Boundaries of Nonpositively Curved Groups of the Form $G \times \mathbb{Z}^n$

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Abstract. We prove that whenever G is negatively curved and $\Gamma = G \times \mathbb{Z}^n$ acts geometrically on two $\mathsf{CAT}(0)$-spaces X and X', their visual boundaries are Γ-equivariantly homeomorphic, answering affirmatively a question of Gromov in this special case. However, we give a simple example that shows that the natural Γ-equivariant quasi-isometry from X to X' does ***not*** continuously extend to a homeomorphism of visual boundaries, a situation that does occur if Γ is negatively curved. We show further that the Γ-rational endpoints of X form a dense subset of the visual boundary, generalizing the analogous result that holds in the context of negative curvature.

Introduction

The introduction of curvature considerations in the past decade into Combinatorial Group Theory has had a profound effect on the study of infinite discrete groups. In particular, the theory of negatively curved groups has enjoyed significant and extensive development since Cannon's seminal study of cocompact hyperbolic groups in the early eighties [**7**]. Unarguably the greatest influence on the direction of this development has been Gromov's *tour de force*, his foundational essay in [**12**] entitled *Hyperbolic Groups*. Therein Gromov hints at the prospect of developing a corresponding theory of "nonpositively curved groups" in his non-definition (Gromov's terminology) of a semihyperbolic group as a group that "looks as if it admits a discrete cocompact isometric action on a space of nonpositive curvature"; [**12**, p. 81]. Such a development is now occurring and is closely related to the other notable outgrowth of the theory of negatively curved groups, that of automatic groups [**10**]; we mention here the works [**3**] and [**6**] as developments of a theory of nonpositively curved groups along with Chapter 6 of Gromov's more recent treatise [**13**]. A natural question that serves both to guide and organize the developing theory is: to what extent is the well-developed theory of negatively curved groups reflected in and subsumed under the developing theory of nonpositively curved groups? Our overall interest is in one aspect of this question — namely, as the question relates to the boundaries of groups and spaces: can one define the boundary of a nonpositively curved group intrinsically in a way that generalizes that of negatively curved groups and retains some of their essential features?

Recall that the boundary of a negatively curved group (or more generally, space) G is defined in terms of equivalence classes of sequences convergent at infinity in the Gromov inner product (precise definitions are recalled later) and is compact, metrizable and finite-dimensional [**2,12**]. The boundary also may be described in terms of equivalence classes of geodesic rays that fellow travel in the Cayley graph of G with word metric [**8**], and as such parameterizes the "directions towards infinity" in the Cayley graph. The "visual" or "geodesic" boundary of a $\mathsf{CAT}(0)$-space X parameterizes the geodesic rays issuing from a basepoint and, if X also is negatively curved, then these two concepts of boundary agree. Whenever a group G acts ***geometrically*** (i.e., properly discontinuously, cocompactly, and isometrically) on a negatively curved geometry X, the group G is negatively curved and the natural map $g \mapsto g \cdot x_0$ (x_0 is a fixed basepoint in X) of G to X extends continuously to an equivariant homeomorphism of boundaries. For this reason, if G acts geometrically on two negatively curved geometries X and X', then the boundaries of X and X' are equivariantly homeomorphic. To what extent the situation as outlined here generalizes to groups acting geometrically on $\mathsf{CAT}(0)$-spaces is a question of considerable current interest. In fact, Gromov [**13**, Chapter 6] specifically asks whether the visual boundaries of X and X' are Γ-equivariantly homeomorphic whenever the group Γ acts geometrically on the two $\mathsf{CAT}(0)$-spaces X and X', and it is natural to suggest that implicit in Gromov's question is the question of whether the natural quasi-isometry from X to X' that quasi-isometrically factors through the Cayley graph of Γ extends continuously to a homeomorphism of visual boundaries. As noted, this is exactly what occurs in the presence of negative curvature and, were it true, would lead to an unambiguously defined analogue of the boundary of a group in the presence of nonpositive curvature. This paper presents a case study of these questions in the context of groups of the form $\Gamma = G \times \mathbb{Z}^n$, where G is negatively curved.

Consider then a group Γ acting geometrically on the $\mathsf{CAT}(0)$-geodesic spaces X and X'. In Section 1, we use a lemma of Bridson and Haefliger to show that if Γ is of the form $G \times \mathbb{Z}^n$ with G negatively curved, then the visual boundaries of X and X' are equivariantly homeomorphic, though not canonically so. On the other hand, the following simple but enlightening example shows that in general the question implicit in Gromov's has a negative answer; i.e., in general, the natural quasi-isometry

$$X \cong \Gamma \cdot x_0 \leftrightarrow \Gamma \leftrightarrow \Gamma \cdot x_0' \cong X',$$

where $x_0 \in X$ and $x_0' \in X'$ are basepoints and "$\cong$" indicates a quasi-isometry, does ***not*** extend continuously to a map of $X \cup \partial X$ to $X' \cup \partial X'$.

(In the sequel, any quasi-isometry $X \to X'$ such that $\gamma \cdot x_0 \mapsto \gamma \cdot x'_0$ for $\gamma \in \Gamma$ will be called a ***Γ-equivariant quasi-isometry***.) We elaborate upon and verify the claims of the example in Section 2.

Example. Let $F_2 = |a, b : \phi|$ be the rank 2 free group with basis $\{a, b\}$ and let $X = T \times \mathbb{R}$, where T is the unique nonempty 4-valent tree. We may identify T with the Cayley graph of F_2 with respect to the generating set $\{a, b\}$ and so regard F_2 as the vertex set of T. Endowed with the ***ℓ_2-product metric*** ($(d_1^2 + d_2^2)^{\frac{1}{2}}$ for metrics d_1 and d_2), where T is given a path metric wherein each edge is isometric to the unit interval and $\mathbb{R}$ has its usual absolute value metric, X becomes a $\mathsf{CAT}(0)$-space. We consider two geometric actions of $F_2 \times \mathbb{Z}$ on X. The first is just the product action defined on the generating set $\{(a, 0), (b, 0), (1, 1)\}$ of $F_2 \times \mathbb{Z}$ by

$$(a, 0) \cdot (t, r) = (a \cdot t, r), \quad (b, 0) \cdot (t, r) = (b \cdot t, r), \quad (1, 1) \cdot (t, r) = (t, r + 1).$$

Here of course $(a, t) \mapsto a \cdot t$, $(b, t) \mapsto b \cdot t$ describes the natural action of F_2 on its Cayley graph T. The second is obtained by a simple change in the action of $(b, 0)$ on X and is defined by

$$(a, 0) * (t, r) = (a \cdot t, r), \quad (b, 0) * (t, r) = (b \cdot t, r + 2), \quad (1, 1) * (t, r) = (t, r + 1).$$

In Section 2, we verify that this example satisfies the following properties:

(i) For a fixed basepoint x_0 of X, $(F_2 \times \{0\}) \cdot x_0$, the copy of F_2 realized as the orbit of x_0 under $F_2 \times \{0\}$ by the first action, obviously lies in a horizontal slice of $X = T \times \mathbb{R}$ and is a quasi-convex (see Section 2 for definitions) subset of X; however, $(F_2 \times \{0\}) * x_0$, the copy of F_2 realized under the second action, is ***not*** quasi-convex.

