Singhof Fillings of Closed 3-Manifolds. *

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ABSTRACT:

We give a complete classification of all closed, connected 3-manifolds which admit a Singhof filling with any number of solid tori.

0. Introduction

Let M^n be a closed, connected n-manifold. The Lusternik-Schnirelman category $\mathrm{cat}(M^n)$ of M^n is the smallest number of sets, open and contractible in M^n, needed to cover M^n. While studying the long standing conjecture (see e.g. [Ga] or [VR]) that $\mathrm{cat}(M^n \times S^1) = \mathrm{cat}(M^n)+1$, W. Singhof ([Si], see also [T] and [M]) introduced the concept of fillings, which in this paper, we will call Singhof fillings. He used these (categorical) fillings to study $\mathrm{cat}(M^3)$, for example he shows (Theorem 5.1 of [Si]) that a closed, connected, orientable 3-manifold M with $H_1(M) \neq 0$ and $H_2(M) = 0$ has $\mathrm{cat}(M) = 4$. Later in [GG] it was shown that $\mathrm{cat}(M^3)$ depends only on $\pi_1(M^3)$: $\mathrm{cat}(M^3) = 2$ iff M is simply connected, $\mathrm{cat}(M^3) = 3$ iff $\pi_1(M)$ is free (non-trivial), and $\mathrm{cat}(M^3) = 4$ otherwise. Then

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in [GG1], Singhof fillings of 3-manifolds were used to study the homological category [F] in dimension 3 (see also [GGT]).

In [GGH], it was pointed out that any closed, connected 3-manifold M^3 has a Singhof filling with three orientable handlebodies. In fact, this observation gave rise to the definition of the tri-genus of M^3 (see [GHN]) and some results about the tri-genus of 3-manifolds have been obtained in [N] and [HNG].

In [G], a classification of all closed 3-manifolds that admit a Singhof filling of three solid tori was obtained (see also [H], Theorem 4]). This result has been used to study the round complexity of 3-manifolds [KS]. The main goal of the present paper is to give a classification of all closed, connected 3-manifolds that admit Singhof fillings of any number of solid tori. The main result is that these manifolds are connected sums of Seifert fiber spaces. Therefore, we see that the class of 3-manifolds that admit a decomposition into solid tori (together with the local information given by the definition of a Singhof filling) is precisely the class of 3-manifolds that are connected sums of Seifert fiber spaces.

1. Definitions and Preliminaries

Throughout this paper we work in the PL-category. Every closed 3-manifold M admits a Singhof filling [Si]; a ***Singhof filling*** of M is a decomposition of M into orientable handlebodies $V_1, \dots, V_m$ such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ for $i \neq j$. If a point P of M meets k different $V_{i_1}, \dots, V_{i_k}$ for some $2 \leq k \leq m$, $P \in V_{i_1} \cap \cdots \cap V_{i_k} = \partial V_{i_1} \cap \cdots \cap \partial V_{i_k}$, then a neighborhood of P is as in Figure 1 (here k=4). In this case we say that P has order k. Note that the set of points in M of order two forms the interior of (orientable) compact surfaces whose boundary components are empty or consist of simple closed curves. All points in one such boundary component are of the same order $k \geq 3$. We call these components C *intersection curves of order* k.

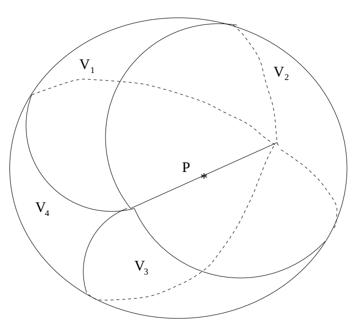

Figure 1

We claim that a regular heighborhood V of C is a solid torus. This can be seen as follows (see Figure 2): V can be obtained from $D^2 \times I$ by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ by a homeomorphism h and $V_{i_j} \cap V$ is obtained from a segment $D^2_{i_j} \times I$ by identifying $D^2_{i_j} \times \{1\}$ with $D^2_{i_j} \times \{0\}$ by h. Since $V_{i_j} \cap V \subset V_{i_j}$ is orientable, it follows that $h : \partial D^2 \times \{0\} \to \partial D^2 \times \{1\}$ is isotopic to the identity.

