Chapter 1

COMMUTATIVE RINGS OF DIMENSION 0

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1. INTRODUCTION

All rings considered in this paper are assumed to be commutative and unitary. If R is a subring of S, we assume that the unity element of S belongs to R, and hence is the unity of R. All allusions to the dimension of a ring refer to its Krull dimension. Thus $\dim R = n$ if there exists a chain $P_0 < P_1 < \cdots < P_n$ of proper prime ideals of R, but no longer such chain; $\dim R = \infty$ if there exist arbitrarily long chains of prime ideals of R. This paper is concerned primarily with rings of dimension 0, hence rings whose proper prime ideals are maximal. There has been a surge of recent work in this area; the work seems to have originated with the papers [2]–[4] of M. Arapovic. The main focus of Arapovic's work is the question of embeddability of a ring in a zero-dimensional ring, the subject of our Section 5. This paper contains an account of some of the work that has been done in the area of zero-dimensional commutative rings since the publication of Arapovic's papers.

Section 2 of the current paper provides a background of some developments in the theory of zero-dimensional rings preceding Arapovic's work. This includes comments on the pioneering work of M. H. Stone and J. von Neumann in the 1930's. In particular, Stone's papers [40] and [41] showed that the theories of Boolean algebras, Boolean rings, and Boolean spaces[1] are equivalent. We also record in Section 2 some

[1]Stone introduced the term *Boolean space* to refer to a topological space that is Hausdorff, compact, and totally disconnected.

basic results in the theory of zero-dimensional rings that are used in the sequel.

Section 3 considers the set $\mathcal{Z}(R,S)$ of zero-dimensional subrings of a ring S that contain a fixed subring R of S. In the case where R is the prime subring of S, $\mathcal{Z}(R,S)$ consists of all zero-dimensional subrings of S and is denoted by $\mathcal{Z}(S)$. Two extreme cases are investigated: (1) $\mathcal{Z}(R,S)$ is empty, and (2) $\mathcal{Z}(R,S)$ contains each subring of S containing R. We give attention in Section 3 to the subclasses $\mathcal{V}(R,S)$ and $\mathcal{A}(R,S)$ of $\mathcal{Z}(R,S)$ consisting, respectively, of the elements of $\mathcal{Z}(R,S)$ that are von Neumann regular and Artinian, respectively. Theorem 3.1 is a key result in Section 3. It shows that $\mathcal{Z}(R,S)$ is nonempty if and only if, for each element x of R, some power of the ideal xS of S is idempotent.

With notation as in Section 3, Theorem 4.1 of Section 4 shows that if $\mathcal{Z}(R,S)$ is nonempty, then it is closed under arbitrary intersection. It follows that $\mathcal{V}(R,S)$ is also closed under arbitrary intersection if it is nonempty. Theorem 4.1 implies that if R is contained in a zero-dimensional subring of S, then there exists a unique minimal zero-dimensional subring $R^0(S)$ of S containing R. Theorem 4.7 provides a set of generators for $R^0(S)$ as an extension ring of R, and Theorem 4.8 records some properties of $R^0(S)$ as an extension of R.

In Section 5 we address the question of embeddability of a ring R in a zero-dimensional ring. Theorem 5.1, which is due to Arapovic, gives equivalent conditions in order for R to be so embedded. In considering the embedding question, products $\prod_{a\in A} R_a$ of families of zero-dimensional rings R_a naturally arise. Theorem 5.5 shows that $\prod R_a$ is either zero-dimensional or infinite-dimensional, and in the latter case, Theorem 5.8 shows that $\prod R_a$ is not embeddable in a zero-dimensional ring. Theorem 5.10 and Remark 5.11 deal with the question of whether a given ring admits an Artinian extension ring.

Corollary 3.10 states that each subring of the ring R is zero-dimensional if and only if R/P is an absolutely algebraic field of nonzero characteristic for each proper prime ideal P of R. This result partially motivated the question considered in Section 6: What indexed families of fields can be realized as the family of residue fields of a zero-dimensional ring? While there has been significant progress toward an answer to this question, resolution seems distant from the current state of knowledge, and at this point there is no conjectured answer to the question.

As for notation and terminology, we use $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{Q}$, respectively, to denote, respectively, the sets of integers, positive integers, and rational numbers. If S is a ring, then $J(S)$ and $N(S)$ denote, respectively, the Jacobson radical and the nilradical of S. The *spectrum* of R, denoted $\mathrm{Spec}(R)$, is the set of proper prime ideals of R, provided with the

Zariski topology. The *maximal spectrum* of R, denoted $\mathrm{MSpec}(R)$, is the subspace of $\mathrm{Spec}(R)$ consisting of all maximal ideals of R; if R is zero-dimensional, then $\mathrm{Spec}(R) = M$. We use $\mathrm{char}(R)$ to denote the characteristic of R, and we sometimes abbreviate *von Neumann regular* by vNr.

The theme of the 1994 John H. Barrett Memorial Lectures was Zero-Dimensional Commutative Rings. The author has four papers [10]–[13] appearing in the published proceedings of the conference associated with those lectures. There is naturally some overlap between the contents of this paper and the four Barrett papers. However, whereas proofs were provided for almost all of the results in [10]–[13], few proofs are given here. We do list citations to sources that provide proofs, and when one of the Barrett papers meets that criterion, it is usually the source we cite in order to minimize the number of sources a reader needs to consult. The result is that fewer than half of the references in the extensive bibliography of [10] are duplicated here, but an interested reader may well wish to consult some of the [10] references not explicitly cited here.

2. HISTORICAL REMARKS

In this section we provide a background of some developments in the theory of zero-dimensional rings that preceded publication of Arapovic's papers [2]–[4]. Our remarks are sketchy, and a reader who wishes more detail should consult some of the sources cited.

