_0^3k_1 + 48j_0j_1k_0^3k_1 \\
+ 18j_1^2k_0^3k_1 + 6j_0^4k_1^2 + 18j_0^3j_1k_1^2 + 36j_0^2j_1^2k_1^2 + 60j_0j_1^3k_1^2 + 90j_1^4k_1^2 + 36j_0^3k_0k_1^2 + 81j_0^2j_1k_0k_1^2 \\
+ 108j_0j_1^2k_0k_1^2 + 90j_1^3k_0k_1^2 + 36j_0^2k_0^2k_1^2 + 54j_0j_1k_0^2k_1^2 + 36j_1^2k_0^2k_1^2 + 9j_0^3k_1^3 + 36j_0^2j_1k_1^3 + 90j_0j_1^2k_1^3 \\
+180j_1^3k_1^3 + 18j_0^2k_0k_1^3 + 48j_0j_1k_0k_1^3 + 60j_1^2k_0k_1^3 + 6j_0^2k_1^4 + 30j_0j_1k_1^4 + 90j_1^2k_1^4\big)\,\frac{H^8}{8!}
\end{gathered}$$

This polynomial is symmetric in the vertices of Σ; unfortunately, we do not have a more intrinsic interpretation for it.

Proposition 3.4. *The correction term due to the selected line ℓ in the tangent cone to C at p is*

$$-\sum_{\Sigma} R(\Sigma)\,(L(\Sigma) - G(\Sigma)) \quad .$$

Proof. This follows from Lemma 3.3.1 and Fact 4(ii). Using the notations of Fact 4(ii), for each side Σ we need the coefficient of the term of degree 7 in the predegree polynomial for limit curves C_σ with equation

$$x^{\overline{q}} y^r z^q \prod_{j=1}^{S} \left(y^c + \alpha_j x^{c-b} z^b\right) \quad ,$$

where

$$\gamma_0 \xi^S + \gamma_1 \xi^{S-1}\eta + \dots + \gamma_S \eta^S = \gamma_0 \prod_j (\xi - \alpha_j \eta) \quad .$$

These are precisely the curves studied in [A-F3]; the predegree polynomial for such curves is computed in Theorem 1.1 of [A-F3]. In our situation, we have

$$r = j_0, \quad q = k_1, \quad \overline{q} = d - (j_1 + k_1)$$

(hence we use $\rho = j_1 + k_1$ when applying Lemma 3.3.1), and

$$b = \frac{k_0 - k_1}{S} \quad , \quad c = \frac{j_1 - j_0}{S} \quad ;$$

applying Lemma 3.3.1 to the polynomial in $\overline{q}$ obtained from Theorem 1.1 in [A-F3] gives the expression

$$-\frac{S\delta}{A}\left(L(\Sigma) - G(\Sigma)\right) \quad ,$$

where A denotes the number of components of the stabilizer of C_σ, and δ is as in Lemma 3.3.1.

According to Fact 4(ii), the contribution to the multiplicity of this component due to Σ is

$$(Sbc + rb + qc)\frac{A}{\delta} = \frac{j_1k_0 - j_0k_1}{S}\frac{A}{\delta} = R(\Sigma)\,\frac{A}{S\delta} \quad ;$$

the correction term is therefore as stated. □

Example 3.4. Suppose p is a k-flex of C, that is, a nonsingular point of C at which C and its tangent line ℓ meet with multiplicity k. (For example, an ordinary inflection point of C is a 3-flex in this terminology). The Newton polygon at ℓ has only one side Σ with slope between -1 and 0, with vertices $(0,1)$ and $(k,0)$. We have $S = 1$, and the expressions given above evaluate to

$$R(\Sigma) = k \quad , \quad G(\Sigma) = \frac{4H^6}{6!} - \frac{36H^7}{7!} + \frac{192H^8}{8!}$$

$$L(\Sigma) = \frac{k^2H^6}{6!} - \frac{(3k^2+3k^3)H^7}{7!} + \frac{(6k^2+9k^3+6k^4)H^8}{8!} \quad ,$$

giving a correction term of

$$k(k-2)\left(\frac{(k+2)H^6}{720} - \frac{(k^2+3k+6)H^7}{1680} + \frac{(2k^3+7k^2+16k+32)H^8}{13440}\right) \quad .$$

For $k = 3$, this recovers the term L_F used in Example 1.1.

The analysis presented up to this point suffices already to compute the predegree of an arbitrary plane curve with ordinary multiple points; this case is analyzed in §4.

§**3.5. Type V contributions.** We are left with the case of components of the projective normal cone E of type V, arising from the interaction of different formal branches with the same tangent line at a point p of C. As pointed out in §2, contributions corresponding to these components arise from truncations of power series with fractional exponents representing the different branches: roughly, a contribution arises when two branches agree up to a certain exponent c, but differ at that exponent. Truncating there determines a germ $\sigma(t)$, centered at $\sigma = \sigma(0)$, and a limit C_σ; the corresponding component consists of the orbit closure of (σ, C_σ). Further, the germ determines two numbers ℓ, W (see Fact 5 in §2).

Limits C_σ obtained by this procedure consist of unions of 4-tangent conics, and a multiple of the distinguished tangent, which is supported on $\ker\sigma$. We let s_i denote the multiplicities with which the conics appear in C_σ, and write $S = \sum s_i$.

Proposition 3.5. *With notations as above, the corresponding correction term is*

$$-\ell W\left(\frac{4(S^5-\sum_i s_i^5)H^6}{6!}-\frac{36(S^6-\sum_i s_i^6)H^7}{7!}+\frac{192(S^7-\sum_i s_i^7)H^8}{8!}\right) \quad .$$

Proof. This is obtained from Lemma 3.3.1 and Fact 5 in §2, by the procedure applied in Propositions 3.3 and 3.4. The main ingredient is the predegree of the curves C_σ, which is given in §4.1 of [A-F3]. □

Example 3.5. As an illustration, we take the origin $(1:0:0)$ in the curve

$$(y^2-xz)^2=y^3z \quad .$$

As seen in Example 2.2, only one truncation needs to be considered for this point; the corresponding limit is a pair of distinct conics; further, $\ell=1$ and $W=5$. With notations as above we have $s_1=s_2=1$, so according to Proposition 3.5 the corresponding correction term is

$$-5\left(\frac{H^6}{6}-\frac{31H^7}{70}+\frac{3H^8}{5}\right) \quad .$$

Applying Proposition 1.1, this yields a contribution to the a.p.p. of

$$-\left(\frac{5H^6}{6}+\frac{47H^7}{42}+\frac{17H^8}{21}\right) \quad ;$$

in particular, the contribution due to this limit to the predegree of the curve is

$$-8!\,\frac{17}{21}=-5\cdot 6528 \quad .$$

This example belongs to a class of singular points which can be realized on a quartic curve, and are analytically isomorphic to the singularity $z^2=y^k$, $k=5$ (as in this example), 6, 7, or 8. The corresponding contribution to the predegree of the quartic turns out to be $-k\cdot 6528$ in all cases (cf. Example 5.4).

