Seifert unions of solid tori *

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Abstract

We obtain a list of all 3-manifolds that can be obtained by gluing 3-balls and solid tori along mutually disjoint surfaces in their boundaries.

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0. Introduction

The Lusternik-Schnirelman category $\mathrm{cat}(M^n)$ of a closed n-manifold M^n is the smallest number of sets, open and contractible in M needed to cover M. It is well known that $2 \leq \mathrm{cat}(M^n) \leq n+1$, but it is an open problem whether $\mathrm{cat}(M^n)$ can be realized by a covering with open balls. For example, if $\mathrm{cat}(M^4) = 4$ then M^4 can be covered by four balls [Si, 6.2], but it is not known whether a 3-manifold M^3 with $\mathrm{cat}(M^3) = 3$ can be covered by three balls. If $\mathrm{cat}(M^n) = 2$ then M^n is a homotopy n-sphere. For $n = 3$, $\mathrm{cat}(M^3)$ depends only on $\pi_1(M)$: in [GG] it is shown that $\mathrm{cat}(M^3) = 3$ if and only if $\pi_1(M)$ is free (non-trivial). If M^n can be covered by two balls then $M^n = S^n$, see e.g. [Z] or [R,p.35]. Hence an affirmative answer to realizing $\mathrm{cat}(M^3)$ by open balls would imply the Poincaré

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Conjecture. Hempel and McMillan showed that a simply connected 3-manifold M is S^3 if M can be covered by three open balls. Work by Singhoff [Si] (see also Montejano [M]) implies that if $\mathrm{cat}(M^3) = 3$ then M^3 admits a 3-filling; i.e., M^3 can be expressed as a union of three orientable handlebodies V_1, V_2, V_3 such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ is a surface for $i \neq j$, and such that each handlebody V_i is a contractible in M^3.

To investigate whether such an M^3 can be covered by three balls we consider more generally the classification of 3-manifolds M that can be expressed as the union of three orientable handlebodies V_1, V_2, V_3 whose interiors are pairwise disjoint, but such that V_i is not necessarily contractible in M. If g_i denotes the genus of ∂V_i and $g_1 \leq g_2 \leq g_3$, then we say that *M has a Singhof splitting of type* (g_1, g_2, g_3) and we define the *tri-genus* of M to be the triple (g_1, g_2, g_3) corresponding to a splitting of minimal type in lexicographical ordering. This is a generalization of the notion of Heegaard splitting.

The study of the class of closed manifolds which posses round functions leads to the question of which manifolds are unions of solid tori. Here a real-valued function is a *round* function if its critical set is a disjoint union of smooth simple loops. Khimshiashvili and Siersma [K-S] used decompositions of 3-manifolds into solid tori to prove the existence of round functions on closed 3-manifolds. They define the *round complexity* $roc(M)$ of an orientable closed 3-manifold M to be the minimal possible number of critical loops of round functions on M. They show that if $roc(M)$=2 then M is a lens space and that $roc(M)$=3 if and only if M has a Singhof splitting of type $(1, 1, 1)$. These manifolds, together with the 3-manifolds that have tri-genus $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$ are classified in [G]. (The simplest case of tri-genus $(0, 0, 0)$ is still open for 4-manifolds: it is not known what 4-manifolds can be obtained as a union of three 4-balls whose interiors are pairwise disjoint.) In [GHN] we obtained the list of 3-manifolds of tri-genus $(0, 0, n)$ and $(0, 1, n)$. The case of tri-genus $(1, 1, n)$ for $n > 1$ seems difficult and is the motivation for the present paper.

A 3-manifold that admits a splitting of type $(1, 1, n)$ is obtained as a union of a handlebody of genus n and a manifold M that is a union of two solid tori such that ∂M is an orientable surface of genus n. More generally, we study the class of 3-manifolds that are ***Seifert unions*** of solid tori V_1, V_2; i.e., that are obtained from V_1, V_2 by identifying a collection of disjoint surfaces $\{F_i'\}$ in ∂V_1 with a collection of disjoint surfaces $\{F_i''\}$ in ∂V_2 under homeomorphisms. Note that if the collection $\{F_i'\}$ and $\{F_i''\}$ consist of essential annuli then such a Seifert union is a Seifert fiber space. In obtaining the classification of Seifert unions of two tori we are lead to consider Seifert unions of any collection of punctured balls and punctured tori. At first one might think that this would include all compact 3-manifolds with boundary; however, it turns out that this collection of Seifert unions yields a very restricted class. Indeed, our main result is

Theorem 8. *A Seifert union of punctured solid tori and punctured balls is a connected sum of Seifert webs, of S^2-bundles over S^1, of lens spaces, and of handlebodies.*

Here a ***Seifert web*** is a connected manifold obtained from Seifert fiber spaces by attaching 1-handles. For example the closure of the complement of a regular neighborhood of a knot K in S^3 is in our collection of Seifert unions if and only if K is a torus knot. As another example note that $P^2 \times I$ is not in our collection. Thus, even though $P^2 \times I$ admits a decomposition into three balls B_1, B_2, B_3 such that $B_i \cap B_j = \partial B_i \cap \partial B_j$ is a surface (for $i \neq j$), it cannot be obtained as a Seifert union of balls and solid tori.

Outline of the paper

In Section 1 we introduce web-manifolds and in particular Seifert webs. Lemmas proved in Section 1 are needed to obtain a classification of π_1-trivial or trivial Seifert unions in Section 2. The main results about Seifert unions are proved in Section 3. In Section 4 we give some applications of Seifert unions of two solid tori. Finally we classify in Section 5 the complements of handlebodies of genus 2 in S^3 that are Seifert webs.

1. Definitions and Preliminaries

Throughout this paper we work in the PL-category. All n-manifolds considered are compact.

A handlebody W is obtained from the 3-ball B^3 by attaching 1-handles: $W = B^3 \cup$ 1-handles.

Lemma 1. *Let F be a (not necessarily connected) surface in the boundary of a handlebody W such that inclusion induces the zero homomorphism $\pi_1(F_j) \to \pi_1(W)$ for each component F_j of F. Then there is a ball B^3 in W such that $W = B^3 \cup$ 1-handles with $F \subset \partial B^3$ and such that F misses all the 1-handles.*

Note that each component F_j of F is planar. Otherwise there would be two simple closed curves α and β on F (and hence on ∂W) with algebraic intersection number 1. But since both $\alpha \simeq \beta \simeq 0$ in W this cannot happen (see e.g. [S]).

Proof. (a) First assume that there is a component c of ∂F with $c \not\simeq 0$ on ∂W. We proceed by induction on genus(∂W) = the number of 1-handles of W. The lemma is trivial for genus$(\partial W) = 0$.

