Residue Fields of Zero-Dimensional Rings*

ROBERT GILMER Department of Mathematics 3027, Florida State University, Tallahassee, Florida 32306-3027 USA. email: gilmer@math.fsu.edu.

We begin by defining a couple of terms. If $\mathcal{F} = \{F_i\}_{i \in I}$ is a family of fields, we say that $\mathcal{F}$ *is the family of residue fields of the ring* T if there exists a bijection $g : I \to \mathrm{MaxSpec}(T)$ such that $F_i \simeq T/g(i)$ for each $i \in I$. Note that the definition takes into account the multiplicity with which a given field occurs in $\mathcal{F}$. We say that $\mathcal{F}$ is *realizable* if $\mathcal{F}$ is the family of residue fields of a zero-dimensional ring. This paper is concerned with the following question (RF).

(RF) *What families of fields are realizable?*

At the outset we can say that while much is known in regard to (RF), the general case of the question remains open, and at this point there is no conjectured answer to (RF). Questions of this type have not received much attention in the study of commutative rings, and it seems worthwhile to ask how (RF) arose. For Heinzer and me, there were two primary sources of motivation. The first of these was Theorem 12 of [I], which states that a ring S is hereditarily zero-dimensional

* This paper contains material presented in the third 1994 Barrett Lecture, given on April 9. Most of its results come from the paper [GH6] (see the list of references for the paper *Background and Preliminaries* ... in this volume), and represent work done jointly with W. Heinzer.

if and only if S/P is an absolutely algebraic field of nonzero characteristic for each $P \in \mathrm{Spec}(S)$. In relation to this result, we wondered about families of such fields (that is, absolutely algebraic of nonzero characteristic) that could be realized as the family of residue fields of a zero-dimensional ring (see [GH4, p. 670]); such rings would, of course, be hereditarily zero-dimensional. A second source of motivation arose from a question mentioned in [II] and considered in [GH7]: What rings admit an Artinian subring? Proposition 1 and Corollary 2 yield the connection between this question and (RF). The statement of Proposition 1 uses the following notation. If T is a ring, $\mathcal{C}(T) = \{\mathrm{Char}(T/M) \ : \ M \in \mathrm{MaxSpec}(T)\}$.

Proposition 1. *If R is a subring of the ring S, then $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$; the inclusion may be proper, but if R is zero-dimensional, $\mathcal{C}(S) = \mathcal{C}(R)$.*

Proof. If $p \in \mathcal{C}(S) - \{0\}$, say $p = \mathrm{Char}(S/M)$ where $M \in \mathrm{MaxSpec}(S)$, then $M \cap R$ is contained in a maximal ideal P of R and $\mathrm{Char}(R/P) = p$. Therefore $p \in \mathcal{C}(R)$ and $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$.

The example $R = \mathbb{Z}$, $S = \mathbb{Q}$ shows that the inclusion $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$ may be proper, and equality of $\mathcal{C}(S)$ and $\mathcal{C}(R)$ in the case where R is zero-dimensional follows easily from the fact that each maximal ideal of R is contracted from S in that case. ∎

Corollary 2. *If the ring S is such that $\mathcal{C}(S)$ is infinite, then $\mathcal{C}(R)$ is infinite for each subring R of S. In particular, S admits no Artinian subring if $\mathcal{C}(S)$ is infinite.*

The next result includes some basic observations related to question (RF). In the statement of Theorem 3, we use the symbol $\dot\cup$ to denote disjoint union. For a ring R, we denote by $\mathcal{F}(R)$ the family $\{R/M \ : \ M \in \mathrm{MaxSpec}(R)\}$. If $\mathcal{F} = \{F_i\}_{i \in I}$ and $\mathcal{G} = \{K_j\}_{j \in J}$ are families of fields, then we write $\mathcal{F} = \mathcal{G}$ if there exists a bijection $f \ : \ I \to J$ such that F_i and $K_{f(i)}$ are isomorphic for each $i \in I$; this relation $=$ is an equivalence relation.

Theorem 3. ([GH6, Sect. 2]) *Let R be a ring with nilradical $N(R)$.*

(1) $\mathcal{F}(R) = \mathcal{F}(R/N(R))$.

(2) If $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are realizable families of fields, then so is $\dot\bigcup_{i=1}^{n} \mathcal{F}_i$. In particular, any finite nonempty family of fields is realizable.

(3) $\mathcal{G} = \{q(R/P_\lambda) \ : \ P_\lambda \in \mathrm{Spec}(R)\}$ is realizable; here $q(\bullet)$ denotes the quotient field.

(4) *If R is zero-dimensional and $\mathcal{C}(R)$ is infinite, then $0 \in \mathcal{C}(R)$.*

(5) *Suppose $\mathcal{F}$ is a nonempty family of fields and let $\mathcal{F} = \cup_{i\in I} \mathcal{F}_i$ be the partition of $\mathcal{F}$ into nonempty subfamilies $\mathcal{F}_i$ according to characteristic. If $\mathcal{F}$ is realizable, each $\mathcal{F}_i$ is realizable, but not conversely.*

Proof. (1) is clear. In (2), if $\mathcal{F}_i = \mathcal{F}(R_i)$ for the zero-dimensional ring R_i, then $R = R_1 \oplus \ldots \oplus R_n$ is zero-dimensional, and the known prime ideal structure of R implies that $\mathcal{F}(R) = \dot{\cup}_{i=1}^{n} \mathcal{F}(R_i)$.

