Triaxial Galaxy Models With Separable Potentials

Balša Terzić[1,2] and Christopher Hunter

Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510

P. Tim de Zeeuw

Sterrewacht Leiden, 2300 RA Leiden, The Netherlands

Abstract. We present a numerical scheme for the construction of three-integral phase-space distribution functions (DF) for triaxial galaxy models with a gravitational potential of Stäckel form, and an arbitrary triaxial luminous density distribution. The intrinsic velocity moments are obtained with little extra effort. The orbits all have three independent integrals of motion and form four different families, the inner and outer long axis tubes, short axis tubes and boxes.

Our models are not unique. Our method of construction begins with a choice of how to split the density between the inner and outer long axis tubes on a plane in which these are the only orbital families. Also, the dependence of the distribution function of each of the tube families on the radial action is chosen in advance. Once these choices have been made, no other arbitrariness remains.

The numerical scheme presented takes the chosen density, computes the thin-tube distribution functions (maximum streaming) and 'thickens' them according to their prescribed radial dependence. The numerical scheme represents a triaxial generalization of the scheme implemented by Robijn & de Zeeuw for oblate models.

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[1] Visiting Researcher, Sterrewacht Leiden

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1. Introduction

Giant elliptical galaxies have triaxial shapes, and are supported by anisotropic velocity distributions. Their luminosity profiles have cores or at most very shallow central cusps. The observed velocity and dispersion fields show a rich variety of structure (see for example the recent `SAURON` maps).

Few dynamical models have been constructed for these systems. One approach is Schwarzschild's numerical orbit superposition technique. Here we follow an alternative route, and construct general distribution functions for the special triaxial systems in which the motion is separable.

While the separable models are not applicable to the full set of elliptical galaxies, they do provide the most general set of inhomogeneous triaxial models which can be studied semi-analytically, and hence we expect to gain insight in the structure of the more general systems.

2. Triaxial Stäckel Models

The main reason separable potentials of Stäckel form are of interest is that they are fully integrable and the equations of motion separate in ellipsoidal coordinates (λ, μ, ν), defined as the roots for τ of

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1, \tag{1}$$

where (x, y, z) are Cartesian coordinates and α, β and γ are constants determining the triaxial shape of the model. We adopt the convention $0 \le -\gamma \le \nu \le -\beta \le \mu \le -\alpha \le \lambda$. The coordinate surfaces for λ are ellipsoids, for μ they are one-sheeted hyperboloids and for ν they are two-sheeted hyperboloids. There is a set of foci for each of these surfaces, three in total: $z = \pm\sqrt{\gamma - \beta}$, $z = \pm\sqrt{\gamma - \alpha}$ and $y = \pm\sqrt{\beta - \alpha}$. A potential is of Stäckel form if it can be written as

$$V = \frac{U(\lambda)}{(\lambda-\mu)(\lambda-\nu)} + \frac{U(\mu)}{(\mu-\nu)(\mu-\lambda)} + \frac{U(\nu)}{(\nu-\mu)(\nu-\lambda)} \tag{2}$$

where

$$U(\tau) = -(\tau+\alpha)(\tau+\gamma)G(\tau), \tag{3}$$

with $G(\tau)$ an arbitrary smooth function. The corresponding density is computed using Kuzmin's formula (Kuzmin 1973, de Zeeuw 1985) or from Poisson's equation.

2.1. Orbits and Integrals of Motion

The momenta conjugate to the coordinates λ, μ and ν are

$$p_\lambda = P^2\dot{\lambda}, \qquad p_\mu = Q^2\dot{\mu}, \qquad p_\nu = R^2\dot{\nu}, \tag{4}$$

with metric coefficients defined as

$$\begin{aligned} P^2 &= \frac{(\lambda-\mu)(\lambda-\nu)}{4(\lambda+\alpha)(\lambda+\beta)(\lambda+\gamma)}, \\ Q^2 &= \frac{(\mu-\lambda)(\mu-\nu)}{4(\mu+\alpha)(\mu+\beta)(\mu+\gamma)}, \\ R^2 &= \frac{(\nu-\lambda)(\nu-\mu)}{4(\nu+\alpha)(\nu+\beta)(\nu+\gamma)}. \end{aligned} \tag{5}$$