A basic theory of some important classes of zero-dimensional rings was developed in the 1930's, primarily in works of M. H. Stone and J. von Neumann. In particular, [40] and [41] are major works of Stone related to Boolean rings, and von Neumann developed a theory of what he called *regular rings* in [34] (such rings need not be commutative); [39] and [42] are expository papers of Stone in the same area, and Chapters X and XI of [5] includes details of some of this work, as well as coverage of related work by others. Stone used the term *Boolean ring* in [40] to describe a ring in which each element is idempotent. Such a ring is commutative of characteristic 2, and each of its residue fields is isomorphic to $GF(2)$, the field with two elements. Stone chose the adjective Boolean because he showed that the theory of unitary Boolean rings is equivalent to the theory of Boolean algebras. The latter had arisen in logic, in set theory, and in the theory of lattices. More precisely, in his book The Mathematical Analysis of Logic [7], George Boole developed an algebra of propositions using the binary operations of conjunction and disjunction and the unary operation of negation. Abstracted, a set closed under three such operations and satisfying an appropriate set of postulates is called a Boolean algebra. Various formulations of the postulate sets are known; two of the earliest are due to Huntington [26], [27] (see also [5], Ch. X, Sect. 4, and [6], Ch. XI, p. 342). The power set $\mathcal{P}(X)$ of a set X, under union, intersection, and complementation, is a well-known Boolean algebra; its associated unitary Boolean ring is the set $\mathcal{P}(X)$ under the operations of symmetric difference and intersection, and this example serves as a model of the association of a Boolean ring with a Boolean algebra. On the other hand, if R is a Boolean ring and if operations $\vee$, $\wedge$, and $'$ and defined on R by $x \vee y = x + y - xy$, $x \wedge y = xy$, and $x' = 1 - x$, then under these three operations R is a Boolean algebra associated with R as a Boolean ring.

In [41] Stone showed that the theory of unitary Boolean rings is equivalent to the theory of topological spaces that are compact, Hausdorff, and totally disconnected (Stone called such spaces *Boolean spaces*). There are several ways to describe the association between Boolean rings and spaces implicit in this statement. If R is a Boolean ring, then Spec(R) under the Zariski topology is a Boolean space associated with R. And if X is a Boolean space, the set of continuous functions $f : X \to GF(2)$ with compact support is a unitary Boolean ring associated with X; here $GF(2)$ has the discrete topology and addition and multiplication of functions is defined pointwise (cf. [29], p. 168).

After the thirties, relatively few papers dealt primarily with zero-dimensional rings or commutative von Neumann regular rings. There were some, but most new results concerning zero-dimensional rings were either special cases of a more general theory or were proved *ad hoc* within a broader context. Among the papers from the period 1940-83 that we cite in the sequel are [33], [35], and [36]; [44] and [37] are also relevant to our considerations.

We conclude this section by listing a few known results that are needed in Sections 3-6. The first of these is a characterization theorem from [43] (see also [25], Thm. 3.1).

Theorem 2.1. ([43], Lem. 5.6) *The following conditions are equivalent in a ring R.*

(1) *A power of each principal ideal of R is idempotent.*

(2) *For each $r \in R$ there exists $y \in R$ and $n \in \mathbb{Z}^+$ such that $r^n = r^{n+1}y$.*

(3) *$R/N(R)$ is a von Neumann regular ring.*

(4) *R is zero-dimensional.*

The equivalence of (1) and (4) in Theorem 2.1 leads easily to the following characterization of zero-dimensional rings via their prime spectra.

Theorem 2.2. ([33], Thm. 1.3) *The ring R is zero-dimensional if and only if* $\operatorname{Spec}(R)$ *is a Boolean space — that is,* $\operatorname{Spec}(R)$ *is compact, Hausdorff, and totally disconnected.*

Also in [33], Maroscia considered a product $R = \prod_{a \in A} R_a$ of a family $\{R_a\}_{a \in A}$ of zero-dimensional rings. He observed that R need not be zero-dimensional, that $\prod_{i=1}^{\infty} k[X]/(X^{i+1})$ is not zero-dimensional for a field k, for example. For the general case Maroscia proved:

Theorem 2.3. ([33], Prop. 2.6) *If* $\{R_a\}_{a \in A}$ *is a family of zero-dimensional rings, then* $R = \prod_{a \in A} R_a$ *is zero-dimensional if and only if* $N(R) = \prod_a N(R_a)$.

Finally, if R is a ring with $\operatorname{MSpec}(R) = \{M_\lambda\}_{\lambda \in \Lambda}$ and if I is an ideal of R, then Krull referred to I_λ, the contraction of the extension of I to the quotient ring R_{M_λ}, as the isolated component ideal of I with respect to M_λ. Krull proved (cf. [9], (4.10)) that $I = \bigcap_\lambda I_\lambda$. If R is zero-dimensional, then each ideal of each R_{M_λ} is primary, so either $I_\lambda = R$, or else I_λ is M_λ-primary. Thus we have proved the following result.

Proposition 2.4. *Each ideal of a zero-dimensional ring is expressible as an intersection of primary ideals.*

3. ZERO-DIMENSIONAL SUBRINGS

Let S be a ring, and denote by $\mathcal{Z}(S)$, $\mathcal{V}(S)$ and $\mathcal{A}(S)$, respectively, the sets of zero-dimensional, von Neumann regular, and Artinian subrings of S. We begin with a consideration of (necessary and/or sufficient) conditions under which a specified one of these three sets is nonempty. Recalling that we consider only subrings of S containing the prime subring Π of S, we may in fact consider the relative case of a subring R of S. Thus, let $\mathcal{Z}(R, S)$ be the set of zero-dimensional S-overrings[2] of R, and let $\mathcal{V}(R, S)$ and $\mathcal{A}(R, S)$ be analogously defined. Under what conditions is a specified one of these three sets nonempty? Clearly $\mathcal{Z}(R, S)$ is nonempty if $\mathcal{V}(R, S)$ or $\mathcal{A}(R, S)$ is nonempty, and there are no other implications of this type among these sets. A sufficient condition for nonemptiness of one of the sets is that it contains R or S; in particu-

[2]Recall that an *S-overring* of R is a subring of S containing R.

lar, $\mathcal{A}(S)$ is nonempty if $\mathrm{char}(S) \neq 0$, and $\mathcal{V}(S)$ is nonempty if $\mathrm{char}(S)$ is a square-free positive integer. The next result provides equivalent conditions for nonemptiness of $\mathcal{Z}(R,S)$.

Theorem 3.1. ([10], Thm. 2.1) *Suppose R is a subring of the ring S. Then R is a subring of a zero-dimensional subring of S if and only if a power of the ideal xS is idempotent for each x in R.*

Remark 3.2. The proof of Theorem 3.1 uses the notion of the *pointwise inverse* of an element. The definition is as follows. If aS is an idempotent ideal of the ring S, then there exists a unique element $b \in S$ such that $a = a^2b$ and $b = ab^2$; this element b is the pointwise inverse of a. Existence and uniqueness of b follow because aS itself is a ring with unity, and a is invertible in aS; the pointwise inverse of a in S is the inverse of a in the ring aS. The proof of Theorem 2.1 of [12] shows that if $x^{m(x)}S$ is idempotent for $x \in R$ and if s_x is the pointwise inverse of $x^{m(x)}$ in S, then $R[\{s_x \mid x \in R\}]$ is zero-dimensional.