(ii) $F_2 \times \{0\}$ is a quasi-convex negatively curved subgroup of $F_2 \times \mathbb{Z}$ with a Cantor set boundary. Of course, the quasi-isometric embedding $(g, 0) \mapsto (g, 0) \cdot x_0$ of $F_2 \times \{0\}$ extends continuously to a map of the boundary of $F_2 \times \{0\}$ into the visual boundary of X; however, the quasi-isometric embedding $(g, 0) \mapsto (g, 0) * x_0$ of $F_2 \times \{0\}$ does ***not*** extend continuously to a map of the boundary of $F_2 \times \{0\}$. In fact, the natural extension of $(g, 0) \mapsto (g, 0) * x_0$ to the rational points (see definition below) of the boundary of $F_2 \times \{0\}$ is not even continuous.

(iii) The maps $F_2 \times \mathbb{Z} \to X$ given by $\gamma \mapsto \gamma \cdot x_0$ and $\gamma \mapsto \gamma * x_0$ are quasi-isometries [**8**] that induce a quasi-isometry $X \to X$ for which $\gamma \cdot x_0 \mapsto \gamma * x_0$. This Γ-equivariant quasi-isometry of X induced by the two actions of $F_2 \times \mathbb{Z}$ does ***not*** extend continuously to a map of visual boundaries.

(iv) There is an automorphism of $F_2 \times \mathbb{Z}$ that sends $(a, 0)$ to $(a, 0)$, $(b, 0)$ to $(b, 2)$, and $(1, 1)$ to $(1, 1)$, showing that the product action $\cdot$ and the "twisted" action $*$ are equivalent up to automorphism of $F_2 \times \mathbb{Z}$. This shows that the relationship of the group $F_2 \times \mathbb{Z}$ (or its Cayley graph) to visual boundary is not even automorphism-invariant; i.e., an automorphism of $F_2 \times \mathbb{Z}$ does not induce a homeomorphism of visual boundaries, as does occur in the context of negatively curved groups.

(v) It is an open question as to whether all 1-ended negatively curved groups have locally connected boundaries. This simple example provides a negative answer to the analog of this question in the setting of nonpositive curvature. Indeed, our main result guarantees that the group $F_2 \times \mathbb{Z}$ has a well-defined boundary as the suspension of a Cantor set, a connected nonlocally connected space. Furthermore, Mike Mihalik has pointed out to the second author that $F_2 \times \mathbb{Z}$ sits as a finite-index subgroup of a Coxeter group, and this furnishes an example of a 1-ended Coxeter group with nonlocally connected boundary.

The example shows that it is impossible to generalize from the context of negatively curved groups to that of nonpositively curved groups the notion of boundary that parameterizes the directions to infinity in a way that equivariantly identifies the boundary of the group with the visual boundary of any and every $\mathsf{CAT}(0)$-space on which the group might geometrically act. The example, though very simple, is revealing of the lack of structure that results in a move from negatively to nonpositively curved groups.

Section 3 closes the paper on a positive note. Recall that an element g of infinite order in the negatively curved group G determines a ***rational point*** $g^{\infty} = \lim g^n$ in the boundary (since the powers of g form a quasi-geodesic [**2**]), and the set of these rational points is dense in the boundary of G [**12,5**]. This denseness has important implications for the dynamics of the natural action of G on its boundary [**9**]. If a group acts on a negatively curved space (or a $\mathsf{CAT}(0)$-space) X with basepoint x_0, the ***G-rational points*** of the (visual) boundary are the limiting values of sequences of the form $\{g^n \cdot x_0\}$ for infinite order elements g. Easily, from our previous discussion, these rational points form a dense subset of the boundary of X whenever the group G acts geometrically on the negatively curved space X. We verify that the corresponding result holds true in our particular nonpositively curved setting: if X is $\mathsf{CAT}(0)$ and Γ is of the form $G \times \mathbb{Z}^n$ with G negatively curved, then the rational points determined by Γ still form a dense subset of the visual boundary of X.

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1. The Boundary Determined by $G \times \mathbb{Z}^n$

Our goal in this section is to prove that the visual boundaries of the $\mathsf{CAT}(0)$-spaces X and X' are equivariantly homeomorphic whenever X and X' both admit a geometric action by the same group $G \times \mathbb{Z}^n$, where G is negatively curved. The proof depends on a result of Bridson and Haefliger that dissects the effect of a hyperbolic isometry on a $\mathsf{CAT}(0)$-space, which generalizes the corresponding effect of a hyperbolic isometry on a Hadamard manifold [**4**]. We prove the main result after stating the pertinent results of Bridson and Haefliger, but it is fitting first to recall some definitions and terminology.

A ***geodesic space*** X, d is a complete, locally compact path metric space and is ***negatively curved*** if it has a negatively curved Gromov inner product. Recall that the ***Gromov inner product*** (or ***overlap function***) based at $w \in X$ is given by

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y))$$

and is ***negatively curved*** if there is a nonnegative constant δ such that

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta$$

for all $x, y, z \in X$. The definition of negatively curved geodesic space is basepoint independent since the negative curvature of the inner product based at one point of X guarantees that based at every other point of X. A group is negatively curved if its Cayley graph with respect to some (finite) generating set with word metric is negatively curved. Negative curvature of a geodesic space is equivalent to the more geometric condition of having ***thin triangles***; see for instance [**8**]. Because negative curvature of a geodesic space is a quasi-isometric invariant, any group that acts geometrically on a negatively curved space is itself negatively curved. We refer the reader to [**2,8,9,12**] for an account of the theory of negatively curved groups and spaces, and to [**8,11**] for general information on groups acting geometrically.

