TWO ENDPOINT BOUNDS FOR GENERALIZED RADON TRANSFORMS IN THE PLANE

Jong-Guk Bak Daniel M. Oberlin Andreas Seeger

1. Introduction

The purpose of this note is to prove $L^p \to L^q$ inequalities for averaging operators in the plane (also known as generalized Radon transforms). To describe our setup let Ω_L and Ω_R be open sets in $\mathbb{R}^2$ and let $\mathcal{M}$ be a submanifold in $\Omega_L \times \Omega_R$ which will contain the singular support of the kernel of our operator. We assume that the projections $\mathcal{M} \to \Omega_L$ and $\mathcal{M} \to \Omega_R$ have surjective differential; thus the varieties

$$\begin{aligned} \mathcal{M}_x &= \{y \in \Omega_R; (x,y) \in \mathcal{M}\} \\ \mathcal{M}^y &= \{x \in \Omega_L; (x,y) \in \mathcal{M}\} \end{aligned} \tag{1.1}$$

are smooth immersed curves in Ω_L and Ω_R, respectively.

Let $\chi \in C^\infty(\Omega_L \times \Omega_R)$ be compactly supported. We consider the operator

$$\mathcal{R}f(x) = \int_{\mathcal{M}_x} \chi(x,y) f(y)\, d\sigma_x(y); \tag{1.2}$$

where $d\sigma_x$ is a smooth density on $\mathcal{M}_x$ depending smoothly on $x \in \Omega_L$.

The regularity properties of $\mathcal{R}$ depend on certain finite type conditions, formulated in [15]. We recall that a vector field V on $\mathcal{M}$ is of type $(1,0)$ on an open subset U of $\mathcal{M}$ if for every $P \in U$ we have $V_P \in T_P\mathcal{M} \cap (T_P\Omega_L \times \{0\})$. V is of type $(0,1)$ on U if $V_P \in T_P\mathcal{M} \cap (\{0\} \times T_P\Omega_R\})$ for every $P \in U$. The $C^\infty(U)$ modules of vector fields of type $(1,0)$ and $(0,1)$ on U are denoted by $\mathcal{V}^{1,0}(U)$ and $\mathcal{V}^{0,1}(U)$, respectively. Since $\mathcal{M}$ is three-dimensional there is a nonvanishing one-form ω which annihilates $(1,0)$ and $(0,1)$ vectors. If X and Y are nonvanishing vector fields of type $(1,0)$ and $(0,1)$, respectively, then the quantity $\langle \omega, [X,Y]\rangle$ is comparable to the rotational curvature introduced by Phong and Stein. In fact if $\mathcal{M}$ is given by the equation $\Phi(x,y) = 0$ with $\Phi_x \neq 0$, $\Phi_y \neq 0$ and if we choose $X = \Phi_{x_2}\partial_{x_1} - \Phi_{x_1}\partial_{x_2}$, $Y = \Phi_{y_2}\partial_{y_1} - \Phi_{y_1}\partial_{y_2}$ and $\omega = \Phi_x dx - \Phi_y dy$, then $\langle \omega, [X,Y]\rangle/2$ is equal to

$$J = \det \begin{pmatrix} \Phi_{xy} & \Phi_x^t \\ \Phi_y & 0 \end{pmatrix},$$

the rotational curvature. The generalized Radon transform $\mathcal{R}$ is a Fourier integral operator of class $I^{-1/2}(\Omega_L, \Omega_R; N^*\mathcal{M}')$ in the sense of [5], and $N^*\mathcal{M}'$ is a local canonical graph if and only if J does not vanish.

We now recall the notion of finite type (μ,ν). We write $\mathrm{ad}V(W) = [V,W]$ for the commutator of V and W and for integers $\mu \geq 1$, $\nu \geq 1$, we let $\mathcal{V}^{\mu,\nu}(U)$ denote the $C^\infty(U)$-module generated by all vector fields in $\mathcal{V}^{1,0}(U) \cup \mathcal{V}^{0,1}(U)$ and all vector fields of the form $g\,\mathrm{ad}V_1 \cdots \mathrm{ad}V_{n-1}(V_n)$, where

Research supported in part by KOSEF grant 1999-2-102-003-5 and the BK21 Project (J.B.), by NSF grant DMS 9986804 (D.O.) and by NSF grant DMS 9970042 (A.S.)

g is smooth, $V_i \in \mathcal{V}^{1,0}(U) \cup \mathcal{V}^{0,1}(U)$, at most μ of the V_i are in $\mathcal{V}^{1,0}(U)$ and at most ν of the V_i are in $\mathcal{V}^{0,1}(U)$. We say that $\mathcal{M}$ is of type (μ, ν) at P if there is an open neighborhood U and a vector field $V \in \mathcal{V}^{\mu,\nu}(U)$ so that $\langle \omega_P, V_P \rangle \neq 0$ but $\langle \omega_P, W_P \rangle = 0$ for all $W \in \mathcal{V}^{\mu-1,\nu}(U) \cup \mathcal{V}^{\mu,\nu-1}(U)$. Thus type $(1,1)$ corresponds to the nondegenerate situation of nonvanishing rotational curvature.

Let $n \geq 2$, $m \geq 2$. Following [14] we also say that $\mathcal{M}$ satisfies a *left finite type condition of degree* n in U if $\mathcal{M}$ is of finite type $(1,k)$ for some k with $k \in \{1, \ldots, n-1\}$, for every $P \in U$. We note (see [15]) that $\mathcal{M}$ satisfies this condition if only if for all $(x_0, y_0) \in \mathcal{U}$ the quantity $J(x_0, y)$ when restricted to the curve $\mathcal{M}_{x_0}$ vanishes of order at most $n-2$ at $y = y_0$. Likewise $\mathcal{M}$ satisfies a *right finite type condition of degree* m in U if $\mathcal{M}$ is of finite type $(j,1)$ at P for some $j \in \{1, \ldots, m-1\}$, for every $P \in U$. Again an equivalent formulation is that for all $P_0 = (x_0, y_0) \in \mathcal{U}$ the quantity $J(x, y_0)$ when restricted to the curve $\mathcal{M}^{y_0}$ vanishes of order at most $m-2$ at $x = x_0$.

