Convolution with measures on hypersurfaces

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§0 Introduction

Let S be a smooth hypersurface in $\mathbb{R}^n$ with surface area measure ds and Gaussian curvature $\kappa(s)$. Define the convolution operator T by

$$Tf(x) = \int_S f(x-s)\ \kappa(s)^{\frac{1}{n+1}} ds, \quad x \in \mathbb{R}^n,$$

for suitable functions f on $\mathbb{R}^n$. We are interested in the $L^p - L^q$ mapping properties of T. Write $\mathcal{S}$ for the type set of T, the set

$$\{(\frac{1}{p}, \frac{1}{q}) \in [0,1]^2 : \ T(L^p(\mathbb{R}^n)) \subseteq L^q(\mathbb{R}^n)\}.$$

It is well known (see, e.g., [O1]) that $\mathcal{S}$ is contained in the closed triangle $\mathcal{T}$ with vertices $(0,0)$, $(1,1)$, and $(\frac{n}{(n+1)}, \frac{1}{(n+1)})$. This paper is concerned with estimates of the form

$$\|T\chi_E\|_{n+1} \leq c \cdot |E|^{\frac{n}{n+1}}. \tag{1}$$

The estimate (1) is interesting because it serves as a weak substitute for the $L^{(n+1)/n} - L^{n+1}$ boundedness of T. For example, if S is compact and (1) holds, then well known arguments show that $\mathcal{S}$ differs from the full triangle $\mathcal{T}$ by at most the point $(\frac{n}{n+1}, \frac{1}{n+1})$. Our main result is a condition sufficient to imply (1). Its statement requires the following definition.

Definition. Suppose $\Omega \subseteq \mathbb{R}^p$. Say that a mapping $F : \Omega \to \mathbb{R}^p$ has generic multiplicity ℓ if card $F^{-1}(y) \leq \ell$ for almost all $y \in \mathbb{R}^p$.

Theorem 1. *Suppose ϕ is a real-valued $C^{(2)}$ function defined on a subset Ω of $\mathbb{R}^{n-1}$. Define $\gamma : \Omega \to \mathbb{R}^n$ by $\gamma(x) = (x, \phi(x))$. There is a positive constant C depending only on n and the generic multiplicities of the maps* grad ϕ *and*

$$\Psi : \Omega^n \to (\mathbb{R}^n)^{n-1}, \quad \Psi(x_1, \ldots, x_n) = (\gamma(x_1) - \gamma(x_2), \ldots, \gamma(x_1) - \gamma(x_n))$$

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such that if the operator T is defined on $\mathbb{R}^n$ by

$$Tf(y) = \int_\Omega f\left(y - \gamma(x)\right) \left|\det H_\phi(x)\right|^{\frac{1}{n+1}} \, dx,$$

then

$$\|T\chi_E\|_{L^{n+1}(\mathbf{R}^n)} \leq C|E|^{\frac{n}{n+1}}.$$

Theorem 1 is a generalization of the result in [O4]. The idea of choosing the measure $\kappa(s)^{1/(n+1)}ds$ to compensate for degeneracy of the surface S is due to Drury ([D2]). The next theorem shows that no better choice is possible.

Theorem 2. *Suppose S is a $C^{(2)}$ hypersurface in $\mathbb{R}^n$ with surface area measure ds. Suppose λ is a positive measure on S of the form $d\lambda(s) = \omega(s)ds$ where $\omega(s)$ is, say, continuous on S. If λ satisfies the convolution estimate*

$$\|\lambda * \chi_E\|_{L^{n+1}(\mathbf{R}^n)} \leq c \cdot |E|^{\frac{n}{n+1}}$$

for measurable $E \subseteq \mathbb{R}^n$, then

$$\omega(s) \leq C(n) \cdot c \cdot (\kappa(s))^{\frac{1}{n+1}},$$

where $\kappa(s)$ is the Gaussian curvature of S at s.

The following result is an immediate consequence of Theorem 1 and the proof of Theorem 2.