Let X, d be a negatively curved geodesic space with inner product $(\quad\cdot\quad)$. A sequence $\{x_i\}$ in X is ***convergent at infinity*** if $(x_i \cdot x_j) \to \infty$ as $i, j \to \infty$, and two sequences $\{x_i\}$ and $\{y_i\}$ convergent at infinity are equivalent if $(x_i \cdot y_i) \to \infty$ as $i \to \infty$. The ***boundary*** of X, denoted as ∂X, consists of all equivalence classes of sequences in X convergent at infinity. We refer the reader to [**2**] for a precise description of the topology on $X \cup \partial X$; for our purposes, it suffices to note that the sequence $\{b_i\}$ in ∂X converges to the boundary point b if and only if $\varepsilon_i \to \infty$ as $i \to \infty$, where

$$\varepsilon_i = \liminf_j (x_{i,j}, x_j),$$

for any choice of sequences $\{x_{i,j}\}$ representing b_i and $\{x_j\}$ representing b. Each geodesic ray $\sigma : [0, \infty) \to X$ determines a point $\sigma(\infty)$ in ∂X; indeed, given any two sequences $\{s_i\}$ and $\{t_i\}$ of positive real numbers with $s_i \to \infty$ and $t_i \to \infty$, the sequences $\{\sigma(s_i)\}$ and $\{\sigma(t_i)\}$ are equivalent sequences convergent at infinity. Each point of ∂X is determined in this way by some geodesic ray, and two arclength parameterized geodesic rays σ and τ determine the same point $\sigma(\infty) = \tau(\infty)$ if and only if they ***fellow travel*** (i.e., $\{d(\sigma(s), \tau(s)) : s \geq 0\}$ is bounded above); thus, ∂X parameterizes the equivalence classes of fellow-traveling geodesic rays. Finally, $X \cup \partial X$ is a finite-dimensional metrizable compactification of X, and we refer the reader to the following sources for more extensive and informative discussion: [**2,8,12**].

$\mathsf{CAT}(0)$-spaces are the metric geometry version of nonpositively curved, simply-connected riemannian manifolds. A geodesic space X, d is $\mathsf{CAT}(0)$, or satisfies the ***0-comparison inequality of Alexandrov-Toponogrov***, if every geodesic triangle in X has no larger Alexandrov (upper) angles than the corresponding ones of a comparison triangle of the same edge lengths in the flat euclidean plane. $\mathsf{CAT}(0)$-geodesic spaces have unique geodesic segments between pairs of points, geodesic rays issuing from the same point diverge at least as fast as those in euclidean space, and geodesic triangles are at least as thin as comparison triangles of the same edge lengths in flat euclidean space. Instead of Alexandrov angles, appropriate reformulations of this last property are often used in the definition of $\mathsf{CAT}(0)$-spaces; see [**4,6,13**]. What is likely to become the standard reference on $\mathsf{CAT}(0)$-geometry is the as yet unpublished book of Bridson and Haefliger [**6**], but for now the reader is referred to [**1**] for a good standard exposition of the theory. The ***visual boundary*** of the $\mathsf{CAT}(0)$-geodesic space X, d based at $x_0 \in X$ is the collection of geodesic rays issuing from x_0 with the topology of uniform convergence on bounded subsegments. Since the visual boundaries based at different points are canonically homeomorphic, we delete any reference to basepoints in notation and use ∂X to denote any visual boundary of X. If X happens to be simultaneously negatively curved and $\mathsf{CAT}(0)$, the "evaluation map" $\sigma \mapsto \sigma(\infty)$ describes a homeomorphism of the visual boundary of X to the negatively curved boundary of X, justifying our use of the notation ∂X for both.

Reflecting the classification of isometries of hyperbolic n-space, the isometries of the $\mathsf{CAT}(0)$-geodesic space X, d fall into three categories — hyperbolic, elliptic, and parabolic. Our interest here is only in the hyperbolic isometries; general information in the setting of Hadamard manifolds appears in [**4**, Lecture II] and generalizations to the setting of $\mathsf{CAT}(0)$-spaces will appear in [**6**]. If λ is an isometry of X, then the ***translation length*** of λ is

$$|\lambda| = \inf\{d(x, \lambda(x)) : x \in X\},$$

and λ is ***hyperbolic*** if $|\lambda| > 0$ and λ assumes the infimum; i.e., there exists $x_0 \in X$ such that $|\lambda| = d(x_0, \lambda(x_0))$. Following [**4**], the ***minimal set*** $\mathsf{MIN}(\lambda)$ of the hyperbolic isometry λ consists of the set of points x for which $|\lambda| = d(x, \lambda(x))$. Two (bi-infinite) geodesic lines $\sigma, \tau : \mathbb{R} \to X$ are ***asymptotic*** (or ***fellow travel***) if $\{d(\sigma(t), \tau(t) : t \in \mathbb{R}\}$ is bounded, and the images $\sigma(\mathbb{R})$ and $\tau(\mathbb{R})$ ***cobound a flat strip*** in X if there is an isometric embedding $\psi : [0, D] \times \mathbb{R} \to X$, for some positive number D, whose restrictions to $\{0\} \times \mathbb{R}$ and $\{D\} \times \mathbb{R}$ are respective reparameterizations of σ and τ. Observe that the ***flat strip*** $\psi([0, D] \times \mathbb{R})$ is a convex subset of X. The next theorem is a restatement in the context of $\mathsf{CAT}(0)$-spaces of a basic property of Hadamard manifolds. Its proof is essentially that of its Hadamard-analog found in [**4**] and will appear in [**6**].

Flat Strip Theorem. *If geodesic lines σ and τ in the $\mathsf{CAT}(0)$-geodesic space*

X are asymptotic, then their images $\sigma(\mathbb{R})$ and $\tau(\mathbb{R})$ cobound a (unique) flat strip.

The next result, the aforementioned lemma of Bridson and Haefliger that is applied in the proof of Proposition 1.1, illuminates the effect of a hyperbolic isometry on a $\mathsf{CAT}(0)$-space. It generalizes [**4**, Lemma 6.2] and is a fairly direct corollary of the Flat Strip Theorem. Since [**6**] has yet to appear, even in preprint form, and their quick proof of item (iii) of the proposition relies on properties of Busemann functions developed early therein, we have decided to include a self-contained proof.

Decomposition Lemma [Bridson and Haefliger]. *Let λ be a hyperbolic isometry of the $\mathsf{CAT}(0)$-geodesic space X, d.*

*(i) There exists a geodesic line $\sigma : \mathbb{R} \to X$ such that $\lambda(\sigma(t)) = \sigma(t + |\lambda|)$ for all $t \in \mathbb{R}$. For such σ, the set $\sigma(\mathbb{R})$ is called an **axis** for λ.*

(ii) Any two axes for λ are asymptotic and the union of all the axes for λ is exactly $\mathsf{MIN}(\lambda)$. Furthermore, $\mathsf{MIN}(\lambda)$ is a closed convex subset of X, hence a $\mathsf{CAT}(0)$-space when metrized by the restriction of d.