We now state an endpoint $L^p \to L^q$ estimate for two-sided finite type conditions. In fact a sharper statement can be obtained by working with Lorentz-spaces $L^{p,q}$; note that $L^p \subset L^{p,r}$, if $r \geq p$, with continuous embedding.

Theorem 1.1. *Suppose that $\mathcal{M}$ satisfies a left finite type condition of degree n and a right finite type condition of degree m.*

(i) Suppose that $(1/p, 1/q)$ belongs to the closed trapezoid $\mathcal{T}(m,n)$ with corners $(0,0)$, $(1,1)$, $(\frac{m}{m+1}, \frac{m-1}{m+1})$, $(\frac{2}{n+1}, \frac{1}{n+1})$. Then $\mathcal{R}$ maps L^p boundedly to L^q.

(ii) $\mathcal{R}$ maps $L^{\frac{n+1}{2}, n+1}$ to L^{n+1} and $L^{\frac{m+1}{m}}$ to $L^{\frac{m+1}{m-1}, \frac{m+1}{m}}$.

(iii) If there is a point P such that $\chi(P) \neq 0$ and $\mathcal{M}$ is of type $(1, n-1)$ at P then $\mathcal{R}$ does not map $L^{\frac{n+1}{2}, r}$ to L^{n+1} if $r > n+1$. If there is a point P such that $\chi(P) \neq 0$ and $\mathcal{M}$ is of type $(m-1, 1)$ at P then $\mathcal{R}$ does not map $L^{\frac{m+1}{m}}$ to $L^{\frac{m+1}{m-1}, s}$ for $s < (m+1)/m$.

Remarks.

(a) Let $\mathcal{G}(P)$ be the graph connecting $(0,0)$ and $(1,1)$ with the points $(\frac{\mu+1}{\mu+\nu+1}, \frac{\mu}{\mu+\nu+1})$ for which $\mathcal{M}$ is of type (μ, ν) at P and suppose that $(1/p, 1/q)$ lies above $\mathcal{G}(P)$. Then a result in [15] states that $\mathcal{R}$ maps L^p to L^q provided that the cutoff function has sufficiently small support close to P; see also Phong-Stein [6], [7] for sharp endpoint bounds in several model cases. If $(1/p, 1/q)$ lies below $\mathcal{G}(P)$ and $\chi(P) \neq 0$ then $L^p \to L^q$ boundedness fails ([15]). In the present situation this implies the following: If there is a point P with $\chi(P) \neq 0$ such that $\mathcal{M}$ is of type $(1, n-1)$ and of type $(m-1, 1)$ and if $\mathcal{M}$ is not of type (μ, ν) at P for all (μ, ν) with $(\frac{\mu+1}{\mu+\nu+1}, \frac{\mu}{\mu+\nu+1}) \notin \mathcal{T}(m,n)$ then the result in part (i) of Theorem 1.1 is sharp. In particular, the $L^{(n+1)/2, n+1} \to L^{n+1}$ estimate is best possible if $\mathcal{M}$ is of type $(1, n-1)$ and of type of type $(m-1, 1)$ for some m.

(b) The sharp bounds for $p > (n+1)/2$, $q = 2p$, and $p < m/(m-1)$, $1/q = 2/p - 1$ are in [14], [15]. The $L^{(n+1)/2, n+1} \to L^{n+1}$ endpoint inequality for polynomial surfaces of the form $\mathcal{M} = \{(x,y) : y_2 = x_2 + \sum_{j+k \leq n} a_{j,k} x^j y^k\}$, with $a_{1,n-1} \neq 0$ was obtained by the first author in [1] based on multilinear arguments in [3], [11]; our proofs of Theorem 1.1 and Theorem 1.2 below rely on this technique as well.

(c) Let $\mathcal{M}$ be defined by a polynomial as in (b) . Then $\mathcal{M}$ is of type (μ, ν) at the origin if $a_{\mu,\nu} \neq 0$ but $a_{j,k} = 0$ whenever $j \leq \mu$ and $k \leq \nu - 1$ or $j \leq \mu - 1, k \leq \nu$.

Our second result concerns weighted Radon transforms which incorporate the rotational curvature J as an improving factor (see *e.g.* [16]), namely for $\gamma > 0$ one defines

$$\mathcal{R}_\gamma f(x) = \int_{\mathcal{M}_x} \chi(x,y) |J(x,y)|^\gamma f(y) \, d\sigma_x(y).$$

It is known ([15]) that $\mathcal{R}_\gamma$ maps L^2 into the Sobolev space $L^2_{1/2}$, provided that $\gamma > 1/2$. By standard arguments combining Littlewood-Paley theory and (complex) interpolation (*cf.* [2]) one can see that

$\mathcal{R}_\gamma : L^p \to L^{p'}_\alpha$ if $\alpha \le 2 - 3/p$, $\gamma > 1/p'$ and $1 < p \le 2$, in particular it maps $L^{3/2} \to L^3$ for $\gamma > 1/3$. In various cases the endpoint bounds for $\gamma = 1/3$ are known. If $\mathcal{M}$ is given by the equation $y_2 = x_2 + S(x_1, y_1)$ then $J = S_{x_1 y_1}$ and for real analytic S the endpoint $L^{3/2} \to L^3$ estimate can be deduced from the endpoint L^2 estimates for damped oscillatory integrals in Phong-Stein [9]. We shall prove an $L^{3/2} \to L^3$ endpoint estimate for the case where S is a polynomial of degree $\le N$, which will have the added feature that the operator norms depend only on N. In the translation invariant case such theorems were obtained by the second author in [10], [13]. As in [7] our operator is now globally defined (without inserting cutoff-functions) and we obtain an improved inequality using Lorentz-spaces. We note that the standard interpolation argument alluded to above does not seem to yield this estimate since one uses analytic interpolation with changing powers of γ.

