Stiefel–Whitney surfaces and the tri–genus of non–orientable 3–manifolds. *

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0. Introduction

To a compact 3–manifold M there are associated several different numerical invariants that turn out to be closely related. If M is endowed with a (unique) differentiable structure and $f : M \to R$ is a smooth function, one obtains information about the structure of M from the critical points of f. The invariant $F(M)$ is the minimum of $\{\mu_M(f)\}$ over all f, where $\mu_M(f)$ denotes the number of critical points of f. Some other invariants of M that are defined without a differentiable structure are

$C(M)$ = the smallest number of open cells needed to cover M,

$cat(M)$ = the smallest number of sets, open and contractible in M needed to cover M (This is the Lusternik–Schnirelman category),

$N_0(M)$ = the minimal number of charts (i.e., open subsets of R^3) needed to cover M.

Work by Hempel and McMillan [HM] on $C(M)$ and by Takens [T] on $F(M)$ implies

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that $F(M) = C(M)$, and in [GG] it is shown that, modulo the Poincaré conjecture, that $C(M) = cat(M)$. Furthermore $cat(M)$ depends only on $\pi_1(M)$; in fact $cat(M) = 2$ if $\pi_1(M) = 1$, $cat(M) = 3$ if $\pi_1(M)$ is free (non–trivial), and $cat(M) = 4$ otherwise.

For the study of $N_0(M)$ it turns out that a useful concept is that of categorical filling in the sense of Singhof [Si]. In particular, if M is closed and connected, $N_0(M)$ can be realized by orientable handlebodies; every M can be expressed as a union of three orientable handlebodies V_1, V_2, V_3 such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ is a surface for $i \neq j$. Of course, if M is orientable we need only two orientable handlebodies (Heegaard splitting), so the interesting case is when M is non–orientable. Thus $N_0(M) = 2$ or 3. It is shown in [BE] and [GGH] that $N_0(M) = 2$ if and only if M is the union of two orientable 3–manifolds W_1, W_2. In fact each W_i can be choosen to be obtained from $F \times [0,1]$, where F is a closed orientable surface, by attaching 1–handles along $F \times \{0\}$.

$N_0(M)$ is related to the first Stiefel–Whitney class $w_1(M) \in H^1(M; Z_2)$. If $\beta : H^1(M; Z_2) \to H^2(M; Z)$ is the Bockstein connecting homomorphism induced by the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$, then (see[BE], [GGH]) $N_0(M) = 2$ if and only if $\beta w_1(M) = 0 \in H^2(M; Z)$.

If M is expressed as the union of three orientable handlebodies V_1, V_2, V_3 whose interiors are pairwise disjoint, and g_i denotes the genus of ∂V_i, and if $g_1 \leq g_2 \leq g_3$, then we say that *M has a splitting of type* (g_1, g_2, g_3). Ordering types lexicographically we consider splittings of minimal type (g_1, g_2, g_3) and call this ordered triple the *tri–genus* of M.

In [GHN] we showed that every M has tri–genus $(0, g_2, g_3)$ or $(1, g_2, g_3)$; the former if $\beta w_1(M) = 0$, the latter if $\beta w_1(M) \neq 0$. This was shown by considering *Stiefel–Whitney surfaces* of M, i.e., closed connected surfaces F in M that represent the non–trivial element of the torsion subgroup of $H_2(M; Z)$. The homology class $[F] \in H_2(M; Z_2)$ is the Poincaré dual of $w_1(M)$ and $M - F$ is an orientable 3–manifold. Such a Stiefel–Whitney surface may be 1–sided or 2–sided in M and it is shown in [GHN] (see also [BE]) that M contains a 2–sided Stiefel–Whitney surface if and only if $\beta w_1(M) = 0$.

The genus g of a Stiefel–Whitney surface F gives useful information about g_2. In [GHN] it is shown that if F is 2–sided then M has a splitting of type $(0, 2g, g_3)$ and if F is 1–sided then M has a splitting of type $(1, 2g-1, g_3)$, for some g_3. Furthermore, in the case $\beta w_1(M) = 0$, the converse holds: If M has a splitting of type $(0, g_2, g_3)$, then M contains a 2–sided Stiefel–Whitney surface of genus g with $2g \leq g_2$. The idea of the proof is to start with a decomposition $M = B^3 \cup V_2 \cup V_3$ of type $(0, g_2, g_3)$ and to change the decomposition by cutting B^3 into subballs, that are attached to V_2 and V_3, to $M = W_1 \cup W_2$ where

W_1, W_2 are orientable 3–manifolds and where the union is along two orientable connected surfaces F_1, F_2, at least one of which has genus $\leq g_2/2$.

The main goal of this paper is to prove the converse in the case $\beta w_1(M) \neq 0$. The proof is considerably more difficult than in the above proof in the case $\beta w_1(M) = 0$.

Theorem A. *Let $M = V_1 \cup V_2 \cup V_3$ correspond to a splitting of type $(1, g_2, g_3)$. If $\beta w_1(M) \neq 0$, then M contains a (1–sided) Stiefel–Whitney surface of genus g with $2g - 1 \leq g_2$.*

Corollary. *(a) If $\beta w_1(M) = 0$, then M has tri–genus $(0, g_2, g_3)$ where $g_2 = 2g$ and g is the minimal genus of a 2–sided Stiefel–Whitney surface in M.*

(b) If $\beta w_1(M) \neq 0$, then M has tri–genus $(1, g_2, g_3)$ where $g_2 = 2g - 1$ and g is the minimal genus of a (1–sided) Stiefel–Whitney surface in M.

The computation of g_3 has been carried out in [N] for all Seifert fiber spaces M that are fiber bundles over S^1 with fiber an orientable surface. In these cases g_3 is much bigger than the (non–orientable) Heegaard genus $h(M)$ of M. In the remaining cases, i.e., for the Seifert fiber spaces M that do not fiber over S^1 with fiber an orientable surface, it is shown in [N] that $g_3 \leq h(m) + 1$ if $\beta w_1(M) = 0$. Furthermore it follows from [N] and Theorem A above that for these Seifert Fiber spaces $g_3 \leq h(M)$ if $\beta w_1(M) \neq 0$. The question of whether in these cases $g_3 = h(M) + 1$, resp. $g_3 = h(M)$, is still open. If M is not a Seifert fiber space we do not know if there is a relation between $h(M)$ and g_3.

Outline of the paper.

In Section 3 we start with a decomposition of $M = V_1 \cup V_2 \cup V_3$ where V_1 is a solid torus, V_2, V_3 are orientable 3–manifolds (not necessarily handlebodies). The goal is to show that for a decomposition with minimal complexity $c = (1, g_2, g_3)$, where $g_i = g(\partial V_i)$, this decomposition is nice, i.e., $V_1 \cap V_2$ and $V_1 \cap V_3$ consists of alternating annuli A_i, B_i on ∂V_1 as in Figure 2, and $V_2 \cap V_3 = G' \cup G''$ consists of 2-manifolds G' and G'' (not necessarily connected) such that $M' = \overline{M - V_1}$ cut along G' and M' cut along G'' is connected (and orientable). Section 1 provides some preliminary lemmas needed in this construction.

In Section 4 we show that we may connect the boundary curves of the 2–manifold G' (or G'') by annuli in V_1 as in Figure 5 to obtain a Stiefel–Whitney surface for M. Section 2 provides the technical support for this construction.

In Section 5 we prove a more general version of Theorem A in that V_2 and V_3 need not to be handlebodies.

Finally in Section 6 we consider as an Example a graph manifold and show how the Theorem applies to compute its tri–genus.

1. Preliminaries

Throughout this paper we work in the PL–category and use the following notations.

For a submanifold F of an n–manifold M^n we denote by $N(F)$ a regular neighborhood of F in M. All n–manifolds in this paper are compact.