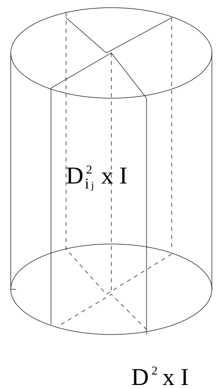

Figure 2

Let $M = V_1 \cup \cdots \cup V_m$ be a Singhof filling for M and let V be a regular neighborhood of an intersection curve $C \subset V_{i_1} \cap \cdots \cap V_{i_m}$ of order $k > 3$. Let $V_i' = V_i$ if $V \cap V_i = \emptyset$ and $V_i' = \overline{V_i - V_i \cap V}$ otherwise. Then V_i' is homeomorphic to V_i and $M = V_1' \cup \cdots \cup V_m' \cup V$ is a Singhof filling with one less intersection curve of order $k > 3$ (see Figure 3).

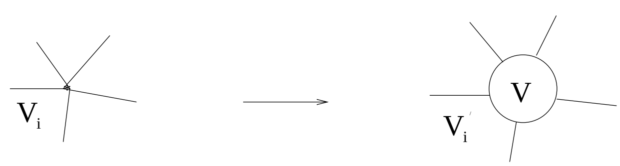

Figure 3

Thus we obtain from any Singhof filling of M a ***special Singhof filling*** of M; i.e., a Singhof filling with no intersection curves of order > 3.

2. Singhof Fillings with Balls and Tori

The goal of this paper is to classify all closed 3-manifolds that admit a (special) Singhof filling $M = V_1 \cup \cdots \cup V_k$, where each V_i is a solid torus or a ball. To do this it is convenient to allow V_i to have punctures, where by a ***punctured*** V_i we mean a manifold obtained from V_i by removing the interiors of (arbitrarily many) 3-balls from int V_i.

Lemma. *If $M = H_1 \cup H_2$ has a (special) Singhof filling with each H_i a punctured ball, then M is S^3 or a connected sum of S^2-bundles over S^1.*

This follows from the observation that since M is closed, each 2-sphere boundary component of H_1 is identified with a 2-sphere boundary component of H_2. $\square$

Let $\mathbb{B}$ denote a connected sum of S^3 and S^2-bundles over S^1 (with arbitrarily many factors). In particular, we allow $\mathbb{B} = S^3$.

Proposition 1. *If $M = H_1 \cup \cdots \cup H_m$ has a special Singhof filling with each H_i a punctured ball then $M = \mathbb{B}$.*

For $m = 2$ this is the Lemma; assume $m \geq 3$. Suppose there is a disk component D of $H_i \cap H_m$, for some $1 \leq i \leq m-1$. Then ∂D is a component of $H_k \cap H_i \cap H_m$ for some $k \neq i, m$. Let N be a regular neighborhood of D in H_i and let $H_i' = \overline{H_i - N}$, $H_k' = H_k \cup N$, $H_j' = H_j$ for $j \neq i, m$. Note that H_i' is homeomorphic to H_i and H_k' is obtained from H_k by attaching a 2-handle. Hence $M = H_1' \cup \cdots \cup H_{m-1}' \cup H_m$ is a Singhof filling with punctured balls and the number of components of $H_m \cap H_i'$ $(1 \leq i \leq m-1)$ is reduced by one. Thus after finitely many steps we may assume that for each $i = 1, \ldots, m-1$ there are no disk components of $H_i \cap H_m$. But then the components of $H_m \cap H_i$ consists of boundary spheres of H_i' and H_m.

Consider $M_1 = \overline{M - H_m} = H_1 \cup \cdots \cup H_{m-1}$. Note that ∂M_1 consists of boundary spheres of (some of) the H_i's and the manifold $\widehat{M}_1$, obtained from M_1 by filling in all boundary components with 3-balls, has a Singhof filling $\widehat{M}_1 = H_1^* \cup \cdots \cup H_{m-1}^*$, where each H_i^* is obtained from H_i by filling in some boundary spheres of H_i with 3-balls. By induction on m it follows that $\widehat{M}_1$ is a connected sum of S^2-bundles over S^1.