Remark 3.3. Let $S = \prod_{n=1}^{\infty}(\mathbb{Z}/p^n\mathbb{Z})$, where p is a prime integer. The prime subring Π of S is isomorphic to $\mathbb{Z}$, and no power of the ideal pS of S is idempotent. Hence S contains no zero-dimensional subring. A less elementary proof of this assertion can be based on the fact that if I is the direct sum ideal of S, then the ring $\Pi + I$ is not embeddable in a zero-dimensional ring [14], Example 5.

Using the reasoning of Remark 3.3 we obtain:

Corollary 3.4. *The ring S contains a zero-dimensional subring if and only if either $\mathrm{char}(S) \neq 0$, or else $\mathrm{char}(S) = 0$ and a power of qS is idempotent for each prime integer q.*

We remark that an affine ring over $\mathbb{Z}$ admits no zero-dimensional subring. This assertion is a corollary to the following result of Gilmer and Heinzer; recall that a ring R is a *Hilbert ring* if each proper prime ideal of R is an intersection of maximal ideals of R.

Proposition 3.5. ([18], Prop. 2.2) *If R is a Hilbert ring of positive dimension and if S is an affine ring over R, then $\mathcal{Z}(R,S) = \emptyset$.*

Turning to the Artinian case, two key properties of an Artinian ring R that come into play are that $\mathrm{Spec}(R)$ is finite and that R has only finitely many idempotents. For a ring R, let $\mathcal{C}(R) = \{\text{char}(R/M) \mid M \in M\text{Spec}(R)\}$. The proof of the next result is elementary.

Proposition 3.6. ([18], Prop. 1, Cor. 2) *Suppose R is a subring of the ring S.*

(a) $\mathcal{C}(S) - \{0\} \subseteq \mathcal{C}(R)$. In particular, $\mathcal{C}(R)$ is infinite if $\mathcal{C}(S)$ is infinite.

(b) If $\mathcal{C}(S)$ is infinite, then S admits no Artinian subring.

Remark 3.7. If $S = \prod_{n=1}^{\infty}(\mathbb{Z}/p^n\mathbb{Z})$ is the ring of Remark 3.3, then the Jacobson radical of S is $J(S) = \prod_{n=1}^{\infty} J(\mathbb{Z}/p^n\mathbb{Z}) = \prod_{n=1}^{\infty}(p\mathbb{Z}/p^n\mathbb{Z}) = pS$. Hence $\mathcal{C}(S) = \{p\}$, but as noted in (3.3), S admits no zero-dimensional, hence no Artinian, subring.

One additional result concerning nonemptiness of the set $\mathcal{V}(R, S)$ is given in Remark 4.10. For the remainder of this section we turn to a consideration of another extreme case — that in which each subring of S (or each S-overring of a fixed subring R) is zero-dimensional, von Neumann regular, or Artinian. There is some established terminology that has been employed in this context. If E is a ring-theoretic property, we say that ring S *has property E hereditarily* if each subring of S has property E. Thus S is *hereditarily zero-dimensional* if $\mathcal{Z}(S)$ is the set of all subrings of S. If R is a subring of S, then we say that (R, S) is an *E-pair* if each S-overring of R has property E. In seeking to characterize zero-dimensional, von Neumann regular, and Artinian pairs, the vNr case follows from the zero-dimensional case: (R, S) is a vNr-pair if and only if S is vNr and (R, S) is a zero-dimensional pair. For integral domains R and S, it is clear that (R, S) is a zero-dimensional pair if and only if R and S are fields and S is algebraic over R. This observation leads in turn to:

Proposition 3.8 ([15], Cor. 4.2) *Suppose R is a subring of the ring S. Then (R, S) is a zero-dimensional pair if and only if R is zero-dimensional and S is integral over R.*

Remark 3.9. Suppose R is a subring of S. The proof of Corollary 4.2 of [15] uses the fact that S is integral over R if (and only if) S/P is integral over $R/(P \cap R)$ for each $P \in \text{Spec}(S)$. A second use of this result enables Gilmer and Heinzer to prove ([15], Prop. 4.5) that if R and S are zero-dimensional and if S is an affine extension of R, then (R, S) is a zero-dimensional pair.

Corollary 3.10. ([15], Cor. 4.3) *For a ring R, the following conditions are equivalent.*

(a) R is hereditarily zero-dimensional.

(b) $\mathrm{char}(R) \neq 0$ and R is integral over its prime subring.

(c) R/P is an absolutely algebraic field[3] *of nonzero characteristic for each $P \in \mathrm{Spec}(R)$.*

Corollary 3.11. *If the ring S is hereditarily zero-dimensional, then S is a directed union of Artinian subrings.*

Another result with the same conclusion as Corollary 3.11 is:

Theorem 3.12. ([15], Cor. 5.5) *If S is a zero-dimensional semiquasilocal ring, then there exists an Artinian subring R of S such that (R, S) is a zero-dimensional pair. In particular, S is a directed union of Artinian subrings.*

Suppose the ring S admits a zero-dimensional subring R. If S has only finitely many idempotents, then R has only finitely many idempotents, so R is semiquasilocal. By Theorem 3.12, R is a directed union of Artinian subrings. In particular we obtain:

Corollary 3.13. *Suppose ring S has only finitely many idempotents and that S admits a zero-dimensional subring. Then S admits an Artinian subring.*

Remark 3.14. Results 3.11 and 3.12 bring us to an interesting open problem area within the theory of zero-dimensional rings. A general problem in the area asks for a characterization of rings of the class $\mathcal{D}$ of rings expressible as a directed union of Artinian subrings. A ring in $\mathcal{D}$ is zero-dimensional, but Proposition 3.6 implies that there exist zero-dimensional rings with no Artinian subrings ($\Pi_{p \text{ prime}}\mathbb{Z}/p\mathbb{Z}$ is such a ring). Besides the rings that belong to $\mathcal{D}$ by virtue of (3.11) and (3.12), $\mathcal{D}$ is closed under taking homomorphic images and finite direct products. Moreover, Theorem 6.7 of [15] shows that an arbitrary product $\prod_{a \in A} R_a$ of elements of $\mathcal{D}$ is in $\mathcal{D}$ if and only if it is zero-dimensional (see Section 5 for more on the dimension of ΠR_a). A specific question raised in Remark

[3] Recall that a field is ***absolutely algebraic*** if it is algebraic over its prime subfield.