For $(n-1)$–submanifolds F and G of ∂M^n and a homeomorphism $h : F \to G$ let M/h denote the quotient manifold $M/(x \sim h(x))$.

For n–submanifolds W_1, W_2 of M^n with $W_1 \cap W_2 = \partial W_1 \cap \partial W_2$ and an $(n-1)$–submanifold C of $W_1 \cap W_2$ we denote by $W_1 \cup W_2/C$ the union of a copy of W_1 and a copy of W_2 along C; i.e., if $N(W_1 \cap W_2) = (W_1 \cap W_2) \times I$ then $W_1 \cup W_2/C = \overline{W_1 \cup W_2 - (W_1 \cap W_2 - C) \times I}$.

Lemma 1. *Let M be a connected oriented n–manifold, F and G connected $(n-1)$-submanifolds of ∂M with the orientations of F and G induced by the orientation of M. Let $h : F \to G$ be a homeomorphism. Then M/h is orientable if and only if h is not orientation preserving.*

The proof is straightforward. □

Let W_1, W_2 be connected orientable n–manifolds and F_i' (resp. F_i''), $i = 1, \ldots, k$, be disjoint connected $(n-1)$–manifolds in ∂W_1 (resp. ∂W_2). Let $h_i : F_i' \to F_i''$ be a homeomorphism and $M_{ij} = W_1 \cup W_2/h_i \cup h_j$ $(i \neq j,\, i, j = 1, \ldots, k)$.

Lemma 2. *(a) If M_{12} and M_{23} are non-orientable then M_{13} is orientable.*

(b) If $M_{12}, M_{23}, \ldots, M_{\ell,\ell+1}$, are non-orientable for some $1 \leq \ell \leq k-1$ then $W_1 \cup W_2/h_1 \cup h_3 \cup \cdots \cup h_{2m+1}$ is orientable for all $2m+1 \leq \ell$.

(c) If M_{12} is non-orientable and M_{13} is orientable then M_{23} is non-orientable.

This follows easily from Lemma 1 . □

The genus $g(G)$ of a 2–manifold of components $G_1, \ldots, G_k$ is defined to be the sum of the genera $g(G_i)$, $i = 1, \ldots, k$. If G_i has non empty boundary then $g(G_i)$ is the genus of the closed surface obtained from G_i by capping off all boundary curves with disks.

For easy reference we state the following lemma in two parts, even though both parts follow from an easy computation of Euler characteristics.

Lemma 3. *Suppose G is a closed orientable 2–manifold of c components.*

(a) If G is obtained from 2-manifolds G', G'' of k', k'' components, resp., by identifying m boundary curves of G' with m boundary curves of G'', then $g(G) = (g(G') - k') + (g(G'') - k'') + c + m$.

(b) If G is obtained from connected surfaces $G_1, \ldots, G_k$ and m annuli by identifying boundary curves in pairs then

$$g(G) = \sum_{i=1}^{k} g(G_i) - k + c + m.$$

□

2. Arcs with colored vertices in a disk.

In this section we show that we can find a collection of disjoint proper arcs in a disk joining colored vertices and satisfying certain properties.

For an even number m we consider $\frac{m}{2}$ arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ properly embedded in a disk D^2 that connect (all) m given colored vertices on ∂D^2 in pairs. The colors of the vertices are labelled $1, 2, \ldots, k, \tau$. A *walk* of length n is an ordered sequence of n of the arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ such that for adjacent arcs α_i, α_j an endpoint of α_i carries the same color as an endpoint of α_j. A walk w is a *τ-avoiding* walk if no arc of w has an endpoint of color τ. A *reduced walk* is a walk of length n that visits $n + 1$ different colors.

A *loop* is a walk of length n that visits n different colors and such that, if $n > 1$, the boundary of each arc in the loop has two different colors.

Note that if we remove an arc from a loop of length $n > 1$ we obtain (after reordering, if necessary) a reduced walk of length $n-1$. Also note that a reduced walk does not contain a loop.

Lemma 4. *Let $v_1, \ldots, v_m \in \partial D^2$ be an ordered set of vertices with k colors, where the ordering is induced by an orientation of ∂D^2. If the set of vertices of color x has at least two elements for each $1 \le x \le k$, then there exist $k-1$ disjoint properly embedded arcs $\alpha_1, \ldots, \alpha_{k-1} \subset D^2$ such that for each i, α_i has its ends v_r, v_s of different colors, $r-s$ is odd, and $\alpha_1, \ldots, \alpha_{k-1}$ is a reduced walk.*

Proof: If $k \ge 2$, find adjacent vertices v_i, v_{i+1} of colors x and y resp., $x \ne y$ (here i is *mod* m). Find a properly embedded arc α_{k-1} with ends v_i, v_{i+1}. Then α_{k-1} splits D^2 into D' and D'', where $int(D'')$ contains no vertex; in D' delete vertices v_i, v_{i+1}, and color the vertices of color y with color x. By assumption this new set of color x has at least two elements. We find now the system $\alpha_1, \ldots, \alpha_{k-1}$ by induction on the number of colors applied to D' and the new colored vertices.

For each $1 \le i \le k-1$, α_i has its ends v_r, v_s of different colors, and, since we always connect adjacent vertices in each subdisk, $r-s$ is odd. Then $\alpha_1, \ldots, \alpha_{k-1}$ is a walk of length $k-1$ that visits k colors, that is, $\alpha_1, \ldots, \alpha_{k-1}$ is a reduced walk. $\square$

Lemma 5. *Let $X_1, \ldots, X_\ell \subset \{v_1, \ldots, v_m\}$ be ℓ subsets such that for each $i \in \{1, \ldots, \ell\}$, X_i contains at least two elements and all elements of X_i have color i.*

Let $Y = \{v_1, \ldots, v_m\} - \bigcup\limits_{i=1}^{\ell} X_i$ be a set of vertices of color $\tau \notin \{1, \ldots, \ell\}$. Then there is a set of properly embedded arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}} \subset D^2$ connecting the m vertices such that

(a) *If $y = \#Y \ge \sum\limits_{i=1}^{\ell} \#X_i - 2(\ell-1)$, then*

(1) there are no τ–avoiding loops,

(2) no arc joins two vertices of the same color $c \in \{1, \ldots, \ell\}$, and

(3) for given colors $c_1, c_2 \in \{1, \ldots, \ell\}$, there is at most one arc joining a vertex of color c_1 to a vertex of color c_2.

(b) *If $y = \#Y < \sum\limits_{i=1}^{\ell} \#X_i - 2(\ell-1)$, then*

(1) there are $\ell-1$ arcs $\alpha_1, \ldots, \alpha_{\ell-1}$ that do not contain color τ and the system $\alpha_1, \ldots, \alpha_{\ell-1}$ is a reduced walk,

(2) there are y arcs $\alpha_\ell, \ldots, \alpha_{\ell+y}$ each of which has exactly one endpoint of color τ, and

(3) each of the remaining arcs does not contain color τ.

Proof: The proof is by induction on $\ell + y$.

In case (a) the induction starts with $\ell + y = 2$, i.e., $\ell = 0$, $y = 2$, since $\#Y \geq \sum_{i=1}^{\ell} \#X_i - 2(\ell - 1) \geq 2\ell - 2(\ell - 1) = 2$. Then $m = 2$, and we find and arc $\alpha_1 \subset D^2$ with $\partial \alpha_1 = \{v_1, v_2\}$.

In case (b) the induction starts with $\ell + y = 1$, i.e., $\ell = 1$ $y = 0$. We find here a set of arcs joining adjacent vertices.

Assume $\ell \geq 1$.

First case: There is a pair of vertices v_i, v_{i+1} of colors $c_1, c_2 \in \{1, \ldots, \ell\}$, resp., and $c_1 \neq c_2$. An arc $\alpha \subset D^2$ which joins v_i and v_{i+1} splits D^2 into D' and D'', where $int(D'')$ contains no vertex.