(3): Part (1) shows that in proving (3), there is no loss of generality in assuming that R is reduced. Thus R is imbedded in $T = \prod_{\lambda\in\Lambda} q(R/P_\lambda)$, where $\mathrm{Spec}(R) = \{P_\lambda\}_{\lambda\in\Lambda}$. The ring T is zero-dimensional, so there exists a minimal zero-dimensional extension $S = R^0(T)$ of R in T. Each P_λ is contracted from T, and hence from S. Parts (2) and (3) of Theorem 2.5 of [II] then show that for each $\lambda \in \Lambda$, there exists a unique prime M_λ of S lying over P_λ in R; moreover, $\mathrm{Spec}(S) = \{M_\lambda\}_{\lambda\in\Lambda}$ and $S/M_\lambda \simeq q(R/P_\lambda)$ for each λ. Therefore $\mathcal{G}$ is realizable as $\mathcal{F}(S)$, where S is zero-dimensional.

(4): Since $\mathcal{C}(R)$ is infinite, $\mathrm{Char}(R) = 0$ and an isomorphic copy of $\mathbb{Z}$ is contained in R. The zero ideal of $\mathbb{Z}$ is contracted from R, so there exists $M \in \mathrm{Spec}(R)$ such that $M \cap \mathbb{Z} = (0)$. Then $0 = \mathrm{Char}(R/M) \in \mathcal{C}(R)$.

In (5) we assume that $\mathcal{F}$ is the family of residue fields of the zero-dimensional ring R, and for each $i \in I$ we denote by c_i the characteristic of the elements of $\mathcal{F}_i$. If $c_i \neq 0$, then $\mathcal{F}_i = \mathcal{F}(R/c_iR)$, where R/c_iR is zero-dimensional. If $c_i = 0$, then $\mathrm{Char}(R) = 0$ and the elements of $\mathcal{F}_i$ are precisely the maximal ideals of R that do not meet the multiplicative set N of nonzero integers in R. Therefore $\mathcal{F}_i = \mathcal{F}(R_N)$, where again $\dim(R_N) = 0$. If $p_1, p_2, \ldots$ is the sequence of positive primes, then for $\mathcal{F}_i$ the singleton set $\{\mathbb{Z}_{p_i}\}$, each $\mathcal{F}_i$ is realizable, but by (4) above, $\cup_1^\infty \mathcal{F}_i$ is not realizable. ∎

The family $\mathcal{F} = \{\mathbb{Z}_{p_i}\}_{i=1}^{\infty}$ just mentioned illustrates a couple of interesting observations. First, $\mathcal{F}$ is the family of residue fields of a ring, but it is not the family of residue fields of a zero-dimensional ring. Second, part (3) of Theorem 3 does not generalize to the statement that $\{q(R/P_a) : a \in A\}$ is realizable if $\{P_a : a \in A\} \subseteq \mathrm{Spec}(R)$ is such that $\cap_{a\in A} P_a = N(R)$, for $\cap_{i=1}^{\infty} p_i\mathbb{Z} = (0) = N(\mathbb{Z})$.

Theorem 4 represents one of our first efforts toward a general realizability theorem.

Theorem 4. ([GH6, Thm. 2.7]) *Suppose $\mathcal{F} = \{K\} \dot{\cup} \{K_a\}_{a\in A}$ is a family of* *fields such that, for each $a \in A$, there exists a monomorphism ϕ_a of K into K_a. Then $\mathcal{F}$ is realizable.*

Proof. Let $T = \prod_{a \in A} K_a$, let $I = \sum \oplus K_a$ be the direct sum ideal of T, let $K^* \simeq K$ be the image of K under the product map $\phi = \prod \phi_a$, and let $R = K^* + I$. We show that R is zero-dimensional and $\mathcal{F}(R) = \mathcal{F}$. Toward this end we note that the a-th projection map $p_a : T \to K_a$ restricts to a surjective homomorphism $p_a^* = p_a\big|_R$ on R with kernel $(1-e_a)T \cap R = (1-e_a)R = M_a$, where e_a is the primitive idempotent associated with a. Thus $\{M_a\}_{a \in A} \subseteq \operatorname{Spec}(R)$, where $R/M_a \simeq K_a$ for each $a \in A$. It is clear that $I \in \operatorname{Spec}(R)$, where $R/I \simeq K^* \simeq K$. Thus to complete the proof of Theorem 4, it suffices to show that an arbitrary maximal ideal M of R belongs to the set $W = \{I\} \cup \{M_a : a \in A\}$. If $1 - e_a \in M$ for some $a \in A$, this is clear; otherwise M contains each e_a, so M contains $\{e_a : a \in A\}R = I$, and hence $M = I$. ∎

In the notation of the proof of Theorem 4, we observe that $\{M_a : a \in A\}$ is a discrete subspace of $\operatorname{Spec}(R)$ under the Zariski topology and $\operatorname{Spec}(R)$ is the one-point compactification of the subspace $\{M_a : a \in A\}$.

Remark 5. Theorem 4 can be used to show that a von Neumann regular ring is not determined by its family of residue fields. For example, if F is a field and if $\mathcal{F}$ is the infinite family consisting of countably many copies of F, the proof of Theorem 4 shows that there exists a von Neumann regular ring R such that $\mathcal{F}(R) = \mathcal{F}$ and R has a unique non-principal maximal ideal. Thus R and $S = R \oplus R$ are nonisomorphic von Neumann regular rings with the same family of residue fields. We note that in this example, $\operatorname{Spec}(R)$ and $\operatorname{Spec}(S)$ are not homeomorphic as topological spaces, for $\operatorname{Spec}(R)$ has a unique non-isolated point, but $\operatorname{Spec}(S)$ has two such points. Popescu and Vraciu in [PV1] have considered the question suggested by this remark: Suppose R_1 and R_2 are von Neumann regular rings and $g : \operatorname{Spec}(R_1) \to \operatorname{Spec}(R_2)$ is a homeomorphism such that $R_1/M \simeq R_2/g(M)$ for each $M \in \operatorname{Spec}(R_1)$. Must R_1 and R_2 be isomorphic? Example 5.3 of [PV1] purports to give a negative answer to this question, but we discovered that the proposed rings $A[a], A[b]$ of that example are not only isomorphic, but are set theoretically equal. Roger Wiegand subsequently gave an example of rings R_1, R_2 that provide a negative answer to the question; he allowed us to include his example in [GH6] as Example 6.4.