Corollary 3. *For positive measures λ as in Theorem 2 on hypersurfaces S as in Theorem 1, an estimate*

$$\|\lambda * \chi_E\|_{L^{n+1}(\mathbf{R}^n)} \leq c_1 \cdot |E|^{\frac{n}{n+1}}$$

for measurable $E \subseteq \mathbb{R}^n$ obtains if and only if there is a constant c_2 such that the measure λ satisfies the estimate

$$\lambda(R) \leq c_2 \cdot |R|^{\frac{n-1}{n+1}}$$

for every rectangle $R \subseteq \mathbb{R}^n$.

Corollary 3 is interesting in light of the following problem of Stein ([S], pp. 122-123): "characterize (if possible, in terms of the size of the measure $d\mu$, whatever that means) the condition of $f \to f * d\mu$ yielding a bounded operator from an L^p space to an L^r space".

This paper is organized as follows: Section 1 contains the proof of Theorem 1; Section 2 contains the proof of Theorem 2; Section 3 contains a few remarks on the problem of replacing (1) with a strong-type estimate.

§1 Proof of Theorem 1

Write $\omega(x) = |\det H_\phi(x)|^{\frac{1}{n+1}}$. Then we need to establish the inequality

$$\int_{\mathbf{R}^n} \left[\int_\Omega \chi_E\left(y - \gamma(x)\right) \omega(x)\, dx \right]^{n+1} dy \le C\,|E|^n\,.$$

It is enough to show that

$$\int_{\mathbf{R}^n} \int_\Omega \chi_E\left(y - \gamma(x_1)\right) \left[\int_{\{\omega(x_2) \ge \omega(x_1)\}} \chi_E\left(y - \gamma(x_2)\right) \omega(x_2)\, dx_2 \right]^n \omega(x_1)\, dx_1\, dy \le C|E|^n.$$

Replacing y by $y + \gamma(x_1)$, this will follow from

$$\int_\Omega \left[\int_{\{\omega(x_2) \ge \omega(x_1)\}} \chi_E\left(\gamma(x_1) - \gamma(x_2)\right) \omega(x_2)\, dx_2 \right]^n \omega(x_1)\, dx_1 \le C|E|^{n-1}. \tag{2}$$

Let $J(x_1, \ldots, x_n)$ be the Jacobian of the map

$$\Psi(x_1, \ldots, x_n) = \left(\gamma(x_1) - \gamma(x_2), \ldots, \gamma(x_1) - \gamma(x_n)\right).$$

By our assumption on the generic multiplicity of Ψ,

$$\int_{\Omega^n} \prod_{j=2}^n \chi_E\left(\gamma(x_1) - \gamma(x_j)\right) J(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \le C|E|^{n-1}.$$

Thus (2) will follow from

$$\begin{aligned} &\omega(x_1) \left[\int_{\{\omega(x_2) \ge \omega(x_1)\}} \chi_E\left(\gamma(x_1) - \gamma(x_2)\right) \omega(x_2)\, dx_2 \right]^n \\ &\le C \int_{\Omega^{n-1}} \prod_{j=2}^n \chi_E\left(\gamma(x_1) - \gamma(x_j)\right) J(x_1, \ldots, x_n)\, dx_2 \cdots dx_n. \end{aligned} \tag{3}$$

To establish (3), we need to compute $J(x_1, \ldots, x_n)$. Now

$$\Psi(x_1, \ldots, x_n) = (x_1 - x_2, \phi(x_1) - \phi(x_2); x_1 - x_3, \phi(x_1) - \phi(x_3); \ldots; x_1 - x_n, \phi(x_1) - \phi(x_n))$$

and so $\Psi(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_n)$ is equal to

$$(-x_2, \phi(x_1) - \phi(x_1 + x_2); -x_3, \phi(x_1) - \phi(x_1 + x_3); \ldots; -x_n, \phi(x_1) - \phi(x_1 + x_n)).$$

