TRIANGULATIONS AND HOMOLOGY OF RIEMANN SURFACES

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ABSTRACT. We derive an algorithmic way to pass from a triangulation to a homology basis of a (Riemann) surface. The procedure will work for any surfaces with finite triangulations. We will apply this construction to Riemann surfaces to show that every compact hyperbolic Riemann surface X has a homology basis consisting of curves whose lengths are bounded linearly by the genus g of X and by the homological systole.

This work got started by comments presented by Y. Imayoshi (see [9]) in his lecture at the 37th Taniguchi Symposium which took place in Katinkulta near Kajaani, Finland, in 1995.

1. INTRODUCTION

The topology of a surface X is determined by the combinatorial structure of any of its triangulations. Here we apply that observation to the study of the homology of a compact Riemann surface of genus g, $g > 1$.

Given a triangulated Riemann surface X it appears possible that a homology basis for X can be built out of closed curves that are unions of edges of the triangles. The formation of these curves may be quite complicated. It appears also possible that the curves forming a homology basis can be chosen in such a way that any edge of any triangle of the triangulation appears at most twice in any given curve of the homology basis. This means that every curve in the homology basis consists of at most $2n_1 = 3n_2$ edges. Here n_1 is the number of edges of the triangulation and n_2 the number of triangles.

Now if the Riemann surface X does not have short geodesics, then one can always find a triangulation of X for which the lengths of the edges are universally bounded and for which $n_2 \leq Cg$, where C is a constant that will be specified below. This means that a Riemann surface which does not have short geodesics always carries a homology basis consisting of curves whose lengths are linearly bounded by the genus.

In order to prove the above we derive an *algorithmic* way to obtain an explicit canonical homology basis for a triangulated Riemann surface X. By a *canonical homology basis* we mean any family of homologically non-trivial simple closed curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, on X such that each α_i intersects β_i in exactly one point and such that no other intersections occur. The method leads to a homology basis whose curves are unions of at most $n_2/2$ edges of the triangulation.

The construction is general and can be applied to any topological surface. One complication in the process is to make sure that we find curves satisfying the required intersection property.

As indicated above, we apply this construction to find a short canonical homology basis for a compact Riemann surface together with the Poincaré metric of constant curvature -1. Such homology bases are important for numerical computations on Riemann surfaces.

In [6] we have shown that given a family $\mathcal{P}$ of simple closed geodesics on X that decomposes X into three-holed spheres then one can find such a canonical homology basis where $\alpha_1, \ldots, \alpha_g$ belong to $\mathcal{P}$ and where any β_i intersects $\mathcal{P}$ at most $2g - 2$ times. The bound $2g - 2$ is the best possible one. If we combine this with Bers' partitioning theorem ([1], [5]) then we also get an upper bound of the lengths of $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$, with respect to the hyperbolic metric. That bound is of order g^2 and it also depends on the homological systole (see below).

Starting with a suitable triangulation rather than with a pants decomposition we show, as indicated above, that there always exists a canonical homology basis consisting of curves whose lengths are bounded *linearly* by the genus.

To state the result let us introduce the following terminology. The *homological systole* $\sigma = \sigma(X)$ is the length of a shortest homologically non-trivial simple closed geodesic γ on X (there may be several such curves). Any homology basis must contain at least one curve that intersects γ transversely. By the Collar Theorem ([3], [4], [7]), this curve becomes long if the length of γ is small. Our result is now as follows,

Theorem 1. *Let X be a compact Riemann surface of genus g with homological systole σ. There exists a canonical homology basis on X such that any curve has length*

$$(2) \qquad\qquad \ell < (g - 1)(45 + 6 \operatorname{arcsinh}(\tfrac{2}{\sigma})).$$

The constants 45 and 6 are not optimal. We point out, however, that in the example given in [6], any homology basis must contain at least one curve whose length exceeds $(g - 1)(2 + 2 \operatorname{arcsinh}(\tfrac{2}{\sigma}))$. For convenience we repeat this example at the end of the paper. Hence the result obtained here is of the best possible order of magnitude.

2. From triangulations to homology

Consider a triangulation $\mathcal{T}$ of a surface X. Let V be the set of vertices of $\mathcal{T}$ and let $\mathcal{G}$ be the graph whose vertex set is V and whose edges are the edges occurring in $\mathcal{T}$. On $\mathcal{G}$ we select a *spanning tree*, i.e. a connected subgraph Γ as shown in Fig. 1 which has the same vertices as $\mathcal{G}$ but which is such that no non-trivial closed edge path occurs.