The component c bounds an essential disk D in W. We cut W along a copy D' parallel to D that does not meet F into one handlebody W_1 or two handlebodies W_1, W_2, depending on whether D' is non-separating or separating. Since D' is essential, genus$(\partial W_i) <$ genus(∂W). Each component F_j of F lies in some W_i and, since inclusion induces an injection $\pi(W_i) \to \pi(W)$, it follows that $\pi(F_j) \to \pi(W_i)$ is the zero homomorphism. By induction, $W_i = B_i^3 \cup$ 1-handles where each component F_j of F that lies in W_i misses the 1-handles.

If D' does not separate W then $F \subset W_1$ and F is disjoint from the two copies of D' in W_1. By an isotopy we can assume that the copies of D' are disjoint to the 1-handles. Hence W is obtained from W_1 by attaching a 1-handle disjoint from F; i.e., F misses the 1-handles of $W = B_1^3 \cup$ (1-handles) $\cup$1-handle.

If D' separates W then W is obtained from W_1 and W_2 by identifying the copies of D' in ∂W_1 and ∂W_2 and again after an isotopy we may assume that the copies of D' are disjoint from the 1-handles of W_1 and W_2. Then $W = W_1 \underset{D'}{\cup} W_2 = (B_1^3 \underset{D'}{\cup} B_2^3) \cup$ 1-handles $= B^3 \cup$ 1-handles, where F misses the 1-handles.

(b) Now suppose that each component C of ∂F is $\simeq 0$ on ∂W. Then each component F_j of F is a punctured disk on ∂W; i.e., there is a disk D_j on ∂W with $F_j \subset D_j$ and such that ∂D_j is a component of ∂F_j. Construct a subcollection $D_{i1}, \ldots, D_{ir}$ of this collection $\{D_j\}$ by deleting D_i from the collection if $D_i \subset D_j$ for some j. This disjoint collection of disks can be isotoped off the 1-handles of W, and since $F \subset D_{i1} \dot{\cup} \cdots \dot{\cup} D_{ir}$, this isotopy moves F off the 1-handles of W. $\square$

We will use the following notations throughout this paper:

For a 3-manifold M we denote by $M_{\propto\{D_i\}}$ a manifold obtained from M by attaching 1-handles. In particular the manifold obtained from M by attaching exactly one 1-handle is denoted by $M_{\propto D}$

$S^1 \tilde{\times} S^2$ denotes an S^2-bundle over S^1 (orientable or non-orientable)

B denotes a connected sum of S^3 and S^2-bundles over S^1 (with arbitrarily many factors).

H denotes a punctured ball with arbitrarily many punctures.

L denotes an arbitrary lens space (including S^3)

$S(m)$ denotes a Seifert fiber space (orientable or non-orientable), with arbitrary orbit surface and at most m exceptional fibers, and ***with non-empty boundary***.

$\mathcal{L}$ denotes the collection of all 3-manifolds that are connected sums of lens spaces L_i.

$\mathcal{F}$ denotes the collection of all 3-manifolds of the form $\mathbf{B}\sharp W_1\sharp\cdots\sharp W_n\sharp H$, where W_i is a handlebody (orientable or non-orientable) and n is an arbitrary non-negative integer.

For collections $\mathcal{A}_1$ and $\mathcal{A}_2$ of manifolds we denote by $\mathcal{A}_1 \sharp \mathcal{A}_2$ the collection of all manifolds of the form $M\sharp N$ for $M \in \mathcal{A}_1$ and N in $\mathcal{A}_2$. For a manifold M we denote by $M\sharp \mathcal{A}$ the collection of all manifolds of the form $M\sharp N$ for $N \in \mathcal{A}$.

By a ***punctured*** M we mean a manifold obtained from M by removing interiors of disjoint balls in $\mathrm{int}(M)$. Note that a punctured $M = M\sharp H$, for some punctured ball H.

A ***web-manifold with nodes*** $\boldsymbol{N_1,\ldots,N_k}$ is a connected manifold $M = W(N_1,\ldots,N_k)$ that is obtained from (connected) 3-manifolds $N_1,\ldots,N_k$ by identifying mutually disjoint disks on their boundaries. Note that M can be obtained from a disk sum and attaching 1-handles:

$$W(N_1,\ldots,N_k) = (N_1 \triangle \cdots \triangle N_k)_{\propto\{D_i\}}.$$

For 3-manifolds $N_1,\ldots,N_n$ let $\mathcal{W}(N_1,\ldots,N_n\}$ be the collection of all connected sums of web-manifolds $W(N_{i_1},\ldots,N_{i_k})$ such that each N_i is a node of exactly one connected sum factor, for $i = 1,\ldots,n$.

A ***Seifert-web*** $\boldsymbol{W(S(m_1),\ldots,S(m_k))}$ is a web-manifold with nodes $N_i = S(m_i)$, $i = 1,\ldots,k$.

As an example consider the embedding G of a dumbbell into S^3 as shown in Figure 1. The dumbbell space $\overline{S^3 - N(G)}$ is a Seifert-web $W(S(2),S(2))$, which is the disk sum of two Seifert fiber spaces, each with 2 exceptional fibers: the knot space of the trefoil $K_{2,3}$ and the Solomon's seal $K_{5,6}$. This type of example is considered in more detail in Section 5.

By $\mathcal{S}$ we denote the collection of all connected sums of Seifert-webs.

Remark. The collections $\mathcal{F}$, $\mathcal{S} \sharp \mathcal{F}$, $\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$ are closed under connected sum operations; i.e., if M and N are in $\mathcal{F}$ (or $\mathcal{S} \sharp \mathcal{F}$, $\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$, respectively) then $M\sharp N$ is in $\mathcal{F}$ (or $\mathcal{S} \sharp \mathcal{F}$,

$\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$, respectively).

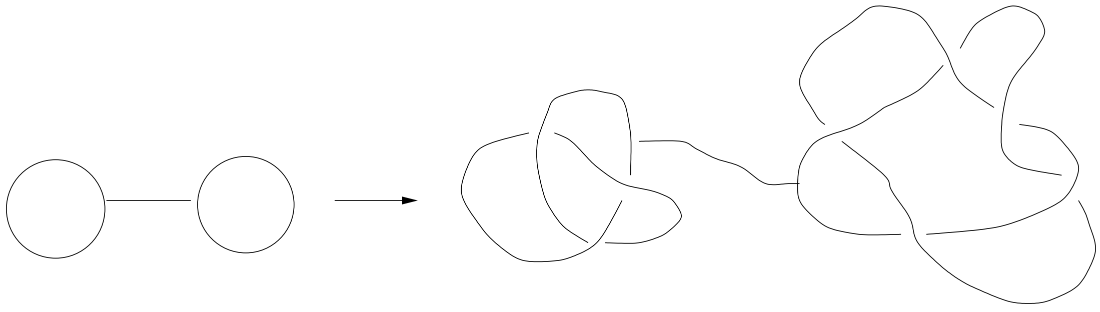

Figure 1

Lemma 2. *(a) The collections $\mathcal{F}$ and $\mathcal{S} \sharp \mathcal{F}$ are closed under the operation of attaching 1-handles, i.e. attaching 1-handles to manifolds $M_1, \ldots, M_s$ in one of these collections results in manifolds $X_1, \ldots, X_t$, each of which is in the same collection.*

(b) If $M_1, \ldots, M_s$ are in $\mathcal{W}(N_1, \ldots, N_m) \sharp \mathcal{F}$ such that the nodes of M_i are distinct from the nodes of M_j for each $i \neq j$, then attaching 1-handles to the collection $\{M_1, \ldots, M_s\}$ results in manifolds each of which is in $\mathcal{W}(N_1, ..., N_m) \sharp \mathcal{F}$.