The jacobian $J(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_n)$ of $\Psi(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_n)$ is the same as the jacobian of the map

$$x_1 \mapsto (\phi(x_1) - \phi(x_1 + x_2), \phi(x_1) - \phi(x_1 + x_3), \ldots, \phi(x_1) - \phi(x_1 + x_n))$$

from Ω into $\mathbb{R}^{n-1}$. Thus, writing ϕ_j for the j^{th} partial of ϕ,

$$J(x_1, x_1 + x_2, \ldots, x_1 + x_n) =$$

$$\left| \det \begin{bmatrix} \phi_1(x_1) - \phi_1(x_1 + x_2) & \phi_1(x_1) - \phi_1(x_1 + x_3) & \cdots & \phi_1(x_1) - \phi_1(x_1 + x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n-1}(x_1) - \phi_{n-1}(x_1 + x_2) & \phi_{n-1}(x_1) - \phi_{n-1}(x_1 + x_3) & \cdots & \phi_{n-1}(x_1) - \phi_{n-1}(x_1 + x_n) \end{bmatrix} \right|$$

and therefore

$$J(x_1, x_2, \ldots, x_n) =$$

$$\left| \det \begin{bmatrix} \phi_1(x_1) - \phi_1(x_2) & \phi_1(x_1) - \phi_1(x_3) & \cdots & \phi_1(x_1) - \phi_1(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{n-1}(x_1) - \phi_{n-1}(x_2) & \phi_{n-1}(x_1) - \phi_{n-1}(x_3) & \cdots & \phi_{n-1}(x_1) - \phi_{n-1}(x_n) \end{bmatrix} \right|.$$

With x_1 fixed, define $X(x)$ for $x \in \mathbb{R}^{n-1}$ by

$$X(x) = (\phi_1(x_1) - \phi_1(x), \phi_2(x_1) - \phi_2(x), \ldots, \phi_{n-1}(x_1) - \phi_{n-1}(x))$$

and write $X_j = X(x_j)$, $j = 2, 3, \ldots, n$, to get

$$J(x_1, \ldots, x_n) = \left| \det (X_2, \ldots X_n) \right|.$$

Note that the jacobian of the map $x \mapsto X$ is just $|\det H_\phi(x)| = \omega^{n+1}(x)$.

We return to (3) and write M for the map $x \mapsto X$. The left hand side of (3) is

$$\begin{aligned}
&\omega(x_1)\left[\int\limits_{\{\omega(x)\geq\omega(x_1)\}} \chi_E\left(\gamma(x_1)-\gamma(x)\right)\omega^{-n}(x)\omega^{n+1}(x)\,dx\right]^n \\
&= \omega(x_1)\left[\int \sum_{\substack{M(x)=X\\ \omega(x)\geq\omega(x_1)}} \chi_E\left(\gamma(x_1)-\gamma(x)\right)\omega^{-n}(x)\,dX\right]^n
\end{aligned} \tag{4}$$

while the integral in the right hand side of (3) is

$$\int\limits_{M(\Omega)^{n-1}} \prod_{j=2}^{n}\left[\sum_{M(x_j)=X_j} \chi_E\left(\gamma(x_1)-\gamma(x_j)\right)\omega^{-(n+1)}(x_j)\right] |\det(X_2,\ldots,X_n)|\,dX_2\cdots dX_n. \tag{5}$$

Now, for almost all X,

$$\begin{aligned}
&\sum_{M(x)=X} \chi_E\left(\gamma(x_1)-\gamma(x)\right)\omega^{-n}(x) \\
&\leq \left[\sum_{M(x)=X} \chi_E\left(\gamma(x_1)-\gamma(x)\right)\omega^{-(n+1)}(x)\right]^{\frac{n}{n+1}} \cdot \ell^{\frac{1}{n+1}}
\end{aligned} \tag{6}$$

if ℓ is a bound for the generic multiplicity of $\operatorname{grad}\phi$ (and so for the generic multiplicity of M). Define

$$g(X) = \sum_{\substack{M(x)=X\\ \omega(x)\geq\omega(x_1)}} \chi_E\left(\gamma(x_1)-\gamma(x)\right)\omega^{-n}(x).$$