Remark. As an immediate application of the results obtained above, we can measure the effect on the contribution of a point p due to taking a 'multiple' of the curve on which p lies.

If C has ideal $(F(x:y:z))$ and m is a positive integer, we let mC denote the curve with ideal (F^m). Let $p\in C$, and assume the contribution of p to the a.p.p. of C is $K(H)$.

Claim. *Then the contribution of p to mC is $K(mH)$.*

Proof. This follows from the homogeneity of the various correction terms. The effect of replacing C by mC is that of replacing e_i by m^ie_i in correction terms of type III; (j_i,k_i) by (mj_i,mk_i), W by mW, and S, $\sum s_i^5$, $\sum s_i^6$, $\sum s_i^7$ by mS, $m^5\sum s_i^5$, $m^6\sum s_i^6$, $m^7\sum s_i^7$ respectively in correction terms of type IV and V. The claim follows. □

A similar homogeneity holds for global correction terms as well, so that if $P(H)$ is the a.p.p. of a curve C, then $P(mH)$ is the a.p.p. of its multiple mC. This can also be deduced by considering the map $\mathbb{P}^{d(d+3)/2}\to\mathbb{P}^{md(md+3)/2}$ defined by $C\mapsto mC$, a projection of the m-th Veronese embedding.

§3.6. Summary. The results obtained in this section, together with the discussion in §2, give an algorithm to compute the adjusted predegree polynomial of an arbitrary plane curve. This will be illustrated in §4 and §5 by applying it to several classes of curves.

For reference we list here the contributions to the predegree of a curve (with orbit of dimension 8) due to its features. Each of these is obtained by applying Proposition 1.1 to the results obtained in Propositions 3.1–3.5, obtaining corresponding additive contributions to the a.p.p., then reading the coefficient of $\frac{H^8}{8!}$.

Assume C has degree d. The predegree of its orbit closure is obtained then by subtracting various contributions from d^8, indexed here according to the corresponding type:

(I) A line appearing in C with multiplicity m, meeting the rest of the curve along a $(d-m)$-tuple of points with multiplicities r_i, gives a contribution of

$$m^3\left(d^3\left(10d^2-15dm+6m^2\right)+10\left(28d^2-48dm+21m^2\right)\left((d-m)^3-\sum r_i^3\right)\right.$$
$$\left.-45\left(8d-7m\right)\left((d-m)^4-\sum r_i^4\right)+126\left((d-m)^5-\sum r_i^5\right)\right) \quad .$$

(II) A component of C of degree $e>1$ and appearing with multiplicity m contributes

$$16dem^5\left(7d^2-18dm+12m^2\right) \quad .$$

Points $p \in C$ may contribute different terms:

(III) Let e_i be the elementary symmetric functions in the multiplicities of the distinct lines in the tangent cone to C at p. Then the corresponding contribution is

$$30e_1(e_2e_3-e_1e_4-e_5)\left(28d^2-48de_1+21e_1^2\right) \quad .$$

(In particular, no such contribution is present if the tangent cone consist of <3 distinct lines.)

(IV) Let ℓ be a line of the tangent cone of C at p, and let Σ denote the sides of slope strictly between -1 and 0 of the corresponding Newton polygon. With notations as in Proposition 3.4, the contribution due to each Σ is obtained by adding

$$\frac{16(j_1-k_0)}{S}\left(7d^2\sum s_i^5-18d\sum s_i^6+12\sum s_i^7\right)$$

and

$$\begin{aligned}
&(j_1k_0-j_0k_1)\left(90j_0^4k_0^2+180j_0^3k_0^3+90j_0^2k_0^4+60j_0^3k_0^2j_1+90j_0^2k_0^3j_1+30j_0k_0^4j_1+36j_0^2k_0^2j_1^2\right.\\
&+36j_0k_0^3j_1^2+6k_0^4j_1^2+18j_0k_0^2j_1^3+9k_0^3j_1^3+6k_0^2j_1^4-240j_0^3k_0^2d-240j_0^2k_0^3d-144j_0^2k_0^2j_1d\\
&-96j_0k_0^3j_1d-72j_0k_0^2j_1^2d-24k_0^3j_1^2d-24k_0^2j_1^3d+168j_0^2k_0^2d^2+84j_0k_0^2j_1d^2+28k_0^2j_1^2d^2+30j_0^4k_0k_1\\
&+90j_0^3k_0^2k_1+60j_0^2k_0^3k_1+48j_0^3k_0j_1k_1+108j_0^2k_0^2j_1k_1+48j_0k_0^3j_1k_1+54j_0^2k_0j_1^2k_1+81j_0k_0^2j_1^2k_1\\
&+18k_0^3j_1^2k_1+48j_0k_0j_1^3k_1+36k_0^2j_1^3k_1+30k_0j_1^4k_1-96j_0^3k_0dk_1-144j_0^2k_0^2dk_1-144j_0^2k_0j_1dk_1\\
&-144j_0k_0^2j_1dk_1-144j_0k_0j_1^2dk_1-72k_0^2j_1^2dk_1-96k_0j_1^3dk_1+84j_0^2k_0d^2k_1+112j_0k_0j_1d^2k_1\\
&+84k_0j_1^2d^2k_1+6j_0^4k_1^2+36j_0^3k_0k_1^2+36j_0^2k_0^2k_1^2+18j_0^3j_1k_1^2+81j_0^2k_0j_1k_1^2+54j_0k_0^2j_1k_1^2+36j_0^2j_1^2k_1^2\\
&+108j_0k_0j_1^2k_1^2+36k_0^2j_1^2k_1^2+60j_0j_1^3k_1^2+90k_0j_1^3k_1^2+90j_1^4k_1^2-24j_0^3dk_1^2-72j_0^2k_0dk_1^2-72j_0^2j_1dk_1^2\\
&-144j_0k_0j_1dk_1^2-144j_0j_1^2dk_1^2-144k_0j_1^2dk_1^2-240j_1^3dk_1^2+28j_0^2d^2k_1^2+84j_0j_1d^2k_1^2+168j_1^2d^2k_1^2\\
&+9j_0^3k_1^3+18j_0^2k_0k_1^3+36j_0^2j_1k_1^3+48j_0k_0j_1k_1^3+90j_0j_1^2k_1^3+60k_0j_1^2k_1^3+180j_1^3k_1^3-24j_0^2dk_1^3\\
&\left.-96j_0j_1dk_1^3-240j_1^2dk_1^3+6j_0^2k_1^4+30j_0j_1k_1^4+90j_1^2k_1^4\right)
\end{aligned}$$

