A NOTE ON FOURIER RESTRICTION FOR CURVES IN $\mathbb{R}^3$

Jong-Guk Bak Daniel M. Oberlin

Abstract. We obtain estimates for restriction of the Fourier transform to certain curves in $\mathbb{R}^3$.

The purpose of this note is to prove the following restriction theorem for the Fourier transform in $\mathbb{R}^3$:

Theorem. *Suppose $y(t)$ and $z(t)$ are $C^{(3)}$ functions defined on an interval $I \subseteq \mathbb{R}$. Suppose that $z^{(3)}$ never vanishes on I. Let γ be the curve in $\mathbb{R}^3$ defined by $\gamma(t) = \big(t, y(t), z(t)\big)$ and suppose that for some $\delta > 0$ there is the inequality*

$$\big|\det\big[\gamma'(s), \gamma''(t), \gamma^{(3)}(u)\big]\big| \geq \delta, \tag{1}$$

for $s, t, u \in I$. Suppose that $1 \leq p < \frac{7}{6}$ and that $\frac{1}{p} + \frac{1}{6}\frac{1}{q} = 1$. Then there is $C(\delta, p) > 0$ such that the estimate

$$\Big(\int_I |\hat{f}\big(\gamma(t)\big)|^q dt\Big)^{\frac{1}{q}} \leq C(\delta, p)\ \|f\|_p$$

holds for $f \in L^p(\mathbb{R}^3)$.

Comments: The first Fourier restriction estimates for curves in $\mathbb{R}^3$ were due to Prestini [8]. The subsequent paper [1] of Christ, dealing with estimates for degenerate curves in $\mathbb{R}^n$, was followed by Drury's [2], where the method of "offspring curves" was introduced to extend the range of p from $1 \leq p < \frac{15}{13}$ to $1 \leq p < \frac{7}{6}$ in the case of (t, t^2, t^3). There are also the papers [4] and [5] of Drury and Marshall, and [3] of Drury, the last containing some interesting comments on the subsequent history of the method of "offspring curves". Our hypothesis (1) is a strong form of nondegeneracy for the curve γ and is reminiscent of the hypothesis of Theorem 3 in [6]. The novelty of our result is that our constant depends only on δ and p. Corollaries are restriction estimates for curves (t, t^b, t^c) with $b < 2$, $b \neq 0, 1$, $b + c = 5$ on $(0, \infty)$ or for flat curves such as $\big(t, y(t), z(t)\big)$ with $y''(t) = e^{1/t}$, $z''(t) = e^{-1/t}$ on $(0, \epsilon)$ for small enough $\epsilon > 0$. Our method of proof is just an adaptation of Drury's method in [2]. The (small) novelty we bring to it is contained in some technical observations and in the exposition.

We begin with two lemmas. These concern the so-called "offspring curves" of γ, curves Γ of the form $\Gamma(t) = \sum_{j=0}^{N} \gamma(t + h_j)$ where $h_0 = 0$, $h_j > 0$ if $1 \leq j \leq N$, and $t \in I_0 \doteq \cap_j (I - h_j)$. We will also consider the mapping B, defined for $t_1, t_2, t_3 \in I_0$ by $B(t_1, t_2, t_3) = \Gamma(t_1) + \Gamma(t_2) + \Gamma(t_3)$.

1991 *Mathematics Subject Classification.* 42B10.

Key words and phrases. Fourier transform, restriction.

Research supported in part by KOSEF grant 1999-2-102-003-5 and the BK21 Project (J.B.) and by NSF grant DMS 9986804 (D.O.)

Lemma 1. *Let $J = J(t_1, t_2, t_3)$ be the absolute value of the determinant of the Jacobian of B, and let $V = V(t_1, t_2, t_3)$ be the absolute value of the Vandermonde determinant $(t_2 - t_1)(t_3 - t_2)(t_3 - t_1)$. Then, with δ as in (1), $J \geq \delta \cdot V$*

Proof of Lemma 1: As in [2], an extension of the mean-value theorem ([7], part V, Chapter 1, problem 95) implies that if $t_1, t_2, t_3 \in I_0$ then there are $u, v, w \in I_0$ such that

$$J(t_1, t_2, t_3) = V(t_1, t_2, t_3) \cdot |\det[\Gamma'(u), \Gamma''(v), \Gamma^{(3)}(w)]|.$$

The proof is then an immediate consequence of the definition of Γ, the multilinearity of the determinant, and (1).

