Moduli spaces of surfaces and real structures.

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This article is dedicated to the memory of Boris Moisezon

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Abstract

We give infinite series of groups Γ and of compact complex surfaces of general type S with fundamental group Γ such that

1) any surface S' with the same Euler number as S, and fundamental group Γ, is diffeomorphic to S

2) the moduli space of S consists of exactly two connected components, exchanged by complex conjugation.

Whence,

i) on the one hand we give simple counter-examples to the DEF = DIFF question whether deformation type and diffeomorphism type coincide for algebraic surfaces

ii) on the other hand we get examples of moduli spaces without real points.

iii) Another interesting corollary is the existence of complex surfaces S whose fundamental group Γ cannot be the fundamental group of a real surface.

Our surfaces are surfaces isogenous to a product, i.e., they are quotients $(C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G.

They resemble the classical hyperelliptic surfaces, in that G operates freely on C_1, while the second curve is a **triangle curve**, meaning that $C_2/G \equiv \mathbf{P}^1$ and the covering is branched in exactly three points.

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1 Introduction

Let S be a minimal surface of general type, then to S we attach two positive integers $x = \chi(\mathcal{O}_S)$, $y = K_S^2$ which are invariants of the oriented topological type of S.

The moduli space of the surfaces with invariants (x, y) is a quasi-projective variety defined over the integers, in particular it is a real variety (similarly for the Hilbert scheme of 5-canonical embedded canonical models, of which the moduli space is a quotient , cf. [Bo], [Gie]).

Fixed (x, y) we have several possible topological types, but (by the result of [F]) indeed at most two if moreover the surface S is assumed to be simply connected (actually by [Don1], [Don2], related results hold more generally for the topological types of simply connected compact oriented differentiable 4-manifolds: cf. [Don4], [Don5] for the so-called 11/8 conjecture stating that a similar result to the one below should hold true, if we allow also to take $K3$ surfaces with reversed orientation.).

These two cases are distinguished as follows:

- S is EVEN, i.e., its intersection form is even : then S is a connected sum of copies of a $K3$ surface and of $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$.
- S is ODD : then S is a connected sum of copies of $\mathbf{P}^2_{\mathbf{C}}$ and ${\mathbf{P}^2_{\mathbf{C}}}^{opp}$.

Remark 1.1 *${\mathbf{P}^2_{\mathbf{C}}}^{opp}$ stands for the same manifold as $\mathbf{P}^2_{\mathbf{C}}$, but with reversed orientation.*

It is rather confusing in my opinion that some authors use the symbol $\bar{\mathbf{P}}^2_{\mathbf{C}}$ for ${\mathbf{P}^2_{\mathbf{C}}}^{opp}$.

In fact, our notation is that the conjugate of a complex manifold X, denoted by $\bar{X}$, is just the same differentiable manifold, but with complex structure $-J$ instead of J.

Thus, if X has odd dimension, then $\bar{X}$ acquires the opposite orientation of X, but if X has even dimension, X and $\bar{X}$ are orientedly diffeomorphic.

Recall moreover

Definition 1.2 *A real structure σ on a complex manifold X is the datum of an isomorphism $\sigma : X \to \bar{X}$ such that $\sigma^2 = Identity$. One moment's reflection shows then that σ yields an isomorphism between the pairs (X, σ) and $(\bar{X}, \sigma)$.*

In general, the fundamental group is a powerful topological invariant. Invariants of the differentiable structure have been found by Donaldson, by Seiberg -Witten and several other authors (cf. [Don3], [D-K], [Witten], [F-M3], [Mor]) and it is well known that on a connected component of the moduli space the differentiable structure remains fixed.

Up to recently, the converse question DEF = DIFF ? was open, a "speculation" by Friedman and Morgan [F-M1, pag.12] that deformation type and diffeomorphism type should coincide for algebraic surfaces. The speculation was inspired by the successes of gauge theory, and as I read the question I thought the answer should be negative, but not easy to find.