A spanning tree can be obtained by the following procedure. First one selects an edge which does not separate $\mathcal{G}$ into two parts, and erases this edge from $\mathcal{G}$. If on the remaining graph there is again a non-separating edge then one erases this one also, and so on. Note that Γ has $|V| - 1$ edges where $|V|$ is the number of vertices of V.

In the next step we construct a fundamental polygon for X using the triangulation. For this we cut X open along all the edges of Γ. Since Γ is a tree this produces a connected surface X_g of signature $(g, 1)$. On X_g the triangulation has now the property that all vertices lie on the boundary. Let us denote by E the set of all edges of $\mathcal{G}$ which are not subsets of the boundary of X_g. We now iteratively
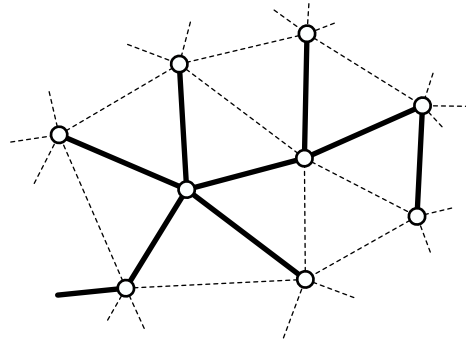
FIGURE 1.   A spanning tree

cut open X_g along certain edges of E until we get the desired polygon. This can be done by the following procedure.

First we look for an edge $u \in E$ which does not separate X_g into two parts. Such an edge exists because X_g has Euler characteristic $\chi = 2 - 2g - 1$. We recall that u has both end points on the boundary of X_g. Cutting X_g open along u we therefore obtain a connected surface X'_g with Euler characteristic $\chi' = \chi + 1$. Since X_g is orientable, X'_g has *two* connected boundary components, say ω' and ω''. Let us denote by u' and u'' the two copies of u on the boundary of X'_g, the notation being such that $u' \subset \omega'$ and $u'' \subset \omega''$.

Next, we look for an edge $v \in E$ that connects ω' with ω''. To see that such an edge exists we consider a continuous curve c going from ω' to ω''. The initial point of c lies on a triangle which has a vertex on ω'; the end point of c lies on a triangle which has a vertex on ω''. If the initial triangle has a vertex on ω'' also, we are done. Otherwise we go along c until we reach for the first time a triangle which does not have all three vertices on ω'. This triangle shares a vertex with the preceding one and, therefore, has an edge v going from ω' to ω''.

Cutting X'_g open along v we obtain a connected surface X_{g-1} which has now again a *connected* boundary, say ω and which has Euler characteristic $\chi' + 1$. Let us denote by v' and v'' the two copies of v on ω. Note that along ω the pair v', v'' is separated by the pair u', u''.

Since X_{g-1} has signature $(g-1, 1)$ we can repeat the procedure to obtain successively X_{g-2}, X_{g-3}, etc. until the fundamental polygon, X_0, is reached after g steps.
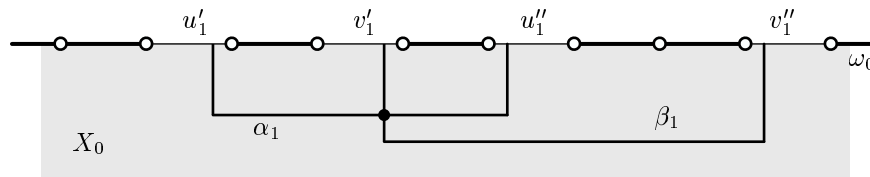

FIGURE 2.   Constructing the homology basis

For the convenience of later reference we remark next that the triangulation of X induces a triangulation of X_0 with all the vertices on the boundary of X_0.

The following observation now follows from the fact that X_0 can be modelled as a polygon in the plane.

Lemma 3. *The boundary of the fundamental polygon X_0 for the Riemann surface X consists of $n_2 + 2$ edges of the triangulation of X.*

The construction of the homology basis is now fairly easy. Starting on X_0 we let u_1', u_1'' and v_1', v_1'' be the sides on the boundary ω_0 of X_0 obtained when X_1 was cut open. We draw a curve α_1 in a small neighborhood of ω_0 from a point of u_1' to an equivalent point of u_1'' (with respect to the pasting) and then a similar curve β_1 from some point of v_1' to the corresponding point of v_1'' as shown in Fig. 2. Then we paste the pairs u_1', u_1'' and v_1', v_1'' together again to obtain X_1 on which α_1 and β_1 are now a pair of simple closed curves intersecting each other in exactly one point.