6.10 of [15] asks whether a zero-dimensional ring R belongs to $\mathcal{D}$ if its associated reduced ring $R/N(R)$ belongs to $\mathcal{D}$. An affirmative answer to this question would focus attention on the subclass $\mathcal{D}_v$ of von Neumann regular rings in $\mathcal{D}$. But a characterization of the elements of $\mathcal{D}_v$ (which is considered by A. Magid in [32]) is of independent interest, even if the question of Gilmer and Heinzer has a negative answer.

The next two results provide characterizations of hereditarily Artinian rings and Artinian pairs.

Theorem 3.15. ([15], Thm. 4.10) *Let R be a hereditarily zero-dimensional ring. The following conditions are equivalent.*

(a) R is hereditarily Artinian.

(b) R is hereditarily Noetherian.

(c) The nilradical $N(R)$ is finite and $R/N(R)$ is a finite direct sum of absolutely algebraic fields of nonzero characteristic.

(d) $R = R_1 \oplus \ldots \oplus R_m \oplus S$, where each R_i is an absolutely algebraic field of nonzero characteristic and S is a finite ring.

Theorem 3.16. ([15], Thm. 4.12) *Let (R, S) be a zero-dimensional pair. The following conditions are equivalent.*

(a) (R, S) is an Artinian pair.

(b) (R, S) is a Noetherian pair.

(c) R is Noetherian and $S = S_1 \oplus \ldots \oplus S_n$ is a finite direct sum of ideals that, as rings, are Artinian and local and are such that each S_i that is not a field is a finitely generated R-module.

(d) R is Noetherian, $\mathrm{Spec}(R)$ is finite, and the nilradical $N(S)$ of S is a finitely generated R-module.

4. INTERSECTIONS OF ZERO-DIMENSIONAL SUBRINGS

As in Section 3, we continue to use $\mathcal{Z}(S)$, $\mathcal{V}(S)$, and $\mathcal{A}(S)$, respectively, to denote the sets of zero-dimensional, von Neumann regular, and Artinian subrings of the ring S. In this section we consider properties of these three sets under intersection. A major result in this regard is Theorem 4.1, which shows that $\mathcal{Z}(S)$ is closed under arbitrary intersection. Thus, if R is a subring of S that is contained in a zero-dimensional subring of S, then there is a unique minimal zero-dimensional subring $R^0(S)$ of S containing R. Properties of $R^0(S)$ as an extension ring of R are considered in this section.

Theorem 4.1. ([12], Thm. 2.1) *Suppose $\{R_a\}_{a\in A}$ is a nonempty family of zero-dimensional subrings of the ring S. If $R = \bigcap_{a\in A} R_a$, then R is zero-dimensional.*

Remark 4.2. In the notation of Theorem 4.1, if $x \in R$ and $a \in A$, then x^{n_a} generates an idempotent ideal of R_a for some positive integer n_a. A key point in the proof of (4.1) is that x^{n_a} generates an idempotent ideal of R_b for each $b \in A$. Once this has been shown, uniqueness of the pointwise inverse implies that if s_x is the pointwise inverse of x^{n_a} in S, then s_x belongs to each R_b, hence to R, and thus $x^{n_a}R$ is idempotent and R is zero-dimensional. This proof shows that if some R_c is von Neumann regular (so that n_c can be taken to be 1), then R is also vNr; alternately, this statement is an easy corollary of Theorem 4.1.

Corollary 4.3. *Suppose $\{R_a\}_{a\in A}$ is a nonempty family of zero-dimensional subrings of the ring S, and let $R = \bigcap_{a\in A} R_a$. If R is reduced, then R is von Neumann regular; in particular, R is vNr if some R_a has this property. If $\mathcal{V}(S)$ is nonempty, it is closed under taking arbitrary intersections.*

Remark 4.4. In contrast with (4.1) and (4.3), the intersection of two Artinian subrings of a ring need not be Artinian. For example, let K be a field containing subfields K_1, K_2 of finite codimension in K such that $K_1 \cap K_2$ is not of finite codimension. (A classical example of this form is where $K = F(t)$, $K_1 = F(t^2 + t)$, and $K_2 = F(t^2)$, where F is a field of characteristic 0 and t is an indeterminate over F.) If $S = K[Y]/(Y^2)$, where Y is an indeterminate over K, then S is a two-dimensional vector space over K with basis $\{1, y\}$, where $y = Y + (Y^2)$. The ring S is Artinian, as are its subrings $R_1 = K_1 + Ky$ and $R_2 = K_2 + Ky$. However, $R_1 \cap R_2 = (K_1 \cap K_2) + Ky$ is not Artinian since the ideal Ky of this ring is not finitely generated. We note that Gilmer and Heinzer in [18], Sect. 4 have given examples of rings T such that $\mathcal{A}(T)$ is closed under finite intersection, but not under arbitrary intersection. Because a von Neumann regular ring with only finitely many idempotents is a finite direct sum of fields, hence Artinian, there is an Artinian analogue of Corollary 4.3 for a reduced ring S.

Corollary 4.5. *Let R_a, S, and R be as in the statement of Corollary 4.3, and assume that S is reduced. If R has only finitely many idempotents, then R is Artinian, in particular, R is Artinian if some R_a is Artinian. If $\mathcal{A}(S)$ is nonempty, it is closed under taking arbitrary intersections.*

Remark 4.6. Suppose R is a subring of the ring S. If R admits a zero-dimensional S-overring (that is, if $\mathcal{Z}(R, S)$ is nonempty), then Theorem 4.1 shows that $\mathcal{Z}(R, S)$ has a unique minimal element. We denote this element by $R^0(S)$, and call it the *minimal zero-dimensional extension of R in S*. It is useful to know that a set of generators for $R^0(S)$ as an extension ring of R (as in Remark 3.2) can be described. Theorem 4.7 provides a precise statement.

Theorem 4.7. ([12], Thm. 2.5) *Suppose R is a subring of a zero-dimensional subring of the ring S. For $x \in R$, assume that $x^{m(x)}S$ is idempotent, and let s_x be the pointwise inverse of $x^{m(x)}$ in S. Then $R^0(S) = R[\{s_x \mid x \in R\}]$.*

The next result records some properties of $R^0(S)$ as an extension ring of R.