Recently, ([Man4]) Manetti was able to find elaborated counterexamples for simply connected surfaces, and about at the same time of this paper Kharlamov and Kulikov ([K-K]) have given a counterexample for rigid surfaces, in the spirit of the work of Jost and Yau ([J-Y1, J-Y2]). Here, we have found rather simple series of examples:

Theorem 1.3 *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G. Then any surface with the same fundamental group as S and the same Euler number of S is diffeomorphic to S. The corresponding moduli space $M_S^{top} = M_S^{diff}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation. There are infinitely many examples of the latter case, and moreover they are almost all of general type.*

Remark 1.4 *The last statement is a direct consequence of the results of Harris-Mumford ([H-M])*

Corollary 1.5 *1) DEF $\neq$ DIFF.*

2) There are moduli spaces without real points

We deduce moreover as a byproduct of the same arguments the following

Theorem 1.6 *There are infinite series of groups Γ which are fundamental groups of complex surfaces but which cannot be fundamental groups of a real surface.*

One word about the construction of these examples:

we imitate the hyperelliptic surfaces, in the sense that we take $S = (C_1 \times C_2)/G$ where G acts freely on C_1, whereas the quotient C_2/G is $\mathbf{P}^1_{\mathbf{C}}$. Moreover, we assume that the projection $\phi : C_2 \to \mathbf{P}^1_{\mathbf{C}}$ is branched in only three points, namely, we have a so called TRIANGLE CURVE.

It follows that if two surfaces of such sort are antiholomorphic, then there would be an antiholomorphism of the second triangle curve (which is rigid).

Now, giving such a branched cover ϕ amounts to viewing the group G as a quotient of the free group with two elements. Let a, c be the images of the two generators, and set $abc = 1$.

We find such a G with the properties that the respective orders of a, b, c are distinct, whence we deduce that an antiholomorphism of the triangle curve would be a lift of the standard complex conjugation if the 3 branch points are chosen to be real, e.g. $-1, 0$ and $+1$.

But such a lifting exists if and only if the group G admits an automorphism τ such that $\tau(a) = a^{-1}, \tau(c) = c^{-1}$.

Appropriate semidirect products do the game for us.

Remark 1.7 *It would be interesting to classify the rigid surfaces isogenous to a product which are not real. Examples due to Beauville ([Bea], [Cat6]) yield real surfaces.*

2 A non real triangle curve

Consider the set $B \subset \mathbf{P}^1_{\mathbf{C}}$ consisting of three real points $B := \{-1, 0, 1\}$.

We choose 2 as a base point in $\mathbf{P}^1_{\mathbf{C}} - B$, and we take the following generators α, β, γ of $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2)$:

- α goes from 2 to $-1-\epsilon$ along the real line, passing through $+\infty$, then makes a full turn counterclokwise around the circumpherence with centre -1 and radius ϵ, then goes back to 2 along the same way on the real line.
- γ goes from 2 to $1+\epsilon$ along the real line, then makes a full turn counterclokwise around the circumpherence with centre $+1$ and radius ϵ, then goes back to 2 along the same way on the real line.
- β goes from 2 to $1+\epsilon$ along the real line, makes a half turn counterclokwise around the circumpherence with centre $+1$ and radius ϵ, reaching $1-\epsilon$, then proceeds along the real line reaching $+\epsilon$, makes a full turn counterclokwise around the circumpherence with centre 0 and radius ϵ, goes back to $1-\epsilon$ along the same way on the real line, makes again a half turn counterclokwise around the circumpherence with centre $+1$ and radius ϵ, reaching $1+\epsilon$, finally it proceeds along the real line returning to 2.

An easy picture shows that α, γ are free generators of $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2)$ and

$$\alpha\beta\gamma = 1.$$

With this choice of basis , we have provided an isomorphism of $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2)$ with the group

$$T_\infty := < \alpha, \beta, \gamma | \ \alpha\beta\gamma = 1 > .$$

For each finite group G generated by two elements a, b, passing from Greek to latin letters we obtain a tautological surjection

$$\pi : T_\infty \to G.$$

I.e., we set $\pi(\alpha) = a, \pi(\beta) = b$ and we define $\pi(\gamma) := c$. (then $abc = 1$).