By (4), (5) and (6), (3) will follow from

$$\omega(x_1)\left(\int\limits_{M(\Omega)} g(X)\,dX\right)^n \leq C \int\limits_{M(\Omega)^{n-1}} \prod_{j=2}^{n} g(X_j)^{\frac{n+1}{n}} |\det(X_2,\ldots,X_n)|\; dX_2\cdots dX_n.$$

Since $\|g\|_\infty \leq \ell\cdot\omega^{-n}(x_1)$ so that $\|g\|_\infty^{\frac{1}{n}} \leq \ell^{\frac{1}{n}}\cdot\omega^{-1}(x_1)$, it is enough to establish the inequality

$$\left(\int\limits_{\mathbf{R}^{n-1}} g(X)\,dX\right)^n \leq C\,\|g\|_\infty^{\frac{1}{n}} \int\limits_{(\mathbf{R}^{n-1})^{n-1}} \prod_{j=2}^{n} g(X_j)^{\frac{n+1}{n}} |\det(X_2,\ldots,X_n)|\; dX_2\cdots dX_n \tag{7}$$

for nonnegative measurable functions g on $\mathbb{R}^{n-1}$. Inequality (7) follows from Lemmas 1 and 2 below.

Lemma 1. *Given $\eta > 0$ there exist nonzero real numbers η_j $(j = 2, \ldots, n)$ with $|\eta_j| < \eta$ and $\sum_2^n \eta_j = 0$ such that, for some positive constant $C = C(n; \eta_2, \ldots, \eta_n)$ and every nonnegative measurable function λ on $(0, \infty)$, there is the inequality*

$$\left(\int_0^\infty \lambda(r) r^{n-2}\, dr\right)^n \le C \int_0^\infty \cdots \int_0^\infty \prod_{j=2}^n \lambda(r_j)^{\frac{n}{n-1}+\eta_j} r_2^{n-1} dr_2 \cdots r_n^{n-1} dr_n.$$

Lemma 2. *Write Σ_{n-2} for the unit sphere in $\mathbb{R}^{n-1}$. There is $\eta > 0$ such that if $|\eta_j| < \eta$ $(j = 2, \ldots, n)$, $\sum_{j=2}^n \eta_j = 0$, then there is a positive constant $C = C(n; \eta_2, \ldots, \eta_n)$ such that the following holds: if $h_2, \ldots, h_n$ are nonnegative functions on Σ_{n-2} with $\|h_j\|_\infty \le 1$, then*

$$\prod_{j=2}^n (\|h_j\|_{L^1(\Sigma_{n-2})})^{\frac{n}{n-1}+\eta_j} \le C \int_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^n h^{\frac{n+1}{n}}(\sigma_j)\ |\det(\sigma_2, \ldots, \sigma_n)|\ d\sigma_2 \cdots d\sigma_n.$$

Proof of (7). Fix g with $\|g\|_\infty = 1$ and put $\lambda(r) = \int_{\Sigma_{n-2}} g(r\sigma) d\sigma$. For appropriate η_j, Lemma 1 gives

$$\left(\int_{\mathbb{R}^{n-1}} g(X) dX\right)^n = \left(\int_0^\infty \lambda(r) r^{n-2} dr\right)^n \le C \int_0^\infty \cdots \int_0^\infty \prod_{j=2}^n \lambda(r_j)^{\frac{n}{n-1}+\eta_j} r_2^{n-1} dr_2 \cdots r_n^{n-1} dr_n.$$

By Lemma 2 this last term is dominated by

$$\int_0^\infty \cdots \int_0^\infty \int_{(\Sigma_{n-2})n-1} \prod_{j=2}^n g^{\frac{n+1}{n}}(r_j\sigma_j)\ |\det(\sigma_2, \ldots, \sigma_n)|\ d\sigma_2 \cdots d\sigma_n\ r_2^{n-1} dr_2 \cdots r_n^{n-1} dr_n$$

$$= \int_{(\mathbb{R}^{n-1})_{n-1}} \prod_{j=2}^n g^{\frac{n+1}{n}}(X_j)\ |\det(X_2, \ldots, X_n)|\ dX_2 \cdots dX_n.$$

This is (7) since $\|g\|_\infty = 1$.