Lemma 2. *The restriction of the mapping B to $\{t_1 < t_2 < t_3\}$ is one-to-one.*

Proof of Lemma 2: Let $\gamma_0(t) = \big(t, t^2, t^3\big)$ and $\gamma_1(t) = \big(t, t^2, z(t)\big)$. For $j = 0, 1$ let Γ_j and B_j be defined as above, but with γ replaced by γ_j and let J_j be the absolute value of the determinant of the Jacobian of B_j. (Then $J_0 = 6V$.) Assume that $t_1 < t_2 < t_3$. We will first show that, given $w \in \mathbb{R}^3$, the equation $B_1(t_1, t_2, t_3) = w$ determines (t_1, t_2, t_3). Drury ([4], pp. 119–120) has observed that $B_0(t_1, t_2, t_3) = v$ determines (t_1, t_2, t_3), so we may think of $w = (w_1, w_2, w_3)$ as a function of $v = (v_1, v_2, v_3)$. Since Γ_0 and Γ_1 agree in the first two slots, so do v and w. Thus the absolute value of the Jacobian of $v \mapsto w$ is equal to both of $\frac{\partial w_3}{\partial v_3}$ and $\frac{J_1}{J_0}$. But

$$\det\big[\gamma_1^{'}(s), \gamma_1^{''}(t), \gamma_1^{(3)}(u)\big] = 2z^{(3)}(u),$$

which is never zero. So it follows from the proof of Lemma 1 that J_1 and hence $\frac{\partial w_3}{\partial v_3}$ are never 0. Thus w_3 determines v_3, w determines v, and hence w determines (t_1, t_2, t_3). Now a repetition of this argument, with B in place of B_1 and B_1 in place of B_0, proves the lemma.

By duality, the theorem will follow from the estimate, uniform in the offspring curve Γ,

$$\Big\| \int_{I_0} g(t) e^{-i\Gamma(t)\cdot\zeta} dt \Big\|_{L^q(d\zeta)} \leq C(\delta, p) \Big(\int_{I_0} |g(t)|^p dt \Big)^{\frac{1}{p}} \tag{2}$$

for $1 \leq p < 7$ and $\frac{1}{p} + \frac{6}{q} = 1$. As in [2], this estimate will be established by induction on p starting with $p = 1$. By an interpolation argument it is enough to show that if (2) holds for $p = p_0$, then (2) holds for characteristic functions g and for $1 \leq p < p_1$ where $\frac{1}{p_1} = \frac{2}{15} + \frac{1}{15p_0}$. So fix $p = p_0$ such that (2) holds and fix a Borel set $E \subseteq I_0$. Now

$$\Big\| \int \chi_E(t) e^{-i\Gamma(t)\cdot\zeta} dt \Big\|^3_{L^q(d\zeta)}$$

$$\leq 6 \cdot \Big\| \int_{\{t_1 < t_2 < t_3\}} \chi_E(t_1)\chi_E(t_2)\chi_E(t_3) e^{-i\big(\Gamma(t_1)+\Gamma(t_2)+\Gamma(t_3)\big)\cdot\zeta} dt_1 dt_2 dt_3 \Big\|_{L^{q/3}(d\zeta)}.$$

Making the convention that $\int\int\cdots\int\cdots dt\cdots dh_2 dh_1$ should henceforth be interpreted as $\int_0^\infty \int_{h_1}^\infty \cdots \int_{I_1} \cdots dt \cdots dh_2 dh_1$ where $I_1 = I_0 \cap (I_0 - h_1) \cap (I_0 - h_2)$, we will establish the inequality

$$\Big\| \int\int\int \chi_E(t)\chi_E(t+h_1)\chi_E(t+h_2) e^{-i\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)\cdot\zeta} dt dh_2 dh_1 \Big\|_{L^{q/3}(d\zeta)} \leq C(\delta, p)\, |E|^{\frac{3}{p}} \tag{3}$$

for $\frac{1}{p} + \frac{6}{q} = 1$ and $p_1 - \epsilon \leq p < p_1$, where $\epsilon = \epsilon(p_0) > 0$. Establishing (3) will involve the interpolation of two estimates, (4) and (5) below. For the first of these note that, for fixed h_1 and h_2, $t \mapsto \Gamma(t) + \Gamma(t + h_1) + \Gamma(t + h_2)$ is an offspring curve to which the inductive hypothesis (2) applies. Thus it follows that

$$(4)\qquad \Big\| \int\int\int f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)e^{-i\left(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\right)\cdot\zeta}dtdh_2dh_1\Big\|_{L^{q_0}(d\zeta)}$$

$$\leq C(\delta,p_0)\int\int\Big(\int \big|f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)\big|^{p_0}dt\Big)^{\frac{1}{p_0}}dh_2dh_1,$$