Definition 2.1 *We let the triangle curve C associated to π be the Galois covering $f : C \to \mathbf{P}^1_{\mathbf{C}}$, branched on B and with group G, determined by the chosen isomorphism $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2) \cong T_\infty$ and by the group epimorphism π.*

Remark 2.2 *Under the above notation, we set m, n, p the periods of the respective elements a, b, c of G (n these are the branching multiplicities of the covering f). Composing f with a projectivity we can assume that $m \le n \le p$.*

Notice that the Fermat curve $C := \{(x_0, x_1, x_2) \in \mathbf{P}^2_{\mathbf{C}} | x_0^n + x_1^n + x_2^n = 0\}$ is in two ways a triangle curve, since we can take the quotient of C by the group $G := (\mathbf{Z}/n)^2$ of diagonal projectivities with entries n-th roots of unity, but also by the full group $A = Aut(C)$ of automorphisms, which is a semidirect product of the normal subgroup G by the symmetric group exchanging the three coordinates. For G the three branching multiplicities are all equal to n, whereas for A they are equal to $2, 3, 2n$.

One can get many more examples by taking unramified coverings of the above curves (associated to characteristic subgroups of the fundamental group).

The following natural question arises then: which are the curves which admit more than one realization as triangle curves?

We are not aware whether the answer is already known in the literature, but (although this is not strictly needed for our purposes) we will show in the next lemma (and its proof) that this situation is rather exceptional if the branching multiplicities are all distinct:

Lemma 2.3 *Let $f : C \to \mathbf{P}^1_{\mathbf{C}} = C/G$ be a triangle covering where the branching multiplicities are all distinct $m < n < p$ and we assume $m \geq 4$. Except possibly in the cases where $m = 2s, n = 3s, p = 6s$ or $m = 4, p = 2n$ the group G equals the full group A of automorphisms of C.*

Proof.

I. By Hurwitz's formula the cardinality of G is in general given by the formula

$$|G| = 2(g-1)(1 - 1/m - 1/n - 1/p)^{-1}.$$

II. In general, assume that $A \neq G$ and let $F : \mathbf{P}^1_{\mathbf{C}} = C/G \to \mathbf{P}^1_{\mathbf{C}} = C/A$ be the induced map. Then $f' : C \to \mathbf{P}^1_{\mathbf{C}} = C/A$ is again a triangle covering, otherwise the number of branch points would be ≥ 4 and we would have a non trivial family of such Galois covers with group A. Whence, also a non trivial family of G -covers, a contradiction.

III. We claim then that the three branch points of f cannot have distinct images through F: otherwise the branching multiplicities $m' \leq n' \leq p'$ for f' would be not less than the respective multiplicities for f, and by the analogous of formula I for $|A|$ we would obtain $|A| \leq |G|$, a contradiction.

IV. Note that if the branching multiplicities m, n, p are all distinct, then G is equal to its normalizer in A: because if $\phi \in A$, $G := \phi G \phi^{-1}$, then ϕ induces an automorphism of $\mathbf{P}^1_{\mathbf{C}}$, fixing B, and moreover such that it sends each branch point to a branch point of the same order. Since the three orders are distinct, this automorphism must be the identity on $\mathbf{P}^1_{\mathbf{C}}$, whence $\phi \in G$.

V. Let x_1, x_2, x_3 be the branch points of f of respective multiplicities m_1, m_2, m_3. Suppose now that $F(x_1) = F(x_2) \neq F(x_3)$: we may clearly assume $m_1 < m_2$. Thus the branching multiplicities for f' are greater than $2, n_2, m_3$, where n_2 is a common multiple of m_1, m_2, whence $m_2 \leq n_2, n_2 \geq 2m_1$.

We obtain

$$|A|/|G| \leq (1 - 1/m_3 - 3/n_2)(1 - 1/2 - 1/m_3 - 1/n_2)^{-1} = 2\frac{n_2 m_3 - n_2 - 3m_3}{n_2 m_3 - 2n_2 - 2m_3} =$$

$$2 + \frac{2n_2 - 2m_3}{n_2(m_3 - 2) - 2m_3}$$

thus $|A|/|G| \leq 2$ if $m_3 \geq 4$, $|A|/|G| \leq 3$ if $m_3 = 4$.

VI. However, if $|A|/|G| \leq 2$ then G is normal in A, thus, by our assumption and by IV, $G = A$. Thus we need only to take care of the possibility $|A|/|G| \geq 3$.