In the same way we may now continue on X_1, then on X_2, and so on. Since on each X_k the curves $\alpha_{k+1}, \beta_{k+1}$ can be drawn arbitrarily close to the boundary they do not intersect the preceding curves. Hence, the curve system obtained in this way has the required intersection properties.

3. Length estimates for surfaces without short geodesics

To find a short homology basis using the construction of Section 2 we need to start with a triangulation whose triangles are not too small. Such a triangulation can be found if the Riemann surface X does not have short closed geodesic curves. We use the following result from [4], pages 116–121, respectively[2], which goes back to [8], [10] and others.

Lemma 4. *Let X be a compact Riemann surface of genus $g \geq 2$ such that the length of the shortest closed geodesic is greater or equal $\ln 4$. Then there exists a geodesic triangulation $\mathcal{T}$ on X with the following properties*

1. *All triangles of $\mathcal{T}$ have sidelengths between $\ln 2$ and $2 \ln 2$.*
2. *The number of triangles in $\mathcal{T}$ is at most $64(g - 1)$.*

(The statement for $\mathcal{T}$ in [4] is that the triangles have area ≥ 0.196. Since X has area $4\pi(g - 1)$ this yields (2).)

As a preparation for the length estimates observe first that in drawing α_1 and β_1 in Section 2 we have two choices for the directions along ω_0. By choosing them properly we can achieve that the curves have lengths $< \frac{1}{2}\ell(\omega_0)$. Likewise, we may choose the directions such that α_1, respectively β_1, is homotopic to an edge path of $\mathcal{G}$ consisting of at most $\frac{1}{2}|\omega_0| - 1$ edges, where $|\omega_0|$ is the number of sides of ω_0.

For Riemann surfaces X of genus g, $g \geq 2$, without short geodesics we now have the following result.

Theorem 5. *Assume that the Riemann surface X has no geodesics shorter than $\ln 4$, then it has a canonical homology basis consisting of simple closed curves of length less than $45(g - 1)$.*

Observe that this result implies Theorem 1 for surfaces without short (closed) geodesics.

Proof. To estimate the lengths we note that X_0 is triangulated with $n_2 \leq 64(g-1)$ triangles. By Lemma 3, the boundary ω_0 consists of $n_2 + 2$ edges. The boundaries of the X_k have fewer edges. As any edge has length $\leq \ln 4$, all α_k and β_k have, by the remarks preceding the lemma, length $\leq \ln 4(\frac{1}{2}|\omega_0| - 1) < 45(g - 1)$. $\square$

This proves Theorem 1 for surfaces without short geodesics. In this proof, the fact that the decomposition of X consisted of *triangles* was not used in an essential way and only played a role in the estimate of the number of edges. If we use any other cell decomposition $\mathcal{D}$ instead, say with n_2 cells and n_1 edges, then the proof works as well. The change in the arithmetic is that now the boundary ω_0 of X_0 has $2(n_1 - n_2 + 1)$ edges. Hence, we have the following combinatorial result, where we include the case $g = 1$.

Theorem 6. *Let $\mathcal{D}$ be a cell decomposition of a compact orientable surface M of genus $g \geq 1$ with n_2 cells and n_1 edges. Then there exists a canonical homology basis for M such that any curve in this basis is homotopic to an edge path of $\mathcal{D}$ having at most $n_1 - n_2$ edges.*

4. The general case

In the neighborhood of a short geodesic the triangulation of the preceding section cannot be applied, and we need a slightly different cell decomposition of X.

