HOMOLOGICAL SYSTOLES, HOMOLOGY BASES AND PARTITIONS OF RIEMANN SURFACES

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1. Introduction

A compact Riemann surface of genus g, $g > 1$, can be decomposed into pairs of pants, i.e., into three hole spheres, by cutting the surface along $3g-3$ simple closed non–intersecting geodesic curves. These curves can always be chosen in such a way that their hyperbolic lengths are bounded by $21g$ ([7]).

First length controlled decompositions of Riemann surfaces into pairs of pants were found by Lipman Bers ([3]). His method did, however, yield a bound that was much larger than the above mentioned $21g$.

The same question can be asked about homology bases of Riemann surfaces: is it possible to estimate lengths of closed geodesic curves constituting a basis for the homology of a given genus g, $g > 1$, Riemann surface? More precisely, one would like to find a canonical homology basis $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ consisting of curves that are as short as possible.

A canonical homology basis is characterized by the property that the curves α_j and β_j are simple closed curves, each α_j intersects β_j exactly at one point, and there are no other intersection points. Such bases are needed when computing period matrices of Riemann surfaces, or when forming a fundamental domain for a uniformizing group. If the curves α_j and β_j are short, then the sides of the corresponding fundamental domain are also short. This, on the other hand, has potential applications to various computational problems.

After having posed the problem of finding a short homology basis for a given Riemann surface, one observes immediately, that it is not possible to find a universal bound that would depend only on the genus of the Riemann surface in question. For if γ is a short non–separating simple closed geodesic curve, then any homology basis contains a curve that intersects γ. By the Collar Theorem ([6]), any such curve is necessarily long (cf. also Example 3). As the length of γ goes to zero, the length of any closed intersecting curve grows towards infinity. Hence one cannot find any length controlled homology basis in which the bound for the lengths of the curves would depend only on the genus.

This leads one to define the *homological systole* of a Riemann surface as the minimal length of simple closed non–separating geodesic curves. The main result of this paper is that one can always find a homology basis consisting of curves whose lengths are bounded by an expression depending only on the homological systole and on the genus of the Riemann surface.

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More precisely, we prove, in Section 3:

Theorem 7. Let X be a compact Riemann surface of genus $g \geq 2$ which has a partition with longest geodesic of length L and whose homological systole is ε. Then there exists a canonical homology basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ on X such that any α_i belongs to the partition and the length $\ell(\beta_i)$, of any curve β_i, satisfies

$$\ell(\beta_i) \leq (2g-2)(2L + 2\,\mathrm{arcsinh}\,\tfrac{4}{\varepsilon}) \tag{1}$$

Observe that in the above the constant L can be replaced by $21g$ to get a bound that depends only on the genus g and on the homological systole ε of the Riemann surface in question.

Bounds for the lengths of *individual* homologically trivial and non trivial shortest closed curves have been studied for quite a long time. The name "systole" for such curves is due to Berger [1]. We refer to Gromov [10] and [9] for a broad bibliography on the subject and for a large number of new results. For systoles in connection with the Schottky problem we refer to [4], for systoles in connection with arithmetic Fuchsian groups we refer to [12]. As for curve *systems*, short partitions have been investigated in [2], [3], [5], [7], [13], [14], [15].

2. Homology bases and partitions

In this section we study homology bases and partitions from a purely topological point of view. We begin with the following definition.

A *topologically marked pair of pants* is a compact bordered surface Y of signature $(0, 3)$ with boundary curves c_1, c_2, c_3 together with three pairwise disjoint simple arcs p_{12}, p_{13}, p_{23}, where p_{ij} has its initial point on c_i, its end point on c_j and all other points in the interior of Y. We call these arcs *connectors*. In the final section, where Y carries again a hyperbolic metric with geodesic boundary, the connectors will be the usual common perpendiculars decomposing Y into right angled geodesic hexagons.