Under the hypothesis of V, this can only occur if $|A|/|G| = 3$, $m_3 = m = 4, n_2 = p = 2n$.

VII. There remains the case where $F(x_1) = F(x_2) = F(x_3)$. Then the branching order of f' in $F(x_i)$ is a common multiple ν of m, n, p, and we get the estimate

$$|A|/|G| \leq (1-1/m-1/n-1/p)(1-1/2-1/3-1/\nu)^{-1} = (1-1/m-1/n-1/p)\frac{6\nu}{\nu - 6}.$$

Now, if $p < \nu$, then $\nu \geq 2p, \nu \geq 3n, \nu \geq 4m$, thus $|A|/|G| \leq \frac{6(\nu-9)}{\nu-6} < 6$.

However, looking at the inverse image of $F(x_i)$ under F, we obtain

$$(*)|A|/|G| \geq \nu/m + \nu/n + \nu/p,$$

whence $|A|/|G| \geq 9$, a contradiction.

Thus $p = \nu$, then $\nu = p, \nu \geq 2n, \nu \geq 3m$ and we get $|A|/|G| \leq 6$ from the first inequality, and from $(*)$ we derive that $|A|/|G| \geq 6$.

The only possibility is : $|A|/|G| = 6$, $p = 3m, p = 2n$. *Q.E.D.*

We come now to our particular triangle curves.

Let $r,\ m$ be positive integers $r \geq 3, m \geq 4$ and set

$$p := r^m - 1,\ n := (r-1)m\ .$$

Notice that the three integers $m < n < p$ are distinct.

Let G be the following semidirect product of $\mathbf{Z}/p$ by $\mathbf{Z}/m$:

$$G := < a, c \mid a^m = 1,\ c^p = 1,\ aca^{-1} = c^r\ >$$

The definition is well posed (i.e., the semidirect product of $\mathbf{Z}/p$ by $\mathbf{Z}$ given by $G" := < a, c \mid c^p = 1,\ aca^{-1} = c^r\ >$ descends to a semidirect product of $\mathbf{Z}/p$ by $\mathbf{Z}/m$) since

$$a^i c a^{-i}\ = c^{r^i}$$

and, by very definition of m, $r^m \equiv 1\ (mod\ p)$.

Lemma 2.4 *Define $b \in G$ by the equation $abc = 1$. Then the period of b is exactly n.*

Proof.

The elements of G can be uniquely written as $\{c^i a^j |\ 0 \leq i \leq p-1,\ 0 \leq j \leq m-1\}$.

The period of b equals the one of its inverse, namely, ca.

Now,

$$(ca)^i = c\ (aca^{-1})\ (a^2ca^{-2})\ ...\ (a^{i-1}ca^{-(i-1)})\ a^i\ = c^{1+r+..r^{i-1}}\ a^i = c^{\frac{r^i-1}{r-1}}\ a^i.$$

Whence, $b^i = 1$ if and only if $m|i$ and $p|\frac{r^i-1}{r-1}$.

Let therefore mk be the period of b: then k is the smallest integer with $\frac{r^{mk}-1}{r-1} \equiv 0\ mod\ (r^m-1)$.

Since $\frac{r^{mk}-1}{r-1} = \frac{r^{mk}-1}{r^m-1}\frac{r^m-1}{r-1}$, all we want is $\frac{r^{mk}-1}{r^m-1} \equiv 0\ (mod(r-1))$; however, $\frac{r^{mk}-1}{r^m-1} \equiv \sum_{j=0}^{k-1} r^{mj} \equiv k\ (mod(r-1))$.

Therefore $k = r-1$ and the period of b equals n.

Q.E.D.

Proposition 2.5 *The triangle curve C associated to π is not antiholomorphically equivalent to itself (i.e., it is not isomorphic to its conjugate).*

Proof.

We shall derive a contradiction assuming the existence of an antiholomorphic automorphism σ of C.

STEP I : $G = A$, where A is the group of holomorphic automorphisms of C, $A := Bihol(C, C)$.

Proof. This follows from the previous lemma 2.3, since in this case we assumed $m \geq 4$, and since $p = r^m - 1, n = m(r-1)$, obviously $p > (r-1+m)(r-1)^{m-1} \geq (2+m)9(r-1) > 2n$.