The proof is easy and consists of an examination of all cases. We illustrate just one case each for $\mathcal{F}$ and $\mathcal{S} \sharp \mathcal{F}$.

(1) Let $M = W_1 \sharp W_2 \in \mathcal{F}$ and suppose we attach a 1-handle $D^2 \times I$ to M with $D^2 \times 0$ attached to W_1 and $D^2 \times 1$ attached to W_2. Then $W_1 \sharp W_2 = (W_1 - \overset{\circ}{B}{}^3_1) \cup (W_2 - \overset{\circ}{B}{}^3_2)$ where B^3_i is a 3-ball in intW_i and the union is along the 2-spheres $S^2_i = \partial B^3_i$. Hence $(W_1 \sharp W_2) \cup (D^2 \times I)$ is obtained from the twice punctured handlebody $W = (W_1 - \overset{\circ}{B}{}^3_1) \cup (D^2 \times I) \cup (W_2 - \overset{\circ}{B}{}^3_2)$ by identifying the two 2-spheres S^2_1, S^2_2 in ∂W; i.e., $M \cup$ (1-handle)$\approx \hat{W} \sharp (S^2 \mathbin{\tilde{\times}} S^1) \in \mathcal{F}$, where $\hat{W}$ is obtained from W by filling in the punctures with 3-balls.

(2) Note that for a Seifert-web $W(S(m_1), \ldots, S(m_k))$ and a handlebody W, the union $W(S(m_1), \ldots, S(m_k)) \cup (D^2 \times I) \cup W$ with $D^2 \times 0$ attached to $\partial W(S(m_1), \ldots, S(m_k))$ and $D^2 \times 1$ attached to ∂W is again a Seifert-web of type $W(S(m_1), \ldots, S(m_k))$.Now if $M = W(S(m_1), \ldots, S(m_k)) \sharp W \in \mathcal{S} \sharp \mathcal{F}$ then as in case (a) it follows that $M \cup (D^2 \times I)$ with $D^2 \times 0$ attached to $\partial W(S(m_1), \ldots, S(m_k))$ and $D^1 \times 1$ attached to ∂W is of type

$W(S(m_1), \ldots, S(m_k))\sharp(S^2 \tilde{\times} S^1) \in \mathcal{S} \sharp \mathcal{F}$. $\square$

2. Trivial and π_1-Trivial Seifert Unions

Let $M_1, \ldots, M_n$ be a collection of 3-manifolds and $F_1, \ldots, F_k$ a collection of surfaces. Suppose that for each F_i there are two copies F_i', F_i'' on $\partial(M_1 \dot{\cup} \cdots \dot{\cup} M_n)$ such that all the surfaces F_i', F_j', F_i'', F_j'' are ***mutually disjoint*** ($i = 1, \ldots, k$, $j = 1, \ldots, k$, $i \neq j$). (Here $\dot{\cup}$ denotes disjoint union). Note that a pair F_i', F_i'' may lie on the boundary of one M_j or of two different M_j's.

Definition. The connected manifold $M = M_1 \cup \cdots \cup M_n/\{F_i\}$ is a ***Seifert union of*** $M_1, \ldots, M_k$ ***along*** $\{F_i\}$ if M is obtained from $M_1, \ldots, M_n$ by identifying F_i' with F_i'' under some homeomorphism h_i, for $i = 1, \ldots, k$. We refer to the collection $\{F_1, \ldots, F_k\}$ as the ***collection of gluing surfaces***. We also use the notation $M = M_1 \underset{\{F_1,\ldots,F_k\}}{\cup \cdots \cup} M_n$.

We say that M is a ***trivial*** Seifert union if every $F_i' \subset M_j$ lies in a disk on ∂M_j and every $F_i'' \subset M_\ell$ lies in a disk on ∂M_ℓ. M is a ***π_1-trivial*** Seifert union if every curve on $F_i' \subset M_j$ and $F_i'' \subset M_\ell$ is contractible in M_j and M_ℓ, respectively ($i = 1, \ldots, k$).

For a 3-manifold N we denote by $\widehat{N}$ the 3-manifold obtained from N by filling in all 2-sphere components of ∂N with 3-balls.

Theorem 3. *Let $M_1, \ldots, M_n$ be 3-manifolds and let M_i^* be a punctured M_i. Then a trivial Seifert union of $M_1^*, \ldots, M_n^*$ is in $\mathcal{W}\{\widehat{M}_1, \ldots, \widehat{M}_n\}\sharp \mathcal{F}$.*

Proof. (a) Let $M = M_1^* \cup \cdots \cup M_n^*/\{F_1, \ldots, F_k\}$. First suppose $k = 1$, hence $n = 1$ or 2. If $n = 1$ and if the two copies of the planar surface F_1 are not nested, i.e., F_1' and F_1'' lie in disjoint disks D_1, D_2 of ∂M_1^* , then we obtain $M_1^*/\{F_1\}$ from M_1^* by first identifying D_1 and D_2 and then removing the interiors of m balls, where m is one less than the number of boundary components of F_1 , see Figure 2. Hence $M = M_1^*/\{F_1\}$ is a punctured $M_1^* \cup$ 1-handle.

If the 1-handle is attached to components of $\partial \widehat{M}_1$ then we write the puncture $M_1^* \cup$ 1-handle as $(\widehat{M}_1 \cup$1-handle$)\sharp H$ for some punctured ball H, hence $M \in W(\widehat{M}_1)\sharp \mathcal{F}$.

If the 1-handle is attached to different components of ∂ (punctured M_1^*) such that at least one is a sphere S^2, let $S^2 \times I$ be a collar on S^2 with $S^2 = S^2 \times \{0\}$ and let $M_1' = \overline{M_1^* - S^2 \times I}$. We obtain M from a disk sum $M_1'' = M_1' \triangle (S^2 \times I)$ by identifying the two copies of $S^2 \times \{1\}$ in $\partial M_1''$. Since M_1'' is homeomorphic to a punctured M_1^*, it follows that M is homeomorphic to $\widehat{M}_1 \sharp H \sharp (S^2 \tilde{\times} S^1)$, for some punctured ball H.