Since M is obtained from $\widehat{M}_1$ by removing balls and then identifying each resulting 2-sphere with a 2-sphere of ∂H_m, it follows that $M = \mathbb{B}$. $\square$

We now consider the case that M has a Singhof filling with punctured balls and punctured solid tori. We may assume that this Singhof filling $M = V_1 \cup \cdots V_n$ is special. Removing from M the interiors of regular neighborhoods of intersection curves of order 3 we obtain a 3-manifold M' that is a Seifert union in the sense of [GH], of punctured balls and solid tori. It follows [GH, Theorem 8] that M' is a connected sum of Seifert-webs, lensspaces, S^2-bundles over S^1, and handlebodies. Here a Seifert-web is obtained from Seifert fiber spaces by attaching 1-handles.(The standard reference for Seifert fiber spaces is [S]). Since $\partial M'$ consist of tori, the Seifert-webs that occur are in fact Seifert fiber spaces and the handlebodies are solid tori. Now M is obtained from M' by Dehn filling with solid tori. By [H1] each Dehn filling of a Seifert fiber space is again a Seifert fiber space or a connected sum of lensspaces and S^2-bundles over S^1. Hence M is a connected sum of Seifert fiber spaces, lensspaces, and S^2-bundles over S^1.

This argument does not control the number of lensspace summands or the number and type of the Seifert fiber space summands and, therefore, we consider the case of a special Singhof filling with punctured balls and punctured solid tori in more detail.

By $S(m)$ we denote a closed Seifert fiber space (orientable or non-orientable) with arbitrary orbit surface and at most m exceptional fibers, and different from a lensspace. By L (or L_i) we denote a lensspace, possibly S^3, but different from $S^2 \times S^1$.

Theorem 1. *If M has a special Singhof filling, $M = \bigcup_{i=1}^{n} T_i \cup \bigcup_{j=1}^{m} H_j$, with n punctured solid tori T_i and m punctured balls H_j, then M is of the form*

$$M = S(m_1)\sharp \cdots \sharp S(m_k)\sharp L_1 \sharp \cdots \sharp L_r \sharp \mathbb{B}$$

where each $m_i \geq 3$ and $\sum_{i=1}^{k} m_i + r \leq n$.

Proof. We use induction on n, where $n = 0$ corresponds to Proposition 1. Write $M = \bigcup_{i=1}^{u+m} V_i$, where each V_i is a punctured ball or a punctured solid torus.

(1) Suppose there is a disk component D of $V_i \cap V_j$, with ∂D a component of $V_i \cap V_j \cap V_k$. As in the proof of Proposition 1, we let $N = D \times I$ be a regular neighborhood of D in V_i and change the special Singhof filling to $M = \bigcup_{i=1}^{u+m} V_i'$, where $V_i' = \overline{V_i - N}$, $V_k' = V_k \cup N$, and $V_\ell' = V_\ell$ for $\ell \neq i, k$; see Figure 4.

Note that $V_i' \approx V_i$ and $V_k' = V_k \cup$ 2-handle and the splitting $V_1' \cup \cdots \cup V_{n+m}'$ of M is a Singhof filling, except that V_k' may be a punctured lensspace.

Case 1(a) V_k' is a punctured lensspace. Fill in the punctures with balls $B_1, \ldots, B_s$ to get $\widehat{V}_k' = L$, let B be a ball in $\widehat{V}_k'$ that contains all the punctures and let H be

$\overline{B - B_1 \cup \cdots \cup B_s} \cup$ (a ball attached to ∂B). Then $M = L \sharp M'$, where $M' = \bigcup\limits_{\substack{\ell=1 \\ \ell \neq k}}^{n} V_\ell \cup H$ has a special Singhof filling with punctured balls and $n-1$ punctured solid tori.

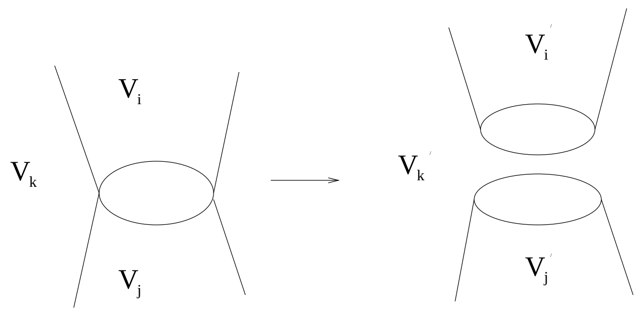

Figure 4

By induction M' is of the form $S(m'_1)\sharp \cdots \sharp S(m'_{k'})\sharp L_1 \sharp \cdots L_{r'} \sharp \mathbb{B}$, with $\sum {m'}_i + r' \leq n-1$ and each $m'_i \geq 3$, and hence M is as in Theorem 1 (with $m'_i = m_i$ and $r = r' + 1$).