(V) Finally there are contributions from truncations, as explained in Fact 5 of §2 and Proposition 3.5. A truncation determines two numbers ℓ, W, and germs whose limits C_σ consist of unions of 4-tangent conics and a multiple of the distinguished tangent line; let s_i denote the multiplicities of the conics in C_σ, and write $S=\sum s_i$. Then the contribution of the germ is

$$\ell W\left(192(S^7-\sum s_i^7)-288\,d(S^6-\sum s_i^6)+112\,d^2(S^5-\sum s_i^5)\right) \quad .$$

§4. Ordinary multiple points, and multiplicativity of adjusted predegree polynomials

In this section we give an illustration of the results of §3 by obtaining explicit expressions for contributions accounting for ordinary multiple points. We say that p is an ordinary multiple point for C if C has nonsingular branches with distinct tangent directions at p; in particular, we allow branches to have flexes of arbitrary order at p, or to be (reduced) lines. We also discuss to what extent adjusted predegree polynomials are multiplicative with respect to union of transversal curves.

§4.1. Ordinary multiple points. It is clear that ordinary multiple points do not contribute components of type V, since there is only one branch along any direction of the tangent cone. The contribution of an ordinary multiple point is therefore due to 1-PS germs, that is, components of type III and IV.

Proposition 4.1. *Let p be an ordinary multiple point of C, of multiplicity m, and for all lines ℓ tangent to a non-linear branch of C at p let r_ℓ be the intersection multiplicity of ℓ and C at p. Then the multiplicative contribution to the adjusted predegree polynomial of C due to p is given by*

$$\begin{aligned}&\left(1-m^2(m-1)(m-2)(m^2+3m-3)\left(\frac{H^6}{720}-\frac{mH^7}{840}+\frac{m^2H^8}{1920}\right)\right)\\ &\quad\cdot\prod_\ell\Big(1-r_\ell(2-3r_\ell+r_\ell^2-12m+3r_\ell m+6m^2)\frac{H^6}{6!}+3r_\ell(-12+2r_\ell-2r_\ell^2+r_\ell^3\\ &\quad+10m-8r_\ell m+3r_\ell^2m-20m^2+6r_\ell m^2+10m^3)\frac{H^7}{7!}-3r_\ell(-64+2r_\ell^2-3r_\ell^3+2r_\ell^4\\ &\quad+10r_\ell m-12r_\ell^2m+6r_\ell^3m+30m^2-30r_\ell m^2+12r_\ell^2m^2-60m^3+20r_\ell m^3+30m^4)\frac{H^8}{8!}\Big)\end{aligned}$$

where the $\prod$ is over all lines ℓ tangent to non-linear branches of C at p.

Note that linear branches do not appear directly in this formula, although they have impact on the contribution by affecting m and the intersection multiplicities.

Proof. The first factor is the contribution of type III, as in Example 3.3. According to Fact 4(ii) in §2, the other contributions from p are due to the individual tangent lines to the branches. Let ℓ be a line in the tangent cone to C at p, and consider the branch of C tangent to ℓ at p. We note that

—if the branch is a line, ℓ does not contribute to the a.p.p.; indeed, the corresponding Newton polygon has no sides of slope strictly between -1 and 0;

—if the branch is not a line, and has intersection multiplicity k with ℓ, then the corresponding Newton polygon has exactly one side of slope strictly between -1 and 0; this side has vertices $(m-1,1)$ and $(r_\ell,0)$, where $r_\ell=m-1+k$ is the intersection multiplicity of ℓ and C at p.

Applying Proposition 3.4 gives the contribution of type IV due to ℓ in terms of m and r_ℓ: this is the factor corresponding to ℓ in the statement. $\square$

To state the result differently, let e_i be the elementary symmetric functions in the intersection multiplicities of C with the tangent lines to the non-linear branches to C at p. Then the multiplicative contribution of p to the a.p.p. of C is

$$\Big(1+(-2e_1+3e_1^2-e_1^3-6e_2+3e_1e_2-3e_3+12e_1m-3e_1^2m+6e_2m+6m^2-6e_1m^2-15m^3$$
$$+10m^4-m^6)\frac{H^6}{6!}+(-36e_1+6e_1^2-6e_1^3+3e_1^4-12e_2+18e_1e_2-12e_1^2e_2+6e_2^2-18e_3+12e_1e_3-12e_4$$
$$+30e_1m-24e_1^2m+9e_1^3m+48e_2m-27e_1e_2m+27e_3m-60e_1m^2+18e_1^2m^2-36e_2m^2-36m^3$$
$$+30e_1m^3+90m^4-60m^5+6m^7)\frac{H^7}{7!}+(192e_1-6e_1^3+9e_1^4-6e_1^5+18e_1e_2-36e_1^2e_2+30e_1^3e_2+18e_2^2$$
$$-30e_1e_2^2-18e_3+36e_1e_3-30e_1^2e_3+30e_2e_3-36e_4+30e_1e_4-30e_5-30e_1^2m+36e_1^3m-18e_1^4m+60e_2m$$
$$-108e_1e_2m+72e_1^2e_2m-36e_2^2m+108e_3m-72e_1e_3m+72e_4m-90e_1m^2+90e_1^2m^2-36e_1^3m^2$$
$$-180e_2m^2+108e_1e_2m^2-108e_3m^2+180e_1m^3-60e_1^2m^3+120e_2m^3+126m^4-90e_1m^4-315m^5$$
$$+210m^6-21m^8)\frac{H^8}{8!}\Big)$$