The proof of Lemma 2 depends on the next lemma.

Lemma 3. *There is $\eta > 0$ such that if $|\eta_j| < \eta$ and $\sum_{j=2}^{n} \eta_j = 0$, then there is $C = C(n; \eta_1, \ldots, \eta_n)$ for which the following holds: if the sets $E_j \subseteq \Sigma_{n-2}$ $(j = 2, \ldots, n)$ are measurable, then*

$$\prod_{j=2}^{n} |E_j|^{\frac{n}{n-1}+\eta_j} \leq C \int_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) \ |\det(\sigma_2, \ldots, \sigma_n)| \ d\sigma_2 \cdots d\sigma_n.$$

Proof of Lemma 2. Let $\lambda_j(s_j) = |\{\sigma \in \Sigma_{n-2} : h_j(\sigma) \geq s_j\}| \doteq |E_j|$ and let

$$A = \int_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} h^{\frac{n+1}{n}}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)| d\sigma_2 \cdots d\sigma_n.$$

Then

$$\left(\prod_{j=2}^{n} s_j\right)^{\frac{n+1}{n}} \prod_{j=2}^{n} (\lambda_j(s_j))^{\frac{n}{n-1}+\eta_j} \leq C \cdot A$$

by Lemma 3. Thus

$$\lambda_2(s_2) \leq \left(\frac{C \cdot A}{\left(\prod_{j=2}^{n} s_j\right)^{\frac{n+1}{n}} \prod_{j=3}^{n} (\lambda_j(s_j))^{\frac{n}{n-1}+\eta_j}} \right)^{\frac{1}{\frac{n}{n-1}+\eta_2}}.$$

Since $\frac{n+1}{n} \cdot \frac{1}{\frac{n}{n-1}+\eta_2} < 1$ if η_2 is small enough, the equality

$$\|h_2\|_{L^1(\Sigma_{n-2})} = \int_0^1 \lambda_2(s_2) ds_2$$

gives

$$(\|h_2\|_{L^1(\Sigma_{n-2})})^{\frac{n}{n-1}+\eta_2} \cdot \left(\prod_{j=3}^{n} s_j\right)^{\frac{n+1}{n}} \prod_{j=3}^{n} (\lambda_j(s_j))^{\frac{n}{n-1}+\eta_j} \leq C \cdot A.$$

Continuing this process leads to the conclusion of Lemma 2.

Proof of Lemma 3. It is a result of Drury ([D1]) that the inequality

$$\int_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} f_j(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)|^{-1+\alpha} d\sigma_2 \cdots d\sigma_n \leq C \prod_{j=2}^{n} \|f_j\|_{L^p(\Sigma_{n-2})}$$

holds for nonnegative functions f_j on Σ_{n-2} if $0 < \alpha < 1$ and $\frac{1}{p} = 1 - \frac{1-\alpha}{n-1}$. The proof (by multilinear interpolation) actually gives

$$\int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} f_j(\sigma_j)\ |\det(\sigma_2, \ldots, \sigma_n)|^{-1+\alpha} d\sigma_2 \cdots d\sigma_n \leq C(\delta_2, \ldots, \delta_n) \prod_{j=2}^{n} \|f_j\|_{L^{p_j}(\Sigma_{n-2})} \tag{8}$$

if $\frac{1}{p_j} = 1 - \frac{1-\alpha}{n-1} + \delta_j$ where the numbers δ_j are sufficiently small and satisfy $\sum\limits_{j=2}^{n} \delta_j = 0$. Fix $p > 2$, so that if α is defined by the equation $-\frac{1}{p-1} = -1 + \alpha$, then $0 < \alpha < 1$. We have