Case 1(b) Whenever there is a disk component of $V_i \cap V_j$ then V'_k is not a punctured lensspace.

Hence V'_k is a punctured ball or a punctured torus (with one more puncture than V_k) and $M = \bigcup\limits_{i=1}^{n+m} V'_i$ is a Singhof filling with punctured balls and n punctured solid tori. Note that the modification $\bigcup\limits_i V_i \to \bigcup\limits_i V'_i$ reduces the number of intersection curves of order 3. Thus after finitely many steps we obtain

(2) $M = \bigcup\limits_{i=1} V_i$ is a special Singhof filling with punctured balls and n punctured solid tori and there are no disk components of $V_i \cap V_j$, for all $i \neq j$.

It follows that each component of $V_i \cap V_j$ is a torus, a 2-sphere, or an annulus.

If a component of $V_i \cap V_j$ is a torus T, then gluing V_1 and V_2 along T we obtain a punctured lensspace and, as in Case 1(a), we obtain $M = L \sharp M'$, where M' has a special Singhof filling with punctured balls and $n-2$ punctured solid tori, and Theorem 1 follows by induction.

Thus we assume that there are no torus components of $V_i \cap V_j$ and every 2-sphere of ∂V_i is identified with a 2-sphere of ∂V_j, for some $j \neq i$. Filling in all the 2-spheres of ∂V_i with balls we see (as in Case 1(a)) that M is obtained from $M^* = \widehat{V}_1 \cup \cdots \cup \widehat{V}_n$ by removing balls and identifying resulting 2-sphere boundaries, hence $M = M^* \sharp \mathbb{B}$. Here M^* may not be connected, but each component M^*_t of $M^* = M^*_1 \dot{\cup} \cdots \dot{\cup} M^*_\ell$ is a closed

3-manifold with a special Singhof filling of i_t solid tori, where $\sum_{t=1}^{\ell} i_t = n$. Relabel the solid tori $\widehat{V}_1, \ldots, \widehat{V}_n$ such that $M_t^* = \widehat{V}_{t1} \cup \cdots \cup \widehat{V}_{ti_t}$ and note that $\widehat{V}_{ti} \cap \widehat{V}_{tj}$ is empty or consists of (parallel) annuli on $\partial \widehat{V}_{ti}$.

Case 2(a) For some $1 \leq t \leq \ell$, one of the annuli A of $\widehat{V}_{ti} \cap \widehat{V}_{tj}$ is inessential in $\widehat{V}_{ti}$. Then ∂A bounds two disjoint meridian disks in $\widehat{V}_{ti}$ that split $\widehat{V}_{ti}$ into two balls B_1, B_2 and (relabelling) $M_t^* = (B_1 \underset{A}{\cup} \widehat{V}_{t2}) \cup (B_2 \cup \widehat{V}_{t3} \cup \cdots \cup \widehat{V}_{ti_t})$. Since $B_1 \underset{A}{\cup} \widehat{V}_{t2}$ is a once punctured lensspace (possibly S^3 or $S^2 \times S^1$) it follows that $M_t^* = L \sharp \widetilde{M}_1$, where $\widetilde{M}_1 =$ ball $\cup B_2 \cup \widehat{V}_{t3} \cup \cdots \cup \widehat{V}_{ti_t}$ is a special Singhof filling. By induction $\widetilde{M}_t = S(m_{t_1}) \sharp \cdots \sharp S(m_{t_{k_t}}) \sharp L_1 \sharp \cdots \sharp L_{r'_t} \sharp \mathbb{B}$ with $\sum m_{t_j} + r'_t \leq i_t - 2$, each $m_{t_i} \geq 3$, and hence M_t^* is of this form with $\sum m_{t_j} + r_t \leq i_t - 1$, where $r_t = r'_t + 1$, $m_{t_j} \geq 3$.

Since each of the other components M_s^* (if $\ell > 1$) has a Singhof filling with $i_s < n$ solid tori it follows by induction that each M_s^* is a connected sum of Seifert fiber spaces, lensspaces, and S^2-bundles over S^1, where the multiplicities of the exceptional fibers for all Seifert fiber space summands and the number of lensspaces add up to at most i_s.