Theorem 4 admits an extension, which we label as Theorem 4*.

Theorem 4*. *Suppose $\mathcal{F} = \{K_a\}_{a \in A}$ is a family of fields. If there exists a finite subset $\{a_i\}_{i=1}^n$ of A such that each K_a contains an isomorphic copy of at least one of the fields K_{a_i}, then $\mathcal{F}$ is realizable.*

Proof. Partition the set A into n subsets $A_1, \ldots, A_n$ by defining A_1 to be $\{a \in A : K_a$ contains an isomorphic copy of $K_{a_1}\}$ and, for $i > 1$, defining A_i to be $\{a \in A - (\cup_{j=1}^{i-1} A_j) : K_a$ contains an isomorphic copy of $K_{a_i}\}$. Without loss of generality assume that each A_i is nonempty. If $\mathcal{F}_i = \{K_a\}_{a \in A_i}$, then Theorem 4 shows that each $\mathcal{F}_i$ is realizable, and part (2) of Theorem 3 implies that $\dot{\bigcup}_{i=1}^n \mathcal{F}_i = \mathcal{F}$ is also realizable. ∎

Bill Heinzer and I tended to use the phrase "has only finitely many minimal elements" to describe a family of fields satisfying the hypothesis of Theorem 4*. In using this terminology one must be careful to realize that under the relation $F \leq K$ on fields F and K defined to mean that K contains an isomorphic copy of F, it is possible to have $F \leq K$ and $K \leq F$ while F and K are not isomorphic.[1] At this point I will remark that (RF) is a more general form of the question in this area that Bill and I were originally interested in. That question, which I will refer to here as (RF1), was motivated by the characterization of hereditarily zero-dimensional rings given in Theorem 12 of [I]. Thus (RF1) was the following question.

(RF1) *Suppose $\mathcal{F} = \{F_a\}_{a \in A}$ is a family of fields, where each F_a is absolutely algebraic of nonzero characteristic. Under what conditions is $\mathcal{F}$ realizable?*

We were, in fact, interested in a special case (RF2) of (RF1) and in a special case (RF3) of (RF2); (RF2) was the case of (RF1) where all the fields F_a are of the same characteristic, and (RF3) was the case of (RF2) in which each F_a is finite. I have already stated that there is no current conjectured answer to (RF). In fact, Bill and I never had such a conjecture, but at one point we did conjecture that if $\mathcal{F}$ is as in (RF1), then $\mathcal{F}$ is realizable if and only if it "has only finitely many minimal elements". We show presently that that conjecture is false for both (RF1) and (RF2). Corollary 7 below implies that our conjecture is valid for (RF3).

Theorem 6. ([GH6, Thm. 3.1]) *Suppose K is a field and R is a K-algebra such that R/M is a finite-dimensional K-algebra for each $M \in \mathrm{Spec}(R)$. Then there*

[1] For example, $F = \mathbb{C}$ and $K = \mathbb{C}(X)$, where $\mathbb{C}$ is the field of complex numbers and X is an indeterminate over $\mathbb{C}$, have this property.

exists a finite subset $\{M_i\}_{i=1}^n$ *of* $\mathrm{Spec}(R)$ *such that, for each* $M \in \mathrm{Spec}(R)$, *there exists* i *between* 1 *and* n *such that* R/M_i *is a* K*-subalgebra of* R/M.

Proof. By passage from R to $R/N(R)$ we assume without loss of generality that R is reduced. The hypothesis on R implies that R/M is integral over K for each $M \in \mathrm{Spec}(R)$, so R is integral over K (cf. [GH4, Prop. 4.1]). For each $M \in \mathrm{Spec}(R)$, let $Y(M) = \{u_i\}_{i=1}^k$ be a finite set of residue-class representatives of M in R so that $\{u_i + M\}_{i=1}^k$ is a basis for R/M as a vector space over K. The subring $K[Y(M)]$ of R is Artinian and reduced, and hence $M \cap K[Y(M)]$ is principal, generated by an idempotent e_M. Moreover, by choice of $Y(M)$, the rings R/M and $K[Y(M)]/(M \cap K[Y(M)])$ are isomorphic as K-algebras. Let $\mathcal{D}(1 - e_M) = \{P \in \mathrm{Spec}(R) : e_M \in P\}$; $\mathcal{D}(1 - e_M)$ is an open subset of $\mathrm{Spec}(R)$ containing M, and since e_M generates a maximal ideal of $K[Y(M)]$, each $P \in \mathcal{D}(1 - e_M)$ is such that R/P contains $K[Y(M)]/e_M K[Y(M)] \simeq R/M$ as a K-subalgebra. The set $\{\mathcal{D}(1 - e_M) : M \in \mathrm{Spec}(R)\}$ is an open cover of $\mathrm{Spec}(R)$. Since R is zero-dimensional, $\mathrm{Spec}(R)$ is compact [BP, Thm. 3.6], and hence there exists a finite subcover $\{\mathcal{D}(1 - e_{M_i})\}_{i=1}^n$ of $\mathrm{Spec}(R)$. Since each P in $\mathcal{D}(1 - e_{M_i})$ is such that R/P contains an isomorphic copy of R/M_i as a K-subalgebra, this completes the proof of Theorem 6. ∎

While Theorems 4* and 6 are results in a similar vein, there is a lack of parallelism in their statements that we have been unable to resolve. The point in question is this: In Theorem 6, R itself is required to contain an isomorphic copy of K, but in Theorem 4*, only the residue fields of R are hypothesized to contain copies of one of the fields $R/M_1, \ldots, R/M_n$. Thus, we do not know the answer to the following question (q).