In D' define

$$X'_j := X_j, \text{ for } j \in \{1, \ldots, \ell\} - \{c_1, c_2\},$$

$$X'_{c_1} := X_{c_1} \cup X_{c_2} - \{v_i, v_{i+1}\},$$

$$Y' := (\{v_1, \ldots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup X'_j,$$

$$\ell' := \ell - 1,$$

and paint all the vertices of X'_{c_1} with color c_1. Then $\ell' + \#Y' < \ell + \#Y$.

In case (a),

$$\#Y' = \#Y \geq \sum \#X_j - 2(\ell - 1) = \sum \#X'_j + 2 - 2(\ell - 1) = \sum \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3). Clearly $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}, \alpha$ satisfies (2) and (3) for D^2. A τ–avoiding loop would be of the form $\alpha, \alpha_{i_1}, \ldots, \alpha_{i_r}$ and the colors c_1 and c_2 appear in $\alpha_{i_1}, \ldots, \alpha_{i_r}$. Then $\alpha_{i_1}, \ldots, \alpha_{i_r}$ would be a reduced walk visiting $r + 1$ colors. Since in D' color c_1 equals color c_2, $\alpha_{i_1}, \ldots, \alpha_{i_r}$ would be a loop in D', a contradiction. Hence $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}, \alpha$ satisfies also (1).

In case (b),

$$\#Y' = \#Y < \sum \#X_j - 2(\ell - 1) = \sum \#X'_j + 2 - 2(\ell - 1) = \sum \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha_\ell, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3). Since $\alpha_1, \ldots, \alpha_{\ell-2}$ is a reduced walk that visits $\ell - 1$ colors in D', and since color c_2 was changed to color c_1 in D', it follows that $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha$ visits ℓ colors in D, hence it is a reduced walk in D^2. Clearly (2) and (3) hold for the system $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha, \alpha_\ell, \ldots, \alpha_{\frac{m}{2}}$ in D^2.

Second case: For each i, either v_i and v_{i+1} have the same color $c \in \{1, \ldots, \ell\}$, or one of them has color τ. Find a pair v_i, v_{i+1} one with color τ and the other with color $c \in \{1, \ldots, \ell\}$ (possible, for $\ell \geq 1$). We assume, without loss of generality, that v_i has color τ, and v_{i+1} has color c. Find an arc $\alpha_{\frac{m}{2}} \subset D^2$ connecting v_i with v_{i+1}; then $\alpha_{\frac{m}{2}}$ splits D^2 into D' and D'', where $int(D'')$ contains no vertex.

First subcase: $\#X_c > 2$. In D' we define

$$X'_j := X_j, \text{ for } j \in \{1, \ldots, \ell\} - \{c\},$$

$$X'_c := X_c - \{v_{i+1}\},$$

$$Y' := (\{v_1, \ldots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup X'_j,$$

$$\ell' := \ell.$$

Then $\ell' + \#Y' < \ell + \#Y$.

In case (a),

$$\#Y' = \#Y - 1 \geq \sum \#X_j - 2(\ell - 1) - 1 = \sum \#X'_j + 1 - 2(\ell - 1) - 1 = \sum \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3). Clearly $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) for D^2.

In case (b) we have

$$\#Y' = \#Y - 1 < \sum \#X_j - 2(\ell - 1) - 1 = \sum \#X'_j + 1 - 2(\ell - 1) - 1 = \sum \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_{\ell+1}, \ldots, \alpha_{\ell+y}, \alpha_{\ell+y+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3) in D'. It is easy to see that then $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha, \alpha_{\ell+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) in D^2.

Second subcase: $\#X_c = 2$. Paint vertex v_r with color τ, where $X_c = \{v_{i+1}, v_r\}$. In D' define

$$X'_j := X_j, \text{ for } j \in \{1, \ldots, \ell\} - \{c\},$$

$$Y' := (\{v_1, \ldots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup_{j \neq c} X'_j,$$

$$\ell' := \ell - 1.$$

Then $\ell' + \#Y' < \ell + \#Y$.

In case (a) we have

$$\#Y' = \#Y \geq \sum \#X_j - 2(\ell - 1) = \sum_{j \neq c} \#X'_j + 2 - 2(\ell - 1) = \sum_{j \neq c} \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3), and is easy to see that $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) for D^2.

In case (b) we have

$$\#Y' = \#Y < \sum \#X_j - 2(\ell - 1) = \sum_{j \neq c} \#X'_j + 2 - 2(\ell - 1) = \sum_{j \neq c} \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of $\dfrac{m}{2} - 1$ arcs $\alpha_1, \ldots, \alpha_{\ell-2}, \beta,\ \alpha_{\ell+1}, \ldots, \alpha_{\ell+y}$, $\alpha_{\ell+y+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3) in D'. Here β is an arc that connects vertex v_r to a vertex of a color different from τ. Then the system $\alpha_1, \ldots, \alpha_{\ell-2}, \beta$ is a reduced walk in D^2 and the arcs $\alpha, \alpha_{\ell+1}, \ldots, \alpha_{\ell+y}$ satisfy (2) in D^2, and the system $\alpha_1, \ldots, \alpha_{\ell-2}, \beta, \alpha, \alpha_{\ell+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3). $\square$

3. Minimal decompositions of M.

In this section M is a closed non-orientable 3-manifold that admits a decomposition (V_1, V_2, V_3) as follows:

$$(*) \qquad M = V_1 \cup V_2 \cup V_3$$

V_1 is a solid torus, V_2, V_3 are orientable 3-manifolds, $V_i \cap V_j = \partial V_i \cap \partial V_j$ is a 2-manifold for $i \neq j$.

Let $g_i = g(\partial V_i)$ be the sum of the genera of the boundary surfaces of V_i. We assume that $g_1 = 1 \leq g_2 \leq g_3$ and define the *complexity* c of the decomposition (V_1, V_2, V_3) is the ordered triple $c = (g_1, g_2, g_3) = (1, g_2, g_3)$.

We first choose a decomposition (V_1, V_2, V_3) for M for which c is minimal. Let d be the sum of the number of components of $V_1 \cap V_2$ and $V_1 \cap V_3$. Among all decompositions (V_1, V_2, V_3) of M with complexity c we choose one for which d is minimal.

The goal in this section is to show, step by step, that the given decomposition of M into (V_1, V_2, V_3) has certain additional properties.

Claim (1): *The intersections $V_1 \cap V_2$ and $V_1 \cap V_3$ contain no disks.*

Proof: For suppose a component of $V_1 \cap V_2$ is a disk D. Let $N = D \times I$ be a regular neighborhood of D in V_2 such that $N \cap V_1 = D$, $N \cap V_3 = \partial D \times I$, see Fig. 1. For $V_2' = \overline{V_2 - N}$, $V_3' = V_3 \cup N$ we have $g(\partial V_2') = g_2$ and $g_3' = g(\partial V_3') \leq g_3$, since V_3' is obtained from V_3 by attaching a 2-handle to ∂V_3. Note that $g_2 \leq g_3'$, since otherwise M would have a decomposition (V_1, V_3', V_2') with complexity $c' < c$, which contradicts the minimality of c. For the decomposition (V_1, V_2', V_3') the total number of disk components of $V_1 \cap V_i'$ has not increased $(i = 2, 3)$, the complexity $c' \leq c$, and the number of components of the intersections $d' < d$, which contradicts the minimality of d.

A corresponding argument holds if $V_1 \cap V_3$ contains disk components. $\square$

By Claim 1 $V_1 \cap V_i$ consists only of annuli $(i = 2, 3)$, all parallel and essential in ∂V_1. Note that an annulus A on ∂V_i is *incompressible* in V_i if and only if ∂A is not contractible in V_i.