STEP II : if σ exists, it must be a lift of complex conjugation.

Proof. In fact σ normalizes $Aut(C)$, whence it must induce a antiholomorphism of $\mathbf{P}^1_{\mathbf{C}}$ which is the identity on B, and therefore must be complex conjugation.

STEP III : complex conjugation does not lift.

Proof. This is purely an argument about covering spaces: complex conjugation acts on $\pi_1(\mathbf{P}^1_{\mathbf{C}} - B, 2) \cong T_\infty$, as it is immediate to see with our choice of basis, by the automorphism τ sending α, γ to their respective inverses.

Thus, complex conjugation lifts if and only if τ preserves the normal subgroup $K := ker\pi$. In turn, this means that there is an automorphism $\rho : G \to G$ with

$$\rho(a) = a^{-1},\ \rho(c) = c^{-1}.$$

Recall now the relation $aca^{-1} = c^r$: applying ρ, we would get $a^{-1}c^{-1}a = c^{-r}$, or, equivalently,

$$a^{-1}ca = c^r.$$

But then we would get $aca^{-1} = c^r = a^{-1}ca = a^{m-1}ca^{m-1} = c^{r^{m-1}}$, which holds only if

$$r \equiv r^{m-1} \ (modp).$$

Since $p = r^m - 1$ we obtain, after multiplication by r, that we should have $r^2 \equiv 1 \ (modp)$ but this is the desired contradiction, because $r^2 - 1 < r^m - 1 = p$. $Q.E.D.$

Remark 2.6 *What the above proposition says can be rephrased in the following terms (cf. [Cat6], pages 29-31).*

Denote by Π_g *the fundamental group of a compact curve of genus* g.

The epimorphism $\pi : T_\infty \to G$ *factors through an epimorphism* π' *of the triangle group* $T(m,n,p) := < a,b,c \mid a^m = 1,\ b^n = 1,\ c^p = 1,\ abc = 1 >$ *onto* G, *and once an isomorphism* $ker\pi' \cong \Pi_g$ *is fixed, the pair* $(C,\ ker\pi' \cong \Pi_g)$ *yields a point in the Teichmüller space* $\mathcal{T}_g$. *This point is the only fixpoint for the action of* G *on* $\mathcal{T}_g$ *induced by the natural homomorphism* $G :\to Out(ker\pi' \cong \Pi_g)$. *The pair* $(\bar{C}, ker\pi' \circ \tau')$ *corresponds to the epimorphism* $\pi' \circ \tau' : T_{m,n,p} \to G$. *What we have shown is that* $\bar{C}, C$ *yield different points in the moduli space* $\mathcal{M}_g = \mathcal{T}_g / Out\ (\Pi_g)$.

3 Theorems, and corrigenda

We begin this section by recalling some results of ([Cat6]), and we draw some consequences for real surfaces. For one of the theorems of ([Cat6]) we shall need to make a small correction which, although it amounts to remembering that $(-1)^2 = 1$, will be completely crucial to our argument.

Recall that ([Cat6], 3.1- 3.13):

Definition 3.1 *A projective surface* S *is said to be isogenous to a (higher) product if it admits a finite unramified covering by a product of curves of genus* ≥ 2. *In this case, there exist Galois realizations* $S = (C_1 \times C_2)/G$, *and each such Galois realization dominates a uniquely determined minimal one.* S *is said to be of nonmixed type if* G *acts via a product action of two respective actions on* C_1, C_2. *Otherwise* S *is of mixed type and it has a canonical unramified double cover which is of unmixed type and with* $C_1 \cong C_2$ *(see 3.16 of [Cat6] for more details on the realization of surfaces of mixed type). In the latter case the canonical double cover corresponds to a subgroup* $G^0 \subset G$ *of index* 2.

Proposition 3.2 *Let* S, S' *be surfaces isogenous to a higher product, and let* $\sigma : S \to S'$ *be a antiholomorphic isomorphism. Let moreover* $S = (C_1 \times C_2)/G$, $S' = (C'_1 \times C'_2)/G'$ *be the respective minimal Galois realizations. Then, up to possibly exchanging* C'_1 *with* C'_2, *there exist antiholomorphic isomorphisms* $\tilde{\sigma}_i,\ i = 1,2$ *such that* $\tilde{\sigma} := \tilde{\sigma}_1 \times \tilde{\sigma}_2$ *normalizes the action of* G, *in particular* $\tilde{\sigma}_i$ *normalizes the action of* G^0 *on* C_i.