It follows that M is as claimed in Theorem 1.

Case 2(b) For each t, every annulus of $\widehat{V}_{ti} \cap \widehat{V}_{tj}$ is essential in $\widehat{V}_{ti}$ and $\widehat{V}_{tj}$.

In this case, each M_t^* is a Seifert fiber space with at most i_t exceptional fibers. Hence $M = M^* \sharp B = S(m_{i_t}) \sharp \cdots \sharp S(m_{i_\ell}) \sharp B$ is as in Theorem 1. □

Since every Singhof filling can be reduced to a special Singhof filling by replacing each intersection curve of order > 3 with a solid torus we obtain

Corollary 1. *If M has a Singhof filling, $M = \bigcup_{i=1}^{n} T_i \cup \bigcup_{j=1}^{m} H_j$, with n punctured solid tori T_i and m punctured balls H_j, then M is of the form*

$$M = S(m_1) \sharp \cdots \sharp S(m_k) \sharp L_1 \sharp \cdots \sharp L_r \sharp \mathbb{B}.$$

Each $m_i \geq 3$ and $\sum_{i=1}^{k} m_i + r \leq n + s$, where s is the number of intersection curves of order > 3.

Note that lensspaces and S^2-bundles over S^1 are Seifert fiber spaces; hence we obtain

Corollary 2. *If a closed 3-manifold admits a Singhof filling with (punctured) balls and (punc-*

tured) solid tori, then it is a connected sum of Seifert fiber spaces.

3. Realization

A converse to Theorem 1 also holds:

Theorem 2. *If* $M = S(m_1)\sharp\cdots\sharp S(m_k)\sharp L_1\sharp\cdots\sharp L_r\sharp\mathbb{B}$ *with each* $m_i \geq 3$, *then* M *has a (special) Singhof filling as a union of balls and at most* $m_1 + \ldots m_k + r$ *solid tori.*

The proof follows easily from the proof of Theorem 1. We give two typical examples below.

Example 1. $M = S(m)\sharp L$ for $m \geq 3$.

Represent $S(m)$ as a union of (at most) $m \geq 3$ solid tori $T_1, \ldots, T_m$, so that $T_i \cap T_j$ consists of essential annuli on ∂T_i and ∂T_j $(i \neq j)$. For example, decompose the orbit surface F of $S(m)$ into (at most) m disks $D_1, \ldots, D_m$, each with at most one exceptional point and let $T_i = p^{-1}(D_i)$, where $p : S(m) \to F$ is the projection to the orbit surface. Figure 5 shows an example for $m = 5$ and F the orientable surface of genus 2.

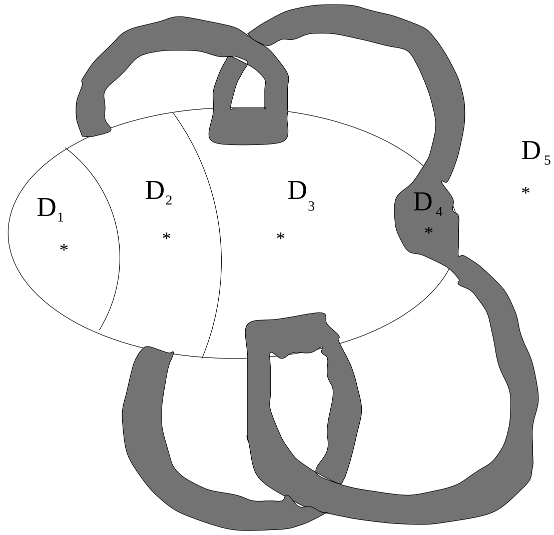

Figure 5

$S(m)$ is obtained by identifying annuli in $\partial T_1, \ldots, \partial T_m$ in pairs. By relabelling we may assume that A is an annulus component of $\partial T_1 \cap \partial T_2$ and we let A_i be a copy of A

