# Constructing stellar dynamic models for elliptical galaxies

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**Abstract.** We review methods for constructing steady-state stellar dynamic models, both axisymmetric and triaxial. Models need to be capable of matching the increasing amount of kinematic and other observational data that is now becoming available. To date, several elliptical galaxies have been well fitted by axisymmetric models. Some of these are galaxies which are believed to have central black holes, because of the interest in providing constraints on the masses of those black holes.

Methods are available for constructing triaxial galaxy models with smooth cores at their centers, though few models of real galaxies have been constructed. More effort has been directed at the more difficult problem of constructing triaxial models with central cusps. Many of these models incorporate stochastic orbits, and hence are not truly steady.

Spectral methods provide a means of greatly improving the accuracy which Schwarzschild's orbit superposition method can attain for galaxy models composed of regular orbits.

## 1. Introduction

All stellar dynamic models are composed of orbits. Orbits range from the wholly regular to the very stochastic. They are confined in phase space by any number from one to three of isolating integrals of motion; integrals which may be known exactly, or approximately, or only from numerical exploration. The Strong Jeans Theorem (Binney & Tremaine 1987, chapter 4) is assumed in model building. It requires that the distribution function (DF) $f$ of a steady state galaxy, in which almost

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all orbits are regular with incommensurable frequencies, is a function only of three independent isolating integrals.

Binney & Tremaine identify two basic approaches to constructing that distribution function: that from $f$ to the spatial density $\rho$, and that from $\rho$ to $f$. Unlike modelers of globular clusters, who have confidence in their knowledge of the correct form of $f$, galactic modelers normally adopt the second basic approach. From $\rho$ to $f$ is no longer an adequate description of it, because more than $\rho$ is now used to constrain $f$. For example, the gravitational potential $\Phi$, which plays an essential role in determining the orbital structure, may include include a contribution from dark as well as the visible matter, and $f$ may be required to fit kinematic as well as density data.

A central and still unresolved issue for the past quarter century is that of whether or not elliptical galaxies are triaxial. Because elliptical galaxies are viewed always in projection, there is no simple way of resolving from observations what their true intrinsic shapes are. Though the problem of intrinsic shape has been around since Hubble's (1926) day, it was assumed that ellipticals are primarily axisymmetric, and the problem did not attract much attention from dynamicists for the next fifty or so years. But Bertola & Capaccioli (1975) found an elliptical galaxy (NGC 4697) which rotates too slowly to account for its flattening, and many other observations confirmed that this case is not unusual. That aroused interest, and led Binney (1976) to argue that the flattening of ellipticals is due to the anistropic "pressure" of stellar motions. Given this mechanism, ellipticals could as well be triaxial as axisymmetric. There is of course observational evidence for triaxiality, such as twisted isophotes and minor axis rotation.

A major landmark in the theory was Schwarzschild's (1979) success in constructing a stellar dynamical model of a triaxial galaxy. His paper is remarkable as well for the path-breaking methods that it introduced, and which continue in use to this day, as for its results. Schwarzschild used a mass model of a modified Hubble profile with no known analytical isolating integral other than the energy $E$ to confine the orbits. His numerics nevertheless found that the motions of most orbits are regular. Lacking sufficient isolating integrals to characterize the constraints on orbits, he computed a large library of orbits on which to draw for his model construction. This library recorded the orbital density of each orbit, that is the density distribution in physical space that it produces as a result of its motion in phase space. Specifically this orbital density was recorded in terms of relative weights in the cells into which he binned the mass model. The model is then constructed by finding some set of orbits whose combination reproduces the same set of masses in the cells as that of the mass model.

Triaxial model building got a further boost from de Zeeuw & Lynden-Bell's (de Zeeuw 1985, de Zeeuw & Lynden-Bell 1985) rediscovery of the classical Stäckel potentials (Eddington 1915, Lynden-Bell 1962, Kuzmin 1973). They realized that Stäckel potentials provide analytical integrals of the type needed to account for the regular orbits of the library which Schwarzschild had generated numerically.