Claim (2): *If $\beta w_1(M) \neq 0$, then all annuli of $V_1 \cap V_i$ are incompressible in V_1 and $V_i (i = 2, 3)$.*

Proof: For if one annulus of $V_1 \cap V_i$ is compressible in V_1 then there is a decomposition of V_1 into 3-balls $B_1, B_1', \ldots, B_m, B_m'$ such that $B_i \cap B_j'$ is empty or a meridian disk and $(B_1 \cup B_2 \cup \cdots \cup B_m) \cap V_2 = V_1 \cap V_2$, $(B_1' \cup B_2' \cup \ldots \cup B_m') \cap V_3 = V_1 \cap V_3$ are the annuli of $V_1 \cap V_2$ and $V_1 \cap V_3$. Then $M = W_2 \cup W_3$ where $W_2 = B_1 \cup \cdots \cup B_m \cup V_2$ and $W_3 = B_1' \cup \cdots \cup B_m' \cup V_3$ are orientable 3-manifolds. It follows from Lemma 8.1 of [BE] that $\beta w_1(M) = 0$, a contradiction.

If an annulus A of $V_1 \cap V_2$ is compressible in V_2 then there are two disjoint disks D_1, D_2 in V_2 such that $(D_1 \cup D_2) \cap V_1 = (\partial D_1 \cup \partial D_2) \cap V_1 = \partial A$ and $D_1 \cup D_2 \cup A$ bounds a 3-ball H

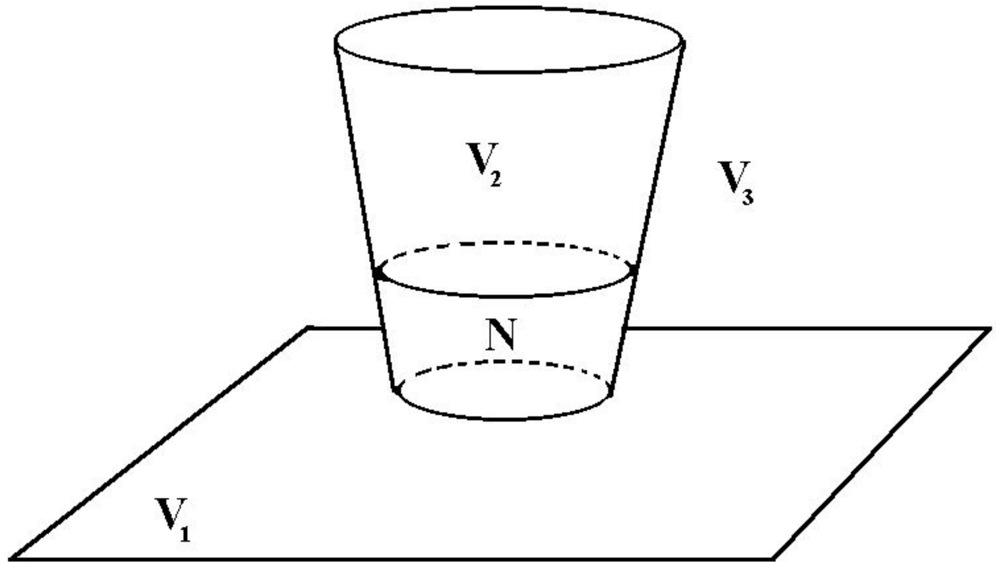

Figure 1.

in V_2. Then $V_2' = \overline{V_2 - H}$ is orientable and $V_1' = V_1 \cup H$ is a punctured lensspace $\overline{L - B_1^3}$. It follows that $\partial(V_2' \cup V_3) = \partial V_1'$ is a 2-sphere and $M = V_1' \cup V_2' \cup V_3 = L\#(B^3 \cup V_2' \cup V_3)$. As in the proof of Lemma 1.5 of [GHN] we cut B^3 into sub-balls $B_1, \ldots, B_s, B_1', \ldots, B_t'$ such that $B_i \cap V_2'$ are connected planar surfaces that meet V_3 only in their boundaries, $B_j' \cap V_3$ are connected planar surfaces meeting V_2' only in their boundaries, and such that $B^3 \cup V_2' \cup V_3 = W_2 \cup W_3$, where $W_2 = B_1 \cup \cdots \cup B_s \cup V_2'$, $W_3 = B_1' \cup \cdots \cup B_t' \cup V_3$ are orientable 3-manifolds. Hence $\beta w_1(M) = \beta w_1(B^3 \cup V_2' \cup V_3) = 0$ which violates the hypothesis.

The same argument shows that the annuli of $V_1 \cap V_3$ are incompressible in V_3. □

We now assume that $\beta w_1(M) \neq 0$ and the decomposition (V_1, V_2, V_3) is such that

$$V_1 \cap V_2 = A_1 \cup \cdots \cup A_m$$

$$V_1 \cap V_3 = B_1 \cup \cdots \cup B_m,$$

where the annuli A_i, B_j are incompressible in V_1 and all A_i are incompressible in V_2, all B_j are incompressible in V_3.

Recall that among all such decompositions with a fixed $g_2 = g(\partial V_2)$ we are assuming that we have chosen one for which m is minimal.

V_1 admits a Seifert fibration such that the annuli A_i, B_j are fibered. In the schematic pictures that follow, V_1 is represented by the orbit surface of this Seifert fibration. We chose an oriented crossing curve c on ∂V_1 (i.e. c intersects each fiber exactly once) and label the annuli A_i, B_j so that, following c, they are encountered in the sequence $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$, see Fig. 2.

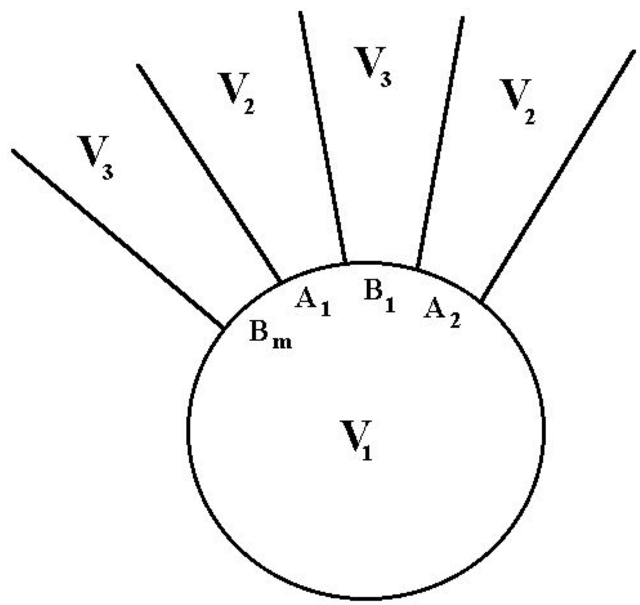

Figure 2.

Claim (3): *$V_1 \cup V_2/(A_i \cup A_{i+1})$ and $V_1 \cup V_3/(B_i \cup B_{i+1})$ are non-orientable.*

Proof: Let $N = (A_i \cup B_i \cup A_{i+1}) \times I$ be a regular neighborhood of $int(A_i \cup B_i \cup A_{i+1})$ in V_1 such that $N \cap V_2 = A_i \cup A_{i+1}$, $N \cap V_3 = B_i$, see Fig. 3. Note that $V_1 \cup V_2/(A_i \cup A_{i+1})$ is homeomorphic to the submanifold $V_2' = V_2 \cup N$ of M. If V_2' is orientable, then for $V_1' = \overline{V_1 - N}$ we have a decomposition (V_1', V_2', V_3) for M of the same complexity c, since V_1' is homeomorphic to V_1, and $g(\partial V_2') = g(\partial V_2)$, because the incompressible annuli A_i and A_{i+1} of ∂V_2 are replaced by the essential annuli B_i and $\overline{\partial N - A_i \cup B_i \cup A_{i+1}}$ of $\partial V_2'$. In this new decomposition the two annuli A_i, A_{i+1} of $V_1 \cap V_2$ are replaced by one annulus