$$\begin{aligned}
\prod_{j=2}^{n} |E_j| &= \int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)|^{\frac{1}{p}} |\det(\sigma_2, \ldots, \sigma_n)|^{-\frac{1}{p}} d\sigma_2 \cdots d\sigma_n \\
&\leq \left(\int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)| d\sigma_2 \cdots d\sigma_n \right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)|^{-\frac{1}{p-1}} d\sigma_2 \cdots d\sigma_n \right)^{1-\frac{1}{p}} \\
&\leq C \left(\int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)| d\sigma_2 \cdots d\sigma_n \right)^{\frac{1}{p}} \left(\prod_{j=2}^{n} |E_j|^{\frac{1}{p_j}} \right)^{(1-\frac{1}{p})},
\end{aligned}$$

where the last inequality is a consequence of (8). Thus

$$\prod_{j=2}^{n} |E_j|^{p - \frac{(p-1)}{p_j}} \leq C \int\limits_{(\Sigma_{n-2})^{n-1}} \prod_{j=2}^{n} \chi_{E_j}(\sigma_j) |\det(\sigma_2, \ldots, \sigma_n)| d\sigma_2 \cdots d\sigma_n.$$

Now

$$p - \frac{p-1}{p_j} = p - (p-1)\left(1 - \frac{1-\alpha}{n-1} + \delta_j\right) = \frac{n}{n-1} - (p-1)\delta_j,$$

where we have used the equation $1 - \alpha = \frac{1}{p-1}$. Thus the conclusion of Lemma 3 holds if $\eta_j = -(p-1)\delta_j$.

Proof of Lemma 1. Since

$$\int\limits_{\{\frac{1}{r} > s\}} r^{n-1} dr = \frac{s^{-n}}{n},$$

it follows that $\frac{1}{r} \in L^{n,\infty}(r^{n-1}dr)$. Thus

$$\int_0^\infty \lambda(r) r^{n-2} dr \leq C \|\lambda\|_{L^{\frac{n}{n-1},1}(r^{n-1}dr)}.$$

Since $(\frac{n}{n-1}+\eta_j)\frac{1}{n} = \frac{1}{n-1}+\frac{\eta_j}{n}$, the conclusion of Lemma 1 is a consequence of the inequality

$$\|\lambda\|_{L^{\frac{n}{n-1},1}} \leq C \prod_{j=2}^{n} (\|\lambda\|_{L^{\frac{n}{n-1}+\eta_j}})^{\frac{1}{n-1}+\frac{\eta_j}{n}}.$$

This plausible convexity result is an easy consequence of the definition of the Lorentz norms in terms of decreasing rearrangements.

§2 Proof of Theorem 2

The estimate

$$\|\lambda * \chi_E\| \leq c \cdot |E|^{\frac{n}{n+1}}$$

implies the inequality

$$\langle \lambda,\ \chi_E * \chi_F \rangle \leq c \cdot (|E| \cdot |F|)^{\frac{n}{n+1}}$$

whenever E and F are measurable subsets of $\mathbb{R}^n$. This leads in turn to the estimate

$$\lambda(R) \leq C \cdot c \cdot |R|^{\frac{n-1}{n+1}} \tag{9}$$

whenever R is a rectangle in $\mathbb{R}^n$. Here (and below) C denotes a constant which depends only on n. We will show that (9) implies the inequality

$$\omega(s) \leq C \cdot c \cdot (\kappa(s))^{\frac{1}{n+1}} \tag{10}$$

if $s \in S$. So fix an interior point s on S. After a linear change of variables we can assume that, in a neighborhood of s, S has a parametrization $(x, \phi(x))$ as the graph of a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. There is no loss of generality in assuming that $s = (0, \phi(0)) = 0$ and that $\operatorname{grad} \phi(0) = 0$. We will establish (10) under the asssumption that $\kappa(s)$ is nonzero. Only a slight adaptation is required if $\kappa(s) = 0$.