A *topologically marked pants decomposition* of a compact orientable surface X of genus $g \geq 2$ is a partition $\mathcal{P}$ of X with topologically marked pairs of pants. Formally, $\mathcal{P}$ is understood as the set formed by the $3g-3$ partitioning curves and the $2g-2$ pairs of pants. For the rest of this section we shall now assume that such a partition $\mathcal{P}$ is given on X.

As no metric is specified on X, there is no measure of shortness of curves. However, one can do the following. We shall say that a simple arc υ on X is *elementary* if υ is contained in one of the pairs of pants Y_k of $\mathcal{P}$ and satisfies the subsequent conditions. A curve will then be considered combinatorially "short" if it can be decomposed into a sequence of elementary arcs where, moreover, the number of these arcs is "small".

We denote by $\check{\upsilon}$ the *interior* of υ, that is, the arc without its end points. The conditions for υ on Y_k are as follows.

1. The end points of υ lie on the boundary of Y_k and $\check{\upsilon}$ is contained in the interior of Y_k.
2. $\check{\upsilon}$ intersects $p_{12} \cup p_{23} \cup p_{31}$ in at most two points.

As a limit case we also accept the connectors p_{ij} themselves as elementary arcs. If the end points of υ lie on different boundary components of of Y_k, we shall say that υ is of *type I*. If the end points lie on the same boundary component and υ is not homotopic to an arc on the boundary, then υ is of *type II*. Finally, if υ is

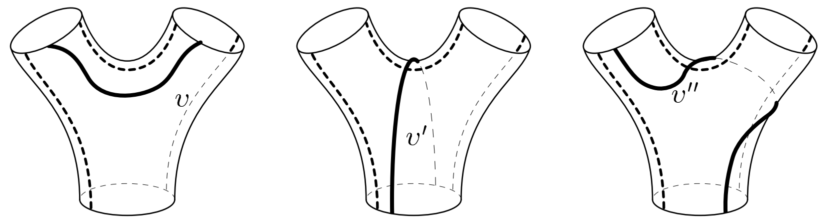

FIGURE 1. Elementary arcs.

homotopic to an arc on the boundary of Y_k then we shall say that v is *trivial*. Fig. 1 shows some cases, the dotted lines are the connectors.

For any closed curve c on X which is the union of non overlapping elementary arcs the number of these arcs will be denoted by $\ell_{\mathcal{P}}(c)$. We shall call $\ell_{\mathcal{P}}(c)$ the *combinatorial length* of c.

We now show that there exist combinatorially short homology bases.

Theorem 1. There exists a canonical homology basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ on X with the following properties.

1. Each α_i belongs to $\mathcal{P}$.
2. Each β_i satisfies $\ell_{\mathcal{P}}(\beta_i) \leq 2g-2$.
3. No β_i intersects a seperating curve of $\mathcal{P}$.

Proof. If $\mathcal{P}$ contains a separating curve η, we cut X open along η to obtain two bordered surfaces Σ', Σ'' of signatures $(g',1)$ and $(g'',1)$ respectively, with $g'+g''=g$. If one of the components Σ' or Σ'' is again separated by some curve in $\mathcal{P}$, then we cut the component open along this curve, and so on. After finitely many steps we obtain surfaces $\Sigma_1, \ldots, \Sigma_n$ of signatures $(g_1, m_1), \ldots, (g_n, m_n)$, where $g_1 + \cdots + g_n = g$ and where for $k = 1, \ldots, n$, no curve of $\mathcal{P}$ in the interior of Σ_k separates Σ_k. (If the above η does not exist, then $n = 1$ and $\Sigma_1 = X$.) This preliminary procedure will guarantee point (3) of the theorem.