Theorem 1.2. *Define*

$$\mathcal{A}f(x_1, x_2) = \int_{-\infty}^{\infty} \Big| \frac{\partial^2 P}{\partial x_1 \partial y_1} \Big|^{1/3} f(y_1, x_2 + P(x_1, y_1))\, dy_1 \tag{1.3}$$

where P is a polynomial in (x_1, y_1) of degree at most N. Then there is a constant $C(N)$ (independent of the particular polynomial) so that for $3/2 \le r \le 3$

$$\|\mathcal{A}f\|_{L^{3,r}} \le C(N) \|f\|_{L^{\frac{3}{2},r}} \tag{1.4}$$

for all $f \in L^{\frac{3}{2},r}(\mathbb{R}^2)$.

If $\partial^2 P / (\partial x_1 \partial y_1)$ does not vanish identically then the operator $\mathcal{A}$ does not map $L^{3/2,r}$ to $L^{3,s}$ for any $s < r$.

In particular $\mathcal{A}$ maps $L^{3/2}$ to L^3.

The proof of Theorem 1.1 will be given in §2, and the proof of Theorem 1.2 in §3. We shall use the notation $\lesssim$ for inequalities involving admissible constants; here the definition of admissibility depends on the context and will be made precise in §2 and §3, respectively.

2. Boundedness under finite type assumptions

In this section we give a proof of the boundedness result in Theorem 1.1. It suffices to establish the $L^{\frac{n+1}{2},n+1} \to L^{n+1}$ inequality. This also implies the $L^{\frac{m+1}{2},m+1} \to L^{m+1}$ inequality for the adjoint operator $\mathcal{R}^*$ and thus the $L^{\frac{m+1}{m}} \to L^{\frac{m+1}{m-1},\frac{m+1}{m}}$ inequality for $\mathcal{R}$.

By compactness arguments it suffices to prove the theorem for the case that our cutoff function χ is supported in a small neighborhood of a fixed point $P \in \mathcal{M}$; by performing translations we may assume that the coordinates vanish at P.

We may assume that $\mathcal{M}$ is given as

$$\mathcal{M} = \{(x, y) : y_2 = G(x_1, x_2, y_1), |x_1|, |x_2|, |y_1| \le 2\}$$

where G is a C^{n+1} function defined on $[-2, 2]^3$ and G satisfies

$$G(0,0) = 0,\ G_{x_1}(0,0) = G_{y_1}(0,0) = 0,\ G_{x_2}(0,0) = 1,\ 1/2 \le G_{x_2}(x, y_1) \le 2. \tag{2.1}$$

We then also have for $x_1, x_2, y_1 \in [-1, 1]$

$$y_2 = G(x, y_1) \iff x_2 = H(y, x_1)$$

where H is defined on $[-1, 1]^3$ and satisfies

$$H(0,0) = 0,\ H_{y_1}(0,0) = H_{x_1}(0,0) = 0,\ H_{y_2}(0,0) = 1,\ 1/2 \le H_{y_2}(y, x_1) \le 2. \tag{2.2}$$

Let $M = \max\{n+1, m+1\}$. We let $\|(G,H)\|_{C^M}$ be the maximum of any derivative of order at most M of G or H in the cube $[-1,1]^4$ and assume that

$$\|(G,H)\|_{C^M} \le B; \tag{2.3}$$

note that $B \ge 1$.

The rotational curvature (with respect to the defining function $\Phi(x,y) = y_2 - G(x,y_1)$) is given by

$$J(x,y_1) = \det\begin{pmatrix} G_{x_1y_1}(x,y_1) & G_{x_1}(x,y_1) \\ G_{x_2y_1}(x,y_1) & G_{x_2}(x,y_1) \end{pmatrix}. \tag{2.4}$$

By our finite type assumptions there are constants $a_L > 0$ and $a_R > 0$ so that

$$\min_x \max_{0\le k\le n-2} \Big|\frac{\partial^k}{(\partial y_1)^k} J(x,y_1)\Big| \ge a_L \tag{2.5-L}$$

$$\min_y \max_{0\le j\le m-2} \Big|\frac{\partial^j}{(\partial x_1)^j} \big[J(x_1, H(y,x_1), y_1)\big]\Big| \ge a_R; \tag{2.5-R}$$

(2.5-L) means that $\mathcal{M}$ is of type $(1,k)$ (some $k \le n-1$) and (2.5-R) means that $\mathcal{M}$ is of type $(j,1)$ (some $j \le m-1$), for any point under consideration, *cf.* the discussion in [15].

In what follows we choose

$$0 < \varepsilon \le \frac{1}{4}\min\{((m+1)!)^{-1} B^{-m} a_R,\ 2^{-n-5} n^{-1} B^{-2} a_L\}. \tag{2.6}$$

We define

$$Rf(x) = \int_{-\varepsilon}^{\varepsilon} \chi(x_1,x_2,y_1,G(x,y_1)) f(y_1, G(x,y_1)) dy_1$$

where χ is the characteristic function of $[-\varepsilon,\varepsilon]^4$. Note that if $x_1,x_2,y_1 \in [-\varepsilon,\varepsilon]$ then $|G(x,y_1)| \le 2\varepsilon$.

It suffices to show that

$$\|Rf\|_{L^{n+1}} \lesssim \|f\|_{L^{\frac{n+1}{2},n+1}}$$

where the notation $\alpha \lesssim \beta$ means $\alpha \le C\beta$ where C depends only on B, m, n, a_L, a_R. Since R is a positive operator we may assume that f is nonnegative.

As in [1] we use a multilinear interpolation argument due to M. Christ [3]. In order to establish that R maps $L^{\frac{n+1}{2},n+1}$ to L^{n+1} one shows the more general multilinear estimate

$$\int \prod_{i=1}^{n+1} Rf_i(x)dx \lesssim \prod_{i=1}^{n+1} \|f_i\|_{L^{\frac{n+1}{2},n+1}}$$

and by symmetry and real interpolation ([3]) this will follow from

$$\int \prod_{i=1}^{n+1} Rf_i(x)dx \lesssim \|f_1\|_1 \prod_{i=2}^{n+1} \|f_i\|_{L^{n,1}}.$$