Let $H(x)$ stand for the Hessian of ϕ at x. Let $v_1, \ldots, v_{n-1}$ be orthonormal eigenvectors for $H(0)$ and let $A_1, \ldots, A_{n-1}$ be the corresponding eigenvalues, so that

$$\prod_{j=1}^{n-1} |A_j| = \kappa(s). \tag{11}$$

Put $a_j = |A_j|^{-1/2}$ and, for $\epsilon > 0$, write r_ϵ for the rectangle in $\mathbb{R}^{n-1}$ with sides $\epsilon a_j v_j,\ j = 1, \ldots, n-1$. Let R_ϵ be the smallest rectangle in $\mathbb{R}^n$ having base r_ϵ and containing the graph of ϕ over r_ϵ. Thus

$$|R_\epsilon| = |r_\epsilon| \cdot \sup\{|\phi(x)| :\ x \in r_\epsilon\}. \tag{12}$$

If ϵ is sufficiently small, as we shall assume, then

$$\sum_{j,k=1}^{n-1} |a_j a_k \langle H(y) v_j, v_k\rangle| \le 2 \sum_{j=1}^{n-1} a_j^2 |\langle H(0) v_j, v_k\rangle| = \sum_{j=1}^{n-1} a_j^2 |A_j| \tag{13}$$

whenever $y \in r_\epsilon$. Fix $x \in r_\epsilon$ and write $x = \epsilon \sum c_j a_j v_j$ with $0 \le c_j \le 1$. Since $\phi(0)$ and $\operatorname{grad} \phi(0)$ are both 0,

$$\begin{aligned}
|\phi(x)| = | \int_0^\epsilon \frac{d^2}{ds^2} \phi(s \sum c_j a_j v_j)(\epsilon - s) ds| \\
&= | \int_0^\epsilon \langle H(s \sum c_j a_j v_j) \sum c_j a_j v_j, \sum c_j a_j v_j\rangle (\epsilon - s) ds| \\
&\le \epsilon^2 \sum_{j=1}^{n-1} a_j^2 |A_j|
\end{aligned}$$

by (13). Thus, by (12), we have

$$|R_\epsilon| \le |r_\epsilon| \cdot \epsilon^2 \sum_{j=1}^{n-1} a_j^2 |A_j| = (n-1)\epsilon^{n+1} \prod_{j=1}^{n-1} |A_j|^{-\frac{1}{2}} \tag{14}$$

since $|r_\epsilon| = \epsilon^{n-1} \prod a_j$ and $a_j = |A_j|^{-1/2}$. Now

$$\omega(s) = \lim_{\epsilon \to 0^+} \frac{1}{|r_\epsilon|} \int_{r_\epsilon} \omega(x, \phi(x))\ dx = \lim_{\epsilon \to 0^+} \frac{\lambda(R_\epsilon)}{|r_\epsilon|} \tag{15}$$

since $\operatorname{grad} \phi(0) = 0$. It follows from (9) that

$$\frac{\lambda(R_\epsilon)}{|r_\epsilon|} \le C \cdot c \cdot \frac{|R_\epsilon|^{\frac{n-1}{n+1}}}{|r_\epsilon|}.$$

Since

$$|r_\epsilon| = \epsilon^{n-1} \prod_{j=1}^{n-1} |A_j|^{-\frac{1}{2}},$$

(14) implies that

$$\frac{\lambda(R_\epsilon)}{|r_\epsilon|} \leq C \cdot c \prod_{j=1}^{n-1} |A_j|^{\frac{1}{2}(1-\frac{n-1}{n+1})} = C \cdot c \cdot (\kappa(s))^{\frac{1}{n+1}},$$

where the last equality follows from (11). Now (15) gives (10).

§3 Improving inequality (1)

It is natural to conjecture that the conclusion of Theorem 1 can be replaced by the strong-type estimate

$$\|Tf\|_{L^{n+1}(\mathbb{R}^n)} \leq C\|f\|_{L^{\frac{n+1}{n}}(\mathbb{R}^n)}. \tag{16}$$