(q) *Suppose* K *is a field and* R *is a ring. Assume that, for each* $M \in \mathrm{Spec}(R)$, *there exists a monomorphism* $\phi_M : K \to R/M$ *such that* $[R/M : \phi_M(K)]$ *is finite. Must there exist a finite subset* $\{M_i\}_{i=1}^n$ *of* $\mathrm{Spec}(R)$ *such that, for each* $M \in \mathrm{Spec}(R)$, *the field* R/M *contains an isomorphic copy of some* R/M_i*?*

In relation to (q), Roger Wiegand asked whether a von Neumann regular ring R must contain an isomorphic copy of a field K if each residue field of R contains a copy of K. Ray Heitmann provided the following example showing that Roger's question has a negative answer. Let t be an indeterminate over the field $\mathbb{Q}$, and for each $i \in \mathbb{Z}^+$, let $F_i = \mathbb{Q}(t)$. Let $T = \prod_{i=1}^{\infty} F_i$, let I be the direct sum ideal of T, and let $S = \mathbb{Q}^* + I$, where $\mathbb{Q}^*$ is the diagonal imbedding of $\mathbb{Q}$ in T. The family of residue fields of S is $\{\mathbb{Q}\} \cup \{F_i\}_{i=1}^{\infty}$. Let $f \in T$ be such that $f(i) = i$ for each $i \in \mathbb{Z}^+$. If R is the smallest zero-dimensional subring of T containing $S[f]$, then R is von Neumann regular and each residue field of R is isomorphic to $\mathbb{Q}(t)$, but $\mathbb{Q}(t)$ is not imbedded in R.

As stated above, the following Corollary 7 answers our question (RF3).

Corollary 7. *Let $\mathcal{F} = \{K_a\}_{a\in A}$ be a family of finite fields. Then $\mathcal{F}$ is realizable if and only if there exists a finite subset $\{a_i\}_{i=1}^n$ of A such that each K_a contains, up to isomorphism, at least one of the fields K_{a_i}.*

Proof. Sufficiency of the stated conditions follows from Theorem 4*. Conversely, if $\mathcal{F}$ is realizable, then part (4) of Theorem 3 shows that the set $\{\mathrm{Char}(K_a) : a \in A\}$ is finite. Thus, the partition of $\mathcal{F}$ according to characteristic yields only finitely many nonempty subsets $\mathcal{F}_1, \ldots, \mathcal{F}_n$, and part (5) of Theorem 3 shows that each $\mathcal{F}_i$ is realizable. To show that the condition of Corollary 7 is satisfied for $\mathcal{F}$, it suffices to show that it is satisfied for each $\mathcal{F}_i$. Thus, we assume $\mathcal{F}_i = \mathcal{F}(R_i)$, where R_i is zero-dimensional and each field in $\mathcal{F}_i$ has nonzero characteristic p_i. In this case the hypothesis of Theorem 6 is satisfied for the field $\mathbb{Z}_{p_i}$ and the ring R_i. That result then implies the desired conclusion. ∎

Suppose p is a prime integer. It follows from Corollary 7 that $\mathcal{F}_p = \{GF(p^q) : q \text{ is prime}\}$ is a family of finite fields that is not realizable. We remark that $GF(p) \simeq \mathbb{Z}_p$ is the unique field F such that $\{F\} \dot\cup \mathcal{F}_p$ is realizable.

Necessity of the condition on $\mathcal{F}$ in Corollary 7 does not extend to the case where $\mathcal{F}$ contains at least one infinite field, and it fails in a rather spectacular way, as indicated by the next result.

Result 8. ([GH6, Examp. 3.6]) Suppose p is prime. There exists a zero-dimensional ring R of characteristic p such that $\mathcal{F}(R) = \{K_i\}_{i=0}^{\infty}$, where no K_i contains an isomorphic copy of K_j for $j \neq i$, K_0 is absolutely algebraic, and K_i, for $i > 0$, is finite.

Result 8 can be obtained via a construction outlined in Result 9. The proof of Result 9 is akin to that of Theorem 4, and we refer the interested reader to [GH6, Thm. 3.3] for details.

Result 9. Suppose $\{K_j\}_{j\in J}$ is a nonempty family of subfields of a field L. For each $j \in J$ let F_j be a subfield of K_j. Let $T = \prod_{j\in J} K_j$, let I be the direct sum ideal of T, and let R be the subring of all sequences $\{a_j\}_{j\in J} \in \prod_{j\in J} F_j \subseteq T$ such that, for some $j_0 \in J$, $\{j \in J \ : \ a_j \neq a_{j_0}\}$ is finite. Then $R + I$ is a von Neumann regular ring and $\mathcal{F}(R+I) = \{K_j\}_{j\in J} \cup \{K\}$, where $K = \{x \in L \ : \ \{j \in J \ : \ x \notin F_j\}$ is finite$\}$.

In the notation of Result 9, take L to be an algebraic closure of $GF(p)$ and, for $i \in \mathbb{Z}^+$, take $K_i = GF(p^{d2^i q_i})$ and $F_i = GF(p^{d2^i})$, where d is a fixed positive integer and $q_1 < q_2 < \ldots$ is a sequence of odd primes. Still in the notion of Result 9, $K = \cup_{i=1}^{\infty} F_i$, an infinite field that contains no K_i. Because $\{K_i\}_{i=0}^{\infty} \cup \{K\}$ is realizable, this yields an example satisfying the conditions of Result 8.