Proof.

Let us view σ as yielding a complex isomorphism $\sigma : S \to \bar{S}'$. Consider the exact sequence corresponding to the minimal Galois realization
$S = (C_1 \times C_2)/G$,

$$1 \to H := \Pi_{g_1} \times \Pi_{g_2} \to \pi_1(S) \to G \to 1.$$

Applying σ_* to it, we infer by theorem 3.4 of ([Cat6]) that we obtain an exact sequence associated to a Galois realization of $\bar{S}'$. Since σ is an isomorphism, we get a minimal one, which is however unique.

Whence,we get an isomorphism $\tilde{\sigma} : (C_1 \times C_2) \to (\bar{C}'_1 \times \bar{C}'_2)$, which is of product type by the rigidity lemma (e.g., lemma 3.8 of [Cat6]). Moreover this isomorphism must normalize the action of $G \cong G'$, which is exactly what we claim.

Q.E.D.

The following is the correction of theorems 4.13 , 4.14 of [Cat6]:

Theorem 3.3 *Let S be a surface isogenous to a product, i.e., a quotient $S = (C_1 \times C_2)/G$ of a product of curves by the free action of a finite group G. Then any surface S' with the same fundamental group as S and the same Euler number of S is diffeomorphic to S. The corresponding moduli space $M_S^{top} = M_S^{diff}$ is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.*

Proof.

The only modification in the proof given in [Cat6] occurs on the last lines of page 30.

As in the previous proposition, an isomorphism between the fundamental groups of S, resp. S' yields a differentiable action of G on the product of curves $(C'_1 \times C'_2)$ yielding the minimal Galois realization of S'. In fact the above isomorphism of fundamental groups , by unicity of the Galois realization, yields an isomorphism of H with H'. This isomorphism yields an orientation preserving diffeomorphism $(C_1 \times C_2) \to (C'_1 \times C'_2)$ which is of product type.

Now, the diffeomorphisms between the respective factors are either both orientation preserving (this was the case we were considering in the argument in loc. cit.), or both orientation reversing.

In the latter case, it means that the topological action of G^0 on the product of the conjugate curves $(\bar{C}'_1 \times \bar{C}'_2)$, which is of product type, yields actions of G^0 on the respective factors $\bar{C}'_1, \bar{C}'_2$ which are of the same type as the respective actions on C_1, C_2 (again here we might have to exchange the roles of C'_1, C'_2 if the genera g_1, g_2 are equal). Therefore we conclude in this case that the conjugate of the surface S' belongs to the irreducible subset of the moduli space containing S.

Q.E.D.

We are now going to explain the construction of our examples :

Let G be the semidirect product group we constructed in section 2, and let C_2 be the corresponding triangle curve.

By the formula of Riemann Hurwitz the genus of C_2 equals
$g_2 = 1 + \frac{1}{2}[(m-1)(r^m - 2) - 1 - \frac{r^m - 1}{r-1}]$.

Let moreover g'_1 be any number greater or equal to 2, and consider the canonical epimorphism ψ of $\Pi_{g'_1}$ onto a free group of rank g'_1, such that in terms of the standard bases $a_1, b_1, \dots a_{g'_1}, b_{g'_1}$, respectively $\gamma_1, ..\gamma_{g'_1}$, we have $\psi(a_i) = \psi(b_i) = \gamma_i$.

Compose then ψ with any epimorphism of the free group onto G, e.g. it suffices to compose with any μ such that $\mu(\gamma_1) = a$, $\mu(\gamma_2) = b$ (and $\mu(\gamma_j)$ can be chosen whatever we want for $j \geq 3$).

For any point C'_1 in the Teichmüller space we obtain a canonical covering associated to the kernel of the epimorphism $\mu \circ \psi : \Pi_{g'_1} \to G$, call it C_1.