Now we use the change of variable $x_2 \mapsto u_2 = G(x_1,x_2,u_1)$ and write

$$\begin{aligned}\int \prod_{k=1}^{n+1} Rf_k(x)dx &= \int\int \chi(x_1,x_2,u_1,G(x,u_1)) f_1(u_1,G(x,u_1)) \prod_{i=2}^{n+1} Rf_i(x)dx\, du_1 \\ &= \int\int \chi(x_1,H(u,x_1),u_1,u_2) f_1(u_1,u_2) \prod_{k=2}^{n+1} Rf_k(x_1,H(u,x_1)) \Big|\frac{\partial H}{\partial u_2}(u_1,u_2,x_1)\Big| du\, dx_1\end{aligned}$$

and, since $|(\partial H)/(\partial y_2)|$ is bounded by B, we may omit this factor. We have reduced matters to the estimate

$$\int \prod_{k=2}^{n+1} Rf_k(x_1, H(u,x_1))dx_1 \lesssim \prod_{i=2}^{n+1} \|f_i\|_{L^{n,1}} \tag{2.7}$$

for every u with $|u_1| \le \varepsilon$, $|u_2| \le 2\varepsilon$. In what follows we fix u. By Hölder's inequality it suffices to show

$$\Big(\int [Rf(x_1, H(u,x_1))]^n dx_1\Big)^{1/n} \lesssim \|f\|_{L^{n,1}}. \tag{2.8}$$

By duality (2.8) is implied by

$$\int Rf(s, H(u,s))g(s)ds \lesssim \|f\|_{L^{n,1}(\mathbb{R}^2)}\|g\|_{L^{n/(n-1)}(\mathbb{R})},$$

for any nonnegative step function g. The left hand side is equal to

$$\int\int \chi(s, H(u,s), y_1, G(s, H(u,s), y_1))f(y_1, G(s, H(u,s), y_1))g(s)dy_1 ds \tag{2.9}$$

and we define

$$\omega^{y_1,u}(s) = G(s, H(u,s), y_1)$$

to change variables in this integral (after interchanging the order of integration).

Lemma 2.1. *(i)*

$$(\omega^{y_1,u})'(s) = \frac{(y_1-u_1)E(s,u,y_1)}{G_{x_2}(s, H(u,s), u_1)}$$

where

$$E(s,u,y_1) = \int_0^1 \det\begin{pmatrix} G_{x_1y_1}(s, H(u,s), u_1+\tau(y_1-u_1)) & G_{x_1}(s, H(u,s), u_1) \\ G_{x_2y_1}(s, H(u,s), u_1+\tau(y_1-u_1)) & G_{x_2}(s, H(u,s), u_1)\end{pmatrix} d\tau. \tag{2.10}$$

(ii) Suppose that $u_1, y_1, s \in [-\varepsilon, \varepsilon]$, $|u_2| \le 2\varepsilon$ and $y_1 \ne u_1$. Then the derivative of $\omega^{y_1,u}$ vanishes at no more than $m-2$ points in $[-\varepsilon,\varepsilon]$.

The elementary proof will be given below. Given y_1, u there are intervals $I_i^{y_1,u}$, $i = 1, \dots, m$ with $\cup_{i=1}^m I_i^{y_1,u} = [-\varepsilon, \varepsilon]$ whose boundary points are measurable functions on (y_1, u) so that $\omega^{y_1,u}$ has nonzero derivative in the interior of $I_i^{y_1,u}$. On each interval $I_i^{y_1,u}$ let $\omega \mapsto s_i^{y_1,u}(\omega)$ be the inverse function of $\omega^{y_1,u}$ and let $\widetilde{I}_i^{y_1,u}$ the image of $I_i^{y_1,u}$ under $\omega^{y_1,u}$. Then the integral (2.9) becomes

$$\begin{aligned}
&\sum_{i=1}^m \int_{-\varepsilon}^{\varepsilon}\int_{I_i^{y_1,u}} \chi(s, H(u,s), y_1, \omega^{y_1,u}(s))f(y_1, \omega^{y_1,u}(s))g(s)dsdy_1 \\
&= \sum_{i=1}^m \int_{-\varepsilon}^{\varepsilon}\int_{\omega\in\widetilde{I}_i^{y_1,u}} \chi(s_i^{y_1,u}(\omega), H(u, s_i^{y_1,u}(\omega)), y_1, \omega)f(y_1,\omega)g(s_i^{y_1,u}(\omega))\Big|\frac{ds_i^{y_1,u}}{d\omega}\Big|d\omega dy_1 \\
&\le \sum_{i=1}^m \|f\|_{L^{n,1}}\|T_{i,u}\|_{L^{\frac{n}{n-1},\infty}}
\end{aligned}$$

where

$$T_{i,u}g(y_1,\omega) = \chi_{[-\varepsilon,\varepsilon]}(y_1)\chi_{\widetilde{I}_i^{y_1,u}}(\omega)g(s_i^{y_1,u}(\omega))\frac{ds_i^{y_1,u}}{d\omega}.$$

In order to finish the proof we have to show that $T_{i,u}$ maps $L^{n/(n-1)}$ to $L^{n/(n-1),\infty}$, that is

$$\operatorname{meas}\big(\{(y_1,\omega):|T_{i,u}g(y_1,\omega)|>\lambda\}\big)\lesssim\frac{\|g\|^{n/(n-1)}_{L^{n/(n-1)}(\mathbb{R})}}{\lambda^{n/(n-1)}}. \tag{2.11}$$

The left hand side of (2.11) is equal to

$$\iint\limits_{\substack{\{(y_1,s)\in[-\varepsilon,\varepsilon]^2,\,s\in I_i^{y_1,u},\\ g(s)\ge\lambda|(\omega^{y_1,u})'(s)|\}}}|(\omega^{y_1,u})'(s)|dy_1ds$$

$$\lesssim\int_{-\varepsilon}^{\varepsilon}\frac{|g(s)|}{\lambda}\operatorname{meas}\big(\{y_1:|y_1-u_1||E(s,u,y_1)|\le 2|g(s)|/\lambda\}\big)\,ds \tag{2.12}$$

where we have used that $|G_{x_2}|\le 2$. We now employ the following standard

Sublevel set estimate [4]. *For any positive integer ℓ there is a constant C_ℓ such that for any interval $I\subset\mathbb{R}$, any $h\in C^\ell(I)$ and any $\gamma>0$ the inequality*

$$\operatorname{meas}\{x\in I:|h(x)|\le\gamma\}\le C_\ell\gamma^{1/\ell}\inf_{x\in I}|h^{(\ell)}(x)|^{-1/\ell}$$

holds.