Theorem 4.8. ([16], Thm. 3.3) *Suppose R is a subring of a zero-dimensional subring of the ring S, and let $R^0 = R^0(S)$.*

(a) If $M \in \mathrm{Spec}(R^0)$, then R^0/M is the quotient field of its subring $(R + M)/M \simeq R/(M \cap R)$.

(b) A primary ideal of R^0 is uniquely determined by its contraction to R.

(c) $|\mathrm{Spec}(R^0)| \leq \mathrm{Spec}(R)|$; if $P \in \mathrm{Spec}(R)$, then P is contracted from R^0 if and only if P is contracted from S.

(d) If $\mathrm{Spec}(R)$ is finite, then R^0 is a finitely generated R-algebra.

Although we sometimes write R^0 instead of $R^0(S)$, the isomorphism class of $R^0(S)$ is easily seen to depend, in general, on S. Theorem 4.9 is a definitive result in this direction.

Theorem 4.9. ([16], Thm. 4.1) *Suppose the ring R is embeddable in a zero-dimensional ring. If $\dim R \neq 0$, then there exist extension rings S_1, S_2 of R with $\mathcal{Z}(R, S_1)$ and $\mathcal{Z}(R, S_2)$ nonempty such that $R^0(S_1) \not\simeq R^0(S_2)$.*

Remark 4.10. In light of Corollary 4.3, it follows that if $\mathcal{Z}(R,S) \neq \emptyset$, then $\mathcal{V}(R,S)$ is nonempty if and only if $R^0(S)$ is von Neumann regular. There is another interesting connection between von Neumann regular rings and minimal 0-dimensional extension rings that is related to work of Olivier in [35]. To wit, Olivier shows that if R is a ring, then there exists a vNr ring R^* and a homomorphism $f : R \to R^*$ such that if $g : R \to T$ is any homomorphism with T vNr, then there exists a unique homomorphism $g^* : R^* \to T$ such that $g = g^* \circ f$. Olivier calls R^* *the regular ring associated to* R. Popescu and Vraciu [37], Sect. 3 show as follows that R^* can be realized as a minimal zero-dimensional extension. Let $\mathrm{Spec}(R) = \{P_i\}_{i \in I}$ and let K_i be the quotient field of R/P_i. Let f_i be the canonical map from R onto R/P_i, and let $f : R \to T = \prod_{i \in I} K_i$ be the product of the maps f_i; we note that f is an imbedding of R if and only if R is reduced. Popescu and Vraciu show that R^* is isomorphic to $(f(R))^0(T)$, with f being the associated map from R into this ring. No meaningful generalization of this concept to that of a zero-dimensional ring associated to R seems to have appeared in the literature.

Remark 4.11. We discuss in Section 5 zero-dimensional extension rings of a given ring R. There is one classical case of an embedding into a zero-dimensional ring that we wish to mention here, however, in connection with minimal zero-dimensional extensions. The case alluded to is that of a ring R whose zero ideal is a finite intersection of primary ideals. If, say, $(0) = \bigcap_{i=1}^n Q_i$, where Q_i is P_i-primary, then R is canonically embedded in $\prod_{i=1}^n (R/Q_i)$, which in turn is embedded via canonical inclusion in the zero-dimensional ring $T = \prod_{i=1}^n (R/Q_i)_{(P_i/Q_i)}$. If ϕ denotes the embedding of R into T via composition of these two canonical maps, then Proposition 4.4 of [16] shows that each proper subring of T containing $\phi(R)$ is of positive dimension, and hence T itself is the minimal zero-dimensional extension of $\phi(R)$ in T. Through identification of R with $\phi(R)$, we refer to T as a minimal zero-dimensional extension of R.

5. ZERO-DIMENSIONAL EXTENSION RINGS

We consider in this section zero-dimensional extension rings of a given ring R. This was the primary focus of Arapovic's work in [2]–[4]. Arapovic was particularly interested in the question of whether R admits a zero-dimensional extension ring. This is a natural question since an integral domain possesses a quotient field, a Noetherian ring is embeddable in a zero-dimensional ring (more generally, each ring whose zero ideal is a finite intersection of primary ideals is so embeddable — see Remark

4.11), and a reduced ring is embeddable in a von Neumann regular ring. As noted in Proposition 2.4, each ideal of a zero-dimensional ring is the intersection of a (possibly infinite) family of primary ideals. Thus, if R is a subring of a zero-dimensional ring, then each universally contracted ideal[4] of R is an intersection of primary ideals of R. This observation alone implies that not every ring is embeddable in a zero-dimensional ring. For example, if R is a chained ring — that is, if the set of ideals of R is linearly ordered under inclusion — then an intersection of primary ideals of R is primary [12], Res. 1.8. Therefore if I is a non-primary ideal of a valuation domain V, then V/I admits no zero-dimensional extension ring. Arapovic's main result in regard to the embedding question is recorded below as Theorem 5.1; we refer to this as Arapovic's Embedding Theorem, abbreviated AET:

Theorem 5.1. ([4], Thm. 7, [25], Thm. 3.5, [12], Thm. 1.7) *The ring R is embeddable in a zero-dimensional ring if and only if there exists a family $\{Q_\lambda\}_{\lambda\in\Lambda}$ of primary ideals of R, where $P_\lambda = \mathrm{rad}(Q_\lambda)$, such that the following conditions (A1) and (A2) are satisfied.*

(A1) $\bigcap_{\lambda\in\Lambda} Q_\lambda = (0)$.

(A2) For each $a \in R$, there exists a positive integer n_a such that $a^{n_a} \notin \bigcup_{\lambda\in\Lambda}(P_\lambda \setminus Q_\lambda)$.

At first encounter condition (A2) may appear contrived. While it is difficult to verify, a natural way to interpret the condition is to consider it in the case where R itself is zero-dimensional. In that case (A2) is satisfied for each family $\{Q_\lambda\}$ of primary ideals satisfying (A1). To see this, for $a \in R$ choose n_a so that $a^{n_a}R$ is idempotent. For $\lambda \in \Lambda$, if $a^{n_a} \in P_\lambda$, then Q_λ contains a power of $a^{n_a}R$, and hence Q_λ contains a^{n_a}. Whence $a^{n_a} \notin P_\lambda - Q_\lambda$. The condition is easily seen to descend to subrings of a zero-dimensional ring.