Now let Σ_k be one of the components with $g_k \neq 0$ and denote, by $\mathcal{P}_k$, the pants decomposition of Σ_k induced by $\mathcal{P}$. We cut Σ_k open along a non separating curve of $\mathcal{P}_k$ into a connected surface Σ_k^1 of signature $(g_k - 1, m_k + 2)$. If $g_k - 1 \neq 0$ then $\mathcal{P}_k$ contains a curve not separating Σ_k^1 and we cut Σ_k^1 open along it to obtain a surface Σ_k^2 of signature $(g_k - 2, m_k + 4)$, and so on. This procedure yields a surface S_k of signature $(0, m_k + 2g_k)$ tesselated with $2g_k - 2 + m_k$ pairs of pants of $\mathcal{P}$. We now denote by $\mathcal{P}_k$ the corresponding pants decomposition of S_k. The combinatorial scheme of $\mathcal{P}_k$ is a three regular graph $\mathcal{G}_k$ without closed edge paths. $\mathcal{G}_k$ has $m_k + 2g_k$ half edges and $2g_k + m_k - 3$ edges.

Since $\mathcal{G}_k$ has no closed edge paths (i.e. $\mathcal{G}_k$ is a tree), the surface S_k may be reconstructed out of the pants in $\mathcal{P}_k$ by starting with a first pair of pants, say Y_1, then paste the neighboring pair of pants Y_2 along the corresponding boundary component of Y_1, say along γ_1, to obtain a surface of signature $(0,4)$ as shown in Fig.2, then paste Y_3 along γ_2 to get a surface of signature $(0,5)$, and so on. S_k is finished after $2g_k + m_k - 3$ steps.

The procedure thus described allows us to construct a sequence of curves on S_k which we shall call a *boundary lane* and which is obtained inductively as follows (see Fig.2). On Y_1 the boundary lane is formed by the three arcs p_{12}, p_{13}, p_{23}. On $Y_1 \cup Y_2$

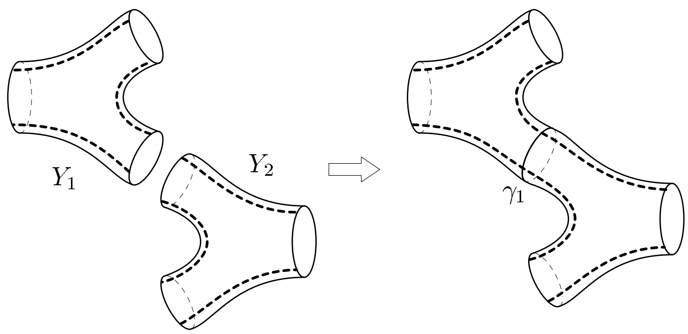

FIGURE 2. Constructing the boundary lane.

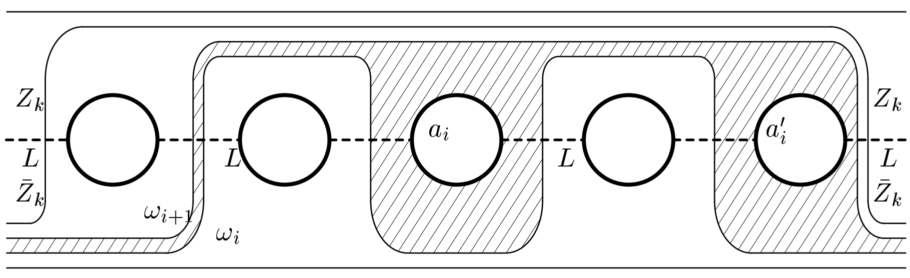

FIGURE 3. Domains of signature $(0, 4)$ along the boundary lane.

we connect the p_{ij} of Y_1 and the p'_{kl} of Y_2 which meet γ_1 by inserting two disjoint arcs on γ_1. The four curves obtained up to now yield the boundary lane on $Y_1 \cup Y_2$. This is shown by the four dotted arcs on the right hand side in Fig.2. Proceeding in this way for each new pair of pants which is added, we obtain a boundary lane on S_k which is a union $L = \lambda_1 + \cdots + \lambda_q$, $q = m_k + 2g_k$, of pairwise disjoint simple arcs, where λ_1 goes from a first boundary component of S_k to a second one, λ_2 goes from the second boundary component to a third one, etc. Finally, λ_q goes back to the first boundary component. Since S_k has genus 0, L separates S_k into two topological disks Z_k and $\bar{Z}_k$. Observe that with an arbitrarily small homotopy L may be deformed such that the deformed lane consists of $3(2g_k + m_k - 2)$ arcs of type I. Fig.3 shows part of L schematically together with additional curves which we define next.