Definition 3.4 *Let S be the surface $S := (C_1 \times C_2)/G$ (S is smooth because G acts freely on the first factor).*

Theorem 3.5 *For any two choices $C'_1(I), C'_1(II)$ of C'_1 in the Teichmüller space we get surfaces $S(I), S(II)$ such that $S(I)$ is never isomorphic to $\bar{S}(II)$. Varying C'_1 we get a connected component of the moduli space, which has only one other connected component, given by the conjugate of the previous one.*

Proof.

By theorem 3.3 it suffices to show the first statement, because we know already, by the rigidity of the second triangle curve, that we get a connected component of the moduli space varying C'_1. By proposition 3.2 follows that if $S(I)$ were isomorphic to $\bar{S}(II)$, then there would be an antiholomorphism of C_2 to itself. This is however excluded by proposition 2.3. *Q.E.D.*

We come now to the last result:

Theorem 3.6 *Let S be a surface in one of the families constructed above. Assume moreover that X is another complex surface such that $\pi_1(X) \cong \pi_1(S)$. Then X does not admit any real structure.*

Proof.

Observe that since S is a classifying space for the fundamental group of $\pi_1(S)$, then by the isotropic subspace theorem of ([Cat3]) the Albanese mapping of X maps onto a curve $C'(I)_2$ of the same genus as C'_2.

Consider now the unramified covering $\tilde{X}$ associated to the kernel of the epimorphism $\pi_1(X) \cong \pi_1(S) \to G$.

Again by the isotropic subspace theorem, there exists a holomorphic map $\tilde{X} \to C(I)_1 \times C(I)_2$, where moreover the action of G on $\tilde{X}$ induces actions of G on both factors which either have the same topological types as the actions of G on C_1, resp. C_2 , or have both the topological types of the actions on the respective conjugate curves.

By the rigidity of the triangle curve C_2, in the former case $C(I)_2 \cong C_2$, in the latter $C(I)_2 \cong \bar{C}_2$.

Assume now that X has a real structure σ: then the same argument as in [C-F] section 2 shows that σ induces a product antiholomorphic map $\tilde{\sigma} : C(I)_1 \times C(I)_2 \to C(I)_1 \times C(I)_2$. In particular, we get a non costant antiholomorphic map of C_2 to itself, contradicting proposition 2.3 .

Q.E.D.

Finally, in ([Cat6]), motivated by some examples by Beauville ([Bea], p. 159) we gave the following

Definition 3.7 *A Beauville surface is a rigid surface isogenous to a product.*

However, in commenting in five lines where lies the problem of the classification of such surfaces, I confused together the nonmixed type and the mixed type (which is more difficult to get).

Therefore , I would simply like here to comment that to obtain a Beauville surface of non mixed type it is equivalent to give a finite group G together with two systems of generators $\{a, b\}$ and $\{a', b'\}$ which satisfy a further property, denoted by (*) in the sequel.

In fact, the choice of the two systems of generators yields two epimorphisms $\pi, \pi' : T_\infty \to G$ where we recall that $T_\infty :=< \alpha, \beta, \gamma | \ \alpha\beta\gamma = 1 >$ is the fundamental group of the projective line minus three points.

We get corresponding curves C, C' with an action of G, and the product action of G on $C \times C'$ is free if and only if , defining c, c' by the properties $abc = a'b'c' = 1$, and letting Σ to be the union of the conjugates of the cyclic subgroups generated by a, b, c respectively, and defining Σ' analogously, then we have

$$(*) \ \Sigma \cap \Sigma' = \{1_G\}.$$

In the mixed case, one requires instead that the two systems of generators be related by an automorphism ϕ of G which should satisfy the further conditions:

- i) ϕ^2 is an inner automorphism, i.e., $\phi^2 = Int_\tau$ for some $\tau \in G$
- $(*) : \quad \Sigma \cap \phi(\Sigma) = \{1_G\}$.
- There is no $g \in G$ such that $\phi(g)\tau g \in \Sigma$.

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Note . Before turning to these examples, I tried to look at rigid surfaces, trying in particular to construct non real Beauville surfaces. V. Kharlamov had independently a similar idea, and we spent one afternoon together trying to make it work with several examples. Later on Kharlamov and Kulikov found the right examples ([K-K]), one of them a quotient of a Hirzebruch covering of the plane.

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