*(iii) Let $\sigma(\mathbb{R})$ be an axis for λ and let Y be the **perpendicular bisector** (in $\mathsf{MIN}(\lambda)$) of $\sigma(\mathbb{R})$ at $\sigma(0)$; i.e.,*

$$Y = \{x \in \mathsf{MIN}(\lambda) : d(x, \sigma(\mathbb{R})) = d(x, \sigma(0))\}.$$

Then Y is a closed convex subset of X and therefore a $\mathsf{CAT}(0)$-geodesic space when metrized by the restriction of d; moreover, $\mathsf{MIN}(\lambda)$ is isometric to $Y \times \mathbb{R}$ (ℓ_2-product metric) and the restriction of λ to $\mathsf{MIN}(\lambda)$ is of the form $(y, t) \mapsto (y, t + |\lambda|)$.

(iv) Every isometry μ that commutes with λ leaves $\mathsf{MIN}(\lambda) \cong Y \times \mathbb{R}$ invariant and its restriction to $Y \times \mathbb{R}$ is of the form $\mu' \times \mu''$, where μ' is an isometry of Y and μ'' is an isometry of $\mathbb{R}$.

Proof. (i) Choose $x_0 \in \mathsf{MIN}(\lambda)$ and let $\sigma : [0, |\lambda|] \to X$ be the unique geodesic segment for which $\sigma(0) = x_0$ and $\sigma(|\lambda|) = \lambda(x_0)$. Extend σ to a map, still called σ, from $\mathbb{R}$ to X by

$$\sigma(n|\lambda| + p) = \lambda^n(\sigma(p)),$$

for all $n \in \mathbb{Z}$ and $p \in [0, |\lambda|]$. It suffices to prove that σ is locally geodesic since local geodesics are global geodesics in $\mathsf{CAT}(0)$-spaces. That σ is a local geodesic follows from the fact that the distance from $\sigma(|\lambda|/2)$ to $\sigma(3|\lambda|/2)$ is precisely $|\lambda|$, which follows easily from the fact that $x_0 \in \mathsf{MIN}(\lambda)$.
(ii) The two claims of the first sentence are easy, the second following from the proof of item (i). That $\mathsf{MIN}(\lambda)$ is convex is a direct consequence of the Flat Strip Theorem; indeed, for $x_0, x_1 \in \mathsf{MIN}(\lambda)$, let $\sigma_0, \sigma_1 : \mathbb{R} \to X$ be geodesic lines for which $\sigma_i(0) = x_i$ and $\lambda(\sigma_i(t)) = \sigma(t + |\lambda|)$ for all $t \in \mathbb{R}$. Then σ_0 and σ_1 are asymptotic, hence the axes $\sigma_0(\mathbb{R})$ and $\sigma_1(\mathbb{R})$ cobound a flat strip S. Since λ acts as a Clifford translation (i.e., every element is moved a distance of $|\lambda|$ by λ) on $\sigma_0(\mathbb{R}) \cup \sigma_1(\mathbb{R})$, the boundary of S, it, being an isometry, must do so also on S. It is immediate that $S \subset \mathsf{MIN}(\lambda)$, hence $\mathsf{MIN}(\lambda)$ is convex.
(iii) Let $\sigma_0(\mathbb{R})$, $\sigma_1(\mathbb{R})$, and $\sigma_2(\mathbb{R})$ be three distinct axes for λ that do not lie on a common flat strip, and for $\{i, j, k\} = \{0, 1, 2\}$, let S_i be the flat strip cobounded by $\sigma_j(\mathbb{R})$ and $\sigma_k(\mathbb{R})$. Let ρ be the intrinsic metric on the union $S = S_0 \cup S_1 \cup S_2$ determined by d; i.e., for $u, v \in S$, $\rho(u, v)$ is the infinum (in this case, minimum) of the d-lengths of all paths in S between u and v. Easily, ρ is complete and locally euclidean and as such makes S into a flat cylinder. Now S is foliated by axes for λ, namely, the euclidean lines in the flat strips that are parallel to the cobounding axes. The important observation about these foliating geodesic lines is that they are ρ-***lines*** — i.e., not only does each locally minimize arclength (= geodesic), but also each globally minimizes ρ-distance between any pair of its points (= line). The pertinent point is that it follows (for instance, by writing S as the quotient of the euclidean plane by a translation) that the ρ-geodesic foliation on S whose leaves are perpendicular to these axes consists entirely of closed curves, circles that go once around S. Let x_0 and x_1 be points on the axes $\sigma_0(\mathbb{R})$ and $\sigma_1(\mathbb{R})$, respectively, such that the segment $[x_0, x_1]$ is perpendicular to the cobounding axes of S_2. The

conclusion above guarantees that the unique point on the axis $\sigma_2(\mathbb{R})$ closest to x_0 is equal to the unique point on $\sigma_2(\mathbb{R})$ closest to x_1. It is now easy to verify the convexity of Y. Indeed, let $\sigma_0(\mathbb{R}) = \sigma(\mathbb{R})$ and $x_0 = \sigma(0)$. For elements x_1 and x_2 of Y, let $\sigma_1(\mathbb{R})$ and $\sigma_2(\mathbb{R})$ be axes for which $\sigma_1(0) = x_1$ and $\sigma_2(0) = x_2$. Since x_2 is by definition the unique point on $\sigma_2(\mathbb{R})$ closest to x_0, the observation above guarantees that x_2 is also the unique point on $\sigma_2(\mathbb{R})$ closest to x_1; hence, the segment $[x_1, x_2]$ is perpendicular to the cobounding axes of S_0. For an arbitrary point z of $[x_1, x_2]$, the same argument applied to the three points z, x_1, and x_0 implies, since x_0 is the unique point on $\sigma_0(\mathbb{R})$ closest to x_1, then x_0 also is the unique point on $\sigma_0(\mathbb{R})$ closest to z. But this says $z \in Y$, so $[x_1, x_2] \subset Y$ and Y is convex. The remaining claims of item (iii) now can be verified routinely.

