The H^p-Norm of a Quasiconformal Mapping

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1. Introduction and Main Result

An analytic function $f(z)$ in the unit disk $\mathbb{D}$ is in the Hardy space H^p, $0 < p < \infty$, if

$$\sup_{0<R<1} \int_0^{2\pi} \left| f\left(Re^{i\theta}\right)\right|^p \, d\theta < \infty. \tag{1.1}$$

This important class of analytic functions has an extensive literature. In 1927, Prawítz proved the following result [P].

Theorem A. *If f is conformal in $\mathbb{D}$, then $f \in H^p$ for all $p < \frac{1}{2}$.*

The Koebe function $k(z) = z/(1-z)^2$ shows this is sharp since $k \notin H^{\frac{1}{2}}$.

The main result of this paper, Theorem B, gives an analogue of Theorem A for quasiconformal mappings in the unit ball $\mathbb{B}^n$. This improves a previous result, Theorem C, due to Jerison and Weitsman [JW], by identifying an exponent, namely $(n-1)/(2K)^{1/(n-1)}$ where K is the dilatation of the quasiconformal mapping. We use an integrability theorem for quasiconformal mappings due to P. Koskela in the proof [K] (Theorem 4).

2. Quasiconformal mappings and the main result

Definition 2.1. A homeomorphism $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is called K-quasiconformal in Ω if

a) f is in the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$ (the space of functions locally L^n-integrable with distributional first derivatives that are locally L^n-integrable).

b) There is a constant K, $1 \le K < \infty$, such that

$$|Df|^n \le K J_f \qquad \text{a.e.}$$

Here $|Df|$ is the norm of the derivative as a linear mapping and J_f is the Jacobian of f.

Definition 2.2. If f is K-quasiconformal in the unit ball $\mathbb{B}^n$, of $\mathbb{R}^n$, then we write $f \in H^p(\mathbb{B}^n)$ when

$$\sup_{0<R<1} \int_{S^{n-1}} |f(R\zeta)|^p \, d\sigma(\zeta) < 0.$$

Here S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is surface area measure on S^{n-1}.

We now state the main result of this paper.

Theorem B. *If f is K-quasiconformal in $\mathbb{B}^n$, then $f \in H^p(\mathbb{B}^n)$ for all*

$$p < \frac{(n-1)}{(2K)^{1/(n-1)}}. \tag{2.3}$$

Moreover, if also $f(\mathbb{B}^n)$ is contained in a half-space, then $f \in H^p(\mathbb{B}^n)$ for all

$$p < \frac{(n-1)}{(K)^{1/(n-1)}}. \tag{2.4}$$

We mention that this identifies an exponent in the theorem of Jerison and Weitsman [JW]:

Theorem C. *There exists $p_0(n,K) > 0$ such that every K-quasiconformal mapping in $\mathbb{B}^n$ is in $H^p(\mathbb{B}^n)$ for all $p < p_0$.*

Our proof of Theorem B uses the following theorem of Koskela [K]:

Theorem D. *If f is K-quasiconformal in $\mathbb{B}^n$, then*

$$\int_{\mathbb{B}^n} |f|^q dx < \infty \qquad \text{for all} \qquad q < \frac{n}{(2K)^{1/(n-1)}}.$$

If also $f(\mathbb{B}^n)$ is contained in a half-space, then

$$\int_{\mathbb{B}^n} |f|^q \, dx < \infty \quad \text{for all } q < \frac{n}{K^{1/(n-1)}}.$$

Here dx is the volume measure in $\mathbb{R}^n$.

3. Proof of Theorem B

The proof is a modification of the proof of Theorem C, in order to accomodate the use of Theorem D. Following [JW], we differentiate in the radial direction,

$$\frac{\partial |f|^p}{\partial r} = p|f|^{p-2} \left(\sum_{j=1}^{n} f_j \frac{\partial f_j}{\partial r} \right).$$

Using the triangle inequality we may assume that $f(0) = 0$. Throughout the following argument we assume that $\frac{1}{2} < R < 1$ and $\delta > 1$. Now

$$\begin{aligned} \int_{S^{n-1}} |f(R\zeta)|^p \, d\sigma(\zeta) - \int_{S^{n-1}} |(\zeta/2)|^p \, d\sigma(\zeta) &= \int_{S^{n-1}} \left(\int_{1/2}^{R} \frac{\partial |f|^p}{\partial r} dr \right) d\sigma \\ &= p \int_{S^{n-1}} \int_{1/2}^{R} |f|^{p-2} \left(\sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial r} \right| \right) dr d\sigma. \end{aligned}$$

It follows that

$$\int_{S^{n-1}} |f(R\zeta)|^p \, d\sigma(\zeta) \leq p \int_{S^{n-1}} \int_{\frac{1}{2}}^{R} |f|^{p-1} \left(\sum_{j=1}^{n} \left| \frac{\partial f_j}{\partial r} \right| \right) dr d\sigma + C_1 \tag{3.1}$$

where we may choose $C_1 = \sigma(S^{n-1}) \sup\{|f(x)|^p \big| x \in B(0, \frac{1}{2})\}$.

Next, since f is K-quasiconformal,

$$\left|\frac{\partial f_j}{\partial r}\right| \leq |Df| \leq (KJ_f)^{1/n}.$$

Together with (3.1)

$$\int\limits_{S^{n-1}} |f(R\zeta)|^p \, d\sigma(\zeta) \leq pnK^{1/n} \int\limits_{S^{n-1}} \int\limits_{1/2}^{R} |f|^{p-1} J_f^{1/n} dr d\sigma + C_1.$$

Next, for each $\delta > 1$, by Hölder's inequality, the above is

$$\leq C_2 \left(\int\limits_{S^{n-1}} \int\limits_{1/2}^{R} |f|^{-n((\delta-1)p+1)} r^{(n-1)} J_f dr d\sigma \right)^{1/n} \cdot \left(\int\limits_{S^{n-1}} \int\limits_{1/2}^{R} |f|^{\delta pn/(n-1)} r^{-1} dr d\sigma \right)^{(n-1)/n}.$$

The first integral above, under a change of variable, is

$$\left(\iint\limits_{f(A)} |x|^{-n((\delta-1)p+1)} dx \right)^{1/n}$$

where $A = \{y \in \mathbb{R}^n | 1/2 \leq |y| \leq R\}$. Since $f(0) = 0$, this integral is finite whenever $\delta > 1$.

By Theorem D the second integral is finite if

$$\delta pn/(n-1) \leq n/(2K)^{1/(n-1)}$$

or

$$p < \frac{1}{\delta} \frac{(n-1)}{(2K)^{1/(n-1)}}.$$

Since $\delta > 1$ is arbitrary, Theorem B follows.

4. Examples

We use the quasiconformal dilation map $F(x) = x|x|^{a-1}$, $1 \leq a \leq \infty$. It is well known that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal where $K = a^{(n-1)}$.

Example 4.1. For $n = 2$, the composition $F(k(z)) = k(z)|k(z)|^{K-1}$ is K-quasiconformal while

$$\int\limits_0^{2\pi} |F(k(Re^{i\theta}))|^p d\theta = \int\limits_0^{2\pi} |k(Re^{i\theta})|^{Kp} d\theta,$$

which is finite as $R \to 1$ only if $Kp < \frac{1}{2}$, or $p < \frac{1}{2K}$. This shows that (2.3) is sharp for $n = 2$.

Example 4.2. By composing F in $\mathbb{B}^n$ with an appropriate Möbius transformation, one sees that (2.4) is best possible when the image lies in a half-space.

We do not know if (2.3) is sharp in general.

References

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