Remark 5.2. In work on the embedding problem for a ring R, one frequently considers a canonical diagonal embedding $r \to \{r + I_a\}_{a\in A}$ of R into the product of a family $\{R/I_a\}_{a\in A}$ of residue class rings of R, where $\{I_a\}_{a\in A}$ is a family of ideals of R with intersection (0). If $t(\cdot)$ denotes total quotient ring, then $\prod_{a\in A}(R/I_a)$ is canonically embedded in $\prod_a t(R/I_a)$, so R is also embedded in the latter ring. As in Remark 4.11, $R \hookrightarrow \prod_a t(R/I_a)$ is frequently more relevant for the

[4]An ideal I of R is *universally contracted* if I is contracted from each extension ring of R.

question of embeddability of R into a zero-dimensional ring. For example, if $\{A_\lambda\}_{\lambda\in\Lambda}$ is a family of primary ideals of R satisfying (A1), then $S_\lambda = t(R/Q_\lambda) \simeq R_{P_\lambda}/Q_\lambda R_{P_\lambda}$ is zero-dimensional, so $S = \prod S_\lambda$ is a product of zero-dimensional rings. As noted in Theorem 2.3, such a ring need not itself be zero-dimensional. A naive conjecture might be that condition (A2) implies zero-dimensionality of S. This is incorrect, but (A2) does imply, as follows, that if $\phi : R \to S$ denotes the canonical embedding, then $\phi(R)$ is contained in a zero-dimensional subring of S. Thus, $a \in R$ corresponds to $\phi(a) = \{a + Q_\lambda/1 + Q_\lambda\}_{\lambda\in\Lambda}$ and in $S_\lambda = R_{P_\lambda}/Q_\lambda R_{P_\lambda}$, the element $a_\lambda = a^{n_a} + Q_\lambda/1 + Q_\lambda$ is either 0 or a unit of S_λ. In either case a_λ generates an idempotent ideal of S_λ, so $\phi(a)^{n_a}$ generates an idempotent ideal of S. It then follows from Theorem 3.1 that $\phi(R)$ admits a zero-dimensional S-overring.

Remark 5.3. For each family $\mathcal{B} = \{Q_\lambda\}$ of primary ideals of R satisfying (A1) and (A2), one obtains a minimal zero-dimensional extension $R_{\mathcal{B}}$ of R via the construction in Remark 5.2. Gilmer and Heinzer in [16] refer to $R_{\mathcal{B}}$ as a *canonical minimal zero-dimensional extension* of R. The family $\mathcal{B}$ is not unique and, in general, $R_{\mathcal{B}}$ is not determined up to isomorphism. However, Remark 4.6 of [16] shows that in a ring with a family of primary ideals satisfying (A1) and (A2), there always exists such a family where distinct ideals have distinct radicals; such a family can be obtained from a given family $\mathcal{B}$ by replacing the ideals of $\mathcal{B}$ having the same radical P by their intersection (condition (A2) implies that the intersection is P-primary).

If $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings and $R = \prod_{a\in A} R_a$, recall the result of Maroscia (Theorem 2.3) which states that R is zero-dimensional if and only if $N(R) = \prod\limits_a N(R_a)$. There is a useful reformulation of this equality that is expressed as an internal property of the family $\{R_a\}$. Thus, if S is a ring and $x \in N(S)$, the *index of nilpotency* of x, denoted $\eta(x)$, is the smallest positive integer k such that $x^k = 0$. The *index of nilpotency* of S, denoted $\eta(S)$, is defined to be $\sup\{\eta(x) \mid x \in N(S)\}$; if the given set is unbounded, we write $\eta(S) = \infty$. Gilmer and Heinzer prove:

Theorem 5.4. ([11], Thm. 6) *If $R = \prod_{a\in A} R_a$, where each R_a is zero-dimensional, then R is zero-dimensional if and only if there exists a positive integer k such that $\{a \in A \mid \eta(R_a) > k\}$ is finite.*

For arbitrary families of rings, the next result is a definitive statement concerning the dimension of the product.

Theorem 5.5. ([11], Thm. 11) *Suppose $\{R_a\}_{a\in A}$ is an infinite family of nonzero rings. Let $R = \prod_{a\in A} R_a$.*

(a) *If $\dim R_a > 0$ for infinitely many elements $a \in A$, then $\dim R = \infty$.*

(b) *If $\dim R_a > 0$ for only finitely many elements $a_1, a_2, \ldots, a_t \in A$, then $\dim R = \sup\{\dim(R_{a_1}), \ldots, \dim(R_{a_t}), \dim(\prod_{b\in B} R_b)\}$, where $B = A \setminus \{a_1, \ldots, a_t\}$.*

(c) *If each R_a is zero-dimensional, then $\dim R \in \{0, \infty\}$, and Theorem 5.4 gives equivalent conditions for R to be zero-dimensional.*

Remark 5.6. Several papers have investigated properties of $\mathrm{Spec}(R)$, where $R = \prod_{a\in A} R_a$ is the product of an infinite family of nonzero rings. For example, Aoyama in [1] considers the case where each R_a is a rank-one discrete valuation domain, and [31] deals with the case where each $R_a = \mathbb{Z}$. If each R_a is zero-dimensional and R is infinite-dimensional, Shapiro in [38], Thm. 2.3 showed that R contains an infinite chain of prime ideals. For each R_a a zero-dimensional chained ring, Gilmer and Heinzer showed that for each $P \in \mathrm{Spec}(R)$ of positive height, there exists an infinite strictly descending chain $P > P_1 > P_2 > \ldots$ of primes of R [20], Thm. 2.3.

Another aspect of $\mathrm{Spec}(R)$ that has been considered is the possible dimension of a homomorphic image of R. Theorem 5.7 is a definitive result in this vein in the case where each R_a is zero-dimensional.

Theorem 5.7. ([19], Rem. 3.9, [20], Cor. 3.8, [21], Thm. 3.8 and Exam. 3.10) *Suppose $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings and that $R = \prod_{a\in A} R_a$ is infinite-dimensional.*

(a) *If each R_a is a chained ring, then a finite-dimensional homomorphic image of R has dimension at most 1; R need not have a homomorphic image of dimension 1.*

(b) *If each R_a is a principal ideal ring, then R admits a one-dimensional homomorphic image.*

(c) *For each positive integer d, there exists a family $\{S_b\}_{b\in B_d}$ of zero-dimensional rings such that the dimensions of nonzero homomorphic images of $S = \prod\limits_{b\in B_d} S_b$ are precisely the integers $0, 1, 2, \ldots, d$ and ∞. Moreover, there is a family $\{T_c\}_{c\in C}$ of zero-dimensional rings such that, for each nonnegative integer d, $T = \prod_{c\in C} T_c$ admits a homomorphic image of dimension d.*

For a family $\{R_a\}_{a\in A}$ of zero-dimensional rings, Gilmer and Heinzer have shown that if $R = \prod_{a\in A} R_a$ is not zero-dimensional (whence $\dim R = \infty$), then R is not embeddable in a zero-dimensional ring [14], Cor. 8, [12], Thm. 1.9. Their proof uses AET. We present here a more elementary proof that has not previously appeared in the literature.

