Fixed Points in Boundaries of Negatively Curved Groups

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An important feature of the natural action of a negatively curved (= word hyperbolic) group on its boundary is that the fixed points of hyperbolic (= infinite order) elements form a dense subset of the boundary. Gromov states this in his influential treatise *Hyperbolic Groups* [4, 8.2 D] and his explanation involves Stalling's theorem on groups with infinitely many ends. This short note provides an elementary proof of this fact. The proof relies on a simple enhancement of the Pumping Lemma from the theory of finite state automata and on the fact that the geodesic words with respect to a fixed generating set for a negatively curved group form a regular language. For background material on negatively curved groups see [1] and on automata and regular languages on groups see [2]. For a different proof see [5].

Definition. Let L be a language over the finite alphabet A. An infinite sequence $\{a_i\}$ of letters from A is an ***L-ray*** provided, for every integer $n > 0$, there exists an integer $N \geq n$ such that $a_1 \cdots a_N$ is a word in the language L.

The following lemma, which we term an ***Ultra-Pumping Lemma*** (UPL), is our aforementioned simple enhancement of the Pumping Lemma.

Ultra-Pumping Lemma. *Let $\{a_i\}$ be an L-ray where L is a regular language over the finite alphabet A. There exists an integer $M \geq 0$ such that, for each integer $n > 0$, there exists an integer $N \geq n$ such that $(a_1 \cdots a_M)(a_{M+1} \cdots a_{M+N})^r \in L$ for every integer $r \geq 0$.*

Proof. Let W be a finite state automaton accepting L. The L-ray $\{a_i\}$ describes an infinite path in W and, as there are only finitely many (accept) states in W and infinitely many initial segments of $\{a_i\}$ in L, this path visits some accept state s_∞ infinitely often. Let M be the step at which this path first visits s_∞ and choose $N \geq n$ such that $a_{M+1} \cdots a_{M+N}$ describes a loop in W based at s_∞. $\diamondsuit$

We use the term ***δ-negatively curved*** in the sense of Gromov's δ-hyperbolic condition [4, p. 89]. A group G is δ-negatively curved with respect to a finite generating set $A = A^{-1}$ if the corresponding Cayley graph $\Gamma_A(G)$ with the usual word metric has a δ-hyperbolic inner product based at the identity; see [1, Definition 1.1, p. 8]. A group G is ***negatively curved*** (or ***word hyperbolic***) if it is δ-negatively curved with respect to some finite generating set for some $\delta \geq 0$. For an infinite order element g of the negatively curved group G, let $g^{\pm\infty}$ denote the two fixed points of the action of g on $G \cup \partial G$, where ∂G denotes the boundary of G. Recall that $g^{\pm\infty}$ are elements of ∂G and $\lim g^i = g^{+\infty}$ and $\lim g^{-i} = g^{-\infty}$.

Theorem [4, 8.2D]. *Let G be a negatively curved group with boundary ∂G. The set*

$$\{g^{\pm\infty} : g \in G,\ \mathrm{order}(g) = \infty\}$$

is a dense subset of ∂G.

Proof. Recall that ∂G is metrizable [1] and as such sequences suffice to verify denseness. Let G be δ-negatively curved with respect to the finite generating set $A = A^{-1}$ and let $x \in \partial G$. Choose an infinite geodesic path $a_1 a_2 \cdots$ based at 1 in the Cayley graph $\Gamma_A(G)$ such that $\lim \overline{a_1 \cdots a_i} = x$. Here, $a_1 \cdots a_i$ denotes a word over A and $\overline{a_1 \cdots a_i}$ denotes the element of G obtained by multiplying. The sequence $\{a_i\}$ is an L-ray, where L is the language of geodesic words of G with respect to A. Recall that L is a regular language [2] and apply the UPL to obtain an integer M and a strictly increasing sequence $0 < N(1) < N(2) < \ldots$ of integers such that for all integers $i > 0$ and $r \geq 0$,

$$(a_1 \cdots a_M)(a_{M+1} \cdots a_{M+N(i)})^r \in L.$$

Let $a = a_1 \cdots a_M$, $b_i = a_{M+1} \cdots a_{M+N(i)}$, and $g_i = \overline{a}\,\overline{b}_i\,\overline{a}^{-1}$. Our claim is that the sequence $\{g_i^{+\infty}\}$ converges to x. The point is that since $ab_i^r \in L$ for all integers $r > 0$, the path $ab_ib_i \cdots$, where b_i is repeated infinitely, is a geodesic path from 1 to $g_i^{+\infty}$ that shares the initial segment ab_i of length $M + N(i)$ with the geodesic path $a_1a_2 \cdots$ from 1 to x. Since $N(i) \to \infty$ as $i \to \infty$, this forces the convergence of $\{g_i^{+\infty}\}$ to x. To be precise, the Gromov inner product (extended to the boundary [1]) applied to x and $g_i^{+\infty}$ yields

$$(x.g_i^{+\infty}) \geq M + N(i) - 2\delta;$$

see [1, Lemma 4.6(4), p. 51]. Hence $(x.g_i^{+\infty}) \to \infty$ as $i \to \infty$ and therefore $\{g_i^{+\infty}\}$ converges to x as desired. $\diamondsuit$

Observe that the proof of the theorem yields the following interesting fact: an infinite geodesic path $a_1 a_2 \cdots$ based at 1 may be approximated in $\Gamma_A(G) \cup \partial G$ by traveling along an initial segment $(a_1 \cdots a_M)$ and then infinitely often repeating large segments of the path.

As an application of the theorem, we give a quick proof of a known result.