In order to apply this we use

Lemma 2.2. *For $u_1,s,y_1\in[-\varepsilon,\varepsilon]$, $|u_2|\le\varepsilon$ we have*

$$\max_{1\le k\le n-1}\Big|\frac{\partial^k}{(\partial y_1)^k}\big[(y_1-u_1)E(s,u,y_1)\big]\Big|\ge 2^{-n-2}n^{-1}a_L.$$

Taking Lemma 2.2 for granted we apply the sublevel estimate for suitable $\ell\le n-1$ and $\gamma=2|g(s)|/\lambda$ if $g(s)/\lambda\le 1$ (otherwise estimate the size of any sublevel set by 2ε). We obtain

$$\begin{aligned}\operatorname{meas}\big(\{y_1\in[-\varepsilon,\varepsilon]:\big|(y_1-u_1)E(s,u,y_1)\big|\le 2|g(s)|/\lambda\}\big)\\ \le\min\{2\varepsilon,\max_{1\le\ell\le n-1}C_\ell(2^{n+3}na_L^{-1}|g(s)|/\lambda)^{1/\ell}\}\lesssim(|g(s)|/\lambda)^{1/(n-1)}\end{aligned} \tag{2.13}$$

and thus by (2.12), (2.13)

$$\operatorname{meas}\big(\{(y_1,\omega):|T_{i,u}g(y_1,\omega)|>\lambda\}\big)\le C\int\frac{|g(s)|}{\lambda}\Big(\frac{|g(s)|}{\lambda}\Big)^{1/(n-1)}ds=C\frac{\|g\|^{n/(n-1)}_{L^{n/(n-1)}(\mathbb{R})}}{\lambda^{n/(n-1)}}.$$

Proof of Lemmas 2.1 and 2.2. We need the following elementary

Sublemma. *Let g, h be functions having N derivatives at a point x and suppose that $\max_{j\le r}|u^{(j)}(x)|\le B_r$, $r\le N$. Suppose that $\max_{0\le j\le N-1}|(uh'-u'h)^{(j)}(x)|\ge\alpha_N$. Then also*

$$\max_{1\le j\le N}|h^{(j)}(x)|\ge 2^{-N}\alpha_N-B_N|h(x)|.$$

Proof. By the Leibniz rule $(h'u-hu')^{(k-1)}=\sum_{l=1}^k b_{kl}h^{(l)}-hu^{(k)}$ where the coefficients are given by $b_{kl}(x)=[\binom{k-1}{l-1}-\binom{k-1}{l}]u^{(k-l)}(x)$ if $1\le l<k$, and $b_{kk}(x)=u(x)$. Thus

$$\begin{aligned}\max_{1\le k\le N-1}|(h'u-hu')^{(k-1)}|&\le\sup_k\sum_l|b_{kl}(x)|\max_{1\le j\le N}|h^{(j)}(x)|+|h(x)|\max_{1\le k\le N-1}|u^{(k)}(x)|\\ &\le 2^{N-1}B_N\max_{1\le j\le N}|h^{(j)}(x)|+B_N|h(x)|\end{aligned}$$

which implies the assertion.

Proof of Lemma 2.1. Note that

$$(\omega^{y_1,u})'(s) = G_{x_1}(s, H(u,s), y_1) + G_{x_2}(s, H(u,s), y_1) H_{x_1}(u,s).$$

The defining equation for H is $x_2 = G(u_1, H(x_1, x_2, u_1), x_1)$. Implicit differentiation yields that $H_{x_1}(u_1, G(x, u_1), x_1) = -(G_{x_1}/G_{x_2})(x, u_1)$ or

$$H_{x_1}(u_1, u_2, x_1) = -\frac{G_{x_1}(x_1, H(u, x_1), u_1)}{G_{x_2}(x_1, H(u, x_1), u_1)}.$$

Thus

$$\begin{aligned}(\omega^{y_1,u})'(s) &= \left[\frac{1}{G_{x_2}(x,u_1)} \det \begin{pmatrix} G_{x_1}(x,y_1) & G_{x_1}(x,u_1) \\ G_{x_2}(x,y_1) & G_{x_2}(x,u_1)\end{pmatrix}\right]_{x=(s,H(u,s))} \\ &= \frac{(y_1-u_1)E(s,u,y_1)}{G_{x_2}(s,H(u,s),u_1)}.\end{aligned}$$

Now we prove (ii). Since G_{x_2} does not vanish it suffices to show that

$$\max_{0\le j\le m-2} \left|\left(\frac{\partial}{\partial s}\right)^j E(s,u,y_1)\right| \ge \frac{a_R}{2}. \tag{2.14}$$

We expand

$$E(s,u,y_1) = E(s,u,u_1) + (y_1-u_1) r(s,u,y_1) \tag{2.15}$$

where $E(s,u,u_1) = J(u_1, H(u,s), s)$ and

$$r(s,u,y_1) = \int_0^1\int_0^1 \Big[G_{x_1y_1y_1}(X,U_1)G_{x_2}(X,u_1) - G_{x_2y_1y_1}(X,U_1)G_{x_1}(X,u_1)\Big]_{\substack{X=(s,H(u,s))\\ U_1=u_1+\sigma\tau(y_1-u_1)}} d\sigma\tau d\tau.$$

By assumption (2.5-R) we have

$$\max_{0\le j\le m-2} \left|\partial_s^j E(s,u,u_1)\right| \ge a_R. \tag{2.16}$$

To get a concrete upper bound for the derivatives of r we need a well known fact about multiple applications of the chain rule. Namely let v be $\mathbb{R}^d$-valued and let η be a scalar function on the range of μ, both in C^k. Then $(\eta\circ v)^{(k)}$ is a sum of at most $\prod_{i=0}^{k-1}(d+i)$ terms each of which is of the form $\xi w_1\cdots w_\ell$ where ξ is a derivative of η, of order $\le k$, the w_i are derivatives of a component of v, of order at most k, and $\ell\le k$. Of course more explicit formulas are known (such as the Faà di Bruno formula) but we don't need these here. Applying this with $d=2$ we see that a derivative of order k of $s\mapsto G_{x_1}(s,H(u,s),y_1)$ can be estimated by $(k+1)!B^{k+1}$, and a similar remark applies to the other terms in the integrand defining r. Thus by the Leibniz rule we have the bound $|\partial^j r/(\partial s)^j| \le \sum_{l=0}^{j}\binom{j}{l}(l+1)!B^{l+1}(j-l+1)!B^{j-l+1} \le (j+3)!B^{j+2}$, $j\le m-2$. Combining this with (2.16) and $|y_1-u_1|\le 2\varepsilon$ we see that the left hand side of (2.14) has a lower bound $a_R - 2\varepsilon(m+1)!B^m$. Thus (2.14) follows by our choice of ε in (2.6).