Example 4.1. Suppose p is an ordinary *node* such that both branches of C at p intersect the respective tangent lines with multiplicity exactly 2 at p. Then p contributes

$$1-\frac{H^6}{6}+\frac{101H^7}{280}-\frac{25H^8}{64}$$

to the a.p.p. (set $m=2$, $e_1=3+3$, $e_2=3\cdot 3$, $e_3=e_4=e_5=0$ in the previous formula). Since p 'absorbs' 6 ordinary inflection points, the adjusted predegree polynomial for a curve of degree $d\geq 3$ with n such nodes and only ordinary flexes is

$$\exp(dH)\cdot\left(1-2d\left(\frac{H^5}{20}-\frac{(5d+18)H^6}{360}+\frac{(9d+8)H^7}{420}-\frac{dH^8}{60}\right)\right)$$
$$\cdot\left(1-\frac{H^6}{48}+\frac{3H^7}{70}-\frac{197H^8}{4480}\right)^{3d(d-2)-6n}\cdot\left(1-\frac{H^6}{6}+\frac{101H^7}{280}-\frac{25H^8}{64}\right)^n$$

(The term following the exponential is the contribution as in Example 3.2; the next term accounts for the flexes, obtained by setting $k=3$ in Example 3.4.) The predegree of such a curve is therefore

$$d^8-1372d^4+7992d^3-15879d^2+10638d-24n\left(35d^2-174d+213\right) \quad .$$

For instance, the degree of the orbit closure of a quartic of this kind is $14280-1848n$; the predegree of the orbit closure of a rational plane curve of this kind is

$$d^8-1792d^4+11340d^3-25539d^2+22482d-5112 \quad .$$

Example 4.2. An ordinary multiple point p of multiplicity m, and such that each branch is smooth, non-linear, and does not have an inflection point at p contributes

$$\left(1-\frac{m(m^3+m^2+m+16)H^6}{6!}+\frac{3(2m^5+2m^4+2m^3+37m^2+16m+11)H^7}{7!}\right.$$
$$\left.-\frac{21(m^6+m^5+m^4+21m^3+13m^2+17m+9)H^8}{8!}\right)^{m(m-1)} \quad .$$

Using that such a point absorbs $3m(m-1)$ flexes, one then sees that the contribution to the predegree of a curve of degree d due to such a point is

$$\begin{aligned}&-m(m-1)(21m^6-48dm^5+21m^5+28d^2m^4-48dm^4+21m^4+28d^2m^3-48dm^3\\&+441m^3+28d^2m^2-888dm^2+273m^2+448d^2m-384dm+357m-1260d^2+4920d-5130).\end{aligned}$$

For instance, a general quartic curve with a triple point has predegree $14280-3\cdot 2\cdot 1890=2940$.

Example 4.3. A *biflecnode* is an ordinary node at which both branches have an ordinary inflection point; its contribution is

$$1-\frac{H^6}{3}+\frac{88H^7}{105}-\frac{15H^8}{14}$$

(set $m=2$, $e_1=4+4$, $e_2=4\cdot 4$, $e_3=e_4=e_5=0$ in the formula given above). Using that such a point absorbs 8 flexes, we get that a biflecnode corrects the predegree for a curve of degree d by

$$-24(140d^2-832d+1209)\quad .$$

For instance, the quartic with equation

$$x^2y^2+x^2z^2+y^2z^2=0$$

has three biflecnodes and 24 automorphisms, hence its orbit closure has predegree $\frac{14280-3\cdot 2904}{24}=232$. As it happens, this orbit closure is isomorphic to the moduli space of semistable vector bundles on $\mathbb{P}^2$ of rank 2 with Chern classes $c_1=-1$ and $c_2=3$, as Hulek proved [H]. It follows that the corresponding Donaldson invariant of $\mathbb{P}^2$ equals 232, in agreement with [K-L].

Example 4.4. Suppose p is an ordinary node for which one branch is a *line,* and the other intersects its tangent line with multiplicity k at p. Then p contributes

$$1-\frac{(k+1)(k+2)(k+3)H^6}{6!}+\frac{3(k+1)(k^3+7k^2+21k+23)H^7}{7!}-\frac{3(k+1)(k+3)^2(2k^2+5k+17)H^8}{8!}$$

(use $m=2$, $e_1=k+1$, $e_2=e_3=e_4=e_5=0$ in the formula given above). For $k=2$, the contribution is

$$1-\frac{H^6}{12}+\frac{101H^7}{560}-\frac{25H^8}{128}\quad ;$$

of course this is the square root (modulo H^9) of the contribution for a node given in Example 4.1.

§4.2. Multiplicativity of adjusted predegree polynomials. It is natural to ask whether the predegree information behaves well with respect to unions of curves. This is another advantage of adjusted predegree polynomials over other ways to assemble this enumerative information: adjusted predegree polynomials are multiplicative under unions of curves, up to correction terms independent of the degree(!), accounting for the ways in which the curves meet. No such structure is visible at the level of degrees or predegrees alone.

As a representative example, we let C_1, C_2 be arbitrary *reduced* curves, meeting *transversally* at nonsingular points, and we further assume that such points are not inflection points for either curve. Let C_i' resp. L_i be the union of the non-linear resp. linear components of C_i. Let $I=\#(C_1'\cap C_2')$, $J=\#((C_1'\cap L_2)\cup(C_2'\cap L_1))$.

Proposition 4.2. *Let $P_{C_1}(H)$, $P_{C_2}(H)$ be the adjusted predegree polynomials of C_1, C_2. Then the adjusted predegree polynomial of their union $C = C_1 \cup C_2$ is*

$$P_C(H) = P_{C_1}(H) \cdot P_{C_2}(H) \\ \cdot \left(1 - \frac{H^6}{9} + \frac{11H^7}{40} - \frac{311H^8}{960}\right)^I \cdot \left(1 - \frac{H^6}{24} + \frac{7H^7}{60} - \frac{13H^8}{80}\right)^J \quad .$$

Proof. The main remark is that the components of the projective normal cone for $C_1 \cup C_2$ arise from features of C_1, C_2 and from the points of intersection of the two curves; an analysis of the components leads to the formula of the statement. We go through this analysis here as a template for similar computations.