(iv) Since for $x \in \mathsf{MIN}(\lambda) = Y \times \mathbb{R}$, $d(\lambda\mu(x), \mu(x)) = d(\mu\lambda(x), \mu(x)) = d(\lambda(x), x) = |\lambda|$, $\mu(x) \in Y \times \mathbb{R}$ and therefore $Y \times \mathbb{R}$ is μ-invariant. For any $y \in Y$, let $\mu'(y)$ be the unique point of $\mu(\{y\} \times \mathbb{R})$ in Y and, for a point y_0 fixed in Y, let T be the $\mathbb{R}$-coordinate of $\mu(y_0, 0)$. Define $\mu'' : \mathbb{R} \to \mathbb{R}$ by $t \mapsto t + T$. A straightforward exercise using the Flat Strip Theorem and item (iii) now shows that $\mu' : Y \to Y$ is an isometry and the restriction of μ to $Y \times \mathbb{R}$ is given by $\mu' \times \mu''$. $\diamondsuit$

A subset M of a metric space X is ***quasi-dense*** if it is N-dense for some $N > 0$; i.e., if each point of X is N-close to some point of M. A subset F of a metric space X is an ***n*-flat** if the restriction of the metric on X to F makes F into a flat n-plane, isometric with $\mathbb{R}^n$. The next proposition is the main technical result used in this paper. The proof is by induction on n.

Proposition 1.1. *Whenever the group $\Gamma = G \times \mathbb{Z}^n$ acts geometrically on the $\mathsf{CAT}(0)$-geodesic space X, there exist subsets $Y \subset M \subset X$ and a map $\pi : M \to Y$ such that the following hold:*

(i) M is a closed, Γ-invariant, convex, and quasi-dense subset of X that is isometric to the ℓ_2-product $Y \times \mathbb{R}^n$, where Y is metrized by the restriction of the metric on X;

(ii) *Y is closed and convex, and for each $y \in Y$, $\pi^{-1}(y)$ is a $\{1\} \times \mathbb{Z}^n$-invariant n-flat, where 1 denotes the unit element of G;*

(iii) *there exists a cocompact action $(h, r) \mapsto h \cdot r$ of $\mathbb{Z}^n$ on $\mathbb{R}^n$ such that the action of $\{1\} \times \mathbb{Z}^n$ on M, identified as $Y \times \mathbb{R}^n$, is of the form $(1, h) \cdot (y, r) = (y, h \cdot r)$;*

(iv) *the natural embedding of visual boundaries, $\partial Y \to \partial X$, extends to a homeomorphism of the join $\partial Y * S^{n-1}$ onto ∂X;*

(v) *though Y is not necessarily $G \times \{0\}$-invariant, there is nonetheless a geometric action of G on Y obtained by projecting the Y-fiber preserving action of $G \times \{0\}$ on $Y \times \mathbb{R}^n$ to Y.*

Proof. Let 1_n denote the element $(0, \ldots, 0, 1)$ of $\mathbb{Z}^n$. Since the action is cocompact and discrete, the isometry λ of X given by the action of the group element $(1, 1_n)$ of $G \times \mathbb{Z}^n$ must be hyperbolic and, hence, we may choose a basepoint x_0 for X in $\mathsf{MIN}(\lambda)$. Let Y' be the perpendicular bisector of the axis $\sigma(\mathbb{R})$ for λ at $\sigma(0) = x_0$. By item (iii) of the Decomposition Lemma, $\mathsf{MIN}(\lambda)$ may be identified with the ℓ_2-product $Y' \times \mathbb{R}$ and so there is a projection map $\pi' : \mathsf{MIN}(\lambda) \to Y'$ such that, whenever $y \in Y'$, then $(\pi')^{-1}(y) = \{y\} \times \mathbb{R}$ is an axis for λ. By item (iv) of the Decomposition Lemma, since each element of Γ commutes with $(1, 1_n)$, the group Γ leaves $\mathsf{MIN}(\lambda)$ invariant. We denote elements of Γ by $(g, k) \in (G \times \mathbb{Z}^{n-1}) \times \mathbb{Z}$ and elements of $\mathsf{MIN}(\lambda)$ by $(y, t) \in Y' \times \mathbb{R}$. For $g \in G \times \mathbb{Z}^{n-1}$ and $y \in Y'$, define $g * y$ by

$$g * y = \pi'((g, 0) \cdot (y, 0));$$

if μ is the map $x \mapsto (g, 0) \cdot x$ for $x \in X$, then the map $y \mapsto g * y$ for $y \in Y'$ is exactly the map μ' of item (iv) of the Decomposition Lemma. Our claim is that the action $(g, y) \mapsto g * y$ of $G \times \mathbb{Z}^{n-1}$ on Y' is a geometric action on the $\mathsf{CAT}(0)$-space Y'. Indeed, that the action is isometric and Y' is a $\mathsf{CAT}(0)$-geodesic space is immediate from items (iii) and (iv) of the Decomposition Lemma. Since each point of the form $g * x_0$ for $g \in G \times \mathbb{Z}^{n-1}$ is $|\lambda|$-close to a point of the form $(g, k) \cdot x_0$ that lies in the orbit of x_0 under Γ, the discreteness of the action of Γ on X guarantees that of $G \times \mathbb{Z}^{n-1}$ on Y'.

Finally, if $N > 0$ is chosen so that the metric ball in X of radius N about x_0 contains a fundamental region for the action of Γ on X, which exists since the action is cocompact, then the intersection of that ball with Y' contains a fundamental region for the action of $G \times \mathbb{Z}^{n-1}$ on Y'. Indeed, if $y \in Y'$, choose $(g,k) \in \Gamma$ such that $(g,k)\cdot(y,0)$ is N-close to x_0. Notice that

$$g * y = \pi'((g,0)\cdot(y,0)) = \pi'((g,k)\cdot(y,0)).$$

Since $Y' \times \mathbb{R}$ is an ℓ_2-product, the projection map π' is distance non-increasing and, since $(g,k)\cdot(y,0)$ is N-close to x_0, it follows that $g*y = \pi'((g,k)\cdot(y,0))$ is N-close to $\pi'(x_0)$. We conclude that there is a fundamental region for the action of $G \times \mathbb{Z}^{n-1}$ on Y' contained in the N-ball about $\pi'(x_0)$ in Y', and it follows that the action is cocompact.