$A = \overline{\partial N - A_i \cup B_i \cup A_{i+1}}$ of $V_1' \cap V_2'$. Since ∂A is not contractible in V_2' and V_3 this contradicts the minimality of m. □

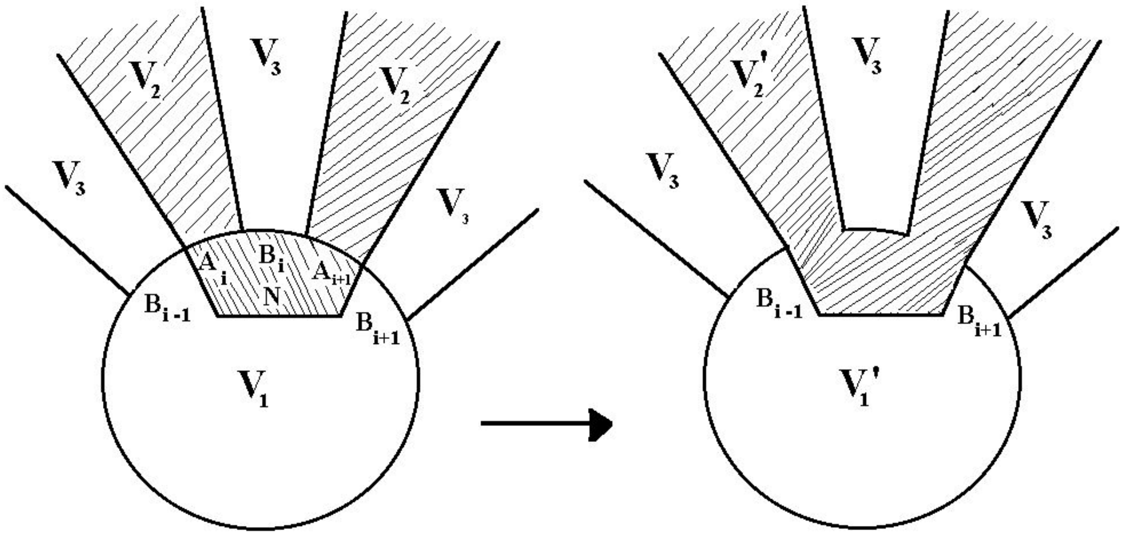

Figure 3.

Claim (4): *Let $A_{i_1}, \ldots, A_{i_l}$ be a subcollection of the annuli $A_1, \ldots, A_m$. Then $V_1 \cup V_2/(A_{i_1} \cup \cdots \cup A_{i_l})$ is orientable if and only if all indices are even or all indices are odd. An analogous statement holds if we identify V_1 and V_3 along a subcollection of $B_1, \ldots, B_m$. In particular m is even.*

Proof: Think of $V_1 \cup V_2/A_i$ as being obtained from $V_1 \cup V_2$ by gluing a copy of A_i on ∂V_1 to a copy of A_i on ∂V_2 with a homeomorphism h_i; i.e. $V_1 \cup V_2/A_i = V_1 \cup V_2/h_i$. Relabelling the annuli we may assume $i_1 = 1$. First we show that $V_1 \cup V_2/(A_1 \cup A_3 \cup \cdots \cup A_{1+2\ell})$ is orientable for each ℓ. By Claim (3) $M_{12} = V_1 \cup V_2/(h_1 \cup h_2), M_{23} = V_1 \cup V_2/(h_2 \cup h_3), \ldots, M_{2\ell,2\ell+1} = V_1 \cup V_2/(h_{2\ell} \cup h_{2\ell+1})$ are non-orientable and Lemma 2 implies that $V_2 \cup V_3/(A_1 \cup A_3 \cup \cdots \cup A_{2\ell+1}) = V_1 \cup V_2/(h_1 \cup h_3 \cup \cdots \cup h_{2\ell+1})$ is orientable.

Now it suffices to show that $V_1 \cup V_2/(A_i \cup A_{i+2k+1})$ is non-orientable for each k. But this follows from Lemma 2(c), since $V_1 \cup V_2/(A_i \cup A_{i+2k})$ is orientable and $V_1 \cup V_2/(A_{i+2k} \cup A_{i+2k+1})$ is non-orientable by step 3. □

We denote components of $V_2 \cap V_3$ that contain a curve of $B_{i-1} \cap A_i$, $A_i \cap B_i$ by G'_t, G''_s, resp. When looking at Fig. 2 from a point in V_1, G'_t is to the left of V_2, and G''_s is to the right of V_2, accordingly we call the curve $c''_i = A_i \cap B_i$ the *right* boundary curve of A_i and $c'_i = A_i \cap B_{i-1}$ the *left* boundary curve of A_i.

Claim (5): *Let G'_t, G''_s be components of $V_2 \cap V_3$ that contain the left curve $c'_i = A_i \cap B_{i-1}$ and right curve $c''_j = A_j \cap B_j$, resp. Then G'_t and G''_s are distinct.*

Proof: For suppose $G'_t = G''_s$. For annuli A and B let $A \times I$ and $B \times I$ be solid tori in V_1 such that $A \times I \cap \partial V_1 = A \times \partial I$, $B \times I \cap \partial V_1 = B \times \partial I$, $A \times 0 = A_i$, $A \times 1 = A_j$, $B \times 0 = B_{i-1}$, $B \times 1 = B_j$, and $A \times I \cap B \times I$ is an anulus C in V_1 with boundary curves c'_i and c''_j (see Fig. 4). Here we assume that i and $i-1$ are different from j since a similar argument holds in these cases. Note that $V_2 \cup A \times I$ and $V_3 \cup B \times I$ are homeomorphic to $V_1 \cup V_2/A_i \cup A_j$ and $V_1 \cup V_3/B_{i-1} \cup B_j$, resp. By Claim (4) one is orientable, the other is non-orientable. Suppose that $V_2 \cup A \times I$ is non-orientable and think of $V_2 \cup A \times I$ as V_2/h, where $h : A_i \to A_j$ is a homeomorphism with $h(c'_i) = c''_j$. By Lemma 1, h is orientation preserving (where A_i, A_j have the orientations induced from V_2). Hence $h' = h|c'_i : c'_i \to c''_j$ is orientation preserving (where c'_i, c''_j receive their orientations from $G'_t = G''_s$) and $G'_t \cup C \cong G'_t/h'$ is a non-orientable surface in the boundary of the orientable manifold $V_3 \cup B \times I$, a contradiction. □

We now call the components G'_t (resp. G''_s) of $V_2 \cap V_3$ that contain a left (resp. right) boundary curve of some A_j, a *left* (resp. *right*) *component.* We let

$$G' = G'_1 \cup \cdots \cup G'_{k'},$$

$$G'' = G''_1 \cup \cdots \cup G''_{k''}$$

where k' (resp. k'') is the number of left (resp. right) components of $V_2 \cap V_3$.

Thus, $V_2 \cap V_3 = G' \cup G''$ and the left and right components alternate around the crossing curve c. Note that $V_2 \cup V_3/G''$ is obtained by cutting the submanifold $V_2 \cup V_3$ of M along G'.