Proof of Lemma 2.2. First

$$\frac{\partial^k}{(\partial y_1)^k}\big[(y_1-u_1)E(s,u,y_1)\big] = \frac{\partial^{k-1}E}{(\partial y_1)^{k-1}}(s,u,y_1) + (y_1-u_1)\frac{\partial^k E}{(\partial y_1)^k}(s,u,y_1).$$

Now we expand the kth derivative of the integrand in (2.10) about u_1 and get $\frac{\partial^k E}{(\partial y_1)^k}(s,u,y_1) = M_k(s,u) + \rho_k(s,u,y_1)$ where

$$M_k(s,u) = \frac{1}{k+1}\Big[G_{x_2}\frac{\partial^{k+1}G_{x_1}}{(\partial y_1)^{k+1}}\Big]_{(s,H(s,u),u_1)}$$

and

$$\rho_k(s,u,y_1) = -\frac{1}{k+1}\Big[G_{x_1}\frac{\partial^{k+1}G_{x_2}}{(\partial y_1)^{k+1}}\Big]_{(s,H(s,u),u_1)} + (y_1-u_1)\times$$
$$\int_0^1\int_0^1\Big[G_{x_2}(x,u_1)\frac{\partial^{k+1}G_{x_1}}{(\partial y_1)^{k+1}}(x,U_1) - G_{x_1}(x,u_1)\frac{\partial^{k+1}G_{x_2}}{(\partial y_1)^{k+1}}(x,U_1)\Big]_{\substack{x=(s,H(u,s))\\ U_1=u_1+\sigma\tau(y_1-u_1)}} d\sigma\tau^k d\tau.$$

Since $|G_{x_1}| \le 8\varepsilon B$ it is easy to see that $|\rho_k(s,u,y_1)| \le 12\varepsilon B^2$, moreover the term $|(y_1 - u_1)\partial^k_{y_1}E(s,u,y_1)|$ above is bounded by $8\varepsilon B^2/(k+1)$. Since $G_{x_2} \ge 1/2$ we obtain by the Sublemma that

$$\begin{aligned}
&\max_{k=0,\dots,n-2}|M_k(s,u)|\\
&\ge (n-1)^{-1}2^{1-n}\max_{k=0,\dots,n-2}\Big|\frac{\partial^k}{(\partial y_1)^k}[G_{x_1y_1}G_{x_2} - G_{x_2y_1}G_{x_1}]_{(s,H(u,s),u_1)}\Big| - B\|G_{x_1}\|_\infty\\
&\ge 2^{1-n}n^{-1}a_L - 8\varepsilon B^2.
\end{aligned}$$

Here the L^∞ norm of G_{x_1} is taken over the cube $[-2\varepsilon, 2\varepsilon]^4$. We finally get

$$\Big|\frac{\partial^k}{(\partial y_1)^k}\big[(y_1-u_1)E(s,u,y_1)\big]\Big| \ge 2^{-n}n^{-1}a_L - 20B^2\varepsilon$$

and the assertion follows from our choice of ε in (2.6).

Remark. For the $L^{(n+1)/2,n+1} \to L^{n+1}$ inequality the lower bound a_R in (2.5-R) enters only in the definition of ε in (2.6), the bounds depend on m but not on a_R. Indeed the type $(m,1)$ assumption can be replaced by an assumption of bounded multiplicity; i.e. there is $\ell \in \mathbb{N}$ so that for almost all u (sufficiently small) the inverse images of the maps $s \mapsto G(s,H(u,s),y_1)$ have cardinality $\le \ell$.

Sharpness of Lorentz exponents. It is well known that the necessary condition $1/q \ge 2/p - 1$ follows by testing $\mathcal{R}$ on characteristic functions of small balls. We assume $1/q = 2/p-1$, $1 < r < \infty$, and verify that $\mathcal{R}$ does not map $L^{p,r} \to L^{q,r-\varepsilon}$. Then applying this to the adjoint operator one also obtains the necessary condition $1/q \le 1/(2p)$ and also that $\mathcal{R}$ does not map $L^{p,r} \to L^{2p,r-\varepsilon}$.

It suffices to consider $1 \le p < 2$. We assume that near the origin $\mathcal{M}$ is defined by $y_2 = G(x,y_1)$ as in (2.1). For a large positive integer ℓ let $f \equiv f_\ell(y) = |y|^{-2/p}$ for $2^{-4\ell} \le |y| \le 2^{-\ell/2}$. Then if $|x_2 - H(0,x_1)| \approx 2^{-k}$ and $\ell \le k \le 2\ell$ then $|\mathcal{R}f(x)| \ge c2^{-k(1-2/p)}$ and this happens on a set of measure $\approx 2^{-k}$. Thus if $\lambda_{\mathcal{R}f}$ denotes the distribution function of $\mathcal{R}f$ then $\lambda_{\mathcal{R}f}(2^{-k(1-2/p)}) \gtrsim 2^{-k}$ and

$$\begin{aligned}
\|\mathcal{R}f\|_{L^{q,s}} &\gtrsim \Big(\int[\alpha\lambda^{\frac1q}_{\mathcal{R}f}(\alpha)]^s\frac{d\alpha}{\alpha}\Big)^{1/s}\\
&\gtrsim \Big(\sum_{k=\ell}^{2\ell}\big[c2^{-k(1-2/p)}\lambda^{1/q}_{\mathcal{R}f}(c2^{-k(1-2/p)})\big]^s\Big)^{1/s} \gtrsim \Big(\sum_{k=\ell}^{2\ell}c'2^{-k(1-2/p+1/q)s}\Big)^{1/s} \gtrsim \ell^{1/s}
\end{aligned}$$

if $1/q = -1 + 2/p$, and by a similar computation $\|f\|_{L^{p,r}} \lesssim \ell^{1/r}$. Thus $\mathcal{R}$ does not map $L^{p,r} \to L^{q,s}$ if $s < r$.