For $i = 1, \ldots, g_k$, we denote by a_i and a'_i the pair of boundary components of S_k which were obtained at step i during the cutting process (i.e. pasting together a_i and a'_i for $i = 1, \ldots, g_k$ yields Σ_k). Then we draw for each $i = 1, \ldots, g_k + 1$ a closed curve $\hat{\omega}_i$ which goes along L and surrounds each boundary curve c of S_k in the following way. If c is none of the a_j, a'_j, then $\hat{\omega}_i$ goes around c in Z_k. If c is a_j or a'_j and $i \leq j$, then $\hat{\omega}_i$ goes around c in Z_k as well. If c is a_j or a'_j and $i > j$, then $\hat{\omega}_i$ goes around c in $\bar{Z}_k$. The $\hat{\omega}_i$ consist of arcs of L and of arcs on the boundary of S_k.

Now we use small homotopies to deform each $\hat{\omega}_i$ into a closed curve ω_i contained in the interior of S_k such that $\omega_1, \ldots, \omega_{g_k+1}$ are pairwise disjoint and such that each ω_i is a union of elementary arcs. It is not difficult to check that this is possible except for the case where S_k consists of a single pair of pants. In this particular case the ω_i are homotopic to one of the boundary components and we shall say that they "consist of 0 elementary arcs".

Since S_k consists of $2g_k + m_k - 2$ pairs of pants and has $2g_k + m_k$ boundary components none of which is intersected by the ω_i, we can perform the above homotopies such that any ω_i consists of

$$3(2g_k + m_k - 2) - (2g_k + m_k) = 4g_k + 2m_k - 6$$

elementary arcs (where some of them may be trivial arcs).

The curves are shown in Fig. 3 where ω_1 is the lowest curve, ω_2 is above ω_1, ω_3 is above ω_2, etc. For $i = 1, \ldots, g_k$, the four curves ω_i, ω_{i+1}, a_i and a_i' bound a domain Ω_i of signature $(0, 4)$ as shown by the shaded area. The domains $\Omega_1, \ldots, \Omega_{g_k}$ do not overlap each other.

For each i we draw a curve b_i in Ω_i going from a point on a_i to an equivalent point (with respect to the pasting) on a_i'. Taking the shorter of the two paths along L from a_i to a_i' we achieve that b_i consists of at most $2g_k + m_k - 2$ elementary arcs.

Since the Ω_i do not overlap, the b_i are pairwise disjoint. It follows that on Σ_k the a_i and b_i are closed curves with the intersection properties as required for a canonical homology basis. Since $2g_k + m_k - 2 \leq 2g - 2$, this proves the theorem. □

Observing that the constructions and length estimates in the preceding proof depend only on the components Σ_k we actually have the following more detailed version of Theorem 1, where we also admit surfaces with boundary.

For the statement of the theorem we note that for a bordered surface S of signature (g, m) a *canonical homology basis* is a curve system $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, $\rho_1, \ldots, \rho_{m-1}$, where $\rho_1, \ldots, \rho_{m-1}$ are boundary components and $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ have the configuration of a canonical homology basis on a compact unbordered surface of genus g.

Theorem 2. Let $\mathcal{P}$ be a pants decomposition of the compact orientable surface Σ of signature (g, m), $g \geq 1$, denote by $\Sigma_1, \ldots, \Sigma_n$ the components obtained by cutting Σ open along all separating curves occurring in $\mathcal{P}$, and let $\#\Sigma_k$ be the number of pairs of pants of $\mathcal{P}$ in Σ_k, $k = 1, \ldots, n$. Then there exists a canonical homology basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \rho_1, \ldots, \rho_{m-1}$ on Σ such that

1. $\alpha_1, \ldots, \alpha_g$ and $\rho_1, \ldots, \rho_{m-1}$ belong to $\mathcal{P}$.
2. each β_i is contained in some Σ_k and satisfies $\ell_{\mathcal{P}}(\beta_i) \leq \#\Sigma_k$. □

We point out that the upper bound $2g - 2$ in Theorem 1 for the combinatorial length of all β_k is the best possible. For this we consider the following.