Basis of induction: If $n = 1$, let $Y = Y'$, $M = \mathsf{MIN}(\lambda)$ and $\pi = \pi'$. Items (i)–(iii) follow from the Decomposition Lemma, except for the quasi-denseness of M. For this, let N be chosen as in the previous paragraph. For any $x \in X$, choose $\gamma \in \Gamma$ such that $\gamma \cdot x$ is N-close to x_0 and observe that since γ commutes with $(1,1)$, the line $\gamma \cdot \sigma(\mathbb{R})$ is an axis for λ and therefore $\gamma \cdot \sigma(0) = \gamma \cdot x_0$ is an element of $M = \mathsf{MIN}(\lambda)$ N-close to x. Thus M is N-dense in X. For item (iv), it is well-known and easily proved that the visual boundary of the ℓ_2-product of two $\mathsf{CAT}(0)$-geodesic spaces is homeomorphic to the join of the visual boundaries of the factors. Therefore, since M is isometric to the ℓ_2-product $Y \times \mathbb{R}$, the visual boundary of M is homeomorphic to the suspension $\partial Y * S^0 = \Sigma(\partial Y)$, and since M is convex and quasi-dense in X, it is immediate that $\partial M = \partial X$. Item (v) is immediate from the previous paragraph.

Inductive step: Assume the result for $n = n-1$. At this point the first paragraph of the proof provides a geometric action $(g,y) \mapsto g * y$ of the group $G \times \mathbb{Z}^{n-1}$ on the $\mathsf{CAT}(0)$-geodesic space Y'. An application of the inductive hypothesis provides subsets $Y \subset M' \subset Y'$ and a map $\pi'' : M' \to Y$ satisfying the appropriate interpretation of items (i)–(v). Let $M = M' \times \mathbb{R} \subset Y' \times \mathbb{R} = \mathsf{MIN}(\lambda) \subset X$, and let $\pi = \pi'' \circ \pi' \big| M$. We leave the straightforward verification that Y, M and π satisfy items (i)–(v) to the reader, noting only that item (iv) of the Decomposition Lemma needs to be used to verify (ii) and (iii). $\diamondsuit$

It is worth noting what Proposition 1.1 does not say: it does not say that the action of $\Gamma = G \times \mathbb{Z}^n$ on the product $M \cong Y \times \mathbb{R}^n$ is a product action nor does it find in X a $G \times \{0\}$-invariant ***convex*** (or even quasi-convex) subset on which the restricted action is cocompact. The example that is presented in the Introduction and discussed in the following section in general precludes both these possibilities.

Theorem 1.2. *Whenever G is negatively curved and $\Gamma = G \times \mathbb{Z}^n$ acts geometrically on the $\mathsf{CAT}(0)$-geodesic space X, there is an embedding $\partial G \to \partial X$ that extends to a homeomorphism of the join $\partial G * S^{n-1}$ onto the visual boundary ∂X. Moreover, if Γ also acts geometrically on the $\mathsf{CAT}(0)$-geodesic space X', then the visual boundaries of X and X' are Γ-equivariantly homeomorphic; however, such a homeomorphism cannot in general be obtained as a continuous extension of a Γ-equivariant quasi-isometry of X to X'.*

Proof. By item (v) of Proposition 1.1, the negatively curved group G acts geometrically on Y and, since negative curvature is a quasi-isometry invariant, we conclude that Y, in addition to being $\mathsf{CAT}(0)$, is also negatively curved. Hence there is a G-equivariant homeomorphism of ∂G onto ∂Y, and item (iv) of Proposition 1.1 implies the first claim. The second claim is a consequence of the fact that the action of the second factor $\mathbb{Z}^n$ of Γ is trivial on ∂X. Indeed, notice that the action of Γ on $\partial X = \partial Y * S^{n-1}$ is given by

$$(g, h) \cdot [y, t, s] = [g * y, t, s]$$

for $(g, h) \in G \times \mathbb{Z}^n$ and $[y, t, s] \in \partial Y * S^{n-1} = Y \times [0, 1] \times S^{n-1}/\sim$, where $(g, y) \mapsto g * y$ denotes the extended action of G on ∂Y. This follows because $(1, h)$ for $h \in \mathbb{Z}^n$ acts trivially on the visual boundary of M since, by item (iii) of Proposition 1.1, the image of any bi-finite geodesic line ℓ under $(1, h)$

is parallel to ℓ. It follows immediately that the homeomorphism

$$\partial Y * S^{n-1} = \partial X \to \partial Y = \partial Y' * S^{n-1}$$

given by $[y, t, s] \mapsto [y', t, s]$, where $y \mapsto y'$ is the natural G-equivariant homeomorphism of ∂Y to $\partial Y'$ that factors through ∂G, is Γ-equivariant. The final claim is a consequence of the Example of the Introduction. $\diamondsuit$

2. The Example

The purpose of this section is to verify the claims of items (i)–(v) of the Example of the Introduction. First, though, we review for the reader our initial strategy for approaching the question of whether ∂X and $\partial X'$ are equivariantly homeomorphic. This strategy, as the Example shows, is untenable but, nonetheless, is quite natural and had the effect of focusing our attention on the possibility of such an example. The second paragraph of the Introduction frames the situation in the presence of negative curvature: whenever a group G acts geometrically on negatively curved geometries X and X', their boundaries are equivariantly homeomorphic because there are natural equivariant homeomorphisms of ∂G itself to both ∂X and $\partial X'$. In the setting of $\Gamma = G \times \mathbb{Z}$ acting geometrically on $\mathsf{CAT}(0)$-geodesic spaces X and X', where G is negatively curved, we do not have the advantage of an intrinsically-defined boundary for the group Γ, but our desire is nonetheless to mimic as much as possible the situation described above by exploiting the availability of the boundary of the negatively curved factor of Γ; namely, ∂G. The overall idea is to find an embedding of ∂G into ∂X that extends to a homeomorphism of the suspension, and, as Proposition 1.1 attests, works out nicely when one exploits the results of Bridson and Haefliger. Our initial strategy for constructing such an embedding, even though a failed one, is perhaps a more natural one to try than the one that actually succeeds in the proof of Proposition 1.1. It is: fix a basepoint $x_0 \in X$ and argue that the quasi-isometric embedding $g \mapsto g \cdot x_0$ of $G \times \{0\}$ into X extends continuously to a map of $\partial(G \times \{0\})$ into X. A first attempt might try to show that the orbit $(G \times \{0\}) \cdot x_0$ is a quasi-convex subset of X, from which the desired result would easily follow. This failing, a direct approach might attempt to verify that the natural extension of $g \mapsto g \cdot x_0$ on $G \times \{0\}$ to the rational points of $\partial(G \times \{0\})$ is continuous, even uniformly so, so that a further extension to $\partial(G \times \{0\})$ would be possible. Item (i) shows that the former attempt cannot succeed while item (ii) shows the same for the latter, and item (iii) shows that any attempt to "quasi-identify" $\Gamma \cup \Sigma(\partial G)$ equivariantly with $X \cup \partial X$ must fail.