The separated equations of motion are

$$p_\tau{}^2 = \frac{B(\tau)}{2(\tau+\alpha)(\tau+\beta)(\tau+\gamma)}, \qquad \tau = \lambda, \mu, \nu, \tag{6}$$

where

$$B(\tau) = (\tau+\alpha)(\tau+\gamma)H - (\tau+\gamma)I_2 - (\tau+\alpha)I_3 - U(\tau), \tag{7}$$

and H, I_2 and I_3 are the three isolating integrals of motion admitted by the potential V. H is the Hamiltonian, with its value equaling the total orbital energy E, while I_2 and I_3 are nonclassical integrals, quadratic in the velocities (de Zeeuw & Lynden-Bell 1985).

The motion separates into an *oscillation* between *turning points* defined by $p_\tau^2 = 0$ or a *rotation* if $p_\tau{}^2 > 0$ for all values of τ ($\tau = \lambda, \mu, \nu$). The number of turning points in each of the coordinates depends on the values of the three integrals H, I_2 and I_3. The four different combinations of oscillation(s) and rotation(s) possible define the four bound orbits allowed: inner long axis tubes (I-tubes), outer long axis tubes (O-tubes), short axis tubes (S-tubes) and boxes (de Zeeuw 1985, Hunter 1995).

2.2. Fundamental Integral Equation

The fundamental integral equation for triaxial Stäckel models is given by

$$\rho(\lambda,\mu,\nu) = \int\int\int f(H,I_2,I_3)\, \rho_{\rm orb}(\lambda,\mu,\nu;H,I_2,I_3)\, dH\, dI_2\, dI_3, \tag{8}$$

where $\rho_{\rm orb}(\lambda,\mu,\nu;H,I_2,I_3)$ is the density of an individual orbit with integrals H, I_2 and I_3 and is defined as

$$\rho_{\rm orb}(\lambda,\mu,\nu;H,I_2,I_3) \equiv \left|\frac{\partial(p_\lambda,p_\mu,p_\nu)}{\partial(H,I_2,I_3)}\right| = \frac{2\sqrt{2}(\gamma-\alpha)}{\sqrt{B(\lambda)}\sqrt{-B(\mu)}\sqrt{B(\nu)}}. \tag{9}$$

The integration in equation (8) is over the values of H, I_2 and I_3 for which the square roots in equation (9) are nonnegative.

We find it useful to work with the DF as a function of the turning points (τ_1,τ_2,τ_3) instead of the integrals of motion (E,I_2,I_3). The fundamental integral equation then becomes

$$\rho(\lambda,\mu,\nu) = \int\int\int F(\tau_1,\tau_2,\tau_3)\ \rho_{\rm orb}(\lambda,\mu,\nu;\tau_1,\tau_2,\tau_3)\ d\tau_1\ d\tau_2\ d\tau_3 \tag{10}$$

where

$$F(\tau_1,\tau_2,\tau_3) = \left|\frac{\partial(H,I_2,I_3)}{\partial(\tau_1,\tau_2,\tau_3)}\right| f(H(\tau_1,\tau_2,\tau_3),I_2(\tau_1,\tau_2,\tau_3),I_3(\tau_1,\tau_2,\tau_3)), \tag{11}$$

and

$$\left|\frac{\partial(H,I_2,I_3)}{\partial(\tau_1,\tau_2,\tau_3)}\right| = \frac{(\tau_3-\tau_1)(\tau_3-\tau_2)(\tau_2-\tau_1)}{(\gamma-\alpha)}\ h(\tau_1,\tau_2,\tau_3), \tag{12}$$

with the positive function h defined as

$$h(\tau_1,\tau_2,\tau_3) = U[\tau_1,\tau_1,\tau_2,\tau_3]\ U[\tau_1,\tau_2,\tau_2,\tau_3]\ U[\tau_1,\tau_2,\tau_3,\tau_3] \tag{13}$$