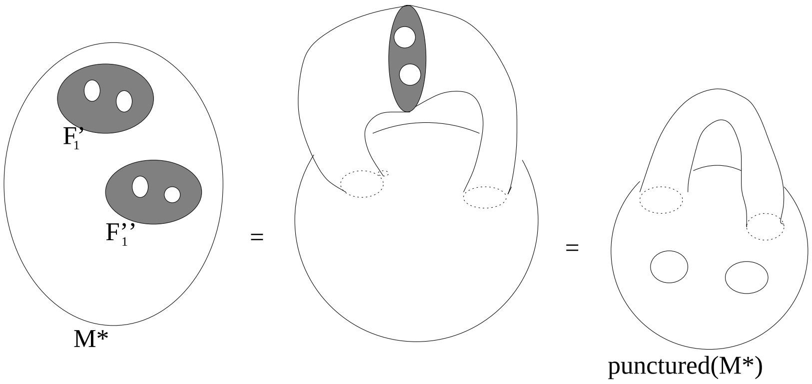

Figure 2

If the 1-handle is attached to the same sphere component S^2 of ∂(punctured M_1^*) then M is a connected sum of M_1' and a solid torus or solid Kleinbottle, where M_1' is obtained from M_1^* by filling in S with a 3-ball. Hence $M \in \widehat{M}_1 \sharp \mathcal{F}$.

If F_1' and F_2'' are nested, assume that F_1' is innermost and let B^3 be a ball in M_1^* such that $B^3 \cap \partial M_1^*$ is a disk G with $F_1' \subset \text{int}G$, $G \cap F_1'' = \emptyset$, and let D be the disk $\overline{\partial B^3 - G}$, (see Figure 3). Now D splits M_1^* into B^3 and $M_1' \cong M_1^*$ and $M_1^*/\{F_1\}$ is obtained from $M_1' \cup B^3/\{F_1\}$ by identifying the two copies of D. Since $M' \cup B^3/\{F_1\}$ is a punctured M_1^* we obtain $M = M_1^*/\{F_1\}$ again as a punctured $M_1^* \cup$ 1-handle, as above.

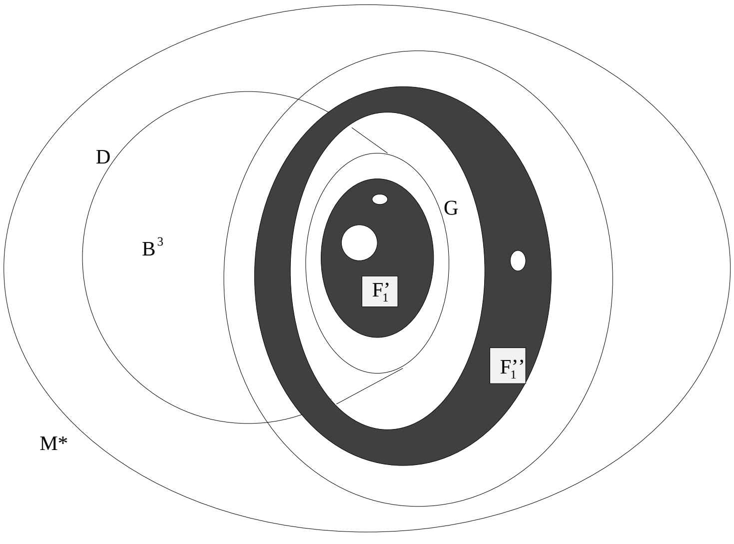

Figure 3

If $n = 2$ then $M = M_1^* \underset{\{F_1\}}{\cup} M_2^*$ is obtained by connecting M_1^* to M_2^* by a 1-handle and then introducing punctures. If the 1-handle connects components of $\partial \widehat{M}_1$ and $\partial \widehat{M}_2$ then M is of type $W(\widehat{M}_1, \widehat{M}_2) \sharp H$, for some punctured ball H.

If the 1-handle connects a component of ∂M_1^* to a sphere component S of ∂M_2^* then M is obtained by first attaching a collar $S \times I$ to M_1^* to get $M_1' = M_1^* \triangle (S \times I)$, and then identifying the 2-sphere $S \times \{1\}$ in M_1' with its copy in $M_2' = \overline{M_2^* - S \times I}$. It follows that $M \cong \widehat{M}_1 \sharp \widehat{M}_2 \sharp H$, for some punctured ball H, hence $M \in \mathcal{W}\{\widehat{M}_1, \widehat{M}_2\} \sharp \mathcal{F}$.

(b) Now suppose $k \geq 2$. If $n = 1$ there is an F_i' or F_i'' that is innermost. Assume that F_1' is innermost and refer to Figure 3. Now $M = ((M_1' \underset{\{F_1\}}{\dot{\cup}} B^3)/\{F_2, \ldots, F_k\})_{\propto \{D\}}$ where $M_1' \underset{\{F_1\}}{\cup} B^3$ is a punctured M_1^*. By induction on k we have $M_1^*/\{F_2, \ldots, F_k\}$ in $\mathcal{W}\{\widehat{M}_1\} \sharp \mathcal{F}$ and hence it follows from Lemma 2 that $M \in \mathcal{W}\{\widehat{M}_1\} \sharp \mathcal{F}$.

If $n \geq 2$ then there is a F_i for which the two copies F_i', F_i'' lie in different M_j^*'s. Assume that F_1' is on ∂M_1^* and $F_1'' \subset \partial M_2^*$.

Let M_1', D be as in Figure 3. (But F_1' need not be innermost on M_1^*.) Then $M = M_1^* \cup \cdots \cup M_n^*/\{F_1, \ldots, F_k\} = ((B^3 \underset{\{F_1\}}{\dot{\cup}} M_2^*) \underset{\{F_2, \ldots, F_k\}}{\cup} (M_1' \cup M_3^* \cup \cdots \cup M_n^*))_{\propto D} = (M_2'' \underset{\{F_2, \ldots, F_k\}}{\cup} (M_1' \cup \cdots \cup M_n^*))_{\propto D}$ where M_2'' is a punctured M_2^*. By induction on k, each component of $M_2'' \underset{\{F_1, \ldots, F_k\}}{\cup} (M_1' \cup \cdots \cup M_n^*)$ belongs to $\mathcal{W}\{\widehat{M}_1 \ldots, \widehat{M}_n\} \sharp \mathcal{F}$ and hence M belongs to this collection by Lemma 2. $\square$

Corollary 4. *(a) A trivial Seifert union of punctured handlebodies and punctured balls is in $\mathcal{F}$.*

(b) A trivial Seifert union of punctured Seifert fiberspaces $S(m_1), \ldots, S(m_k)$, of punctured handlebodies, and punctured balls is in $\mathcal{W}\{S(m_1), \ldots, S(m_k)\} \sharp \mathcal{F}$.