Item (i): Without loss of generality let x_0 be the "origin" $(1,0)$ of $X = T \times \mathbb{R}$, where of course "1" denotes the empty word in the free group F_2. Recall that a subset C of a path metric space is ***quasi-convex*** if there exists a positive number N such that every geodesic segment whose endpoints lie in C is contained in the N-neighborhood of C. We confirm item (i) by exhibiting a sequence $\{g_i\}$ of group elements in F_2 such that the midpoint of the unique segment from x_0 to $x_i = (g_i, 0) * x_0$ lies greater than $\frac{i}{3}$ units from $(F_2 \times \{0\}) * x_0$. For this, set $g_i = a^i b^i$ and observe that $x_i = (a^i b^i, 2i) \in F_2 \times \mathbb{Z} \subset T \times \mathbb{R}$. The point of $T \times \mathbb{R}$ midway between x_0 and x_i is (a^i, i). For an arbitrary element $g = a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k}$ of F_2, where none of the exponents is zero except for possibly m_1 and n_k, $(g, 0) * x_0 = (g, 2n)$, where $n = n_1 + \ldots + n_k$. The distance between $(g, 2n)$ and (a^i, i) is at least as large as the maximum of $|2n - i|$ and $|a^{-i} g|$, the word distance between g and a^i in F_2. If $|2n - i| < \frac{i}{3}$, then $n > \frac{i}{3}$ and therefore

$$|a^{-i} g| \geq |n_1| + \ldots + |n_k| \geq n > \frac{i}{3},$$

showing that the midpoint (a^i, i) lies at least $\frac{i}{3}$ units from $(g, 0) * x_0$. It follows that (a^i, i) is not in the $(\frac{i}{3})$-neighborhood of $(F_2 \times \{0\}) * x_0$ and, hence, $(F_2 \times \{0\}) * x_0$ is not quasi-convex, as claimed.

Item (ii): For an element $1 \neq g \in F_2$, let g^∞ denote the rational point determined by g in the Cantor set boundary ∂F_2. Let $p_T : X = T \times \mathbb{R} \to T$ be the projection map and for a geodesic ray $\sigma : [0, \infty) \to X$ based at x_0, let $\sigma_T = p_T \circ \sigma$. If σ is not parallel to the $\mathbb{R}$-factor, then σ_T is a geodesic ray in T and determines a point $\sigma_T(\infty)$ in ∂F_2. Moreover, the ray σ traces out a straight euclidean line in the flat euclidean half-plane $\sigma_T([0, \infty)) \times \mathbb{R}$ and makes a (signed) euclidean angle $\theta(\sigma)$ with the line $\sigma_T([0, \infty))$. Thus we may parameterize $\partial X = \partial F_2 * S^0 = \Sigma(\partial F_2)$ as

$$\Sigma(\partial F_2) = \frac{\partial F_2 \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}{\partial F_2 \times \left\{\pm\frac{\pi}{2}\right\}},$$

identifying σ with $[\sigma_T(\infty), \theta(\sigma)]$.

Now let $g_i = a^i b^i$ and observe that $g_i^\infty \to a^\infty$ as $i \to \infty$. It is a straightforward exercise to confirm for each i that the point of ∂X determined by the second action by the group element $(g_i, 0)$ is given by

$$\lim_{n\to\infty} (g_i^n, 0) * x_0 = [g_i^\infty, \tfrac{\pi}{4}],$$

where the limit is taken in $X \cup \partial X$. However, the point of ∂X determined by a is

$$\lim_{n\to\infty} (a^n, 0) * x_0 = [a^\infty, 0].$$

This shows that the natural extension of the quasi-isometric embedding $(g, 0) \mapsto (g, 0) * x_0$ of $F_2 \times \{0\}$ does not extend continuously to a map of the boundary of $F_2 \times \{0\}$, nor even to the set of rational points in the boundary.

Item (iii): This follows from the above confirmation of item (ii). Indeed, with $g_i = a^i b^i$,

$$\lim_{n\to\infty} (g_i^n, 0) \cdot x_0 = [g_i^\infty, 0]$$

and

$$\lim_{n\to\infty} (a^n, 0) \cdot x_0 = [a^\infty, 0],$$

so any map $X \cup \partial X \to X \cup \partial X$ that extends the quasi-isometry $\gamma \cdot x_0 \mapsto \gamma * x_0$ continuously to ∂X must send $[g_i^\infty, 0]$ to $[g_i^\infty, \frac{\pi}{4}]$ and fix $[a^\infty, 0]$. However, this condition is incompatible with continuity at $[a^\infty, 0]$, since $[g_i^\infty, 0] \to [a^\infty, 0]$ as $i \to \infty$.

Item (v): Consider the Coxeter group $C = (A * A) \times (A * A * A)$, where $A = \mathbb{Z}/2$, which has presentation

$$\langle a_1, \ldots, a_5 : a_i^2 = 1; (a_1 a_j)^2 = (a_2 a_j)^2 = 1, j = 3, 4, 5\rangle.$$

The first factor $A * A$ contains an infinite cyclic subgroup of finite index and the second $A * A * A$ contains a rank 2 free subgroup of finite index. It follows that C contains a copy of $F_2 \times \mathbb{Z}$ as a subgroup of finite index. Since all Coxeter groups are $\mathsf{CAT}(0)$-groups [**15**], C acts geometrically on some $\mathsf{CAT}(0)$-space. Since $F_2 \times \mathbb{Z}$ is a finite-index subgroup of C, it acts geometrically on every $\mathsf{CAT}(0)$-space that admits a geometric action of C, and Theorem 1.2 applies to show that the visual boundary of every such space is homeomorphic to the suspension of a Cantor set. Thus C has a well-defined boundary as the suspension of a Cantor set, which is not locally connected. Although this provides an example of a Coxeter group with somewhat unexpected behavior, Mike Mihalik has shown that all Coxeter groups are semistable at infinity [**14**].

3. The Rational Points of the Visual Boundary Are Dense

We prove the following theorem, which Gromov observes to hold in the presence of negative curvature [**12**]; for a "slick" proof in the setting of negative curvature see [**5**].