But the issue of the possibility of triaxial galaxies has not been resolved so simply. Gerhard and Binney (1985) showed that central black hole and central density cusps tend to disrupt box orbits, and hence destroy triaxiality. That is because box orbits play a crucial role in maintaining the triaxiality in models such as those of Schwarzschild (1979, 1982) and later Statler (1987), which all have smooth central cores. The repercussions of this paper are still being felt. Though central black holes may have been somewhat speculative in 1985, they are no longer (Kormendy and Richstone 1995), and the increasingly accurate observations of the Hubble Space Telescope (Lauer et al. 1995, Gebhardt et al. 1996) has confirmed how common central density cusps are. Schwarzschild (1993) and Merritt & Fridman (1996) have constructed "quasi-equilibrium" self-consistent models for triaxial models with central cusps, but these require the use of stochastic orbits which evolve in time. That leaves open the question of whether a galaxy with a central cusp can remain triaxial systems, or whether the central cusp will drive it to axisymmetry.

## 2. Steady Axisymmetric Models

Orbits in an axisymmetric potential are confined by the two classical integrals of the energy $E$, and the angular momentum $L_z$ about the $z$–axis of symmetry. While an explicit formula for a third integral $I_3$ is known only for special potentials such as Stäckel ones (Lynden-Bell 1962), there is abundant numerical evidence from ordered surfaces of section that it exists in many other cases too.

### 2.1. Two-integral models

A distribution function $f(E, L_z)$ which depends only on the two classical integrals is dynamically possible, and indeed sufficiently general to fit a given axisymmetric density $\rho(R, z)$, where $R$ measures distance from the axis of symmetry. That density restricts only the component $f_{\rm even}(E, L_z)$ of the DF that is even in $L_z$. The component $f_{\rm odd}(E, L_z)$ that is odd in $L_z$ can be constructed in a similar manner to fit a given distribution of the mean streaming velocity $\overline{v_\phi}$ around the axis of symmetry. The analytical methods for doing this (Fricke

1952, Dejonghe 1986, Hunter & Qian 1993) require some analytical representation, possibly implicit, of the density $\rho(\Phi, R^2)$. Dehnen & Gerhard (1994) have given an alternative numerical method based on double expansion in Chebyshev polynomials of $E$ and the circularity parameter $\eta = L_z/L_{\rm circ}(E)$, where $L_{\rm circ}(E)$ is the maximum value of $L_z$ possible for given $E$.

It is an immediate consequence of the assumed form of the DF that these two-integral models are meridionally isotropic; that is the two components $\overline{v_R^2}$ and $\overline{v_z^2}$ of mean-squared velocity are necessarily equal. The models themselves show that departure from sphericity induces a strong dependence of $f$ on $L_z$. The DF increases strongly with $L_z$ for oblate models even at moderate flattening, while it decreases strongly with $L_z$ for models which are only moderately prolate. Consequently prolate models of more than moderately prolateness are unphysical because $f$ becomes negative for some nearly circular orbits. Dehnen & Gerhard (1994) found that models which are isotropic in the velocity dispersions have velocity profiles of the same form as real galaxies, but that their ratios of maximum rotation velocity to dispersion are too high.

Both Qian et al (1995) and Dehnen (1995) constructed two-integral models with spheroidal density contours and a central black hole which fitted well the then-available surface brightness and line-of-sight velocity profiles for $M32$. The DF determined by Qian et al. is close to a simple separable product of a function of $E$ with one of $\eta$, though this form was not required.

### 2.2. Three-integral models

Binney, Davies & Illingworth (1990) showed that the ratio of line-of-sight velocity dispersion along the minor axis to that along the major axis is a sensitive diagnostic of the importance of a third integral, and that observations of NGC 720 and 4697 require a three-integral DF $f(E, L_z, I_3)$. Three approaches have been used to construct that third integral $I_3$.