Square brackets denote the *divided differences* of $U(\tau)$ given in (3), defined as

$$U[\tau_1,\tau_2] = \frac{U(\tau_1)-U(\tau_2)}{\tau_1-\tau_2}, \tag{14}$$

and the higher-order differences are recursively given by

$$U[\tau_1,\tau_2,...,\tau_n] = \frac{U[\tau_1,\tau_3,...,\tau_n]-U[\tau_2,\tau_3,...,\tau_n]}{\tau_1-\tau_2}. \tag{15}$$

2.3. Distribution Functions

In case of thin tubes, the DF can be written as a delta function in the radial action, which leads to the fundamental integral equation (10) of Abel type. Hunter & de Zeeuw 1992 (HZ92 hereafter) solved this Abel type equation for the thin tube DF for each of the orbital families (equations 3.55, 3.57, 3.59). They are a single quadrature of a function involving the density of the family of orbits on one of the two-dimensional (2D) surfaces ($\nu = -\beta$ for I- and O-tubes and $\mu = -\alpha$ for S-tubes). This is not surprising, as the thin tube orbits are intrinsically 2D.

The box orbit DF is found by solving the Volterra integral equation (2.33) of HZ92, via a method that is a generalization to 3D of the one developed earlier for the elliptic disk (de Zeeuw, Hunter & Schwarzschild 1987, HZ92).

3. Description of the Method

Any physical DF in our models must populate orbits of all four families (HZ92). The special surfaces $\nu = -\beta$ and $\mu = -\alpha$ are reached by some orbital families only in extreme cases. As a result, density of S-tubes and boxes, ρ_s and ρ_b, must vanish on $\nu = -\beta$, while the density of I-tubes and boxes, ρ_i and ρ_b, vanish on $\mu = -\alpha$ (de Zeeuw 1985, HZ92).

Our goal is to construct a DF f which populates not only the thin tube orbits with zero radial action, but also the ones with non-vanishing radial motion. Without loss of generality, a DF can be written as

$$f = f_i + f_o + f_s + f_b, \tag{16}$$

where the subscripts denote each of the four orbital families (I-tubes, O-tubes, S-tubes and boxes). The arguments are $(\lambda, \mu, \nu; p_\lambda, p_\mu, p_\nu)$, or, by Jeans theorem, (E, I_1, I_2), or the relevant turning points (τ_0, τ_1, τ_2).

First, we prescribe the total mass density $\rho(\lambda, \mu, \nu)$ of the model. Next, we split the densities of I- and O-tubes on the $\nu = -\beta$ surface. This gives the densities $\rho_i(\lambda, \mu, -\beta)$ and $\rho_o(\lambda, \mu, -\beta)$, which via equations (3.55) and (3.57) of HZ92 yield $f^0_{ti}(\lambda_0, \mu_0)$ and $f^0_{to}(\mu_0, \lambda_0)$, thin DF for I- and O-tubes which need to be thickened. We do this through the iteration defined in the following section. These iterations can be done in parallel, since they are mutually independent. After the thickening iteration converges, we have a complete thickened DF f_i and f_o. The densities ρ_i and ρ_o in all of 3D space are obtained from their densities on $\nu = -\beta$ surface via equations (3.16) and (3.29) of (HZ92).

The next step is to compute $\rho_s = \rho - \rho_o$ on the $\mu = -\alpha$ surface. This will yield the thin orbit DF $f^0_{ts}(\nu_0, \lambda_0)$ through (3.59) of HZ92. Again, we apply the thickening scheme described below to obtain the thickened DF f_s. The density ρ_s in all of 3D space are obtained from it's density on $\mu = -\alpha$ surface via equations (3.49) of (HZ92).

Finally, after all the tube orbits have been thickened, we simply subtract their combined densities to compute the density provided by the box orbits:

$$\rho_b = \rho - \rho_i - \rho_o - \rho_s \tag{17}$$

from which we can then compute the box DF by solving the Volterra integral equation (HZ92).