3. Polynomial Radon transforms with weights

We now give a proof of Theorem 1.2. Fix a real-valued polynomial $P(s,t)$ of degree $\le N$; we may assume that $(\partial^2 P)/(\partial s\partial t)$ is not identically zero (otherwise there is nothing to prove).

In this section the notation $\alpha \lesssim \beta$ means $\alpha \le C\beta$ where C depends only on N. It suffices to establish the $L^{3/2,3} \to L^3$ boundedness since applying this result to the polynomial $P(y_1,x_1)$ and using duality implies the $L^{3/2} \to L^{3,3/2}$ boundedness and then by real interpolation the $L^{3/2,r} \to L^{3,r}$ boundedness for $3/2 \le r \le 3$. The sharpness assertion is proved as in the previous section (by working close to points with $(\partial^2 P)/(\partial s\partial t) \ne 0$).

We use the argument of the previous section; now $G(x,y_1) = x_2 + P(x_1,y_1)$, $H(y,x_1) = y_2 - P(x_1,y_1)$ and $J(x_1,y_1) = (\partial^2 P)/(\partial x_1\partial y_1)$ are globally defined. For each $s \in \mathbb{R}$, let $I_1^s, I_2^s, \dots, I_{M(N)}^s$ be disjoint intervals with union $\mathbb{R}$ so that $t \mapsto \partial_s\partial_t P(s,t)$ has constant sign on the interior of each I_j^s. For $1 \le j \le M(N)$ let U_j be the set of all (s,t) such that $t \in I_j^s$ and we can choose the I_j^s so that the U_j are measurable. Let χ_j be the characteristic function of U_j and define the operator $\mathcal{A}_j$ by

$$\mathcal{A}_j f(x) = \int f\big(y_1, x_2 + P(x_1,y_1)\big)|J(x_1,y_1)|^{1/3}\chi_j(x_1,y_1)dy.$$

It is enough to prove that $\mathcal{A}_j$ maps $L^{3/2,3}$ to L^3, for any j. The goal is to show

$$\int_{\mathbb{R}^2}\prod_{k=1}^{3}\mathcal{A}_j f_k(x)dx \lesssim \prod_{k=1}^{3}\|f_k\|_{L^{3/2,3}},$$

and the argument in §2 reduces this to the following analogue of (2.8),

$$\sup_{u\in\mathbb{R}^2}\Big(\int |J(x_1,u_1)|^{1/3}|\mathcal{A}_j f(x_1, u_2 - P(x_1,u_1))|^2 dx_1\Big)^{1/2} \lesssim \|f\|_{L^{2,1}(\mathbb{R}^2)},$$

or, with the measure $d\mu_u(s) = |J(s,u_1)|^{1/3}ds$, to

$$\begin{aligned}
&\int |J(s,u_1)|^{1/3}\mathcal{A}_j f(s, u_2 - P(s,u_1))\chi_j(s,u_1)g(s)ds\\
&= \iint \chi_j(s,t)|J(s,t)|^{1/3}|J(s,u_1)|^{1/3} f(t, u_2 + P(s,t) - P(s,u_1))g(s)dsdt\\
&\lesssim \|f\|_{L^{2,1}(\mathbb{R})}\|g\|_{L^2(\mathbb{R},d\mu)}.
\end{aligned} \tag{3.1}$$

In view of the assumption that J is not identically zero it is not hard to see that for every u_1 the function $s \mapsto P(s,t) - P(s,u_1)$ is not constant except for a finite set of values of t. Thus for almost all t there are intervals $I_i^{t,u}$, $i = 1,\dots,N$ with $\cup_{i=1}^N I_i^{t,u} = \mathbb{R}$ whose boundary points are measurable functions on (t,u) so that

$$\omega^{t,u}(s) = u_2 + P(s,t) - P(s,u_1)$$

has nonzero derivative in the interior of $I_i^{t,u}$ and, as in the previous section, we denote by $\omega \mapsto s_i^{t,u}(\omega)$ the inverse function of $\omega^{t,u}$ on $I_i^{t,u}$ and let $\widetilde{I}_i^{t,u}$ be the image of $I_i^{t,u}$ under $\omega^{t,u}$. Let

$$S_{i,j,u}g(t,\omega) = \chi_{\widetilde{I}_i^{t,u}}(\omega)\frac{ds_i^{t,u}}{d\omega}\,\chi_j(s,t)|J(s,t)|^{1/3}|J(s,u_1)|^{1/3}g(s)\Big|_{s=s_i^{t,u}(\omega)}$$

and, arguing as in the proof of Theorem 1.1, we see that (3.1) follows from

$$\operatorname{meas}\big(\{(t,\omega) : |S_{i,j,u}g(t,\omega)| > \lambda\}\big) \lesssim \lambda^{-2}\int |g(s)|^2|J^{1/3}(s,u_1)|^{1/3}ds. \tag{3.2}$$

The left hand side of (3.2) is equal to

$$\iint_{\substack{\{(s,t):s\in I_i^{t,u},(s,t)\in U_j,\\ |J(s,t)|^{1/3}|J(s,u_1)|^{1/3}g(s)\geq\\ \lambda|(\omega^{t,u})'(s)|\}}} |(\omega^{t,u})'(s)|dsdt$$

$$\leq \int_{-\infty}^{\infty} \int_{\substack{\{t\in I_j^s:\\ |J(s,t)|^{1/3}|J(s,u_1)|^{1/3}g(s)\\ \geq\lambda|\frac{\partial P}{\partial s}(s,t)-\frac{\partial P}{\partial s}(s,u_1)|\}}} \Big|\frac{\partial P}{\partial s}(s,t)-\frac{\partial P}{\partial s}(s,u_1)\Big|\,dt\,ds \tag{3.3}$$

and we have to show that the right hand side is controlled by $\lambda^{-2}\int_{\mathbb{R}}|g(s)|^2|J(s,u_1)|^{1/3}ds$, with constant only depending on N. This is accomplished by applying the following lemma to the inner integral in (3.3), with $p(t)=\frac{\partial P}{\partial s}(s,t)$ (which has constant sign on I_j^s).