Example 3. Start with a pair of pants all of whose boundary geodesics have the same length ε and paste two copies of it together in order to obtain a surface Ω of signature $(1, 2)$ with boundary components γ_L and γ_R. Then take $g - 1$ copies $\Omega_1, \ldots, \Omega_{g-1}$ of Ω with respective boundary geodesics $\gamma_{L,i}$, $\gamma_{R,i}$, $i = 1, \ldots, g - 1$.

We construct a "necklace" N out of these copies by pasting Ω_i to Ω_{i+1} along $\gamma_{R,i}$ and $\gamma_{L,i+1}$, for $i = 1, \ldots, g - 2$ and by pasting Ω_{g-1} to Ω_1 along $\gamma_{R,g-1}$ and $\gamma_{L,1}$. The resulting surface N has genus g. We let $\gamma_1, \ldots, \gamma_{g-1}$ be the geodesics in N obtained from $\gamma_{R,1}, \ldots, \gamma_{R,g-1}$ respectively. They are all non separating.

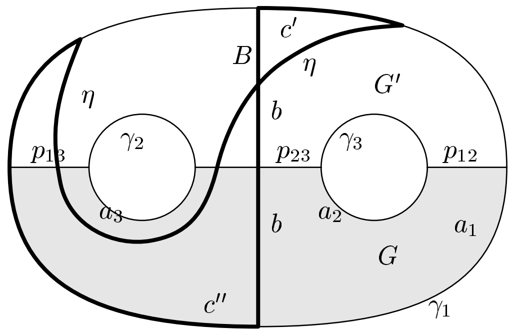

FIGURE 4. Arc of type II on a pair of pants

If $\mathcal{B}$ is a canonical homology basis on N then by the lemma below, at least one curve $b \in \mathcal{B}$ has intersection number $(b, \gamma_1) \neq 0$. On the surface N' obtained by cutting N open along γ_1 the curve b is cut into a number of arcs, one of which connects the two boundary components with each other. This follows from the fact that $(b, \gamma_1) \neq 0$. We conclude that b crosses at least $2g-2$ times a pair of pants.

Lemma 4. Let $\mathcal{B}$ be a homology basis on a compact orientable surface X. For any non separating simple closed curve c on X there exists $b \in \mathcal{B}$ with $(b, c) \neq 0$.

Proof. Otherwise $(c, \beta) = 0$ for any cycle β, but c is non separating and there exists a cycle c' with $(c, c') = 1$. □

3. LENGTH ESTIMATES

Let us now derive metric length estimates. All partitioning curves are assumed to be closed geodesics, and on any pair of pants the connectors are the common perpendiculars between the boundary geodesics.

Replacing the curves constructed in the preceding section by geodesics in their homotopy classes we obtain short homology bases in the sense of the hyperbolic metric. In order to get length estimates we proceed as follows. First we replace any elementary arc of the given curve by a homotopic arc with the same end points. By the minimal intersection property of geodesics, this new arc is also elementary and we shall give a length estimate for it in Proposition 5. Thus we have a piecewise geodesic curve with controlled length and the smooth closed geodesic in its homotopy class is then even shorter.

Proposition 5. Let Y be a pair of pants such that all boundary geodesics have lengths between ε and L, where $0 < \varepsilon \leq L$. Then any elementary geodesic arc η on Y has length $\ell(\eta) \leq 2L + 2 \operatorname{arcsinh} \frac{4}{\varepsilon}$.

Proof. The three common perpendiculars p_{ij} between the boundary geodesics γ_1, γ_2, γ_3 of Y decompose Y into two isometric right angled hyperbolic geodesic hexagons G and G' (cf. [6]), where G has the succession of sides a_1, p_{12}, a_2, p_{23}, a_3, p_{13}. With these symbols we shall also denote the lengths of the sides.