Claim (6): *$V_2 \cup V_3/G''$ and $V_2 \cup V_3/G'$ are orientable.*

Proof: We show the first part. Clearly $V_2 \cup V_3/G''$ is orientable if G'' is connected. If $k'' >$

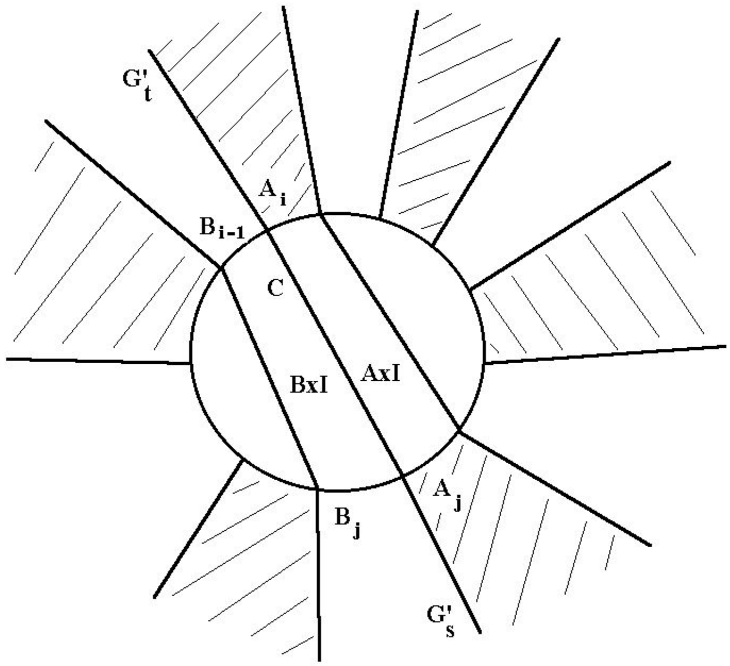

Figure 4.

1, suppose that running around the crossing curve c we encounter different components $G''_{i_1}, G'_{j_2}, G''_{i_2}$. Now $V_2 \cup V_3/(G''_{i_1} \cup G'_{j_2})$ is non-orientable, since it contains a submanifold homeomorphic to $V_1 \cup V_2/(A_i \cup A_{i+1})$ which is non-orientable by step(3). Similarly $V_2 \cup V_3/(G'_{j_2} \cup G''_{i_2})$ is non-orientable. Hence by Lemma 2, $V_2 \cup V_3/\,(G''_{i_1} \cup G''_{i_2})$ is orientable. Think of $V_2 \cup V_3/(G''_{i_1} \cup G''_{i_2})$ as $V_2 \cup V_3/(h''_{i_1} \cup h''_{i_2})$, where h''_i is a homeomorphism from a copy of G''_i in V_2 to a copy of G''_i in V_3. Then h''_{i_1} and h''_{i_2} are orientation reversing by Lemma 1. Now if $k'' > 2$, continuing along c, we let G''_{i_3} be the first component of G'' distinct from G''_{i_1}, G''_{i_2} and let G'_j be the component of G' just before G''_{i_3}. Thus we encounter along c the distinct components $G''_{i_1}, G'_j, G''_{i_3}$ or $G''_{i_2}, G'_j, G''_{i_3}$, say the former. By the above argument $V_2 \cup V_3/(G''_{i_1} \cup G'_j) = V_2 \cup V_3/(h''_{i_1} \cup h'_j)$ is nonorientable, and since h''_{i_1} is orientation reversing, h'_j is orientation preserving; also $V_2 \cup V_3/(G'_j \cup G''_{i_3}) = V_2 \cup V_3/(h'_j \cup h''_{i_3})$ is non-orientable and hence h''_{i_3} is orientation reversing. Therefore $V_2 \cup V_3/G''_{i_1} \cup G''_{i_2} \cup G''_{i_3} = V_2 \cup V_3/h''_{i_1} \cup h''_{i_2} \cup h''_{i_3}$ is orientable. Now the argument can be continued for the remaining components of G''. $\square$

4. Construction of a Stiefel-Whitney Surface.

We now show how to construct a Stiefel-Whitney surface F from the given decomposition (V_1, V_2, V_3) of M obtained in section 3.

Recall that $G' = G'_1 \cup \cdots \cup G'_{k'}$, $G'' = G''_1 \cup \cdots \cup G''_{k''}$ are the components of $V_2 \cap V_3$ such that $G' \cap V_1, (G'' \cap V_1)$ consist of the left, (right,resp.) boundary components of the annuli $A_1, \ldots, A_m$.

Let $C_1, \ldots, C_{\frac{m}{2}}$ be any collection of properly embedded disjoint annuli in V_1 such that $\partial(C_1 \cup \cdots \cup C_{\frac{m}{2}}) = \partial G'$ (see Fig. 5). Let $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$.

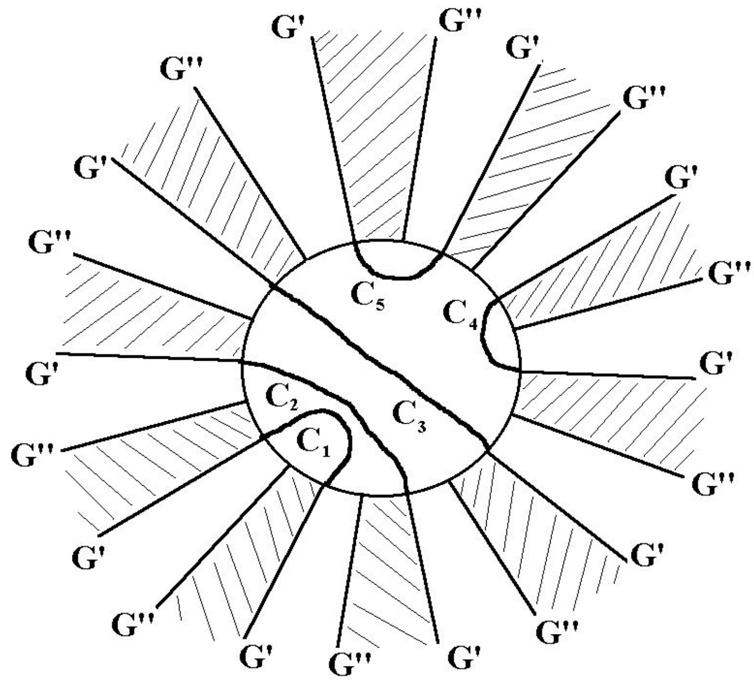

Figure 5.

Lemma 6. *$M - F'$ is orientable.*

Proof: By Claim 6 $M'' = V_2 \cup V_3 / G''$ is orientable. The annuli $C_1, \ldots, C_{\frac{m}{2}}$ cut V_1 into disjoint solid tori $W_1, \ldots, W_\ell$ and $\overline{M - N(F')}$ can be represented as $M'' \cup W_1 \cup \cdots \cup W_\ell$, where $W_i \cap (M'')$ consists of annuli $K_{i_1}, \ldots, K_{i_\ell}$ and where $K_j = A_{u_j} \cup B_{u_j}$ (for each j). If $W_i \cap M''$ consists of only one annulus K_{i_1}, , then $W_i \cup M''$ is orientable. Thus assume that W_i meets M'' in at least two annuli. Then W_i meets M'' in annuli $K_{i_1}, \ldots, K_{i_\ell}$ where the indices $i_1, \ldots, i_\ell$ differ by even numbers, since for each annulus C_i with boundary curves c'_r, c'_s we have that $r - s$ is odd. Note that the manifold $M'' \cup W_i = M'' \cup W_i / K_{i_1} \cup \cdots \cup K_{i_j}$

is homeomorphic to $M'' \cup W_i / A_{u_1} \cup \cdots \cup A_{u_\ell}$, where $K_{i_j} = A_{u_j} \cup B_{u_j}$. By Claim 4 it follows that $M'' \cup W_i$ is orientable. Since $W_1, \ldots, W_\ell$ are disjoint, it follows that $M'' \cup W_1 \cup \cdots \cup W_\ell$ is orientable. $\square$

Note that the same construction can be done with G'' instead of G' to obtain F''.