As pointed out in Example 2.3, the intersection of two lines does not contribute components. Using the formulas given in Examples 4.1–4.4 to evaluate the contribution of the transversal intersections of two curves at non-flex point, and of a line and a curve at a non-flex point, we can write

$$P_C(H) = \exp((d_1 + d_2)H)\left(1 + L_{C_1'}(C) + L_{L_1}(C) + L_{C_2'}(C) + L_{L_2}(C)\right) \\ \cdot (1 + L_{\text{local}}(C_1))\,(1 + L_{\text{local}}(C_2))\left(1 - \frac{H^6}{12} + \frac{101H^7}{560} - \frac{25H^8}{128}\right)^{2I+J}$$

where $d_i = \deg C_i$, and $L_{\dots}$ denote the various correction terms, with hopefully evident notations: for example, $L_{\text{local}}(C_1)$ stands for the term arising from all local features of C_1. It is crucial here to recall (cf. Lemma 3.3.1) that such local terms do not depend on other features of the curve; so the contribution of a local term is the same whether viewed in C_i or in C. (This is not the case for 'global' terms!) With the same notations we can write

$$P_{C_i}(H) = \exp(d_i H)\left(1 + L_{C_i'}(C_i) + L_{L_i}(C_i)\right)(1 + L_{\text{local}}(C_i))$$

and therefore the ratio $\frac{P_C(H)}{P_{C_1}(H)P_{C_2}(H)}$ is expressed by

$$\frac{(1 + L_{C_1'}(C) + L_{L_1}(C) + L_{C_2'}(C) + L_{L_2}(C))}{(1 + L_{C_1'}(C_1) + L_{L_1}(C_1))(1 + L_{C_2'}(C_2) + L_{L_2}(C_2))}\left(1 - \frac{H^6}{12} + \frac{101H^7}{560} - \frac{25H^8}{128}\right)^{2I+J} \quad .$$

Lastly, we note that in evaluating this term we may assume that each line meets the rest of C transversally at non-inflection points: indeed, the terms arising from special positions of the lines can be evaluated locally, so they can be incorporated in the L_{local} terms. All the terms in this expression can then be evaluated very simply by Propositions 3.1 and 3.2, giving the stated result. □

Example 4.5. If both C_1, C_2 are unions of lines, then multiplicativity holds 'on the nose', since $I = J = 0$ in that case. This in fact holds for non-reduced configurations of lines as well, cf. Corollary 2.11 in [A-F4].

Example 4.6. The union of a general curve C of degree $d \geq 2$ and a general transversal line has adjusted predegree polynomial

$$P_C(H) \cdot \left(1 + H + \frac{H^2}{2}\right) \cdot \left(1 - \frac{H^6}{24} + \frac{7H^7}{60} - \frac{13H^8}{80}\right)^d \quad ,$$

where $P_C(H)$ is the adjusted predegree polynomial of a general curve (computed in Example 1.1). For $d = 2$, this yields

$$1 + 3H + \frac{9H^2}{2} + \frac{13H^3}{3} + 3H^4 + \frac{7H^5}{5} + \frac{19H^6}{60} + \frac{H^7}{60} \quad ,$$

detecting that the union of a conic and a transversal line has orbit closure of dimension 7 and degree $\frac{7!}{60 \cdot 4} = 21$. This agrees of course with the naïve combinatorial count, since the orbit of the union of a conic and a transversal line is in fact the set of all such curves; the degree is then the number of curves through 7 general points, that is $\binom{7}{2} = 21$ (the line must contain two of the points, and the conic is then determined by the other five).

Combinatorics would not suffice to compute e.g. the degree for the union of a general *cubic* and a general transversal line; according to the formula given above, this is 8568. Note that these computations do depend on whether the intersection points are or are not inflection points for the branches. Using the formula given in Example 4.4, one obtains that the predegree of the union of a general cubic and a general transversal line *through a flex* of the cubic is 8040.

Example 4.7. The union of two transversal conics has a.p.p. given by

$$\begin{aligned}\left(1 + 2H + 2H^2 + \frac{4H^3}{3} + \frac{2H^4}{3} + \frac{H^5}{15}\right)^2 \cdot \left(1 - \frac{H^6}{9} + \frac{11H^7}{40} - \frac{311H^8}{960}\right)^4 \\ = 1 + 4H + 8H^2 + \frac{32H^3}{3} + \frac{32H^4}{3} + \frac{122H^5}{15} + \frac{64H^6}{15} + \frac{41H^7}{30} + \frac{41H^8}{240} \quad ,\end{aligned}$$

hence predegree 6888.

The reader will have no difficulties adapting the argument in the proof of Proposition 4.2 to compute terms accounting for other kinds of intersections. For example, a point of simple tangency of a line with a curve gives a correction term

$$1 - \frac{H^6}{6} + \frac{7H^7}{15} - \frac{13H^8}{20}$$

to the polynomial of the union of the curve and the line (note that this is the 4-th power (modulo H^9) of the contribution for a point of transversal intersection of a line with a curve. We don't have a conceptual explanation for this phenomenon). Thus, the adjusted predegree polynomial for the union of a smooth conic and a tangent line is

$$\begin{aligned}\left(1 + 2H + 2H^2 + \frac{4H^3}{3} + \frac{2H^4}{3} + \frac{H^5}{15}\right) \cdot \left(1 + H + \frac{H^2}{2}\right) \cdot \left(1 - \frac{H^6}{6} + \frac{7H^7}{15} - \frac{13H^8}{20}\right) \\ = 1 + 3H + \frac{9H^2}{2} + \frac{13H^3}{3} + 3H^4 + \frac{7H^5}{5} + \frac{7H^6}{30} \quad :\end{aligned}$$

the orbit closure has dimension 6 and degree $\frac{6!7}{30 \cdot 4} = 42$, as expected.

§5. Irreducible singularities

Our last and most substantial example illustrating the algorithm implicitly described in §§2-3 will be the computation of the contribution to the adjusted predegree polynomial due to an arbitrary irreducible singularity p on a curve C.

It is well-known that C can be described at such a point by its *Puiseux expansion*

$$\begin{cases} z = (a_n t^n + \cdots +)\ a_{e_1} t^{e_1} + \cdots + a_{e_r} t^{e_r} \\ y = t^m \end{cases}$$

where: $m =$ the multiplicity of C at p; $n =$ the intersection multiplicity of C and the tangent line $z = 0$ at p; all exponents are positive integers, and $m < n \leq e_1 < \cdots < e_r$; and the coefficients a_{e_i} of the 'essential' terms are nonzero. An exponent (or the corresponding term in the expansion) is 'essential' if it is *not* a multiple of the greatest common divisor of m and the exponents preceding it; the () in the expansion collects all non-essential terms. The term $a_n t^n$ will be essential if and only if n is not a multiple of m; note that $e_1 = n$ in that case.

We also need the numbers

$$d_i = gcd(m, e_1, \dots, e_i) \quad ;$$

thus $d_0 = m$, and $d_r = 1$. Note that we allow for the possibility $m = 1$, $r = 0$; that is, there may be *no* essential terms in the expansion.