Result 8 admits an analogue in the case of characteristic 0. To wit, there exists a zero-dimensional ring S containing $\mathbb{Q}$ such that $\mathcal{F}(S) = \{K_i\}_{i=0}^{\infty}$, where each K_i is "minimal" in $\{K_j\}_0^{\infty}$, K_0 is algebraic over $\mathbb{Q}$, and $[K_i \ : \ \mathbb{Q}]$, for $i > 0$, is finite. Again S may be obtained via Result 9, where L can be taken to be the abelian closure of $\mathbb{Q}$ and the fields K_i and F_i can be constructed as in the previous paragraph, using the fact that for each prime p, there exists an ascending sequence $\{L_i^{(p)}\}_{i=1}^{\infty}$ of subfields of L such that $L_i^{(p)}$ is cyclic of degree p^i over $\mathbb{Q}$ for each $i \in \mathbb{Z}^+$ (cf. [Wa, Sect. 7.3 and 13.1] or [GH10, Examp. 2.7]).

We turn now to a question suggested by part (5) of Theorem 3, namely: Suppose $\mathcal{F} = \cup_{i\in I} \mathcal{F}_i$ is the partition of $\mathcal{F}$, a family of fields, into nonempty subfamilies $\mathcal{F}_i$, and assume that each $\mathcal{F}_i$ is realizable. Under what conditions is $\mathcal{F}$ realizable? I will refer to this as the reverse realizability question. We have already observed that, in general, some conditions must be imposed on the subfamilies $\mathcal{F}_i$. A sufficient condition for $\mathcal{F}$ to be realizable is that I is finite, and part (4) of Theorem 3 shows that a necessary condition for realizability of $\mathcal{F}$ in the case where I is infinite is that $\mathcal{F}$ contains a field of characteristic 0. Proposition 10 is a basic result in regard to the reverse realizability question.

Proposition 10. ([GH6, Prop. 4.2]) *Let $\{T_a\}_{a\in A}$ be a family of von Neumann regular rings and let $T = \prod_{a\in A} T_a$. Let I be the direct sum ideal of T, let S be a zero-dimensional subring of T/I, and let R be the inverse image of S under the canonical map of T onto T/I. Then R is von Neumann regular and $\mathcal{F}(R) = \{\dot{\bigcup}_{a\in A} \mathcal{F}(T_a)\} \dot{\cup} \{\mathcal{F}(S)\}$.*

Again the proof of Proposition 10 is similar to that of Theorem 4, and we only indicate a sketch of a proof here. If $P \in \mathrm{Spec}(R)$ and if e_b denotes the primitive idempotent associated with b for $b \in A$, then either (1) $e_a \in P$ for each $a \in A$, or else (2) $1 - e_a \in P$ for some $a \in A$. In the first case $I \subseteq P$ and R/P is isomorphic to a residue field of S. In the second case $P/(1-e_a)R$ is prime in $R/(1-e_a)R \simeq T_a$ and R/P is isomorphic to a residue field of T_a.

Corollary 11. *Let the notation be as in the statement of Proposition 10, and assume that each $T_a = K_a$ is a field. If K is a subfield of T/I, then $\{K_a\}_{a\in A} \dot{\cup} \{K\}$ is realizable.*

Corollary 12. **([GH6, Cor. 4.4])** *Suppose $\mathcal{F} = \{K_a\}_{a\in A}$ is a family of fields, where A is infinite and, for each prime p, the set $\{a \in A : \mathrm{Char}(K_a) = p\}$ is finite. Then $\mathcal{F} \dot{\cup} \{\mathbb{Q}\}$ is realizable.*

Proof. Let $T = \prod_{a\in A} K_a$, let π be the prime subring of T, and let I be the direct sum ideal of T. The hypothesis of Corollary 12 implies that $\pi \simeq \mathbb{Z}$. Moreover, if $\phi : T \to T/I$ is the canonical map, then $\phi\big|_{\pi}$ is an isomorphism. Because $\{a \in A : \mathrm{Char}(K_a) = p\}$ is finite for each prime p, it follows that each nonzero element of $\phi(\pi)$ is a unit of T/I. Consequently, T/I contains an isomorphic copy of $\mathbb{Q}$, and Corollary 11 shows that the family $\mathcal{F} \dot{\cup} \{\mathbb{Q}\}$ is realizable. ∎

Theorem 13 is a broad generalization of the preceding corollary.

Theorem 13. **([W3])** *Let $\{p_i\}_{i\in I}$ be an infinite set of primes, and for $i \in I$, let T_i be a von Neumann regular ring of characteristic p_i. If X is a set of indeterminates over $\mathbb{Q}$ of cardinality $\aleph_1$, then $Q(X)$ imbeds in $\prod_{i\in I} T_i / \sum \oplus_{i\in I} T_i$.*

The case of Theorem 13 where X is finite is stated as Theorem 4.11 of [GH6], but in the Fall of 1992, Bill Heinzer and Roger Wiegand discovered a gap in the published proof of (4.11). Roger subsequently proved Theorem 13, an extension of the questionable result from [GH6], and he has allowed me to add his preprint [W3] containing a proof of Theorem 13 as an Appendix to this paper. The proof of Theorem 13 uses results from Ray Heitmann's thesis [He]. This was interesting to Bill and me for the following reason. When we initially started to work on (RF) (actually, on (RF1)), we tried to think of relevant literature. Heitmann's thesis, which deals with families of fields that can be realized as the family of residue fields of a PID, came to mind. But what is the connection between the PID case and

(RF)? The thing that occurred to us was that a proper homomorphic image T of a PID is zero-dimensional. However, $\mathcal{F}(T)$ is finite, and (RF) is uninteresting in the case of a finite family of fields, so Bill and I recognized little relevance of results of [He] to (RF). The main thing from [He] that R. Wiegand uses in his proof of Theorem 13 is a technique for controlling the way valuation overrings of a countable PID with quotient field K extend to a simple transcendental extension of K. The next result is immediate from Proposition 10 and Theorem 13.