in T_i and choose a disk $D_i \subset \operatorname{int} A_i$ $(i = 1, 2)$. Now construct a punctured $S(m)$, denoted by $S^0(m)$, by identifying all annuli of the ∂T_j in pairs, except that instead of A_1, A_2 we identify $A_1 - \operatorname{int} D_1$ with $A_2 - \operatorname{int} D_2$. Note that $\partial S^0(m) = D_1 \cup D_2$. Construct the punctured lensspace $L^0 = L - \operatorname{int}$ (3-ball) as a solid torus T_{m+1} with a 2-handle $B^3 = D^2 \times I$ attached such that $\partial L^0 = A' \cup (D^2 \times 0) \cup (D^2 \times 1)$, where A' is an annulus on ∂T_{m+1}. We obtain $S(m)\sharp L$ by identifying the boundary sphere $D_1 \cup D_2$ of $S^0(m)$ with ∂L^0 such that $D^2 \times 0 \subset \operatorname{int} D_1$ and $D^2 \times 1 \subset \operatorname{int} D_2$. This gives a Singhof filling $T_1 \cup \cdots \cup T_{m+1} \cup B^3$ for $S(m)\sharp L$.

Example 2. $M = S(m)\sharp S(n)\sharp(S^1 \times S^2)$

As in Example 1, let $S(m) = T_1 \cup \cdots \cup T_m$ and $S(n) = T_{m+1} \cup \cdots \cup T_{m+n}$ be Singhof fillings with solid tori. Assume that A_1, A_2 are copies on $\partial T_1, \partial T_2$ of an annulus component of $\partial T_1 \cap \partial T_2$, and A_{m+1}, A_{m+2} are copies on $\partial T_{m+1}, \partial T_{m+2}$ of an annulus component of $\partial T_{m+1} \cap \partial T_{m+2}$. Choose disjoint disks D_i, E_i, E_i' in A_i for $i = 1, 2$ and disjoint disks D_j in A_j for $j = m+1, m+2$.

Identifying all the annuli of the ∂T_k's, except for these disks, we create a punctured $S^0(m)$ with $\partial S^0(m) = (D_1 \cup D_2) \cup (E_1 \cup E_2) \cup (E_1' \cup E_2')$ and a punctured $S^0(n)$ with $\partial S^0(n) = D_{m+1} \cup D_{m+2}$. Identifying now the boundary spheres $D_1 \cup D_2$ and $D_{m+1} \cup D_{m+2}$ and that $D_1 \subset \operatorname{int} D_{m+1}$, and the spheres $E_1 \cup E_2$ and $E_1' \cup E_2'$ such that $E_1 \subset \operatorname{int} E_1'$, we obtain a Singhof filling $T_1 \cup \cdots \cup T_{m+n}$ of M. $\square$

As an illustration we list all closed 3-manifolds that admit a Singhof filling with at most four solid tori. We say that M has ***Singhof type*** $(g_1, \ldots, g_n)$ if M has a special Singhof filling $M = V_1 \cup \cdots \cup V_n \cup H_1 \cup \cdots \cup H_m$, where V_i is a (punctured) orientable handlebody of genus g_i and H_j is a (punctured) ball (for any number m). Note that if M has Singhof type $(g_1, \ldots, g_n)$ then M also has Singhof type $(g_1, \ldots, g_{n+1})$. From Theorem 1 we then obtain the following table.

Type	M
(0)	$\mathbb{B}$
(1)	manifolds of type (0) and $L\sharp\mathbb{B}$
$(1,1)$	manifolds of type (1) and $L_1\sharp L_2\sharp\mathbb{B}$
$(1,1,1)$	manifolds of type $(1,1)$ and $L_1\sharp L_2\sharp L_3\sharp\mathbb{B}$, $S(3)\sharp\mathbb{B}$

This table generalizes the results in [G] of Singhof fillings $V_1 \cup V_2 \cup V_3$, where each V_i is a ball or a solid torus.

For types $(1,1,1,1)$ and $(1,1,1,1,1)$ the list includes the manifolds of type $(1,1,1)$ and $L_1 \sharp L_2 \sharp L_3 \sharp L_4 \sharp \mathbb{B}$, $L_1 \sharp L_2 \sharp L_3 \sharp L_4 \sharp L_5 \sharp \mathbb{B}$, $S(3) \sharp L \sharp \mathbb{B}$, $S(3) \sharp L_1 \sharp L_2 \sharp \mathbb{B}$, $S(4) \sharp \mathbb{B}$, $S(4) \sharp L \sharp \mathbb{B}$, $S(5) \sharp \mathbb{B}$.

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