The first approach is to use Stäckel models which have an analytical $I_3$. de Zeeuw & Lynden-Bell (1985) and Dejonghe & de Zeeuw (1988) discuss how to construct a best-fitting Stäckel potential. Dejonghe et al. (1996) constructed both two and three-integral Stäckel models for NGC 4697, and found unsurprisingly that the extra flexibility of the three-integral models allowed them to achieve a better fit to the peculiarities in velocity and dispersion profiles. Emsellem, Dejonghe, & Bacon (1999) constructed both two and three-integral Stäckel models for NGC 3115. Though two-integral models fit velocity and dispersion profiles well within 45 arcsec, a three-integral model is again needed for a more precise fit to the line-of-sight velocity distribution.

The second approach is that of Dehnen & Gerhard (1993), who showed how to construct an approximate $I_3$ based on resonant pertubation theory. Such an $I_3$ was used by Matthias & Gerhard (1999) to construct a model for NGC 1600. They express the DF in a separable form as $f = \sum_i \sum_j C_{ij} g_i(E) h_j(L_z, I_3)$, and selected the constants $C_{ij}$ to fit surface brightness, velocity dispersion, and line profile data. They find that no two-integral DF can reproduce the kinematic data, but that a radially anisotropic model flattened by equatorial radial orbits can. Because the problem of inferring the DF from projected data is ill-conditioned, Matthias & Gerhard use a regularization method to limit the sensitivity of their solution to small uncertainties in the observational data, and force it to be smooth.

The third approach is that of incorporating $I_3$ implicity using Schwarzschild's method. van der Marel et al. (1998) used this to construct models for M32 which fit the more recent and more accurate data better than the earlier two-integral one of Qian et al (1995). They fit the observed photometry and kinematics, and allow for effects induced by seeing convolution and pixel binning. The best-fitting models are not two-integral ones, though it is possible that the true DF may be close to two-integral form. (Merritt (1999) argues that a near two-integral form may be a consequence of evolution from a triaxial form due to black-hole-induced chaos). Other galaxies which have been modeled in a similar manner are NGC 4342 (Cretton & van den Bosch 1999) and NGC 2320 (Cretton, Rix & de Zeeuw 2000), and NGC 3379 (Gebhardt et al. 2000). In the latter work, the kinematics was fitted directly, whereas moments of the kinematics were first calculated and then fitted by the other groups. Different methods are used for regularization. Gebhardt et al. maximize entropy, whereas the other groups incorporate smoothing into their non-negative least squares fitting.

## 3. Triaxial Models

### 3.1. Models with a smooth core

The modified Hubble profile which Schwarzschild (1979, 1982) used for his original pioneering studies have a smooth core. So too does the *perfect* ellipsoid (de Zeeuw & Lynden-Bell 1985) for which Statler (1987) constructed a large number of stellar dynamic models. It has a Stäckel potential with known analytical integrals $I_2$ and $I_3$ for use in a three–integral DF $f(E, I_2, I_3)$. Statler did not need to construct an orbit library because he had analytical expressions for all the orbital densities. However he did follow Schwarzschild in binning the mass model into cells, and searching for models by matching masses in cells with those of the mass models. He was able to construct many possible three-integral DFs with different kinematics to populate a given $\rho(x, y, z)$ because of the variety of orbit types.

Only recently has a three-integral DF with a triaxial Stäckel potential been constructed to fit real observational data. Mathieu & Dejonghe (1999) have done this for the E2 galaxy Centaurus A. Their model fits the photometry and the rotation and velocity dispersion on the photometric major and minor axes. It is built using continuous components for the phase space density, and a quadratic programming technique to achieve the fit.

### 3.2. Models with a central cusp

In disrupting box orbits, central black holes and density cusps greatly increase the prevalence of irregular and stochastic orbits. Schwarzschild (1979) actually had found some stochastic orbits in his original smoothly cored model, though he chose not to use them in his model construction. They become increasingly hard to ignore as they become more prevalent, and it becomes increasingly hard to justify ignoring them. It was necessary for Schwarzschild (1993) to incorporate stochastic orbits in his construction of self-consistent models for a triaxial galactic halo with a logarithmically singular potential. Consequently, his models evolve in time, and so are not truly steady. Schwarzschild argued that this may not matter greatly for a halo where only about 50 orbital periods occur in a Hubble time.