Proof. (a) If $M_1, \ldots, M_n$ are punctured handlebodies or punctured balls then a web-manifold $W(\widehat{M}_{i_1}, \ldots, \widehat{M}_{i_k})$ is a handlebody. Hence $\mathcal{W}\{\widehat{M}_1, \ldots, \widehat{M}_n\}$ consists of connected sums of handlebodies and $\mathcal{W}\{\widehat{M}_1, \ldots, \widehat{M}_n\} \sharp \mathcal{F} = \mathcal{F}$.

(b) This follows as in the proof of Lemma 2(b). $\square$

We now consider the case of π_1-trivial (but not necessarily trivial) Seifert unions.

Proposition 5. *A π_1-trivial Seifert union of punctured handlebodies and punctured balls is in $\mathcal{F}$.*

Proof. Each punctured handlebody can be written as a connected sum of a handlebody and a punctured ball. Thus $M = (W_1 \sharp H_1 \cup \cdots \cup W_n \sharp H_n \cup H_{n+1} \cup \cdots \cup H_{n+m})/\{F_i\}$, for some handlebodies W_i and punctured balls H_j. By Lemma 1 we can write each W_j as $B^3_j \cup$(1-handles) where the copies of $\{F_i\}$ on ∂W_j miss the 1-handles. Hence $M = [B^3_1 \sharp H_1 \cup \cdots \cup B^3_n \sharp H_n \cup H_{n+1} \cup \cdots \cup H_{n+m}/\{F_i\}] \cup$ (1-handles) and the proposition follows from Corollary 4(a), the remark before Lemma 2, and Lemma 2. $\square$

Proposition 6. *A π_1-trivial Seifert union of Seifert fiber spaces, punctured handlebodies, and punctured balls is in $\mathcal{S} \sharp \mathcal{F}$.*

Proof. Denote the Seifert fiber spaces by $S(m_i)$, the handlebodies by W_p and the punctured balls by H_t, and the collection of the gluing surfaces by $\{F_1, \ldots, F_k\}$. We may assume that the Seifert fiber spaces $S(m_i)$ have incompressible boundary, since otherwise they are handlebodies (of genus 1). It follows that each F'_j or F''_j in $\partial S(m_i)$ lies in a disk on $\partial S(m_i)$. Therefore the copies of the π_1-trivial gluing surfaces in the boundaries of the Seifert fiber spaces are actually trivial. Also, as in the proof of proposition 5, a π_1-trivial gluing of a surface on the boundary of a (punctured) handlebody can be realized by first doing a trivial gluing of this surface and then attaching 1-handles. Thus we can realize the π_1-trivial Seifert union of Seifert fiber spaces, punctured handlebodies, and punctured balls by first constructing a trivial Seifert union of Seifert fiber spaces and punctured balls and then attaching 1-handles. Hence the proposition follows from Corollary 4(b) and Lemma 2. $\square$

Theorem 7. *The collection $\mathcal{F}$ is closed under π_1-trivial Seifert union.*

Proof. Suppose $M = B \sharp W_1 \sharp \cdots \sharp W_n \sharp H$ and $M' = B' \sharp W'_1 \sharp \cdots \sharp W'_m \sharp H'$ are in $\mathcal{F}$. Then we obtain M (and M', respectively) from the disjoint union $U = B \cup W_1 \cup \cdots \cup W_n \cup H$ (and $U' = B' \cup W'_1 \cup \cdots \cup W'_m \cup H'$, respectively) by connected sum operations; i.e., removing balls and identifying their resulting boundary spheres. Hence a Seifert union of M and M' is obtained by first forming Seifert unions of components of U and U' and then performing connected sum operations. By Proposition 5 each component of a π_1-trivial Seifert union of components of U and U' belongs to $\mathcal{F}$ and since clearly $\mathcal{F}$ is closed under connected sum operations it follows that a π_1-trivial Seifert union of M and M' belongs to $\mathcal{F}$. $\square$

3. Seifert Unions

In this section we consider the general case of a Seifert union of punctured balls and punctured solid tori. We say that a connected surface F in the boundary of a 3-manifold N is ***π_1-trivial in*** N (π_1-trivial in ∂N, respectively) if the inclusion-induced homomorphism $\pi(F) \to \pi(N)$ ($\pi(F) \to \pi(\partial(N)$, respectively) is trivial.

Theorem 8. *A Seifert union of punctured solid tori and punctured balls is in $\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$.*

Proof. Writing a punctured V as $V \sharp H$ we may assume (by the proof of Theorem 7) that $M = (V_1 \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_t)/\{F_1, \ldots, F_k\}$, where each V_i is a solid torus.

Case 1. Each F_i', F_j'' is π_1-trivial. Then M is a π_1-trivial Seifert union and Proposition 5 shows that $M \in \mathcal{F}$.

Case 2. There is a pair, say F_1', F_1'', with F_1' not π_1-trivial in some V_i and F_1'' π_1-trivial in some V_j or F_1'' in some H_j. Suppose F_1' is not π_1-trivial in V_1 and F_1'' is π_1-trivial in V_j (the case $F_1'' \subset H_j$ is similar). Note that F_1', F_1'' are punctured annuli where $\partial F_1'$ has two essential curves on ∂V_1 and all other curves trivial on ∂V_1. (It is possible that $V_i = V_1$.)

(a) If F_1'' is π_1-trivial on ∂V_j then F_1'' lies in the interior of a disk $D' \subset \partial V_j$ (with D' disjoint to F_1') and we choose a ball B^3 in V_j with $B^3 \cap \partial V_j = D'$ and let $D = \overline{\partial B^3 - D'}$, $V_j' = \overline{V_j - B^3} \approx V_j$ and each of the other F_r', F_s'' is either disjoint from B^3 or lies in $\mathrm{int}(D')$, as in Figure 3. If $j \neq 1$ we obtain M as

$$[(V_1 \underset{\{F_1\}}{\cup} B^3) \cup V_2 \cup \cdots \cup V_j' \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_t/\{F_2, \ldots, F_k\}]_{\propto\{D\}}.$$

Since $V_1 \underset{\{F_1\}}{\cup} B^3$ is a punctured lens space $L^* = L\sharp H$, for some punctured ball H, we can write

$$M = L\sharp[(H \cup V_2 \cup \cdots \cup V_j' \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_t)/\{F_2, \ldots, F_k\}]_{\propto D}$$

and the theorem follows by induction on n and Lemma 2. If $j = 1$ the same argument applies, starting with

$$M = ((V_1' \underset{\{F_1\}}{\cup} B^3) \cup V_2 \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_t/\{F_2, \ldots, F_k\})_{\propto\{D\}}.$$