Lemma 3.1. *There is a constant $C(N)$ such that the following is true: If p is a real-valued polynomial of degree $\leq N-1$ and I is an interval with p' of constant sign on I, then for all $t_1\in I$ and all $B>0$ the inequality*

$$\int_{\substack{\{t\in I:B|p'(t)p'(t_1)|^{1/3}\\ \geq|p(t)-p(t_1)|\}}} |p(t)-p(t_1)|dt \leq C(N)\ B^2|p'(t_1)|^{1/3} \tag{3.4}$$

holds.

Proof. Note that the integration in (3.4) is always extended over a finite interval, thus we may assume that I is finite.

We begin by observing that there is $C_1(N)$ such that for $0\leq\theta\leq 1$

$$|b-a||p'(a)|^{1-\theta}|p'(b)|^{\theta}\leq C_1(N)\int_{[a,b]}|p'(u)|du. \tag{3.5}$$

If $a=0$, $b=1$ this is true because the $L^1([0,1])$ and $L^\infty([0,1])$ norms are equivalent on the (finite-dimensional) space of polynomials of degree bounded by $N-2$. For other intervals $[a,b]$ an affine change of variables reduces to the case $a=0$, $b=1$.

Continuing the proof of the lemma, the set $\{t\in I: B|p'(t)p'(t_1)|^{1/3}\geq|p(t)-p(t_1)|\}$ is contained in the union of two minimal subintervals $[t_0,t_1]$ and $[t_1,t_2]$ of I (so that the defining inequality holds for $t=t_0$ and $t=t_2$). It is enough to bound the integral of $|p(t)-p(t_1)|$ over each of these intervals by $C_1(N)\ B^2|p'(t_1)|^{1/3}$. The argument is the same in both cases, so we consider the integral over $[t_0,t_1]$. Clearly

$$\int_{t_0}^{t_1}|p(t)-p(t_1)|dt\leq\int_{t_0}^{t_1}\int_{t}^{t_1}|p'(v)|dvdt\leq(t_1-t_0)\int_{t_0}^{t_1}|p'(v)|dv. \tag{3.6}$$

We apply (3.5) with $\theta=1/3$ and see that the right hand side of (3.6) is dominated by

$$C_1(N)\Big(\int_{t_0}^{t_1}|p'(v)|dv\Big)^2|p'(t_0)|^{-2/3}|p'(t_1)|^{-1/3}\leq C_1(N)B^2|p'(t_1)|^{1/3} \tag{3.7}$$

where the last inequality holds since $B|p'(t_0)p'(t_1)|^{1/3}\geq|\int_{t_0}^{t_1}p'(v)dv|$ and p' is of constant sign on $[t_0,t_1]$. The assertion follows from (3.6), (3.7). $\square$

Remark. Suppose that the polynomial $P(s,t)$ is replaced by a C^2 function $S(s,t)$ with the property that for almost all t_1 the generic multiplicities of the maps $(s,t) \mapsto (S(s,t) - S(s,t_1), t)$ and $s \mapsto S_s(s,t) - S_s(s,t_1)$ are bounded by some number ℓ (here we say that $F : \mathbb{R}^d \to \mathbb{R}^d$ has generic multiplicity bounded by ℓ if $F^{-1}(y)$ has cardinality $\leq \ell$ for almost all $y \in \mathbb{R}^n$). In this case a variant of the argument used by the second author in [12] can be employed to show a slightly weaker inequality, namely that $\mathcal{A}$ is of restricted strong type $(3/2, 3)$; i.e. it maps $L^{3/2,1}$ to L^3, with operator norm depending only on ℓ.

References

1. J.-G. Bak, *An $L^p - L^q$ estimate for Radon transforms associated to polynomials*, Duke Math. J. **101** (2000), 259–269.
2. P. Brenner, *$L_p - L_{p'}$ estimates for Fourier integral operators related to hyperbolic equations*, Math. Z. **152** (1977), 273–286.
3. M. Christ, *On the restriction of the Fourier transform to curves: Endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223-238.
4. ______, *Hilbert transforms along curves, I. Nilpotent groups*, Ann. Math. **122** (1985), 575–596.
5. L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183.
6. D. H. Phong and E.M. Stein, *Radon transforms and torsion*, Int. Math. Res. Notices, appended to Duke Math. J. (1991), 49–60.
7. ______, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. Math. **140** (1994), 703–722.
8. ______, *The Newton polyhedron and oscillatory integral operators*, Acta Math. **179** (1997), 146-177.
9. ______, *Damped oscillatory integral operators with analytic phases*, Advances in Math. **134** (1998), 146-177.
10. D.M. Oberlin, *Oscillatory integrals with polynomial phase*, Math. Scand. **69** (1991), 45–56.
11. ______, *Multilinear proofs for two theorems on circular averages*, Colloq. Math **63** (1992), 187–190.
12. ______, *Convolution with affine arclength measures in the plane*, Proc. Amer. Math. Soc. **127** (1999), 3591–3592.
13. ______, *Convolution with measures on polynomial curves*, Math. Scand. (to appear).
14. A. Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), 685–745.
15. ______, *Radon transforms and finite type conditions*, Journal Amer. Math. Soc. **11** (1998), 869–897.
16. C.D. Sogge and E.M. Stein, *Averages of functions over hypersurfaces: smoothness of generalized Radon transforms*, J. Analyse Math. **54** (1990), 165–188.

J. Bak, Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, and Korea Institute for Advanced Study, Seoul 130-012, Korea

E-mail address: `bak@euclid.postech.ac.kr`

D. M. Oberlin, Department of Mathematics, Florida State University, Tallahassee, FL 32306

E-mail address: `oberlin@math.fsu.edu`

A. Seeger, Department of Mathematics, University of Wisconsin, Madison, WI 53706

E-mail address: `seeger@math.wisc.edu`