Let γ be a simple closed short geodesic of X and consider the following set known as the *collar* around γ,

$$\mathcal{C}_\gamma = \{x \in X \mid \operatorname{dist}(x, \gamma) < w_\gamma\},$$

where w_γ, the *width* of the collar, is defined by the condition $\sinh w_\gamma \sinh \frac{1}{2}\ell(\gamma) = 1$. By the collar theorem ([7], [3], [4]), $\mathcal{C}_\gamma$ is a topological cylinder. Furthermore, any closed geodesic intersecting γ transversely must cross $\mathcal{C}_\gamma$ from one end to the other and therefore has length at least $2w_\gamma$. As short geodesics have length $\leq \ln 4$, this implies that the short geodesics are pairwise disjoint. Moreover, also by the collar theorem, for any distinct short geodesics γ, γ' the collars $\mathcal{C}_\gamma$ and $\mathcal{C}_{\gamma'}$ are disjoint.
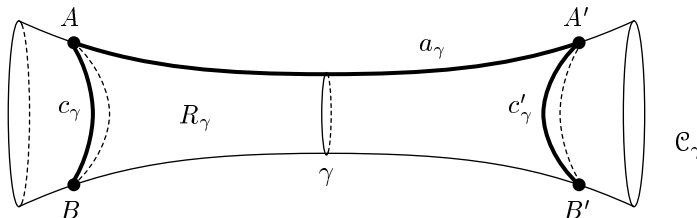

FIGURE 3. Collar around γ

For any short γ a domain $R_\gamma \subset \mathcal{C}_\gamma$ as shown in Fig. 3 will be used which is defined as follows. The four vertices A, A', B, B' have distance $\frac{1}{2}\ln 2$ from the boundary of $\mathcal{C}_\gamma$; the geodesic segments in $\mathcal{C}_\gamma$ from A to A' and from B to B' intersect γ orthogonally and in opposite points. The two minimal geodesics from A to B form a simple closed curve c_γ homotopic to γ; the minimal geodesics from A' to B' form a similar curve c_γ'. Now R_γ is the cylindrical domain bounded by c_γ and c_γ'.

If R_γ is cut open along the geodesic segment a_γ from A to A', then it becomes a geodesic hexagon.

It is shown in [4, p. 116], that X with the interiors of all R_γ removed can be triangulated such that properties (1), (2) as in Lemma 4 hold. This allows us to use the procedure of the preceding section with only minor modifications. First of all we want to avoid intersections with the separating short geodesics. To this

end we remove, for any short geodesic γ, the interior of R_γ from X. The resulting surface falls into connected components $S^1, \ldots, S_p$ with signatures (h_i, m_i) such that $h_1 + \cdots + h_p = g$, and we now consider one of them, say S with signature (h, m).

Let $c_1, \ldots, c_m$ be the boundary components of S, and $\gamma_1, \ldots, \gamma_n$ the non-separating short geodesics of S. We triangulate the closure of $S \setminus (R_{\gamma_1} \cup \cdots \cup R_{\gamma_n})$ as mentioned above with triangles of sidelengths between $\ln 2$ and $\ln 4$. Next we let $\mathcal{G}$ be the graph whose vertices are the vertices of the triangulation and whose edges are the edges of the triangulation plus the geodesic arcs $a_{\gamma_1}, \ldots, a_{\gamma_n}$ which cut the $R_{\gamma_1}, \ldots, R_{\gamma_n}$ into hexagons (see Fig. 3). Since S has boundary, the spanning tree as used in the preceding section needs a minor modification. We take a connected subgraph Γ of $\mathcal{G}$ which has the following properties instead :

1. any vertex of $\mathcal{G}$ is a vertex of Γ;
2. any edge of $\mathcal{G}$ that lies on the boundary of S is an edge of Γ;
3. any edge of Γ that does not lie on the boundary of S separates Γ into two parts.

This is almost a tree, in fact, Γ becomes a tree if the cycles $c_1, \ldots, c_m$ are removed from it.

Cutting S open along the edges of Γ not lying on the boundary we obtain a connected surface S_h which has only one boundary component. Moreover, the triangles and hexagons yield a cell decomposition of S_h all of whose vertices lie on the boundary. Hence, the construction of the preceding section can be carried out on S_h. As S_h has signature $(h, 1)$ we obtain a homology basis $\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h$ (modulo the boundary) of S.

The bases of $S = S^1, \ldots, S^p$ taken together yield our homology basis for X. To estimate the length of a curve η in this basis let us assume that X has q non-separating short geodesics. These are pairwise non-homotopic, and therefore $q \leq 3g - 3$. The domains R_γ around these geodesics are pairwise disjoint and by [4, pp. 116–117], each of them has surface area larger than 2. This implies that the number t of all triangles in the cell decomposition of X satisfies $t \leq 64(g-1) - 10q$.