Corollary 14. *Let $\{p_i\}_{i\in I}$ be an infinite set of primes, and for each $i \in I$, let $\mathcal{F}_i$ be a realizable family of fields of characteristic p_i. If X is a set of indeterminates over $\mathbb{Q}$ of cardinality at most $\aleph_1$, then $(\cup_{i\in I} \mathcal{F}_i) \cup \{\mathbb{Q}(X)\}$ is realizable.*

We remark that some restriction on $|X|$ is necessary in Corollary 14. For example, suppose the family $\{GF(p_i)\}_{i=1}^{\infty} \cup \{\mathbb{Q}(Y)\}$ is realizable as $\mathcal{F}(R)$, where $\{p_i\}_{i=1}^{\infty}$ is the set of primes, Y is a set of indeterminates over $\mathbb{Q}$, and R is von Neumann regular. Let M be the maximal ideal of R with residue field $\mathbb{Q}(Y)$ and let $M_i \in \mathrm{Spec}(R)$ have residue field $GF(p_i)$. Then $M \cap (\cap_{i=1}^{\infty} M_i) = (0)$ since R is reduced. We claim that $\cap_{i=1}^{\infty} M_i = (0)$. If not, then $J = \cap_1^{\infty} M_i \not\subseteq M$, and hence $R = J \oplus M$. The ring R/J is zero-dimensional and $\mathcal{F}(R/J) = \{GF(p_i)\}_{i=1}^{\infty}$ a contradiction to part (4) of Theorem 3. Thus $J = (0)$ and R is imbedded in $\prod_1^{\infty}(R/M_i) \simeq \prod_1^{\infty} GF(p_i) = T$. Consequently, $|Y| \leq |R| \leq |T|$.

We apply Corollary 14 to the reverse realizability question in Theorem 17. The proof of Theorem 17 uses a gluing process for maximal ideals introduced by Doering and Lequain in [DL]. We state and prove a special case of Theorem A of [DL] in Result 15, which is tailored to provide the application of the gluing process that we need.

Result 15. Suppose T is a ring containing maximal ideals M_i, P_i, for $1 \leq i \leq n$, such that $T/M_i \simeq T/P_i$ for each i. Then there exists a subring R of T such that the following conditions are satisfied.

(1) T is integral over R.
(2) For $1 \leq i \leq n$, $M_i \cap R = P_i \cap R = M_i \cap P_i$ and $R/(M_i \cap P_i) \simeq T/M_i$.
(3) For each $Q \in M\mathrm{Spec}(T) - (\{M_i\}_{i=1}^n \cup \{P_i\}_{i=1}^n)$, Q is the unique prime of T lying over $Q \cap R$ in R, and $T/Q \simeq R/(Q \cap R)$.

Thus, if R is zero-dimensional, then $\mathcal{F}(R) = \mathcal{F}(T) - \{T/M_i\}_1^n$.

Proof. The general case of Result 15 can be obtained by repeated application of the case where $n = 1$, so without loss of generality we consider this case, and we write M and P in place of M_1 and P_1. Let $f : T \to K$ and $g : T \to K$ be homomorphisms of T onto a field K with kernels M and P, respectively. Let $R = \{x \in T : f(x) = g(x)\}$. Clearly R is a subring of T containing $M \cap P$. To show that T is integral over R, take $t \in T$. Since M and P are comaximal, there exists $s \in T$ such that $f(s) = g(s) = f(t) + g(t)$. Thus $s \in R$. Moreover, $f(t^2 - st) = f(t)^2 - f(t)^2 - g(t)f(t) = -g(t)f(t)$, and similarly, $g(t^2 - st) = -g(t)f(t)$. Hence $t^2 - st \in R$, so t is integral over R. Therefore (1) is satisfied.

In (2), it suffices to show that $M \cap P = M \cap R$. The inclusion $M \cap P \subseteq M \cap R$ is clear. If $r \in M \cap R$, then $f(r) = 0 = g(r)$, so $r \in P$ and $r \in M \cap P$. Thus $M \cap R = M \cap P$, and to prove $T/M \simeq R/(M \cap P)$, it suffices to show that $T \subseteq R + M$. Toward this end, take $t \in T$. There exists $r \in T$ such that $f(r) = f(t) = g(r)$ by comaximality of M and P. Hence $r \in R$ and $t - r \in M$ since $f(t-r) = 0$. Therefore $t \in R + M$.

(3) Suppose $Q \in M\mathrm{Spec}(T) - \{M, P\}$ and suppose $Q^* \in \mathrm{Spec}(T)$ has contraction $Q \cap R$ in R. We wish to show that $Q^* = Q$. Since T/R is integral, Q^* is maximal in T. If $Q^* \neq Q$, then Q and $I = Q^* \cap M \cap P$ are comaximal. Therefore $1 = q + r$ for some $q \in Q$, $r \in I$. Then $1 - r = q \in Q \cap R = Q^* \cap R$, and this leads to the contradiction that $1 \in Q^*$. Hence Q is the unique prime of T lying over $Q \cap R$ in R. To prove $T/Q \simeq R/(Q \cap R)$, it suffices to show $T \subseteq R + Q$; this inclusion follows for any ideal J of T comaximal with $M \cap P$ since $(M \cap P) + J \subseteq R + J$. ∎

Corollary 16. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are realizable families of fields. Suppose there exist $F_1, \ldots, F_n \in \mathcal{F}$ and $K_1, \ldots, K_n \in \mathcal{G}$ such that $F_i \simeq K_i$ for each i. Then $\{\mathcal{F} - \{F_i\}_1^n\} \dot{\cup} \mathcal{G}$ is realizable.*

Proof. Let $\mathcal{F} = \mathcal{F}(S)$ and $\mathcal{G} = \mathcal{F}(T)$, where S and T are zero-dimensional. Result 15 implies that $S \oplus T$ admits a zero-dimensional subring R such that $\mathcal{F}(R) = \{\mathcal{F} - \{F_i\}_1^n\} \dot{\cup} \mathcal{G}$. ∎