We will see that the contribution of p to the a.p.p. for C *depends only on m, n, and the essential exponents $e_1, \dots, e_r$.*

An alternative terminology to describe the same information is that of *Puiseux pairs:* the singularity is described by the pair (m, n), and by r Puiseux pairs $(m_1, n_1), \dots, (m_r, n_r)$, where

$$\begin{cases} d_i = m_{i+1} \cdots m_r \\ e_i = n_i d_i \end{cases} .$$

Thus for example a non-singular inflection point of order k is described by

$$(1, k)$$

and has no Puiseux pairs ($r = 0$, no essential exponents, $d_0 = 1 = m$); an ordinary cusp $y^n = z^m$ (m, n coprime) is described by

$$(m, n) \quad ; \quad (m, n)$$

and has *one* Puiseux pair ($r = 1$, $e_1 = n$, $d_0 = m$, $d_1 = 1$). The formula given below implies that the correction due to p only depends on m, n, and the Puiseux pairs of C at p.

This result is most easily stated in terms of the numbers d_i, e_i. We let

$$P(a, b) = \frac{a^2 b^2}{(1 + ak)^3 (1 + bk)^3} - \frac{4}{(1 + k)^3 (1 + 2k)^3} \quad ,$$

where k is an indeterminate, and set $e_0 = n$, $e_{r+1} = 0$ for convenience.

Theorem 5.1. *With notations as above, the contribution of p to the adjusted pre-degree polynomial of C is*

$$1 - \left\{ \left(mnP(m, n) + \sum_{j=0}^{r} (e_{j+1} - e_j) d_j P(d_j, 2d_j) \right) \cdot \left(\frac{k^2 H^6}{6!} + \frac{k H^7}{7!} + \frac{H^8}{8!} \right) \right\}_2$$

where $\{\}_2$ *denotes the coefficient of* k^2 *in the expansion of the term within* $\{\}$.

Before proving this formula, we illustrate it with a few explicit examples. For these we will need the number of flexes absorbed by the singularity; remarkably, this number can be expressed by a formula somewhat analogous to the one given in Theorem 5.1:

$$(3mn-2m-2n)+3\sum_{j=0}^{r}(e_{j+1}-e_j)(d_j-1)$$

(cf. [B-K], §9.1, Thm. 2 and [Oka], §2, Proposition 5 and Theorem 7). The correction term that would be due to the flexes absorbed by p if p were not present is, according to Theorem 5.1,

$$1-\left\{\left((3mn-2m-2n)+3\sum_{j=0}^{r}(e_{j+1}-e_j)(d_j-1)\right)3P(1,3)\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2 \quad .$$

Example 5.1. A nonsingular point has no Puiseux pairs, and $(m,n)=(1,k)$, where $k=$ the order of contact with the tangent line. By Theorem 5.1, its contribution is

$$\begin{aligned}&1-\left\{kP(1,k)\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2\\&=1-\frac{k(k-2)(k+2)H^6}{720}+\frac{k(k-2)(k^2+3k+6)H^7}{1680}-\frac{k(k-2)(2k^3+7k^2+16k+32)H^8}{13440}\end{aligned}$$

in agreement with Example 3.4. Note that this contribution is automatically trivial if $k=2$, that is if the point is not an inflection point for C.

Assume next that p has exactly one Puiseux pair (m_1,n_1). With notations as above, necessarily $m_1=m$; and $d_0=m$, $d_1=1$; $e_0=n$, $e_1=n_1$, $e_2=0$. According to Theorem 5.1, the contribution of p is

$$\begin{aligned}1-\frac{m(4m^4(n_1-n)+m^2n^3-4n_1)H^6}{6!}+\frac{3m(12m^5(n_1-n)+m^3n^3+m^2n^4-12n_1)H^7}{7!}\\-\frac{3m(64m^6(n_1-n)+2m^4n^3+3m^3n^4+2m^2n^5-64n_1)H^8}{8!}\end{aligned}$$

Example 5.2. For an ordinary (m,n)-cusp (see above) we find

$$1-\frac{mn(m^2n^2-4)H^6}{6!}+\frac{3mn(m^3n^2+m^2n^3-12)H^7}{7!}-\frac{3mn(2m^4n^2+3m^3n^3+2m^2n^4-64)H^8}{8!}$$

For instance, an ordinary $(2,3)$ cusp contributes

$$1-\frac{4H^6}{15}+\frac{3H^7}{5}-\frac{19H^8}{28} \quad ;$$

using that such a cusp absorbs 8 flexes, we get that an ordinary cusp corrects the predegree of a curve of degree $d\geq 3$ by

$$-72(28d^2-144d+183) \quad .$$

Thus a generic cuspidal quartic has predegree $14280-3960=10320$, etc. Note that for a cuspidal *cubic* this gives a 'predegree' of $216-216=0$; this is because cuspidal cubics have small orbits. According to the formulas given above, the a.p.p. of a cuspidal cubic is

$$1+3H+\frac{9H^2}{2}+\frac{9H^3}{2}+\frac{27H^4}{8}+\frac{69H^5}{40}+\frac{3H^6}{8}+\frac{H^7}{70} \quad ,$$

yielding a degree of $\frac{7!}{70\cdot 3}=24$, as expected.

Example 5.3. *Characteristic numbers.* An enumerative problem that has received a good deal of attention both in the 19th century and in the recent past is that of computing the *characteristic numbers* of various families of plane curves, that is, the number of curves belonging to the family, containing a collection of general points, and tangent to a collection of general lines. This problem is in general surprisingly challenging, even for curves of very low degree.

We note here that the top characteristic number of the (family of curves parameterized by the) orbit closure of C is the degree of the orbit closure of the dual curve $C^\vee$: hence the results of this paper allow us in principle to compute the top characteristic number of the orbit closure of an arbitrary curve, i.e., the number of translates of the curve which are tangent to a maximal number of general lines.

For example, consider the orbit closure of a nonsingular cubic curve C, that is, the closure of the set of cubic curves with a given j-invariant. Its top characteristic number is the degree of the orbit closure of a sextic with 9 cusps; now Example 5.2 lets us compute the predegree of this orbit closure:

$$\begin{aligned}&\text{predegree of a general sextic} - \text{contributions from 9 cusps}\\ &\qquad\qquad = 1119960 - 9\cdot 23544 = 908064 \quad .\end{aligned}$$

For $j \neq 0, 1728$, the stabilizer of C consists of 18 elements; thus there are $\frac{908064}{18} =$ 50,448 cubics with fixed j invariant $\neq 0, 1728$ and tangent to 8 lines in general position. For $j = 0$, resp. $j = 1728$ the extra automorphisms of C correct this number to $\frac{50448}{3} =$ 16,816, $\frac{50448}{2} =$ 25,224, respectively. These results agree with the more direct computations in [A].