Consider now again the above component S and let S_0 be the fundamental polygon obtained in the final step of opening it up. The part of the cell decomposition that lies on S_0 consists of $n'_2 \leq t$ triangles and of $q' \leq q$ hexagons. As all vertices of this decomposition lie on the boundary of S_0, the number of edges in the interior of S_0 is $n'_2 - 1$. Each hexagon is obtained by cutting open some R_γ along the arc a_γ; it has four sides of length $\leq \ln 4$ belonging to the boundary of R_γ, plus two "long" sides corresponding to a_γ. The long sides are shorter than $2w_\gamma$, where in turn $w_\gamma \leq \operatorname{arcsinh}(\frac{2}{\sigma})$. By construction, all long sides lie on the boundary of S_0. Hence, the boundary of S_0 has $2q'$ long sides and $3n'_2 + 4q' - 2(n'_2 - 1)$ triangle sides. Reviewing all inequalities we get the following estimate for any curve η in our homology basis,

$$(7) \qquad \ell(\eta) \leq (32(g-1) - 3q) \ln 4 + 2q \operatorname{arcsinh}(\tfrac{2}{\sigma}).$$

As $q \leq 3g - 3$, this completes the proof of Theorem 1.

Example 8. For comparison we briefly review here the example given in [6]. The surface, N, is obtained by pasting together $g - 1$ copies $\Omega_1, \ldots, \Omega_{g-1}$ of a building block Ω of signature $(1, 2)$ as shown in Fig. 4, where Ω itself is obtained by pasting
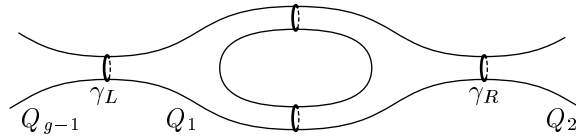
FIGURE 4. A surface of genus g

together two pairs of pants with boundary geodesics of equal lengths ε. By hyperbolic trigonometry, the distance between the two boundary geodesics of $\Omega = \Omega_1$ equals

$$\text{dist}\,(\gamma_L, \gamma_R) = 4\,\text{arcsinh}\,\frac{\cosh(\varepsilon/4)}{\sinh(\varepsilon/2)}.$$

On N, the geodesic γ_L is non-separating. Hence, if $\mathcal{B}$ is a canonical homology basis for N, then $\mathcal{B}$ contains at least one curve β such that γ_L and β have non-zero intersection number. If we cut open N along γ_L, then on the resulting surface N' there must be an arc of β connecting the two boundary components of N' with each other. This implies that $\ell(\beta) \geq (g-1)\,\text{dist}\,(\gamma_L, \gamma_R)$.

For small values of ε we have $\varepsilon = \sigma = \sigma(N)$ and the lower bound for $\ell(\beta)$ is approximately $4(g-1)\,\text{arcsinh}(\frac{2}{\sigma})$. If we take $\varepsilon = \ln 4$, then N has no short geodesics and $\ell(\beta) > 4.58(g-1)$.

References

[1] Lipman Bers. An inequality for Riemann surfaces. In *Differential geometry and complex analysis*, pages 87–93. Springer, Berlin, 1985.

[2] P. Buser, M. Seppälä, and R. Silhol. Triangulations and moduli spaces of Riemann surfaces with group actions. *Manuscripta Math.*, 88(2):209–224, 1995.

[3] Peter Buser. The collar theorem and examples. *Manuscripta Math.*, 25(4):349–357, 1978.

[4] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Birkhäuser Boston Inc., Boston, MA, 1992.

[5] Peter Buser and Mika Seppälä. Symmetric pants decompositions of Riemann surfaces. *Duke Math. J.*, 67(1):39–55, 1992.

[6] Peter Buser and Mika Seppälä. Short homology bases and partitions of Riemann surfaces. *Topology*, 2001.

[7] Isaac Chavel and Edgar A. Feldman. Cylinders on surfaces. *Comment. Math. Helv.*, 53(3):439–447, 1978.

[8] B. Delaunay. Sur la sphère vide. In *Proceedings of the International Mathematical Congress held in Toronto August 11-16*, pages 695–700. University of Toronto Press, Toronto, 1928.

[9] Yoich Imayoshi. An estimate of the number of non-constant holomorphic maps between Riemann surfaces. In *Topology and Teichmüller spaces (Katinkulta, 1995)*, pages 57–78. World Sci. Publishing, River Edge, NJ, 1996.

[10] Fejes Tóth. Kreisausfüllungen der hyperbolischen Ebene. *Acta Math. Acad. Sci. Hungar.*, 4:103–110, 1953.

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