Corollary. *If H is an infinite normal subgroup of the negatively curved group G, then the limit set $\Lambda(H) = \partial G \cap \mathrm{Cl}_{G \cup \partial G}(H)$ is equal to ∂G.*

Proof. Let $x \in \partial G$ and let U be an open neighborhood of x in ∂G. Let h be an infinite order element of H [3] and choose an infinite order element g of G such that $g^{+\infty} \in U$. Since $g^{+\infty}$ is an attractive fixed point of the action of g on $G \cup \partial G$ (induced by left multiplication), there is an integer $N > 0$ such that $g^N \cdot h^{+\infty} \in U$. Since $\lim(g^N h^i g^{-N}) = g^N \cdot h^{+\infty}$ and H is normal, $g^N \cdot h^{+\infty} \in \Lambda(H)$. Since U is an arbitrary neighborhood of x, we conclude that $x \in \Lambda(H)$. $\diamondsuit$

Finally we apply the UPL and the proof of the theorem to verify that an infinite quasi-geodesic path may be approximated by traveling on an initial segment of the path and then repeating a large subpath infinitely often. To set the situation, let $q_1 q_2 \cdots$ be an infinite quasi-geodesic path based at 1 in the Cayley graph $\Gamma_A(G)$ of the negatively curved group G, where $A = A^{-1}$ and $1 \notin A$. Every such path limits to a unique point on ∂G; let $x = \lim \overline{q_1 \cdots q_i}$. Our aim is to produce an integer $M \geq 0$ and a sequence $0 < N(1) < N(2) < \ldots$ such that, if $q = q_1 \cdots q_M$ and $p_i = q_{M+1} \cdots q_{M+N(i)}$, then $x_i \to x$ as $i \to \infty$ where $x_i = \lim \overline{q}\,\overline{p}_i^r$.

To obtain this, let W be a finite state automaton with state set S and start state s_0 that accepts the language of geodesic words of G with respect to A. For simplicity, assume that all dead states of S have been amalgamated into a single state FAIL. For any integer $K \geq 0$, let $B(K)$ be the closed K-neighborhood of the identity in $\Gamma_A(G)$. We describe a finite state automaton $F(K)$ over the alphabet $A \times (A \cup \{1\})$ as follows. The state set of $F(K)$ is $S \times B(K)$ and the start state is $(s_0, 1)$. If $F(K)$ is in state (s, g) and the letter $(a, b) \in A \times (A \cup \{1\})$ is read, then $F(K)$ goes into state $(t, a^{-1}gb)$, where a takes s to t in W, unless $a^{-1}gb \notin B(K)$, in which case $F(K)$ goes to state $(\mathsf{FAIL}, 1)$. Note that the dead states of $F(K)$ are the states in $\{\mathsf{FAIL}\} \times B(K)$, which may be amalgamated into a single fail state if desired. The accept states of $F(K)$ are states of the form (s, g) where s is an accept state of W and $g \in B(K)$. The language $L(F(K))$ accepted by $F(K)$ consists of strings $(a_1, b_1) \cdots (a_n, b_n)$ for which $a_1 \cdots a_n$ is a geodesic word and the path in $\Gamma_A(G)$ starting at 1 described by the string $b_1 \cdots b_n$ K-fellow travels the geodesic path starting at 1 described by the string $a_1 \cdots a_n$.

Choose an infinite geodesic path $a_1 a_2 \cdots$ from 1 to $x = \lim \overline{q_1 \cdots q_i} \in \partial G$. Apply [1, Proposition 3.3, p. 42] to show that the infinite quasi-geodesic path $q_1 q_2 \cdots$ from 1 to x K-fellow travels this infinite geodesic path $a_1 a_2 \cdots$ from 1 to x, for some integer $K \geq 0$. Consequently, the infinite sequence $\{(a_i, q_i)\}$ of letters from $A \times (A \cup \{1\})$ is an $L(F(K))$-ray and the UPL may be applied to obtain an integer $M \geq 0$ and a sequence $0 < N(1) < N(2) < \ldots$ such that

$$(a_1, q_1) \cdots (a_M, q_M)[(a_{M+1}, q_{M+1}) \cdots (a_{M+N(i)}, q_{M+N(i)})]^r \in L(F(K))$$

for every integer $r \geq 0$. It follows that with $a = a_1 \cdots a_M$, $q = q_1 \cdots q_M$, $b_i = a_{M+1} \cdots a_{M+N(i)}$, and $p_i = q_{M+1} \cdots q_{M+N(i)}$, $\lim \overline{a}\, \overline{b}_i^r = \lim \overline{q}\, \overline{p}_i^r = x_i$. Setting $g_i = \overline{a}\, \overline{b}_i\, \overline{a}^{-1}$, we find that $x_i = \lim \overline{a}\, \overline{b}_i^r = \lim g_i^r = g_i^{+\infty}$, and the proof of the theorem now implies that $\lim x_i = x$.

References

[1] J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short (Editor), *Notes on word hyperbolic groups*, in "Group Theory from a Geometrical Viewpoint" (E. Ghys, A. Haefliger, and A. Verjovsky, Editors), World Scientific, Singapore (1991), 3–63.

[2] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, and W.P. Thurston, "Word Processing in Groups", Jones and Bartlett, Boston (1992).

[3] E. Ghys and P. de la Harpe (Eds.), "Sur les Groupes Hyperboliques d'après Mikhael Gromov", Progress in Math. **83**, Birkhäuser (1990).

[4] M. Gromov, *Hyperbolic groups*, in "Essays in Group Theory" (S.M. Gersten, Editor), Springer-Verlag, MSRI Publ, **8** (1987), 75–263.

[5] E. Swensen, *Negatively curved groups and related topics*, PhD Thesis, Brigham Young University, Provo, Utah, 1993.