(b) If F_1'' is not π_1-trivial on ∂V_j then there are meridian disks D_1, D_2 in V_j that split V_j into two balls B_1^3, B_2^3 with $B_i^3 \cap \partial V_j$ an annulus A_i and $F_1'' \subset \mathrm{int}(A_1)$ and all other F_t', F_s'' are disjoint to B_i^3 or lie in $\mathrm{int}(A_i)$. Note that $j \neq 1$. Then we obtain M as

$$((V_1 \underset{\{F_1\}}{\cup} B_1^3) \cup B_2^3 \cup V_2 \cup \cdots \cup V_{j-1} \cup V_{j+1} \cup \cdots V_n \cup H_1 \cup \cdots \cup H_t/\{F_2, \ldots, F_k\}) \propto_{\{D_i\}}$$

and the result follows as in (a).

Case 3. For each i, F_i', F_i'' are both not π_1-trivial in some V_j's or both are π_1-trivial in some V_j's or H_j's.

(a) If there is a pair F_1', F_1'' say, of genus 1, then $F_1' \subset V_1$ and $F_1'' \subset V_2$, say. Then $V_1 \underset{\{F_1\}}{\cup} V_2$ is a punctured lens space $L^* = L\sharp H$ and we can write $M = L\sharp(H \cup V_3 \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_m)/\{F_2, \ldots, F_k\}$ and use induction on n.

(b) Each F_i is planar. Note that all F_i', F_ℓ'' that are not π_1-trivial in one V_t are parallel punctured annuli. Suppose that the not π_1-trivial pairs lie in $\partial V_1, \ldots, \partial V_{j-1}$. Gluing $V_1, \ldots, V_{j-1}$ along all non-trivial pairs we obtain a collection of punctured Seifert fiber spaces $S^*(m_i)$ and then M is a π_1-trivial Seifert union

$$M = S^*(m_1) \cup \cdots \cup S^*(m_q) \cup V_j \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_t/\{F_{i_1}, \ldots, F_{i_p}\}$$

for some subcollection $\{F_{i_1}, \ldots, F_{i_p}\}$ of $\{F_1, \ldots, F_k\}$. Hence by Proposition 6 M belongs to $\mathcal{S}\sharp\mathcal{F}$. $\square$

Let us denote by $\mathbf{L}$ (or $\mathbf{L}_i$) a lens space different from S^3 or $S^2 \times S^1$. Then a manifold from $\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$ is of the form

$$(*) \quad \mathbf{B}\sharp W(S(m_{i_1}), \ldots, S(m_{i_S}))\sharp \cdots \sharp W(S(m_{i_k}), \ldots, S(m_{i_t}))\sharp \mathbf{L}_1 \sharp \cdots \sharp \mathbf{L}_r \sharp W_1 \sharp \cdots \sharp W_m \sharp H$$

where we may assume that each $S(m_j)$ has incompressible boundary.

Conversely we have

Proposition 9. *Every manifold M of $\mathcal{S} \sharp \mathcal{L} \sharp \mathcal{F}$ with $\partial M \neq \emptyset$ can be realized as a Seifert union of tori and balls.*

The proof follows easily from the four examples below.

For a manifold N let N^0 be obtained from N by removing the interior of a 3-ball from $\mathrm{int}(N)$ and denote by $S_0(N)$ the resulting 2-sphere component of ∂N^0. Note that a disk sum of N^0 and M^0, obtained by identifying a disk in $S_0(N)$ with a disk in $S_0(M)$ is homeomorphic to a punctured connected sum: $N^0 \triangle M^0 \cong (N\sharp M)^0$. Note also that for ∂N connected, we may view N^0 as $N\triangle(S^2 \times I)$.

Example 1. $M = W_1\sharp W_2$, a connected sum of two handlebodies.

Represent $W_1\sharp W_2$ as $W_1 \underset{D_1}{\cup} (S^2 \times I) \underset{D_2}{\cup} W_2$, where the disk D_1' in ∂W_1 (respectively, D_2' in ∂W_2) is identified with a disk D_1'' in $S^2 \times \{0\}$ (respectively, D_2'' in $S^2 \times \{1\}$).

Now $S^2 \times I$ is a union of two balls along an annulus F and each W_i is a union of two balls along disks D_{ij}. Choosing the disks D_i', D_i'' disjoint from ∂D_{ij} and $\partial F \subset S^2 \times \partial I$ we see that $W_1\sharp W_2$ is a Seifert union of 6 balls.

Example 2. $M = L\sharp W$, a connected sum of a lens space and a handlebody.

Write W^0 as $W^0 = (S^2 \times I)\triangle W$ and hence $L\sharp W = L^0 \cup W^0 = L^0 \cup (S^2 \times I)\triangle W = L^0\triangle W$. Represent L^0 as a Seifert union of a solid torus and a ball along an annulus and W as a Seifert union of two balls. Hence $L\sharp W$ is a Seifert union of a solid torus and three balls.

Example 3. $M = (L_1\sharp L_2\sharp \cdots \sharp L_n)^0$, a once punctured connected sum of lens spaces.

Since $(L_1\sharp \cdots \sharp L_n)^0 \cong (L_1\sharp \cdots \sharp L_{n-1})^0\triangle L_n^0$ it suffices to consider the case $M = (L_1\sharp L_2)^0 = L_1^0\triangle L_2^0$.

Now L_i^0 is a union of a solid torus and a ball along an annulus A_i. Choosing the disk D for the disk sum $L_1^0\triangle L_2^0 = L_1^0 \underset{D}{\cup} L_2^0$ disjoint from ∂A_i in L_i^0 $(i = 1, 2)$, we see that $(L_1\sharp L_2)^0$ is a Seifert union of two solid tori and two balls.

Example 4. $S(m_1)\sharp S(m_2)$, a connected sum of a two Seifert fiber spaces (with $\partial S(m_i) \neq \emptyset$).

By cutting the orbit surface of $S(m_i)$ along disjoint arcs into disks each of which contains at most one exceptional point we see that $S(m_i)$ is a Seifert union of solid tori F_{ij} along annuli. Then $S(m_1)\sharp S(m_2) = S(m_1)^0 \cup S(m_2)^0 = S(m_1)\triangle(S^2 \times I)\triangle S(m_2) = S(m_1) \underset{D_1}{\cup} S^2 \times I \underset{D_2}{\cup} S(m_2)$, where D_i can be chosen to be disjoint to $\partial F_{ij} \subset \partial S(m_i)$. Hence $S(m_1)\sharp S(m_2)$ is a Seifert union of solid tori and two balls.