The crux of the proof of Theorem 1 was the proof of inequality (2). Thus it is perhaps of interest to note that the inequality

$$\int_\Omega \left[\int_\Omega \chi_E\left(\gamma(x_1) - \gamma(x_2)\right) \omega(x_2)\, dx_2 \right]^n \omega(x_1)\, dx_1 \leq C|E|^{n-1}, \tag{17}$$

which implies (2), also implies (16). This follows from a multilinear argument of Christ as employed in [O2]. Unfortunately, (17) does not follow from the hypotheses of Theorem 1, even in the case $n = 2$. Until after the statement of Theorem 4 below, ϕ will stand for a sufficiently differentiable real-valued function defined on an interval (a, b) and satisfying

$$\phi''(x), \phi^{(3)}(x) > 0 \tag{18}$$

if $a < x < b$. (Thus the generic multiplicity hypotheses of Theorem 1 are fulfilled with bound $\ell = 1$.) We regard (17) (with $n = 2$) as an estimate from $L^{2,1}(\mathbb{R}^2)$ into $L^2((\phi''(x))^{1/3}dx)$. Its dual is an estimate from $L^2((\phi''(x))^{1/3}dx)$ into $L^{2,\infty}(\mathbb{R}^2)$ which can be shown to be equivalent to the inequality

$$\int_{\{|\phi'(t)-\phi'(s)|\leq B|\phi''(t)\phi''(s)|^{1/3}\}} |\phi'(t) - \phi'(s)|ds \leq C \cdot B^2 |\phi''(t)|^{\frac{1}{3}} \tag{19}$$

whenever $t \in (a, b)$ and $B > 0$. And (19) is equivalent to the inequality

$$\phi''(t_1)^{\frac{2}{3}} \phi''(t_2)^{\frac{1}{3}} \int_{t_1}^{t_2} \phi''(s)(s - t_1)\ ds \leq C(\int_{t_1}^{t_2} \phi''(s)\ ds)^2$$

whenever $[t_1, t_2] \subseteq (a, b)$. This last inequality does not follow from only the hypotheses (18), though it is easy to find ad hoc hypotheses which, in addition to (18), will yield (19).

For example, if $\phi^{(4)} < 0$ and (18) holds, then

$$\phi''(t_1)^{\frac{2}{3}}\phi''(t_2)^{\frac{1}{3}}\int_{t_1}^{t_2}\phi''(s)(s-t_1)\ ds$$

$$\leq \phi''(t_1)^{\frac{1}{2}}\phi''(t_2)^{\frac{1}{2}}(t_2-t_1)\int_{t_1}^{t_2}\phi''(s)\ ds \leq \frac{1}{2}(\int_{t_1}^{t_2}\phi''(s)\ ds)^2,$$

thus proving

Theorem 4. *There is a positive constant C such that whenever ϕ is a $C^{(4)}$ function on an interval (a,b) satisfying the inequalities ϕ'', $\phi^{(3)} > 0$, $\phi^{(4)} < 0$ on (a,b), then the operator defined on $\mathbb{R}^2$ by*

$$Tf(x) = \int_a^b f(x-(t,\phi(t))\ (\phi''(t))^{\frac{1}{3}}dt$$

satisfies

$$\|Tf\|_{L^3(\mathbb{R}^2)} \leq C\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}.$$

But our point here is that an analogue of Proposition 4 should hold assuming only, say, that ϕ'' and $\phi^{(3)}$ have constant sign on (a,b) and that neither the method of [O2] nor ideas from the proof of Theorem 1 can be used to establish this. Unfortunately, the same seems to be true of other methods which have been applied to similar problems. The multilinear method of [DG] seems to be less flexible in the present context than that of [O2]. The arguments of [O3] and [Chr] yield only restricted weak-type estimates, while the problem at hand is to obtain a strong-type estimate. Of the methods known to the author, this leaves only complex interpolation as employed, with certain additional hypotheses, by Drury [D2] and then Choi [Cho]. The necessary estimate is

$$|\int_a^b e^{i(\zeta_1 t+\zeta_2\phi(t))}|\phi''(t)|^{\frac{1}{2}+is}dt| \leq C(s)\ (1+|\zeta_2|)^{-\frac{1}{2}},$$

holding uniformly for $a,b\in\mathbb{R}$, $\zeta_1,\zeta_2\in\mathbb{R}$ and ϕ such that ϕ'' and $\phi^{(3)}$ have constant sign on (a,b). But the choice $\phi(t)=\ln(\ln(t))$ shows that such an estimate is impossible.

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