Other workers have run into similar difficulties. Merritt & Fridman (1996) sought self-consistent models for maximally triaxial Dehnen-type (1993) models with central cusps. These models have spatial densities given by the formula

$$\rho(m) = \frac{(3-\gamma)M}{4\pi abc} m^{-\gamma}(1+m)^{\gamma-4}, \tag{1}$$

where $M$ is the total mass, and $m$ is an ellipsoidal variable with semi-axes $a$, $b$, and $c$ given by

$$m = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}. \tag{2}$$

They are cuspy for $\gamma > 0$ because the density is then singular as $m^{-\gamma}$ near the center. Merritt and Fridman too found quasi-equilibrium solutions with stochastic orbits. They also sought fully mixed models, as are described by a distribution function of the form

$$f = \begin{cases} f(E, I_2, I_3), & \text{regular phase space,} \\ f(E), & \text{stochastic phase space.} \end{cases} \tag{3}$$

In this they were successful in the weak cusp ($\gamma = 1$) case, but not when the cusp is strong ($\gamma = 2$). The weak cusp case was re-examined by Siopis (1999) with more orbits and longer integration times. He was able to achieve a self-consistent fully mixed model for the inner 13 of his 20 shells. Merritt (1997) constructed some strongly cusped triaxial models using only regular orbits. He was successful only for a limited range of nearly spheroidal systems. Merritt & Quinlan (1998) showed that a massive central black hole, whose mass exceeds 2.5% that of the galaxy will cause that galaxy to evolve rapidly to axisymmetry. Hence the extent to which triaxiality can persist in the presence of black holes and central cusps is uncertain. Merritt & Fridman may have set themselves a particularly difficult task in seeking to construct models that are (a) maximally triaxial ($a^2 - b^2 = b^2 - c^2$), and (b) have the same degree of triaxiality at all radii. Triaxiality is more likely to be maintained, or at least to persist longer, in the outer regions of galaxies with central singularities.

### 3.3. Singular centers

The need for greater understanding of just how central cusps and black holes modify the dynamics is demonstrated by some recent findings which rule out simplistic explanations. Sridhar & Touma (1997a,b) have shown that central cusps do not necessarily cause chaos. They have found examples of Stäckel potentials in the limit of parabolic or paraboloidal coordinates which are singular and have weak cusps ($0 < \gamma < 1$ for the axisymmetric case) and possibly also a central black hole. Nevertheless the motion in these cuspy potentials is integrable and entirely regular. Chaos is not just a matter of triaxiality. A central cusp can cause chaos even in an axisymmetric potential. This is seen for example in the zero angular momentum surface of section of Fig. 4(a) of Evans (1994) for a strong $\gamma = 2.18$ cusp. Lastly, there can be a lot of order near a central black hole because the superintegrability of the Kepler problem predominates inside a sphere of influence of that black hole (Sridhar & Touma 1999).

## 4. The need for improved methods

Kuijken (1993) used Schwarzschild's method to construct DFs for scale-free elliptical disks with flat rotation curves. His results for this relatively simple problem raises concerns with respect to the accuracy of other implementations of that method, particularly when there is a central singularity. Kuijken had a mixture of regular and chaotic orbits. He found that his success or failure to build a model depended on the fineness of the grid of cells he used for binning the density. Success can turn to failure when a coarser grid is replaced by a finer one. This raises the possibility that no models survive in the limit of an arbitrarily fine grid. For the somewhat similar problem of the flat cuspy potential of Sridhar & Touma (1997a), Syer and Zhao (1998) indeed showed, analytically for larger $\gamma$ and numerically for smaller $\gamma$, that it is impossible to construct self-consistent stellar dynamical models.

It is fortunate and timely that Copin, Zhao & de Zeeuw (2000) have shown how to improve greatly the accuracy with which the orbital densities of regular orbits are computed, and to improve substantially on the binning methods that have served for so long. Their method is based on spectral analysis.