4. Unions of at Most Two or Three Solid Tori

We now consider explicitly the manifolds that occur as a union of at most two tori along mutually disjoint surfaces in their boundaries. An analysis of the proof of Theorem 8 shows that a Seifert union of n (punctured) solid tori and any number of (punctured) balls; i.e., a manifold as in (*), satisfies

$$(**) \qquad \sum m_{i_j} + r \leq n.$$

We say a 3-manifold M with $\partial M \neq \emptyset$ has a ***Seifert splitting of type*** $[\boldsymbol{g_1, \ldots, g_n}]$, if M is a Seifert union of orientable handlebodies $V_1, \ldots, V_n$ and any number of punctured balls, where genus $(V_i) = g_i$.

M has a ***special Seifert splitting of type*** $[\boldsymbol{g_1, \ldots, g_n}]$ if in addition no two copies F_i', F_i'' of F_i that are identified lie in the boundary of the same V_k.

Using the proof of Theorem 8 together with inequality (*) we obtain the following Table 1, which gives a complete list of 3-manifolds M that admit (special) Seifert splittings of types $[g]$ and $[g_1, g_2]$ for $0 \leq g_1, g_2 \leq 1$. Here $\mathbf{W} = \mathbf{B} \sharp W_1 \sharp \cdots \sharp W_m \sharp H$ is any member of $\mathcal{F}$ for some $m \geq 0$.

Table 1

Type	Seifert splitting	special Seifert splitting
$[0]$	$\mathbf{W}$	$\mathbf{W}$
$[1]$	$\mathbf{W}$ $\mathbf{L}\sharp\mathbf{W}$ $W(S(1))\sharp\mathbf{W}$	$\mathbf{W}$ $\mathbf{L}\sharp\mathbf{W}$
$[1, 1]$	$\mathbf{W}$ $\mathbf{L}\sharp\mathbf{W}$ $\mathbf{L}_1\sharp\mathbf{L}_2\sharp\mathbf{W}$ $W(S(2))\sharp\mathbf{W}$ $W(S(1))\sharp W(S(1))\sharp\mathbf{W}$ $W(S(1), S(1))\sharp\mathbf{W}$ $W(S(1))\sharp\mathbf{L}\sharp\mathbf{W}$	$\mathbf{W}$ $\mathbf{L}\sharp\mathbf{W}$ $\mathbf{L}_1\sharp\mathbf{L}_2\sharp\mathbf{W}$ $W(S(2))\sharp\mathbf{W}$

The special Seifert splittings of type [0] (with 3 balls) and [1] (with 2 balls) were obtained in [GHN] in Lemma (1.5) and (1.7) and the list in Table 1 is an extension of these results.

As an application we obtain a different proof of the results of [G], which list all closed 3-manifolds that admit a Singhof splitting (as described in the introduction) of type (g_1, g_2, g_3) with $g_i \leq 1$. As in [GHN] we say that the closed 3-manifold M has ***trigenus*** (g_1, g_2, g_3) if M is a union of three orientable handlebodies V_1, V_2, V_3 with $V_i \cap V_j =$

$\partial V_i \cap \partial V_j$ a surface for $i \neq j$, genus $V_i = g_i$, and the triple (g_1, g_2, g_3) is minimal with respect to lexicographical ordering. Hence M is obtained as a union of three handlebodies $M = V_1 \cup V_2 \cup V_3$ where $V_1 \cup V_2$ is a special Seifert union and $\partial(V_1 \cup V_2) = \partial V_3$.

We now give different proofs of the results of [G].

(1) A closed 3-manifold M of trigenus $(0,0,0)$ is then obtained as $(B_1^3 \cup B_2^3) \cup B_3^3$ where $B_1^3 \cup B_2^3$ is as in Table 1 (special type $[0]$). Hence $M = \mathbf{W} \cup B_3^3$ with $\partial W = S^2$; i.e., $W = \mathbf{B} \sharp B^3$, so $M = \mathbf{B}$.

(2) If M has trigenus $(0,0,1)$ we get $M = (V_1 \cup B_2^3) \cup B_3^3$ with $V_1 \cup B_2^3$ from Table 1 (special type $[1]$) and $\partial(V_1 \cup B_2^3) = S^2$. Hence $M = \mathbf{B} \sharp B^3 \cup B_3^3$ or $\mathbf{L} \sharp \mathbf{B} \sharp B^3 \cup B_3^3$; i.e., $M = \mathbf{B}$ or $M = \mathbf{L} \sharp \mathbf{B}$.

(3) For trigenus $(0,1,1)$ we have $M = (V_1 \cup V_2) \cup B_3^3$ with $V_1 \cup V_2$ from Table 1 (special type $[1,1]$) with $\partial(V_1 \cup V_2) = S^2$ and we get $M = \mathbf{B} \sharp B^3 \cup B_3^3$, $M = \mathbf{L} \sharp \mathbf{B} \sharp B^3 \cup B_3^3$, or $M = \mathbf{L}_1 \sharp \mathbf{L}_2 \sharp \mathbf{B} \sharp (B^3 \cup B_3^3)$, hence $M = \mathbf{B}$, $M = \mathbf{L} \sharp \mathbf{B}$, or $M = \mathbf{L}_1 \sharp \mathbf{L}_2 \sharp \mathbf{B}$.

(4) For trigenus $(1,1,1)$ we have $M = (V_1 \cup V_2) \cup V_3$ with $V_1 \cup V_2$ from Table 1 (special type $[1,1]$) and $\partial(V_1 \cup V_2)$ a torus. Thus $M = \mathbf{B} \sharp W \cup V_3$, $\mathbf{L} \sharp \mathbf{B} \sharp W \cup V_3$, $\mathbf{L}_1 \sharp \mathbf{L}_2 \sharp \mathbf{B} \sharp W \cup V_3$, or $B \sharp S(2) \cup V_3$, where W is a solid torus and $\partial S(2)$ is a torus. The first three cases lead to $M = \mathbf{B}$, $M = \mathbf{L} \sharp \mathbf{B}$, $\mathbf{L}_1 \sharp \mathbf{L}_2 \sharp B$, or $\mathbf{L}_1 \sharp \mathbf{L}_2 \sharp \mathbf{L}_3 \sharp \mathbf{B}$. In the last case M is otained by a Dehn filling on the Seifert fiber space $S(2)$ and it follows that $M = S(3) \sharp \mathbf{B}$ (if a fiber of $\partial S(2)$ is not homotopic to a meridian of ∂V_3) or $M = L_1 \sharp L_2 \sharp L_3 \sharp \mathbf{B}$ (compare Proposition 2 of [H]).

Cases (1), (2), (3), and (4) correspond to Propositions 1, 2, 3 and Theorem 1 of [G].