Consider an elementary geodesic arc η on Y. If η is trivial then η is a simple arc on the boundary and has length less than L.

Now let η be of type II with end points on γ_1 as shown in Fig. 4. η is homotopic with fixed end points to a piecewise geodesic curve $c'Bc''$, where c' and c'' are arcs on γ_1 and B is the arc of length $2b$ formed by the common orthogonals on G and G' from γ_1 to p_{23}. Since the interior of η intersects the union $p_{12} \cup p_{23} \cup p_{13}$ at most twice, the total length of c and c' is at most $\ell(\gamma_1) \leq L$. From the trigonometry of hexagons (cf. [6]) it follows that b becomes maximal when γ_1 assumes the minimal length ε and γ_2 and γ_3 assume the maximal length L. In this extremal case b is given by the formula $\cosh \frac{L}{2} = \sinh b \sinh \frac{\varepsilon}{4}$. After some calculus involving monotonicity of derivatives we then get

$$b \leq \frac{L}{2} + \operatorname{arcsinh} \frac{4}{\varepsilon}. \tag{2}$$

This settles the length estimate for arcs of type II, so let η be of type I, say with end points on γ_2 and γ_3.

Here η is homotopic with fixed end points to the piecewise geodesic curve $c_2 p_{23} c_3$, where c_2, c_3 are arcs on γ_2, γ_3 respectively. The length of p_{23} becomes maximal when γ_2 and γ_3 have length ε and γ_1 has length L. In this extremal case we have the formula $\cosh \frac{L}{4} = \sinh \frac{\varepsilon}{2} \sinh \frac{1}{2} p_{23}$ from which we deduce the inequality

$$p_{23} \leq \frac{L}{2} + 2 \operatorname{arcsinh} \frac{2}{\varepsilon}. \tag{3}$$

If η intersects p_{23} then the interior of η intersects $p_{12} \cup p_{13}$ at most once. Hence the total length of c_2 and c_3 is at most $\frac{3}{2}L$ and we are done in this case. If η does not intersect p_{23} then the total length of c_2 and c_3 may reach $2L$ so that we must refine the estimates a bit. Using the trigonometric formulas for hexagons we get $\cosh \eta = g(c_2, c_3, a_1)$, where

$$g(x, y, z) = \frac{\operatorname{ch} 2x \operatorname{ch} 2y}{\operatorname{sh} x \operatorname{sh} y} (\operatorname{ch} z + \operatorname{ch} x \operatorname{ch} y) - \operatorname{sh} 2x \operatorname{sh} 2y.$$

An elementary argument now shows that for any x, y with $\frac{\varepsilon}{2} \leq x, y \leq \frac{L}{2}$ we have $g(x, y, \frac{L}{2}) \leq 2L + \operatorname{arcsinh} \frac{4}{\varepsilon}$. □

Example 6. If in Example 3 we take pairs of pants with all boundary geodesics of length ε, then the distance λ between two boundary geodesics of a pair of pants satisfies $\operatorname{sh} \frac{\lambda}{2} \operatorname{sh} \frac{\varepsilon}{2} = \operatorname{ch} \frac{\varepsilon}{4}$. It follows that any homology basis on the surface N constructed in the example has at least one curve of length $\geq 4(g-1) \operatorname{arcsinh} \frac{2}{\varepsilon}$. Hence, in certain cases the upper bound in Theorem 7 is close to optimal.

Theorem 1 and Proposition 5 yield now:

Theorem 7. Let X be a compact Riemann surface of genus $g \geq 2$ which has a partition with longest geodesic of length L and whose homological systole is ε. Then there exists a canonical homology basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ on X such that any α_i belongs to the partition and the length $\ell(\beta_i)$, of any curve β_i, satisfies

$$\ell(\beta_i) \leq (2g-2)(2L + 2 \operatorname{arcsinh} \tfrac{4}{\varepsilon}) \tag{4}$$

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