Theorem 5.8. *Suppose $\{R_a\}_{a\in A}$ is a family of zero-dimensional rings such that $R = \prod R_a$ is not zero-dimensional. Then R is not embeddable in a zero-dimensional ring.*

Proof. By Theorem 5.4 there exists a sequence $\{a_i\}_1^\infty$ of distinct elements of A such that $\eta(R_{a_i}) > i$ for each i. The ideal $\prod_{i=1}^\infty R_{a_i}$ of R is a direct summand of R as an R-module, so if R had a zero-dimensional extension ring, so would $\prod_{i=1}^\infty R_{a_i}$ (note that we desire that an extension ring should have the same unity element as the base ring). Therefore without loss of generality assume that $A = \mathbb{Z}^+$ and that $x_i \in R_i$ is such $x_i^i \neq 0$. Let $x = \{x_i\}_1^\infty \in R$ and let $e_i = \{\delta_{ij}\}_{j=1}^\infty$ be the primitive idempotent associated with i; here δ_{ij} is the Kronecker delta. If S is an extension ring of R, we show that no power of xS is idempotent. Suppose not, and assume that $k \in \mathbb{Z}^+$ is such that $x^k S = x^{k+1} S = \ldots$. Since $x_k \in N(R_k)$, $x_k^m = 0$ for some $m > k$. Write $x^k = x^m s$ for $s \in S$. Then $x^k e_k \neq 0$ since $x_k^k \neq 0$. But $x^m e_k = 0$ by choice of m, and $x^m e_k s = 0$. This is a contradiction since $x^k e_k = x^m e_k s$. It follows that no power of xS is idempotent, so S is not zero-dimensional and R is not embeddable in a zero-dimensional ring. ∎

In [21], Thm. 2.1, Gilmer and Heinzer extend Theorem 5.8 to show that if I is the direct sum ideal of R and if J is an ideal of R contained in I, then R/J is not embeddable in a finite-dimensional ring.

Remark 5.9. It is routine to show that ring R is embeddable in a von Neumann regular ring if and only if R is reduced [14], Thm. 1.3. On the other hand, the problem of determining conditions for embeddability in an Artinian ring remains unresolved. Because each Noetherian ring is embeddable in an Artinian ring (see Remark 4.11), a ring is embeddable in an Artinian ring if and only if it is embeddable in a Noetherian ring. An Artinian ring has only finitely many idempotents, is strongly Laskerian[5], and has finite length as a module over itself. Thus if the

[5] A primary ideal Q is *strongly primary* if it contains a power of its radical; a ring S is *strongly Laskerian* if each ideal of S is a finite intersection of strongly primary ideals.

ring T is embeddable in an Artinian ring, then T has only finitely many idempotents (and hence is a finite direct sum of indecomposable rings), each universally contracted ideal of T is a finite intersection of strongly primary ideals, and there is a uniform bound on the length of a chain of universally contracted ideals of T. (We remark that the annihilator ideals of T form a useful subset of the set of universally contracted ideals; in some cases these two sets coincide.) Because (0) is a finite intersection of strongly primary ideals, T embeds in a finite direct sum of zero-dimensional quasilocal rings (T_i, M_i) such that $M_i^{k_i} = (0)$ for some $k_i \in \mathbb{Z}^+$. Most work on Artinian embeddability has dealt with such rings T_i. Theorem 5.10 is in this vein.

Theorem 5.10. ([12], Prop. 1.10) *Suppose* $(0) = \bigcap_{i=1}^{\infty} Q_i$ *in the ring* T, *where* Q_i *is* P_i*-primary and* $P_i^2 \subseteq Q_i$. *Then* T *is embeddable in an Artinian ring.*

Remark 5.11. Theorem 5.4 does not extend to the case where $P_i^3 \subseteq Q_i$; in [30], Kerr constructs a zero-dimensional quasilocal ring (T, M) with $M^3 = (0)$ such that T contains an infinite decreasing chain of annihilator ideals, and hence T admits no Artinian extension ring. Gilmer and Heinzer in [22], Exam. 1.6 modified Kerr's construction to obtain zero-dimensional quasilocal rings (S, N) with $N^3 = (0)$ and for which there is a uniform bound on the lengths of chains of universally contracted ideals (which coincide with annihilator ideals), but which are not embeddable in an Artinian ring.

6. RESIDUE FIELDS OF ZERO-DIMENSIONAL RINGS

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be an indexed family of fields and let R be a ring. We say that $\mathcal{F}$ *is the family of residue fields of* R if there exists a bijection $g : I \to M\mathrm{Spec}(R)$ such that $F_i \simeq R/g(i)$ for each $i \in I$. Proposition 3.6 and Corollary 3.10 involve conditions on the family of residue fields of a zero-dimensional ring. These results motivated Gilmer and Heinzer to consider in [17] the problem of characterizing those indexed families of fields that are realizable as the family of residue fields of a zero-dimensional ring. While significant progress has been made on this problem, it seems far from resolution, and at this point there is no conjectured characterization of the families described. Since R and $R/N(R)$ have the same residue fields for any ring R, in considering the realization problem for zero-dimensional rings, one can restrict to the case of von Neumann regular rings. Theorem 6.1 is a significant positive result concerning realization of an indexed family.

Theorem 6.1. ([13], Thm. 4*) *Suppose $\mathcal{F} = \{F_i\}_{i\in I}$ is an indexed family of fields. If there exists a finite subset $\{i_j\}_{j=1}^n$ of I such that each F_i contains an isomorphic copy of at least one of the fields F_{i_j}, then $\mathcal{F}$ is realizable.*

While the converse of Theorem 6.1 fails in general, a form of the converse is valid for certain zero-dimensional rings that are algebras over a field.