More examples:

(a) If M has trigenus $(0,0,n)$ then $M = B_1^3 \cup B_2^3 \cup V_3$, for a handlebody V_3 of genus n and $B_1^3 \cup B_2^3 = \mathbf{B} \sharp W$ from Table 1 (special type $[0,0]$), for some handlebody W. Hence $M = \mathbf{B} \sharp M_n$, where M_n is any orientable 3-manifold of Heegaard genus at most n.

(b) If M has trigenus $(0,1,n)$ then $M = (V_1 \cup B_2^3) \cup V_3$, for a solid torus V_1 and a handlebody V_3 of genus n. From Table 1 (special type $[0,1]$) we find $V_2 \cup B_2^3 = B \sharp W$ or $L \sharp \mathbf{B} \sharp W$, for some handlebody W of genus n. Hence $M = \mathbf{B} \sharp M_n$ or $\mathbf{B} \sharp \mathbf{L} \sharp M_n$, the latter being of type $\mathbf{B} \sharp M_{n+1}$.

Cases (a) and (b) correspond to Examples (1.4) and (1.6) of [GHN].

(c) If M has trigenus $(1,1,n)$ then $M = (V_1 \cup V_2) \cup V_3$ for solid tori V_1, V_2 and a handlebody V_3 of genus n such that $V_1 \cup V_2$ is as in Table 1 (special type $[1,1]$) and

$\partial(V_1 \cup V_2)$ is a surface of genus n. The cases $V_1 \cup V_2 = \mathbf{B}\sharp W$, $\mathbf{L}\sharp\mathbf{B}\sharp W$, $\mathbf{L}_1\sharp\mathbf{L}_2\sharp\mathbf{B}\sharp W$ with W a handlebody of genus n lead to $M = \mathbf{B}\sharp M_n$, $\mathbf{B}\sharp\mathbf{L}\sharp M_n$, and $\mathbf{B}\sharp\mathbf{L}_1\sharp\mathbf{L}_2\sharp M_n$. For the case when $V_1 \cup V_2 = W(S(2))\sharp\mathbf{B}$ we need to classify the closed manifolds that are obtained as a union of a Seifert web with node a Seifert fiber space (with at most 2 exceptional fibers) and a handlebody. This seems to be a difficult problem.

5. Arc Unions of Knots in S^3

Consider an embedding G of a dumbbell into S^3 (e.g. as in Figure 1). The ***space of G*** is $\overline{S^3 - N(G)}$ for a regular neighborhood $N(G)$ of G. Note that the space of G does not determine G: Figure 4 shows embeddings G_1 and G_2 whose spaces are both handlebodies of genus 2 but there is no homeomorphism $\varphi : S^3 \to S^3$ with $\varphi(G_1) = \varphi(G_2)$. An embedding G is a ***trivial arc sum*** of two knots K_1, K_2 if $G = K_1 \cup A \cup K_2$, $K_i \subset B_i^3 \subset S^3$ for disjoint 3-balls B_1^3, B_2^3, such that $K_i \cap \partial B_i^3$ is one point P_i and A is an arc with $A \cap B_i^3 = P_i$, $\partial A = P_1 \cup P_2$. Figure 1 shows a trivial arc sum of two torus knots.

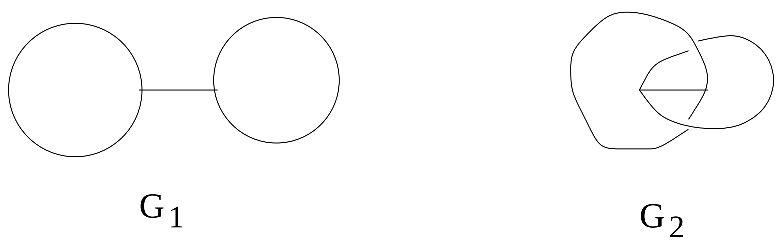

Figure 4

If $M_i = \overline{S^3 - N(K_i)}$ is the knot space of K_i then the space M of the trivial arc sum $G = K_1 \cup A \cup K_2$ is homeomorphic to the disk sum $M_1 \triangle M_2$. This can be seen as follows: represent $S^3 = B_1^3 \cup B_2^3$ as a union of two balls along the boundary sphere $S = \partial B_1^3 = \partial B_2^3$ such that S cuts A into two arcs A_1 , A_2 , and $K_i \cup A_i$ lies in B_i^3 and meets ∂B_i^3 in the point $A_i \cap S$. Note that for a regular neighborhood N_i of $K_i \cup A_i$ in B_i^3 we have $\overline{B_i^3 - N_i} \cong M_i$, and hence M is obtained as $M_1 \underset{D}{\cup} M_2$, where D is the disk $\overline{S - N_i}$.

It is well known that a knot space; i.e., the closure of the complement of a solid torus in S^3, is a Seifert fiber space if and only if the knot is a torus knot [B-Z]. A similar result holds for embeddings of dumbbells:

Proposition 10. *If the closure M of the complement of a handlebody of genus 2 in S^3 is a*

Seifert union of punctured balls and puncturd solid tori, then M is homeomorphic to the space of a trivial arc sum of two torus knots.

Note that M is then of type $W(S(2), S(2))$.

Proof. Since M is irreducible it follows from Theorem 8 that M is a Seifert web or a handlebody. If M is a handlebody (of genus 2) then M is the space of a trivial arc sum of two unknots. If M is a Seifert web then since ∂M is a surface of genus 2 , it follows that M has one or two nodes and exactly one 1-handle.

If M has two nodes then $M = M_1 \cup$ 1-handle $\cup M_2$, where M_i is a Seifert fiber space. Cutting off the 1-handle we see that M_i embeds in S^3 (with ∂M_i a torus) and hence M_i is homeomorphic to $\overline{S^3 - N(K_i)}$, the knot space of a torus knot K_i (possibly the unknot). Hence $M \cong M_1 \triangle M_2$ is homeomorphic to the space of the trivial arc sum $G = K_1 \cup A \cup K_2$.

If M has one node then $M = S(m) \cup$ 1-handle. Cutting off the 1-handle we see that $S(m) \subset S^3$. If $\partial S(m)$ is connected then $S(m) = S(2) = \overline{S^3 - N(K_1)}$ for some torus knot K_1 and M is homeomorphic to the space of a trivial arc sum $G = K_1 \cup A \cup K_2$, where K_2 is the unknot. The case that $\partial S(m)$ has two components cannot occur. Otherwise we construct a simple closed curve from an arc in $S(m)$ and an arc in the 1-handle that intersects each torus component of $\partial S(m)$ in exactly one point. This is a contradiction since every torus in S^3 separates. $\square$

In a future paper we will list the complements of links of handlebodies in S^3 that are Seifert unions of punctured balls and punctured solid tori.

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