### 4.1. Spectral analysis

Fourier analysis of orbits in galactic-type potentials was pioneered by Binney and Spergel (1982,1984). J. Laskar has developed very accurate numerical methods of Fourier analysis which he has named NAFF (Numerical Analysis of Fundamental Frequencies), originally for solar system problems, but then applied by himself and others to triaxial galactic-type potentials (Papaphilippou & Laskar 1996, 1998; Carpintero & Aguilar 1998, Valluri & Merritt 1998, Wachlin & Ferraz-Mello 1998, Merritt & Valluri 1999).

Spectral analysis requires that an orbit be computed long enough to determine a Fourier representation of its position vector

$$\mathbf{x}(t) = \Sigma_{\mathbf{k}} \mathbf{X}_{\mathbf{k}} e^{i(\mathbf{k}.\nu)t}. \tag{4}$$

Here $\nu = (\nu_1, \nu_2, \nu_3)$ is the 3–vector of the three fundamental frequencies of the orbit as it describes its quasi-periodic motion on a 3–torus in phase space, $\mathbf{k} = (\ell, m, n)$ is a 3–vector with integer components, and the Fourier coefficients $\mathbf{X}_{\mathbf{k}}$ are functions of the actions $\mathbf{I}$. The phase space variables of position and velocity are related to the action-angle variables $(\mathbf{I}, \phi)$ via the equations

$$\mathbf{x} = \Sigma_{\mathbf{k}} \mathbf{X}_{\mathbf{k}} e^{i\mathbf{k}.\phi}, \quad \mathbf{v} = \Sigma_{\mathbf{k}} i(\mathbf{k}.\nu) \mathbf{X}_{\mathbf{k}} e^{i\mathbf{k}.\phi}. \tag{5}$$

Copin et al. show that the orbital density of the orbit so described, when normalized to unit total mass, is given simply in terms of the

Fourier expansion as

$$\rho_{\rm orb}(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum \frac{1}{\partial(x_1,x_2,x_3)/\partial(\phi_1,\phi_2,\phi_3)}. \tag{6}$$

That is, it is related directly to the Jacobian of the transformation between position and angles for that particular orbit (for which $\mathbf{I}$ is fixed). Summation is needed because an orbit passes through most of the positions $\mathbf{x}$ which it reaches in more ways than one, and one must sum over all the ways. Figure 5 of Copin et al. shows how accurately the orbital density of a two-dimensional orbit can be computed in this manner, and far better than any binning method can hope to achieve. Moreover the spectral analysis also gives similar formulas for the velocity profile densities. Though Copin et al. have so far applied their method only to two-dimensional and axisymmetric problems, there is no fundamental reason why they should not be equally effective in the more complicated three-dimensional case.

## 5. Conclusion

Not all elliptical galaxies are axisymmetric, though many may be nearly so. We have good methods for constructing elaborate two and three–integral axisymmetric stellar dynamic models of galaxies with the flexibility to match observations. They can be improved considerably with the use of spectral analysis.

Further progress on constructing triaxial galaxies with central cusps and/or central black holes awaits further understanding of how those singularities modify the dynamics, and whether steady triaxial states are even possible.

## References

Bertola, F. & Capaccioli, M. 1975, ApJ, 200, 439

Binney, J. 1976, MNRAS, 177, 19

Binney, J., Davies, R. & Illingworth, G. 1990, ApJ, 361,78

Binney, J. & Spergel, D. 1982, ApJ, 252, 308

Binney, J. & Spergel, D. 1984, MNRAS, 206, 159

Binney, J. & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)