Similarly, the number of nodal cubics tangent to 8 lines in general position is the degree of the orbit closure of the dual of a nodal cubic, that is, a quartic with three cusps:

$$\frac{14280 - 3\cdot 3960}{6} = 400 \quad .$$

Of course this also agrees with the classical result (cf. for example [S]).

It is curious to observe that the dual of a nodal cubic can also be interpreted as a sextic consisting of a quartic with three cusps and a double bitangent line, in the sense that this is what the dual of a nonsingular cubic C degenerates to as C degenerates to a nodal cubic. Arguing as in §4 to account for the contribution of the double line, we compute that the predegree of the orbit closure of such a sextic is 302668; as the stabilizer of a nodal cubic has 6 elements, this gives 50,448 as the top characteristic number of a nodal cubic. This number counts the 400 curves tangent to 8 lines as well as contributions from curves whose node is on one of the lines; the fact that this number agrees with the characteristic number for cubics with $j < \infty$ was already observed in [A], end of §3.

Apart from these and a few other instances (for example conics, or cuspidal cubics), the characteristic numbers that can be obtained by applying the results in this paper are, to our knowledge, new. For example, so is the number 406,758,744 of nonsingular *quartics* with fixed general modulus and tangent to 8 lines in general position.

Example 5.4. The quartic curves

$$(y^2 - xz)^2 = y^3 z \quad ; \quad (y^2 - xz)^2 = yz^3$$

have a singularity at $(1:0:0)$ described by $(m,n)=(2,4)$, and Puiseux pair $(2,k)$ for $k=5$, 7 respectively. Using the formula given above, and that these points absorb $3k$ flexes, we find that these singularities correct the predegree of the quartics on which they lie by $-1785k$.

These singularities are analytically isomorphic to $z^2=y^k$ (cf. Example 3.5). Remarkably, the same correction term applies for quartics with a point analytically isomorphic to $z^2=y^k$ also in the non-irreducible cases $k=4$, 6, 8 (as may be computed explicitly using Propositions 3.4 and 3.5). For $k=8$ the corresponding quartic is $(y^2-xz)^2=z^4$, that is the union of two quadritangent conics (cf. §4.1 in [A-F3]); the formula gives $14280-1785\cdot 8=0$, as expected since unions of quadritangent conics have small orbits.

The case $k=4$ can also be analyzed by the same method, and gives a correction of $-1785\cdot 4=-7140$. Thus a general tacnodal quartic has predegree $14280-7140=7140$, that is, precisely half of the predegree of a general quartic. This latter fact can also be explained conceptually by studying the behavior of the predegree along families of curves, but we will not pursue this approach here.

Proof of Theorem 5.1. The formula given in the theorem is obtained by evaluating explicitly the contributions of type IV and V, using Proposition 3.4 and 3.5. The main subtlety lies in the fact that both these contributions are affected by whether n is an essential exponent or not; as we will see, the amounts by which they are affected precisely compensate each other, so that both cases lead to the same formula.

We consider contributions of type IV first. If $d'=gcd(m,n)$, and $m'=m/d'$, $n'=n/d'$, then the only 1-PS germ giving a contribution is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t^{n'} & 0 \\ 0 & 0 & t^{m'} \end{pmatrix}$$

yielding a limit

$$\left(y^{n'}-*x^{n'-m'}z^{m'}\right)^{d'}x^{d-n}=0 \quad ,$$

corresponding to the side in the Newton polygon joining vertices $(0,m)$ and $(n,0)$. Using Proposition 3.4, this gives a contribution of

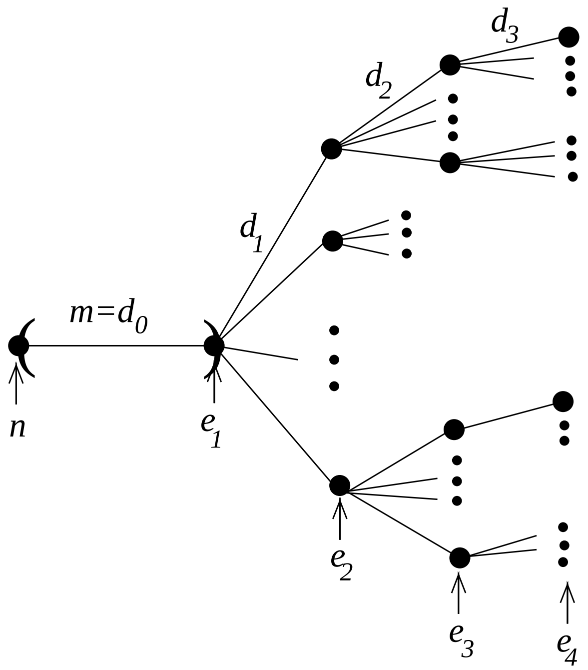
$$1-mn\left(\frac{(m^2n^2-4d'^4)H^6}{6!}-\frac{3(m^3n^2+m^2n^3-12d'^5)H^7}{7!}\right.$$
$$\left.+\frac{3(2m^4n^2+3m^3n^3+2m^2n^4-64d'^6)H^8}{8!}\right)$$

which is checked to equal

$$1-\left\{mn\big(P(m,n)-P(m,2m)\big)\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2$$
$$-mn\left(\frac{(m^4-d'^4)H^6}{180}-\frac{(m^5-d'^5)H^7}{140}+\frac{(m^6-d'^6)H^8}{210}\right) \quad .$$

Here $d'=m$ if n is a multiple of m (in which case the last summand vanishes), while $d'=d_1=gcd(m,e_1)$ if $n=e_1$ is essential.

Moving on to the component of type V, the data describing the singularity determines the structure of the formal branches of the curve at p. Schematically, here is how they group:



If n is not essential, $m = d_0$ branches will run parallel from the beginning of the expansion up to the first essential exponent e_1; if n is essential, the branching starts immediately at $n = e_1$. In both cases, at e_1 the branches divide into d_0/d_1 groups of d_1 parallel branches each; at e_2, each set of d_1 branches splits into d_1/d_2 groups of d_2 parallel branches, and so on. At the last essential exponent e_r, the splitting produces m distinct simple branches.