3.1. Iterative Scheme

The iterative scheme whereby the thick orbit DF is obtained from the thin orbit one is conceptually the same for each of the three tube orbits, so we keep the analysis general.

Let us define for each of the tube orbits the following

ε - radial thickness,
${f_t}^k(\tau_0, \tau_m)$ - thin orbit DF corresponding to ρ^k,
$\tilde{g}(\tau_0, \tau_m, \varepsilon)$ - dependence of DF on the radial thickness,
$f(\tau_0, \tau_m, \varepsilon)$ - thick orbit DF,
$\rho^0 \equiv \rho$ - initial tube mass density distribution,
$f^k(\tau_0, \tau_m, \varepsilon)$ - k^{th} order approximation of f,

where τ_0 and τ_m are the turning points of the thin tube family, and τ_0, $\tau_m - \varepsilon$ and $\tau_m + \varepsilon$ are the turning points of the thick tube family.

Following (Robijn & de Zeeuw 1996), we choose the g-function to be a power-law function in radial thickness (their eq. 3.6). When the function $\tilde{g}$ is sharply peaked, the thick orbit solution f is well approximated by the thin orbit DF f_t. We take the zeroth order DF approximation as

$$f^o(\tau_0, \tau_m, \varepsilon) = {f_t}^o(\tau_0, \tau_m)\tilde{g}(\tau_0, \tau_m, \varepsilon) \tag{18}$$

This approximation gives rise to a residual mass density ρ^1, which is, in a way, a measure of how close the zeroth order solution is to the exact one:

$$\rho^1 = \rho^0 - \int\int\int {f_t}^0 \tilde{g}\, d^3v. \tag{19}$$

The DF is now given by

$$f = f^0 + f_R, \tag{20}$$

where

$$\rho^1 = \int\int\int f_R\, d^3v. \tag{21}$$

f_R is obtained via quadratures given in equations 3.55, 3.57, 3.59 of HZ92 upon substituting ρ^1 for the appropriate mass density.

This scheme can be repeated until the desired accuracy is achieved, and finally takes on the form:

$$\rho^{k+1} = \rho^k - \int\int\int {f_t}^k \tilde{g}\; d^3v, \tag{22}$$

$$f = \sum_{k=0}^{N} f^k. \tag{23}$$

It is important to note that whereas the problem is intrinsically a 3D one, the quadratures are evaluated and convergence sought on 2D surfaces only ($\nu = -\beta$ and $\mu = -\alpha$).

4. Conclusion

We have presented a method to compute the DF for galaxy models with gravitational potential in separable form. The numerical method presented is assured to produce a self-consistent model if the convergence is achieved, and the numerically obtained DF is everywhere non-negative.

Velocity moments are easily obtained once the DF is known. This creates an opportunity to compare our model with the observables. We use this scheme to model giant elliptical galaxies, including the ones with counter-rotating cores.

Our method is not dependent on any particular potential, as long as it is separable. A more extensive study of various potentials and properties of these models is in preparation.

References

Kuzmin, G. G. 1973, in Proc. All-Union Conf. Dynamics of Galaxies and Clusters, ed. T. B. Omarov (Alma Ata: Akad. Nauk Kazakhskoj SSR), 71 (English transl. in IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, ed. P.T. de Zeeuw, Dordrecht: Reidel, 553)

Hunter, C. 1995, Ann. New York Acad. Sci., 751, 76

Hunter, C. & de Zeeuw, P.T. 1992, ApJ, 389, 79

Robijn, F.H.A. & de Zeeuw, P.T. 1996, MNRAS, 279, 673

de Zeeuw, P.T. 1985, MNRAS, 216, 273

de Zeeuw, P.T., Hunter, C. & Schwarzschild, M. 1987, ApJ, 317, 607

de Zeeuw, P.T. & Lynden-Bell, D. 1985, MNRAS, 215, 713