Carpintero, D.D. & Aguilar, L.A. 1998, MNRAS, 298,1

Copin, Y., Zhao, H.S. & de Zeeuw, P.T. 2000, MNRAS, 318, 781

Cretton, N., Rix, H-W. & de Zeeuw, P.T. 2000, ApJ, 536, 319
Cretton, N., & van den Bosch, F.C. 1999, ApJ, 514, 704
Dehnen, W. 1993, MNRAS, 265, 250
Dehnen, W. 1995, MNRAS, 274, 919
Dehnen, W. & Gerhard, O.E. 1993, MNRAS, 261, 311
Dehnen, W. & Gerhard, O.E. 1994, MNRAS, 268, 1019
Dejonghe, H., 1986, Phys. Rept, 133, 218
Dejonghe, H. & de Zeeuw, T. 1988, ApJ, 329, 720
Dejonghe, H., De Bruyne, V., Vauterin, P. & Zeilinger, W.W. 1996, A&A, 306, 363
de Zeeuw, P.T. 1985, MNRAS, 216, 273
de Zeeuw, P.T. & Lynden-Bell, D. 1985, MNRAS, 215, 713
Eddington, A.S. 1915, MNRAS, 76, 37
Emsellem, E., Dejonghe, H., & Bacon, R. 1999, MNRAS, 303, 495
Evans, N.W. 1994, MNRAS, 267, 333
Fricke, W. 1952, Astron. Nachr., 280, 193
Gerhard, O.E. & Binney, J. 1985, MNRAS, 216, 467
Gebhardt, K., Richstone, D., Ajhar, E. A., Lauer, T.R., Byun, Y-I., Kormendy, J., Dressler, A., Faber, S. M., Grillmair, C., & Tremaine, S. 1996, AJ, 112, 105
Gebhardt, K., Richstone, D., Kormendy, J., Lauer, T.R., Ajhar, E. A., Bender, R., Dressler, A., Faber, S. M., Grillmair, C., Magorrian, J. & Tremaine, S. 2000, AJ, 119, 1157
Hubble, E. 1926, ApJ, 64, 321
Hunter, C. & Qian, E. 1993, MNRAS, 262, 401
Kormendy, J. & Richstone, D. 1995, ARA&A, 33, 581
Kuijken, K. 1993, ApJ, 409, 68
Kuzmin, G.G. 1973, in Proc. All-Union Conf. Dynamics of Galaxies and Clusters, ed. T. B. Omarov (Alma Ata: Akad. Nauk Kazakhskoj SSR), 71 (English transl. in IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, ed. P.T. de Zeeuw, Dordrecht: Reidel, 553)
Lauer, T.R., Ajhar, E. A., Byun, Y-I., Dressler, A., Faber, S. M., Grillmair, C., Kormendy, J., Richstone, D., & Tremaine, S. 1995, AJ, 110, 2622
Lynden-Bell, D. 1962, MNRAS, 124, 95
Mathieu, A. & Dejonghe, H. 1999, MNRAS, 303, 455
Matthias, M. & Gerhard, O.E. 1999, MNRAS, 310, 879

Merritt, D. 1997, ApJ, 486, 102
Merritt, D. 1999, Comments on Modern Physics, 1, 39 (astro-ph/9803211)
Merritt, D. & Fridman, T. 1996 ApJ, 460, 136
Merritt, D. & Quinlan, G.D. 1998 ApJ, 498, 625
Merritt, D. & Valluri, M. 1999 AJ, 118, 1177
Papaphilippou, Y. & Laskar, J. 1996, A&A, 307, 427
Papaphilippou, Y. & Laskar, J. 1998, A&A, 329, 451
Qian, E.E., van der Marel, R.P., de Zeeuw, P.T. & Hunter, C., 1995, MNRAS, 274, 602
Schwarzschild, M. 1979, ApJ, 232, 236
Schwarzschild, M. 1982, ApJ, 263, 599
Schwarzschild, M. 1993, ApJ, 409, 563
Siopis, C. 1999, Ph. D. dissertation, University of Florida
Sridhar, S. & Touma, J. 1997a, MNRAS, 287, L1
Sridhar, S. & Touma, J. 1997b, MNRAS, 292, 657
Sridhar, S. & Touma, J. 1999, MNRAS, 303, 483
Statler, T.S. 1987, ApJ, 321, 113
Syer, D. & Zhao, H.S. 1998, MNRAS, 296, 407
Valluri, M. & Merritt, D. 1998, ApJ, 506, 686
van der Marel, R.P., Cretton, N., de Zeeuw, P.T. & Rix, H-W. 1998, ApJ, 493, 613
Wachlin, F.C. & Ferraz-Mello, S. 1998, MNRAS, 298, 22