Theorem 17. (cf. [GH6, Thm. 4.14]) *Let H be a set of distinct primes. For $p \in H$, let $\mathcal{F}_p$ be a realizable family of fields of characteristic p. Let $\mathcal{F}_0$ be a realizable family of fields of characteristic 0 such that $\mathbb{Q}(X) \in \mathcal{F}_0$ for some set X of indeterminates over $\mathbb{Q}$ of cardinality at most $\aleph_1$. Then $\mathcal{F}_0 \cup (\cup\{\mathcal{F}_p : p \in H\})$ is realizable.*

Proof. The statement is clear if the set H is finite. If H is infinite, Corollary 14 shows that the family $\mathcal{F} = \{Q(X)\} \cup (\cup\{\mathcal{F}_p \ : \ p \in H\})$ is realizable. We obtain the conclusion of Theorem 17 by applying Corollary 16 to the families $\mathcal{F}$ and $\mathcal{F}_0$. ■

Throughout this paper we have chosen to not make an issue of the meaning of expressions such as $\mathcal{F} = \mathcal{G}$, $\mathcal{F} = \mathcal{F}(R)$, etc. Nonetheless, there is an issue present in these expressions; primarily it has to do with the multiplicity with which a given field occurs in a given family. There is a sense, however, in which relevance of this multiplicity condition to an answer to (RF) is in doubt. Specifically, suppose $\mathcal{F}$ is a family of fields and $\mathcal{F}^*$ is a family of isomorphism-class representatives of $\mathcal{F}$. What is the relation between realizability of $\mathcal{F}$ and realizability of $\mathcal{F}^*$? This question is treated briefly in Section 4 of [GH6]. While there are a couple of theorems in that section showing, with additional restrictions placed on $\mathcal{F}$, that $\mathcal{F}$ and $\mathcal{F}^*$ are simultaneously realizable, there is no example in the general case showing that one of the families $\mathcal{F}$, $\mathcal{F}^*$ is realizable while the other is not. In families $\mathcal{F}^*$, of course, multiplicity of occurrence of a given field in the family is not an issue.

Again we conclude by mentioning two questions related to material in this paper. The first of these is (RF) itself. As we have already stated, there is no conjectured answer at this point for either (RF) or for its special case (RF1) or (RF2). It is quite likely that more spadework needs to be done in this area before a reasonable conjecture emerges. We have briefly considered a specific question that is related to Remark 5. We label this question as (RF4).

(RF4) *Suppose $\mathcal{F} = \{K_a\}_{a \in A}$ is a family of fields. For what fields K does there exist a von Neumann regular ring R_K such that the following conditions are satisfied?*

(1) R_K has a unique maximal ideal M that is not principal.

(2) $R_K/M \simeq K$.

(3) $\mathcal{F}$ is the family of residue fields of R_K at the set of principal maximal ideals of R_K.

If R_K and M are as in the statement of (RF4), then $\mathrm{Spec}(R_K) - \{M\}$ is a discrete subspace of $\mathrm{Spec}(R_K)$ and $\mathrm{Spec}(R_K)$ is the one-point compactification of $\mathrm{Spec}(R_K) - \{M\}$. A test case of (RF4) is that in which $\mathcal{F} = \{GF(p) \ : \ p$ is prime$\}$. A supplementary field K for $\mathcal{F}$ must be of characteristic 0. In this regard, Corollary 14 shows that we can take $K = \mathbb{Q}(X)$ for any set X of indeterminates over $\mathbb{Q}$ of cardinality at most $\aleph_1$. For a supplementary K that is algebraic over $\mathbb{Q}$, we have shown that $\mathbb{Q}$ is the only admissible choice for K [GH6, p. 143].

The second question we mention is the same as (RF), without the requirement of zero-dimensionality of a realization.

(RF5) *What families of fields can be realized as the family of residue fields of a commutative ring?*

No example is known of a family of fields that cannot be so realized, and a beginning point on (RF5) would be to produce such a family.

Appendix

BIG TRANSCENDENCE DEGREE

Roger Wiegand

Since the proof of Theorem 4.11 in [GH] appears to be invalid, I will show how to use the construction in [H] to prove an even stronger result.

Theorem. *Let $\{q_i\}$ be an infinite set of primes, and let T_i be a von Neumann regular ring of characteristic q_i. Let X be a set of indeterminates with cardinality $\aleph_1$. Then $\mathbf{Q}(X)$ embeds in $\prod_i T_i/\sum_i T_i$.*

Proof. Let $\mathbf{F}_i$ be the prime field with q_i elements, regarded as a subring of T_i. Then $\prod_i \mathbf{F}_i/\sum_i \mathbf{F}_i$ embeds in $\prod_i T_i/\sum_i T_i$, so we may as well replace T_i by $\mathbf{F}_i$. We note that there is a principal ideal domain D_0 with quotient field $\mathbf{Q}$ whose residue fields at maximal ideals are exactly the fields $\mathbf{F}_i$. (Start with $\mathbf{Z}$ and just invert all the primes not in the set $\{q_i\}$.) We will use repeatedly the following lemma, which is a special case of the construction in [H]:

Lemma. *Let D be a countable principal ideal domain, and let $\{P_i\}$ be its set of maximal ideals. Let K be the quotient field of D, and let x be transcendental over D. Then there is a countable PID $\widetilde{D}$ satisfying the following conditions:*

(1) *The maps $I \mapsto I\widetilde{D}$ and $J \mapsto J \cap D$ are reciprocal bijections between {ideals*

of D} and {ideals of $\widetilde{D}$}.