Lemma 7. *Assume that each G'_i has a least two boundary components. There are disjoint properly embedded annuli $C_1, \ldots, C_{k'-1}$ in V_1 such that for each $1 \leq l \leq k'-1$, ∂C_l consists of two left components of $\partial A_i, \partial A_j$ with $i-j$ odd and such that $G' \cup C_1 \cup \cdots \cup C_{k'-1}$ is connected and has the same genus as G'.*

Proof: The orbit surface D^2 of V_1 is a disk. Assign colors $1, 2, \ldots, k'$ to the components $G'_1, \ldots, G'_{k'}$, resp. Then $G'_i \cap V_1$ projects to a set of colored vertices on ∂D^2 with at least two elements. Apply the Lemma 4 and obtain $\alpha_1, \ldots, \alpha_{k'-1} \subset D^2$ a set of disjoint properly embedded arcs with ends of different colors. The lifts of these arcs to V_1 gives a system of disjoint annuli as required. $\square$

Lemma 8. *Assume that each G'_i has at least two boundary components. Then there are annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $\partial(C_1 \cup \cdots \cup C_{\frac{m}{2}}) = \partial G'$ and $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface.*

For if we choose the first $k'-1$ annuli C_i as in Lemma 7, we can complete this collection to a system $C_1, \ldots, C_{\frac{m}{2}}$ as required. $\square$

Now assume that $G'_1, \ldots, G'_l$ each have at least two boundary components and the surfaces $G'_{l+1}, \ldots, G'_{k'}$ each have connected boundary ($l < k'$). Let b_i denote the number of boundary components of G'_i and let $b = \sum_{i=1}^{\ell} b_i - 2(\ell - 1)$.

Lemma 9. *If $\beta w_1(M) \neq 0$, then $k' - \ell < b$.*

Proof: Assume $k' - \ell \geq b$. We will show that $\beta w_1(M) = 0$. For each $i \in \{1, \ldots, \ell\}$, assign color i to the component G'_i, and color each component $G'_{\ell+1}, \ldots, G'_{k'}$ with color τ. Then $G'_i \cap V_1$ projects to a set X_i of vertices of color i on ∂D^2, with $\# X_i \geq 2$

$(1 \leq i \leq \ell)$, and $G'_{\ell+1} \cap V_1, \ldots, G'_{k'} \cap V_1$ projects to a set Y of vertices with color τ, where D^2 is the orbit surface of V_1. We have that $Y = \{v_1, \ldots, v_m\} - \bigcup_{i=1}^{\ell} X_i$. By hypotesis, $\sum \# X_i - 2(\ell - 1) = b \leq k' - \ell = \# Y$, and we can apply Lemma 5 (a) to find disjoint arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}} \subset D^2$ such that the lifts of these arcs gives a system of disjoint annuli $C_1, \ldots, C_{\frac{m}{2}} \subset V_1$ in such a way that the 2-manifold $F' = G'_1 \cup \cdots \cup G'_{k'} \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a 2-manifold with empty boundary and, is obtained by joining different components $G'_1 \cup \cdots \cup G'_{k'}$ together. Since all G'_i are 2-sided in $V_2 \cup V_3$, it follows that F' is orientable and is 2–sided in M. By Lemma 6, $M - F'$ is orientable. Let $W_1 = N(F')$ be a regular neighborhood of F' in M, and $W_2 = \overline{M - W_1}$. Then W_1, and W_2 are both orientable 3–manifolds, and $M = W_1 \cup W_2$. By [BE, Lemma 8.1] we conclude that $\beta w_1(M) = 0$. □

Lemma 10. *If $\beta w_1(M) \neq 0$, there are disjoint annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface and such that*

(a) $F_l = G'_1 \cup \cdots \cup G'_l \cup C_1 \cup \cdots \cup C_{l-1}$ is connected.

(b) Each of $C_l, \ldots, C_{k'-1}$ joins a component G'_i to F_l where $l + 1 \leq i \leq k'$.

(c) Each of $C_{k'}, \ldots, C_{\frac{m}{2}}$ joins two boundary components of F_l.

Proof: As in the proof of Lemma 9, we paint the components $G'_1, \ldots, G'_\ell$ with colors $1, \ldots, \ell$, resp., and we paint the components $G_{\ell+1}, \ldots, G'_{k'}$ all with color τ. Since $\beta w_1(M) \neq 0$, we have that $k' - \ell < b$, and we can apply Lemma 5 (b) to the projections of $\{G'_i \cap V_1\}$ into the orbit space D^2 of V_1. Hence there is a collection of disjoint arcs in D^2 which interconnect the colored vertices in D^2 . Conditions (1), (2) and (3) imply that the surface F' constructed from the G'_i and the lifts C_i of the α_i is connected and satisfies (a),(b), and (c). □

5. Proof of the Theorem

Theorem. *Let (V_1, V_2, V_3) be a decomposition of M as in $(*)$ of Section 3 with $g_1 = 1 \leq g_2 \leq g_3$. If $\beta w_1(M) \neq 0$ then there is a Stiefel-Whitney surface F in M such that $g_2 \geq 2g(F) - 1$.*

Proof: We may assume that (V_1, V_2, V_3) has the properties of Section 3. We look at the surfaces $G' = G'_1 \cup \cdots \cup G'_{k'}$ and $G'' = G'' \cup \cdots \cup G''_{k''}$ obtained in Section 3. Note that $\partial V_2 = G' \cup G'' \cup A_1 \cup \cdots \cup A_m$, hence by Lemma 3(a), $g_2 = (g' - k') + (g'' - k'') + m + c$, where $g' = g(G'), g'' = g(G'')$, and c is the number of components of ∂V_2. Without loss of generality we assume that

$$(**) \qquad g' - k' \leq g'' - k''$$

Case I: Each G'_i has at least two boundary components.

By Lemma 8 there are annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $F = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface. By Lemma 3 (b) we have

$$g(F) = g' - k' + 1 + \frac{m}{2},$$

and by $(**)$

$$\begin{aligned} g_2 =& (g' - k') + (g'' + k'') + m + c \\ \geq& 2(g' - k') + m + c \\ \geq& 2(g' - k') + m + 1 \\ =& 2(g(F) - 1 - \frac{m}{2}) + m + 1 \\ =& 2g(F) - 1 \end{aligned}$$

Case II: There are l components $G'_1, \ldots, G'_l$ each with disconnected boundary and $k' - l$ components $G'_{l+1}, \ldots, G_{k'}$ each with connected boundary.

Let $F = F'$ of Lemma 10. Then F is obtained from F_l by gluing all the components G'_i (for $l + 1 \leq i \leq k'$) to F_l and identifying any remaining boundary components in pairs. In particular, F is connected and as in Case I we obtain $g_2 \geq 2g(F) - 1$. $\square$

Corollary. *Let M be a non–orientable 3–manifold. Then*

(a) $\beta w_1(M) = 0$ if and only if M has tri–genus $(0, 2g, g_3)$, where g is the minimal genus of a 2–sided Stiefel–Whitney surface for M.

(b) $\beta w_1(M) \neq 0$ if and only if M has tri–genus $(1, 2g-1, g_3)$, where g is the minimal genus of a 1–sided Stiefel–Whitney surface for M.

Proof: Part (a) is shown in Corollary (a) of [GHN]. There it is shown that M contains a 2–sided Stiefel–Whitney surface if and only if $\beta w_1(M) = 0$ and that M has splitting of type $(1, 2g-1, g_3)$ if M contains a 1–sided Stiefel–Whitney surface of genus g (Theorem 2.3 of [GHN]). Hence if $\beta w_1(M) \neq 0$, for the tri–genus $(1, g_2, g_3)$ we have $g_2 \leq 2g - 1$. By the Theorem, $g_2 \geq 2g - 1$. □

6. An application.

We show how the Theorem can be applied to compute the tri–genus of a certain graph manifold.