Theorem 3.1. *Whenever G is negatively curved and $\Gamma = G \times \mathbb{Z}^n$ acts geometrically on the $\mathsf{CAT}(0)$-geodesic space X, the set of Γ-rational points of ∂X forms a dense subset of the visual boundary.*

Proof. Apply Proposition 1.1 to obtain subsets $Y \subset M \subset X$ and a map $\pi : M \to Y$ satisfying items (i)–(v). Identifying M with the ℓ_2-product $Y \times \mathbb{R}^n$, let $\pi' : M \to \mathbb{R}^n$ be projection to the second factor, and interpret ∂X as the space of geodesic rays in X issuing from the point $x_0 = (y_0, 0) \in Y \times \mathbb{R}^n$. Fix an element $\sigma : [0,\infty) \to X$ in ∂X whose image lies neither in $Y \times \{0\}$ nor $\{y_0\} \times \mathbb{R}^n$. Note that the image $\sigma(\mathbb{R})$ lies entirely in M, which follows from the convexity and quasi-denseness of the closed set M, as guaranteed by Proposition 1.1(i). Set $\sigma_Y = \pi \circ \sigma$ and $\sigma_{\mathbb{R}^n} = \pi' \circ \sigma$ and observe that since M is the ℓ_2-product of Y and $\mathbb{R}^n$, σ_Y and $\sigma_{\mathbb{R}^n}$ are respective geodesic rays in Y and $\mathbb{R}^n$ based at the respective points y_0 and 0. Let $(g, y) \mapsto g * y$ denote the geometric action of G on Y identified in the proof of Proposition 1.1, and let $C(G)$ denote a Cayley graph for G with respect to some finite generating set and given the word metric. Denote by $q : C(G) \cup \partial G \to Y \cup \partial Y$ the G-equivariant continuous extension of a quasi-isometry of $C(G)$ with Y determined by the orbit map $g \mapsto g * y_0$ of G to Y. Recall that the restriction of q to ∂G is a G-equivariant homeomorphism onto ∂Y. Interpreting ∂Y as the space of geodesic rays in Y issuing from y_0, let $z \in \partial G$ satisfy $q(z) = \sigma_Y \in \partial Y$. Since the rational points of ∂G are dense in ∂G, we may choose an infinite order element $g \in G$ such that $g^\infty = \lim g^i$ lies close to the boundary point z. Since the $*$-action of G on Y is cocompact and discrete, g acts as a hyperbolic isometry on Y. Let $A_g = \alpha(\mathbb{R})$ be an axis for the isometry $y \mapsto g * y$ for $y \in Y$, where of course $\alpha : \mathbb{R} \to Y$ is a bi-infinite geodesic. By a standard argument, the segments from y_0 to $\alpha(i)$, $i \geq 0$, limit to the image of a geodesic ray $\tau_Y : [0,\infty) \to Y$, with $\tau_Y(0) = y_0$, that fellow-travels the ray $\alpha \big| [0,\infty)$. We have

$$q(g^\infty) = q(\lim_{i\to\infty} g^i) = \lim_{i\to\infty} g^i * y_0 = \lim_{i\to\infty} g^i * \alpha(0) = \lim_{i\to\infty} \alpha(i|g|),$$

where $|g|$ is the translation length of the $*$-action of g on Y. Since τ_Y fellow-travels $\alpha \big| [0,\infty)$, this last limit is exactly τ_Y, thought of as an element of ∂Y. Since q is continuous on ∂G, $q(g^\infty) = \tau_Y$ is close to $q(z) = \sigma_Y$. At this point we find it convenient to assume that $y_0 = \alpha(0)$, which causes no loss of generality since visual boundaries based at different points are canonically homeomorphic. With this convenience, the ray τ_Y is just $\alpha \big| [0,\infty)$.

Since A_g is an axis for g in Y, item (ii) of Proposition 1.1 and item (iv) of the Decomposition Lemma imply that the subset $A_g \times \mathbb{R}^n$ of $Y \times \mathbb{R}^n = M$ is a $\langle g \rangle \times \mathbb{Z}^n$-invariant $(n+1)$-flat; moreover, item (iii) of Proposition 1.1 and item (iv) of the Decomposition Lemma imply that the orbit $(\langle g \rangle \times \mathbb{Z}^n) \cdot x_0$ is a cocompact lattice in $A_g \times \mathbb{R}^n$. (Note that we are ***not*** saying that $\langle g \rangle \times \mathbb{Z}^n$ acts as a product action on $A_g \times \mathbb{R}^n$; for instance, $g \cdot A_g$ may be disjoint from A_g while $g * A_g = A_g$, since A_g is an axis for the $*$-action of g on Y.) Parameterizing the visual boundary ∂X as

$$\partial X = \partial Y * \partial \mathbb{R}^n = (\partial Y \times [0,1] \times \partial \mathbb{R}^n) \,/ \sim,$$

we may write the element $\sigma \in \partial X$ as $\sigma = [\sigma_Y, u, \sigma_{\mathbb{R}^n}]$ for some $u \in [0,1]$ and geodesic rays $\sigma_Y \in \partial Y$ and $\sigma_{\mathbb{R}^n} \in \partial \mathbb{R}^n$. Notice that the visual boundary of the $(n+1)$-flat $A_g \times \mathbb{R}^n$ sits naturally in ∂X as $\{[\pm\tau_Y, t, s] : t \in [0,1], s \in \partial \mathbb{R}^n\}$, where $-\tau_Y$ denotes the ray $a \mapsto \alpha(-a)$. Now the $\langle g \rangle \times \mathbb{Z}^n$-rational points of $\partial(A_g \times \mathbb{R}^n)$ are dense in $\partial(A_g \times \mathbb{R}^n)$ and so arbitrarily close to the point $[\tau_Y, u, \sigma_{\mathbb{R}^n}]$ of $\partial(A_g \times \mathbb{R}^n)$ lies a $\langle g \rangle \times \mathbb{Z}^n$-rational point, say $[\tau_Y, t, s]$ determined by the element $(g^k, h) \in \langle g \rangle \times \mathbb{Z}^n$. Then $[\tau_Y, t, s]$ is a $G \times \mathbb{Z}^n$-rational point of ∂X that is close to σ, since τ_Y is close to σ_Y and s and t are close, respectively, to $\sigma_{\mathbb{R}^n}$ and u. $\diamondsuit$

Whenever the group Γ acts geometrically on the $\mathsf{CAT}(0)$-geodesic space X, Gromov [**13**] has suggested that the set of periodic k-flats ought to be dense in the set of k-flats. The proof of Theorem 3.1 may be modified to verify Gromov's expectation for k-flats in the context of $\Gamma = G \times \mathbb{Z}^n$ with G negatively curved. The second author presents, in her PhD thesis, an analysis of Gromov's expectation in more general contexts than ours.

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