(2) *Letting $\widetilde{P}_i = P_i\,\widetilde{D}$, we have $D/P_i = \widetilde{D}\,/\,\widetilde{P}_i$ for all i (that is, the canonical injections are all isomorphisms).*

(3) *The quotient field of $\widetilde{D}$ is $K(x)$.*

Proof. Choose elements $d_i \in D - P_i$ such that every non-zero $f \in D[x]$ belongs to at most finitely many of the ideals $(P_i, x - d_i)$ in $D[x]$, that is, $f(d_i) \in P_i$ for only finitely many i. (This is possible because $D[x]$ is countable and because $|D/P_i| \to \infty$ as $i \to \infty$. See [H, p. 571] for the details. Note that we are not assuming that the residue fields are finite.)

By [H, Proposition 1.2] we can find, for each i, a discrete valuation v_i of $K(x)$, with valuation ring V_i and maximal ideal $\mathbf{m}_i$, such that all of the following good stuff happens:

(1) $D[x] \subseteq V_i$.

(2) $\mathbf{m}_i \cap D[x] = (P_i, x - d_i)$.

(3) $V_i/\mathbf{m}_i = D/P_i$.

(4) $v_i|K = v_{P_i}$ (the P_i-adic valuation).

(5) $v_i(x - d_i) = 1$.

Let $\widetilde{D} = \bigcap_i V_i$. Then $\widetilde{D}$ is a Dedekind domain, its maximal ideals are exactly the ideals $\widetilde{P}_i := \mathbf{m}_i \cap \widetilde{D}$, $D/P_i = \widetilde{D}\,/\,\widetilde{P}_i$, and $\widetilde{P}_i \neq \widetilde{P}_j$ if $i \neq j$. (See Theorem 2.3 of [H] and the *Note* in the middle of page 570.) It follows from (4) that $\widetilde{P}_i \cap D = P_i$. To get the bijections on ideals, it is enough to show (since D and $\widetilde{D}$ are Dedekind domains) that $P_i\,\widetilde{D} = \widetilde{P}_i$. To prove this, choose $t_i \in D$ such that $Dt_i = P_i$. Then $V_i t_i = \mathbf{m}_i$ by (4). We surely have that $\widetilde{D}\,t_i$ is a positive power of $\widetilde{P}_i$. If the power were greater than one, we would have $t_i \in \widetilde{P}_i^2 \subseteq \mathbf{m}_i^2$, contradicting the fact that $v_i(t_i) = 1$ (by (4)). Thus $P_i\,\widetilde{D} = t_i\,\widetilde{D} = \widetilde{P}_i$. We also conclude from this that $\widetilde{D}$ is a principal ideal domain.

To see that the quotient field of $\widetilde{D}$ is $K(x)$, we just observe that (by (1)) $D[x] \subseteq \widetilde{D} \subseteq K(x)$. Finally, note that we have injections of residue fields $D_i/P_i \to \widetilde{D}\,/\,\widetilde{P}_i \to V_i/\mathbf{m}_i$. Since the composition of these injections is a bijection (by (3)), so is the map $D/P_i \to \widetilde{D}\,/\,\widetilde{P}_i$.

Returning to the proof of the theorem, we want to iterate this construction $\aleph_1$ times, taking unions at limit ordinals. In detail, let Ω be the first uncountable ordinal, and build principal ideal domains $D_\alpha, \alpha < \Omega$, according to the following rules:

(1) If $0 \leq \alpha \leq \beta < \Omega$, then $D_\alpha \subseteq D_\beta$, the two rings have isomorphic ideal lattices via extension and contraction, and corresponding maximal

ideals of D_α and D_β have canonically isomorphic residue fields.

(2) $D_{\alpha+1} = \widetilde{D}$ for $0 \le \alpha < \Omega$.

(3) $D_\lambda = \bigcup_{\alpha<\lambda} D_\alpha$ if λ is a limit ordinal less than Ω.

The only thing that needs to be checked is that taking unions doesn't mess up anything. So, let λ be a limit ordinal less than Ω, and assume inductively that all is well for ordinals strictly less than λ. Consider $D_\lambda = \bigcup_{\alpha<\lambda} D_\alpha$. Certainly D_λ is countable. If J is an ideal of D_λ, put $J_\alpha = J \cap D_\alpha$ for each $\alpha < \lambda$. Choose t such that $D_0 t = J_0$. For each $\alpha < \lambda$ we have $J_\alpha \cap D_0 = J_0$, and it follows from (1) that $J_0 D_\alpha = J_\alpha$, that is, $D_\alpha t = J_\alpha$. It follows immediately that $J = D_\lambda t$, and that $J = (J \cap D_\alpha) D_\lambda$ for all $\alpha < \lambda$. Thus D_λ is a PID. To complete the proof that we have a bijection on ideals, let $\alpha < \lambda$ and let I be an ideal of D_α, say, $I = D_\alpha u$. If $w \in ID_\lambda \cap D_\alpha = D_\lambda u \cap D_\alpha$, write $w = du$, with $d \in D_\lambda$. Then $d \in D_\beta$ for some β with $\alpha < \beta < \lambda$, and it follows from (1) that $w \in I$. Finally, D_λ has the right residue fields, since the union of a chain of isomorphisms is an isomorphism.

Put $E = \bigcup_{\alpha<\Omega} D_\alpha$. The arguments above show that E is a principal ideal domain with residue fields $\{\mathbf{F}_i\}$. Moreover, the quotient field of E is clearly $\mathbf{Q}(X)$. The natural map $E \to \prod_i \mathbf{F}_i$ (taking each element of E to its list of residue classes modulo the various maximal ideals) gives us a homomorphism $\phi : E \to \Gamma := \prod_i \mathbf{F}_i / \sum_i \mathbf{F}_i$. Since every non-zero element of E is in only finitely many maximal ideals, each non-zero element of E maps to a unit of Γ. Therefore ϕ induces an embedding of the quotient field $\mathbf{Q}(X)$ into Γ.

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