Let $\hat{M}_1$ and $\hat{M}_2$ be closed Seifert fiber spaces of type $NnII$. $\hat{M}_i$ can be described as follows (with the notations as in [S]). A simple closed curve α on the orbit surface $\hat{G}_i$ of $\hat{M}_i$ has valuation $\kappa(\alpha) = +1$ or -1, depending on whether translation of a fiber h_i in $\hat{M}_i$ along α preserves or reverses the orientation of h_i. The orbit surface $\hat{G}_i$ is a non–orientable surface represented as a connected sum of $k_i \geq 2$ projective planes with generators $v_1^i, \ldots, v_{k_i}^i$ and $\kappa(v_1^i) = +1$, and $\kappa(v_j^i) = -1$ for $j = 2, \ldots, k_i$ $(i = 1, 2)$. Now $p_i^{-1}(v_1^i)$ is a 1–sided torus T_i and $M - T_i$ is orientable, where $p_i : \hat{M}_i \to \hat{G}_i$ is the projection to the orbit surface. It is shown in [GHN] that the tri–genus of $\hat{M}_i$ is $(1, 1, g_{i3})$ for some $g_{i3} \geq 1$.

Let N_i be a regular neighborhood of a regular fiber h_i in $\hat{M}_i$; let $M_i = \overline{\hat{M}_i - N_i}$ and let $\varphi : \partial M_1 \to \partial M_2$ be a homeomorphism that is not fiber preserving. Consider the graph manifold

$$M = M_1 \cup_\varphi M_2.$$

We will compute the tri–genus of M. Notice that without the Theorem there seems to be no easy way to compute the tri–genus of M.

Proposition. *M has tri–genus $(1, 3, g_3)$, for some $g_3 \geq 3$.*

Proof. The proof of the proposition is in three steps.

(1) First we show that $\beta w_1(M) \neq 0$, hence the tri–genus of M is $(1, g_2, g_3)$ for some $1 \leq g_2 \leq g_3$.

(2) Then we show that there is a (1–sided) Stiefel–Whitney surface of genus $g = 2$ in M; hence by [GHN] the tri–genus of M is $(1, g_2, g_3)$, where $g_2 \leq 2g - 1 = 3$.

(3) Finally we show that there is no Stiefel–Whitney surface of genus less than 2 in M. Therefore by the Corollary, the tri–genus of M is $(1, 3, g_3)$ for some $g_3 \geq 3$.

(1). By [GGH] it suffices to show that M contains an orientation reversing curve t of finite order in $H_1(M)$. Now $H_1(M_i)$ has generators

$$v_1^{(i)}, \ldots, v_{k_i}^{(i)}, q_1^{(i)}, \ldots, q_{r_i}^{(i)}, h^{(i)}, d^{(i)},$$

where $d^{(i)}$ corresponds to a crossing curve on ∂N_i, and relations (besides being commutative)

$$\begin{aligned} d^{(i)} = -b^{(i)}h^{(i)} + \sum_{j=1}^{k_i} q_j^{(i)} - 2(v_1^{(i)} + \cdots + v_{k_i}^{(i)}), \\ \alpha_j^{(i)} q_j^{(i)} + \beta_j^{(i)} h^{(i)} =& 0, \\ 2h^{(i)} =& 0 \end{aligned} \tag{a}$$

(see [S]).

The homeomorphism $\varphi : \partial M_1 \to \partial M_2$ is determined, ut to isotopy, by the induced homomorphism $\varphi_* : H_1(\partial M_1) \to H_2(\partial M_2)$ which is described in terms of the basis $d^{(i)}, h^{(i)}$ of $H_1(\partial M_i)$ by

$$\varphi_*(d^{(1)}) = pd^{(2)} + qh^{(2)},$$

$$\varphi_*(h^{(1)}) = rd^{(2)} + sh^{(2)},$$

where p, q, r, s are integers such that $ps - qr = \pm 1$, and $r \neq 0$, since φ is not fiber preserving.

Then $H_1(M)$ is generated by the generators of $H_1(M_1)$ and $H_1(M_2)$ and has the relations of $H_1(M_1)$ and $H_1(M_2)$ and the additional relations

$$d^{(1)} = pd^{(2)} + qh^{(2)}, \tag{b}$$

$$h^{(1)} = rd^{(2)} + sh^{(2)}. \tag{c}$$

Note that $t = v_1^{(1)} + \cdots + v_{k_i}^{(1)}$ is an orientation reversing curve in M_1, hence in M. Since $h^{(i)}$ has finite order in $H_1(M)$ $(i = 1, 2)$, it follows from (c) that $d^{(2)}$ has finite order (since $r \neq 0$), and then from (b) we see that $d^{(1)}$ has finite order. From (a) it follows that $q_j^{(1)}$ has finite order for $j = 1, \ldots, k_1$. The relation $d^{(1)} = -b^{(1)}h^{(1)} + \sum_{j=1}^{k_1} q_j^{(1)} - 2(v_1^{(1)} + \cdots + v_{k_1}^{(1)})$ now shows that t has finite order.

(2). The vertical torus $T_i \subset M_i \subset \hat{M}_i$ is a Stiefel–Whitney surface for M_i. We show that there is a Stiefel–Whitney surface F in M which is homeomorphic to $T_1 \# T_2$. This implies that M has a decomposition of type $(1, 2g-1, g_3)$, where $g = g(F) = 2$.

To obtain F, join T_1 and T_2 by an arc α in M such that $\alpha \cap (T_1 \cup T_2) = \partial\alpha$, let $N(\alpha)$ be a neighborhood of α, rel $T_1 \cup T_2$, such that $N(\alpha) \cap T_i$ is a disk D_i, and let $F = T_1 \cup T_2 \cup \partial N(\alpha) - (int(D_1) \cup int(D_2))$. Note that F is homeomorphic to $T_1 \# T_2$ and $[F] = [T_1] + [T_2]$ in $H_2(M)$. Also F is not homologous to 0, since $T_1 \cup T_2$ does not separate M. In M_i we have $[\partial N(T_i)] = 2[T_i]$ in $H_2(M_i)$ and since $\partial(\overline{M_i - N(T_i)}) = T \cup \partial N(T_i)$ we have $2[T_i] = \pm[T]$ in $H_2(M_i)$, hence in $H_2(M)$. Orient T such that $[T] = 2[T_1]$ in $H_2(M)$; then $[T] = -2[T_2]$. Hence $2[F] = 2([T_1] + [T_2]) = [T] - [T] = 0$ in $H_2(M)$. It follows that $[F]$ generates the torsion of $H_2(M)$ and hence F is a Stiefel–Whitney surface of M.

(3). Let F be a (1–sided) Stiefel–Whitney surface in M. Then $g(F) \neq 0$, since a 2–sphere is 2–sided in M.

Suppose $g(F) = 1$.

Put F into general position with respect to T and let c be an intersection curve of $F \cap T$. Suppose c bounds a disk on T. If c also bounds a disk D in F then, since M is irreducible, we can deform D across c and decrease the number of intersection curves of $F \cap T$. If c does not bound a disk in F then we surger F along c and get a surface F' with $[F'] = [F]$ in $H_2(M)$, i.e., F' is a Stiefel–Whitney surface of M. But this can not be since F' is a 2–sphere.

So assume that all intersections of $F \cap T$ are essential in T. Then they are also essential in F, since T is incompressible in M.

The intersection curves partition F into annuli $A_1, \ldots, A_k, B_1, \ldots, B_k$ where $F \cap M_1 = A_1 \cup \cdots \cup A_k$, $F \cap M_2 = B_1 \cup \cdots \cup B_k$ and the annuli A_i, B_j are π_1–injective in M_1 and in M_2, resp.

The annulus A_1 is 2–sided in M, since otherwise the boundary curves of A_1 would be cores of Moebius bands in T, which is impossible. By [W] (see [H]), we may assume that A_1 is vertical or horizontal in M_1. If A_1 is horizontal, then $p_1 : M_1 \to G_1$ restricts to a branched covering of A_1 over the orbit surface G_1 of M_1. Since G_1 is not a disk, annulus, or a Moebius band, this cannot be. Hence A_1 is vertical in M_1. The same argument shows that ∂B_1 consists of fibers of M_2. But this contradicts the assumption that $\varphi : \partial M_1 \to \partial M_2$ is not fiber preserving. Hence $g(F) \neq 1$. $\square$

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