This gives us the data needed to apply Proposition 3.5. Note that e_1 yields a 'truncation' in the sense of Fact 5 of §2 *only if* n is not an essential exponent: if $n = e_1$ is essential, then the expansion starts at e_1 and in particular e_1 is not greater than the first exponent. If n is not essential, the truncation at e_1 contributes in the terminology of Proposition 3.5 a term with $\ell = 1$, $W = e_1$, $S = m$, and $s_i = d_1$, giving

$$-me_1\left(\frac{(m^4-d_1^4)H^6}{180}-\frac{(m^5-d_1^5)H^7}{140}+\frac{(m^6-d_1^6)H^8}{210}\right) \quad ;$$

if n is essential, there is no such contribution. Adding this to the contribution of type IV computed above, we obtain *in both cases*

$$1-\left\{mn\big(P(m,n)-P(m,2m)\big)\cdot\left(\frac{k^2H^6}{6!}+\frac{kH^7}{7!}+\frac{H^8}{8!}\right)\right\}_2-K_1$$

where

$$K_1 = me_1\left(\frac{(m^4-d_1^4)H^6}{180}-\frac{(m^5-d_1^5)H^7}{140}+\frac{(m^6-d_1^6)H^8}{210}\right) \quad .$$

The contribution due to truncation at e_j, $j \geq 2$, is given by Proposition 3.5, setting $\ell = \frac{m}{d_{j-1}}$ (the least integer such that $\ell\frac{e_1}{m}, \ldots, \ell\frac{e_{j-1}}{m}$ are integers),

$$W = \sum_{k=1}^{j-1}(d_{k-1}-d_k)\frac{e_k}{m}+d_{j-1}\frac{e_j}{m}$$

(keeping track of the exponents at which formal branches start differing), and $S = d_{j-1}$, $s_i = d_j$. If K_j denotes this (additive) contribution, one checks by induction that if there are r Puiseux pairs (so that $d_r = 1$)

$$\sum_{j=2}^{r} K_j = \left\{ \sum_{j=2}^{r} e_j \big(d_{j-1} P(d_{j-1}, 2d_{j-1}) - d_j P(d_j, 2d_j) \big) \left(\frac{k^2 H^6}{6!} + \frac{k H^7}{7!} + \frac{H^8}{8!} \right) \right\}_2$$

(note: this equality does not hold if d_r is not assumed to equal 1!). The whole contribution is therefore given by

$$1 - \left\{ mn \big(P(m,n) - P(m,2m) \big) \cdot \left(\frac{k^2 H^6}{6!} + \frac{k H^7}{7!} + \frac{H^8}{8!} \right) \right\}_2 - \sum_{j=1}^{r} K_j \quad ,$$

and the formula given in the statement is obtained by rearranging this sum. □

Formulas for *reducible* singularities can be obtained by using Propositions 3.3, 3.4, and 3.5. Unfortunately, we haven't been able to find a simple statement in the style of Theorem 5.1 and encompassing the most general case.

As a final comment, we note that a formula in the style of Theorem 5.1 can be concocted to account for some 'global' terms as well. For example, the predegree of the orbit closure of a reduced curve of degree d and (for simplicity) including only points 'of type (t^m, t^n)' (that is, points described by the pair (m,n) as above, with no further Puiseux pairs) is in fact given by

$$d^8 - \left\{ (1+dk)^8 \left[\frac{4d^2}{(1+k)^3(1+2k)^3} + \sum_{p \in C \text{ of type } (t^m, t^n)} mn \left(\frac{m^2 n^2}{(1+mk)^3(1+nk)^3} - \frac{4}{(1+k)^3(1+2k)^3} \right) \right] \right\}_2 \quad ,$$

provided that the orbit closure has dimension 8. This formula should be compared with the formula for the predegree of the orbit closure of a *d-tuple of points in* $\mathbb{P}^1$ (cf. [A-F1]), which can be written

$$d^3 - \left\{ (1+dk)^3 \left[\frac{d}{(1+k)^2} + \sum_{p \in C \text{ of type } (t^m)} m \left(\frac{m}{(1+mk)^2} - \frac{1}{(1+k)^2} \right) \right] \right\}_1$$

(if the orbit closure has dimension 3), where a point 'of type (t^m)' is simply a point of multiplicity m in the d-tuple.

It is tempting to view these two formulas as shadows of a very general, but as yet mysterious, theorem on degrees of orbit closures of hypersurfaces in projective space.

References

[A] P. Aluffi, *How many smooth plane cubics with given j-invariant are tangent to 8 lines in general position?*, Contemp. Math. **123** (1991), 15–29.

[A-F1] P. Aluffi, C. Faber, *Linear orbits of d-tuples of points in $\mathbb{P}^1$*, J. reine angew. Math. **445** (1993), 205-220.

[A-F2] P. Aluffi, C. Faber, *Linear orbits of smooth plane curves*, Jour. of Alg. Geom **2** (1993), 155–184.

[A-F3] P. Aluffi, C. Faber, *Plane curves with small linear orbits I*, Ann. Inst. Fourier (Grenoble) **50** (2000), 151–196.

[A-F4] P. Aluffi, C. Faber, *Plane curves with small linear orbits II*, Internat. J. Math. **11** (2000), 591–608.

[A-F5] P. Aluffi, C. Faber, *Limits of translates of plane curves—On a paper of Aldo Ghizzetti*, math.AG/0306348.

[B-K] E. Brieskorn, H. Knörrer, *Plane algebraic curves*, Birkhäuser, 1986.

[H-M] J. Harris, I. Morrison, *Moduli of curves*, Springer Verlag, 1998.

[H] K. Hulek, *Stable rank-2 vector bundles on* $\mathbb{P}^2$ *with* c_1 *odd*, Math. Ann. **242** (1979), 241–266.

[K-L] D. Kotschick, P. Lisca, *Instanton invariants of* $\mathbb{CP}^2$ *via topology*, Math. Ann. **303** (1995), 345–371.

[Oka] M. Oka, *Geometry of cuspidal sextics and their dual curves*, Singularities–Sapporo 1998, Adv. Stud. Pure Math, 29, Kinokuniya, Tokyo, 2000, pp. 245–277.

[S] H. Schubert, *Kalkül der abzählenden Geometrie (1879)*, reprinted with an introduction by S. L. Kleiman, Springer Verlag, 1979.

Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: `aluffi@math.fsu.edu`

Dept. of Mathematics, Oklahoma State University, Stillwater OK 74078, U.S.A.
E-mail address: `cffaber@math.okstate.edu`

Department of Mathematics, KTH, 100 44 Stockholm, Sweden
E-mail address: `carel@math.kth.se`