Theorem 6.2. ([13], Thm. 6) *Suppose F is a field and R is an F-algebra such that R/M is a finite-dimensional F-algebra for each $M \in \operatorname{Spec}(R)$. Then there exists a finite subset $\{M_i\}_{i=1}^n$ of $\operatorname{Spec}(R)$ such that, for each $M \in \operatorname{Spec}(R)$, there exists i between 1 and n such that R/M_i is an F-subalgebra of R/M.*

Because the realization problem arose in part from Corollary 3.10, the case of a family $\{F_i\}_{i\in I}$ with each F_i absolutely algebraic of nonzero characteristic has received particular attention. Theorems 6.1 and 6.2 resolve that case if each F_i is finite.

Corollary 6.3. ([13], Cor. 7) *Suppose $\mathcal{F} = \{F_i\}_{i\in I}$ is an indexed family of finite fields. Then $\mathcal{F}$ is realizable if and only if there exists a finite subset $\{i_j\}_{j=1}^n$ of I such that each F_i contains, up to isomorphism, at least one of the fields F_{i_j}.*

Remark 6.4. The conclusion of Corollary 6.3 fails if at least one of the fields F_i in $\mathcal{F}$ is not finite. In fact, Example 3.6 of [17] exhibits a realizable family $\{F_i\}_{i=0}^\infty$ of pairwise incomparable fields of the same nonzero characteristic such that F_i is finite for $i > 0$, while F_0 is (infinite) absolutely algebraic.

The realization problem has led to several different constructions of zero-dimensional rings. Several of these have involved the direct sum ideal of a product of zero-dimensional rings. The next result is fairly typical; in the statement of Proposition 6.5 (and hereafter) we use the symbol $\dot{\bigcup}$ to denote disjoint union.

Proposition 6.5. ([17], Prop. 4.2) *Let $\{R_a\}_{a\in A}$ be an indexed family of von Neumann regular rings, let $R = \prod_a R_a$, let I be the direct sum* *ideal of R, and let S be a zero-dimensional subring of R/I. If T is the inverse image of S under the canonical homomorphism from R to R/I, then T is von Neumann regular and $\mathcal{F}(T) = [\uplus \mathcal{F}(R_a)] \uplus \mathcal{F}(S)$.*

Corollary 6.6. ([17], Cor. 4.3 and 4.4) *Suppose $\{F_a\}_{a \in A}$ is a family of fields, $R = \prod_a F_a$, and I is the direct sum ideal of R.*

(1) *If K is a subfield of R/I, then $\{F_a\} \uplus \{K\}$ is realizable.*

(2) *If A is infinite and, for each prime p, the set $\{a \in A | \mathrm{char}(F_a) = p\}$ is finite, then the family $\{F_a\} \uplus \{\mathbb{Q}\}$ is realizable.*

Using results from [24], R. Wiegand proved the following broad generalization of part (2) of Corollary 6.6.

Theorem 6.7. ([13], p. 53) *Let $\{q_i\}$ be an infinite set of primes, and let T_i be a von Neumann regular ring of characteristic q_i. Let X be a set of indeterminates of cardinality $\aleph_1$. Then $\mathbb{Q}(X)$ embeds in $\prod_i T_i / \sum_i T_i$.*

Corollary 6.8. *Let $\{q_i\}_{i \in I}$ be an infinite set of primes, and for each $i \in I$ let $\mathcal{F}_i$ be a realizable family of fields of characteristic q_i. If X is a set of indeterminates over $\mathbb{Q}$ of cardinality at most $\aleph_1$, then the family $(\bigcup_{i \in I} \mathcal{F}_i) \cup \{\mathbb{Q}(X)\}$ is realizable.*

Another construction for zero-dimensional rings uses what Doering and Lequain in [8] call *a gluing process for maximal ideals.* For a zero-dimensional ring R, the process operates as follows. Suppose M_i, N_i are maximal ideals of R such that $R/M_i \simeq R/N_i \simeq K_i$ for $1 \leq i \leq k$. Let μ_i and ρ_i, respectively, be isomorphisms from R/M_i and R/N_i, respectively, onto K_i. Then $S = \{r \in R \,|\, \mu_i(r) = \rho_i(r) \text{ for } 1 \leq i \leq k\}$ is a zero-dimensional subring of R. For $1 \leq i \leq k$, $M_i \cap N_i$ is a maximal ideal of S with residue field K_i; each other maximal ideal of S is contracted from a unique maximal ideal of R, and has the same residue field as the maximal ideal from which it is contracted. Proposition 6.7 is obtained by using the gluing process.

Proposition 6.9. ([13], Cor. 16) *Suppose $\mathcal{F}$ and $\mathcal{G}$ are realizable families of fields. Suppose there exist $F_1, F_2, \ldots, F_n \in \mathcal{F}$ and $K_1, K_2, \ldots, K_n \in \mathcal{G}$ such that $F_i \simeq K_i$ for each i. Then $\{\mathcal{F} - \{F_i\}_1^n\} \uplus \mathcal{G}$ is realizable.*

Remark 6.10. If R is zero-dimensional, then $\mathrm{Spec}(R)$ under the Zariski topology is a Boolean space — that is, a compact, Hausdorff, totally disconnected space. In [23], the authors consider a special case of the realization problem. To wit, given a field F, a Boolean space B, and a family $\{K_b\}_{b\in B}$ of extension fields of F, [23] considers the problem of determining conditions under which there exists a zero-dimensional F-algebra R such that $\{K_b\}$ is, up to F-isomorphism, the family of residue fields of R and the associated bijection from B to $\mathrm{Spec}(R)$ is a homeomorphism.

Remark 6.11. Another question involving residue fields of the zero-dimensional rings and their spectra as Boolean spaces has been considered in [36] and in [17]:

Suppose R and S are von Neumann regular rings and $\phi : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is a homeomorphism such that $R/M \simeq S/\phi(M)$ for each $M \in \mathrm{Spec}(R)$. Are R and S isomorphic?

Example 6.4 of [17], which is due to R. Wiegand, shows that this question has a negative answer.

Remark 6.12. If $\mathcal{C}$ is a class of rings, a more general case of the realization problem for zero-dimensional rings asks for a characterization of the indexed families of fields that can be realized as the family of residue fields of a ring in class $\mathcal{C}$. This section has discussed the case where $\mathcal{C} = \mathcal{Z}$, the class of zero-dimensional rings, and [24] considers the case where $\mathcal{C}$ is the class of principal ideal domains. Other classes of rings where the question is of interest include the class of all rings, the class of integral domains, and the class of Noetherian rings. In particular, no example is known of an indexed family of fields that cannot be realized as the family of residue fields of a commutative ring.

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