where, of course, $\frac{1}{p_0}+\frac{6}{q_0}=1$. Next, with J as in Lemma 1, the Plancherel theorem and Lemma 2 give

$$\Big\| \int\int\int f(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2))e^{-i\left(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\right)\cdot\zeta}J(t,t+h_1,t+h_2)dtdh_2dh_1\Big\|_{L^2(d\zeta)}$$

$$=\Big(\int\int\int \big|f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)\big|^2J(t,t+h_1,t+h_2)dtdh_2dh_1\Big)^{\frac{1}{2}}$$

and so

$$(5)\qquad \Big\| \int\int\int f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)e^{-i\left(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\right)\cdot\zeta}dtdh_2dh_1\Big\|_{L^2(d\zeta)}$$

$$=\Big(\int\int\int \big|f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)J(t,t+h_1,t+h_2)^{-\frac{1}{2}}\big|^2dtdh_2dh_1\Big)^{\frac{1}{2}}.$$

Interpolating (4) and (5) yields

$$(6)\qquad \Big\| \int\int\int f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)e^{-i\left(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\right)\cdot\zeta}dtdh_2dh_1\Big\|_{L^c(d\zeta)}$$

$$\leq C(\delta,p_0)\Bigg(\int\int\Big(\int \big|f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)J(t,t+h_1,t+h_2)^{\alpha}\big|^b dt\Big)^{\frac{a}{b}}dh_2dh_1\Bigg)^{\frac{1}{a}}$$

where $1\leq a\leq 2$,

$$(7)\qquad \Big(1-\frac{2}{p_0}\Big)\frac{1}{a}+\frac{1}{b}=1-\frac{1}{p_0},$$

$$(8)\qquad \frac{5}{a}+\frac{1}{b}+\frac{6}{c}=6,$$

and $\alpha=\frac{1}{a}-1$. To deduce (3) from (6), we begin by noticing that if $\frac{3}{p}=\frac{5}{a}+\frac{1}{b}-3$ and $\frac{3}{q}=\frac{1}{c}$, then (8) implies that $\frac{1}{p}+\frac{6}{q}=1$. By (7) the limits $1\leq a<\frac{5}{3}$ correspond to $\frac{2}{15}+\frac{1}{15p_0}<\frac{1}{p}\leq\frac{12}{15}+\frac{1}{15p_0}$. (The reason for the $\frac{5}{3}$ will be apparent shortly.) With these choices for p and q, define f by $f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)=\chi_E(t)\chi_E(t+h_1)\chi_E(t+h_2)$, which is legitimate by Lemma 2. Then (3) will follow from (6) once we check that

$$\Bigg(\int\int\Big(\int \big|f\big(\Gamma(t)+\Gamma(t+h_1)+\Gamma(t+h_2)\big)J(t,t+h_1,t+h_2)^{\alpha}\big|^b dt\Big)^{\frac{a}{b}}dh_2dh_1\Bigg)^{\frac{1}{a}}\leq C(\delta,p)|E|^{\frac{5}{a}+\frac{1}{b}-3}.$$

Writing $V(h_1,h_2) = h_1 h_2 |h_2 - h_1|$, recalling Lemma 2, and using $\alpha = \frac{1}{a} - 1 < 0$, this will follow from

$$\left(\int\int\left(\int \chi_E(t)\chi_E(t+h_1)\chi_E(t+h_2)dt\right)^{\frac{a}{b}} V(h_1,h_2)^{(1-a)}dh_2dh_1\right)^{\frac{1}{a}} \le C(\delta,p)|E|^{\frac{5}{a}+\frac{1}{b}-3}. \tag{9}$$

Here is Drury's ([2]) proof of (9), repeated for the convenience of the reader: considering Lorentz spaces in the variables h_1 and h_2, one can check that

$$V(h_1,h_2)^{(1-a)} \in L^{\frac{2}{3a-3},\infty} \tag{10}$$

for $1 < a < \frac{5}{3}$. On the other hand, interpolating L^1 and L^∞ estimates for

$$\Phi(h_1,h_2) = \int \chi_E(t)\chi_E(t+h_1)\chi_E(t+h_2)dt$$

leads to

$$\|\Phi^{a/b}\|_{L^{\frac{2}{5-3a}},1} \le C(p)\ |E|^{\frac{a}{b}+5-3a}.$$

Combined with (10), this yields (9).

References

[1] M. Christ, *On the restriction of the Fourier transform to curves: endpoint results and the degenerate case*, Trans. Amer. Math. Soc. **287** (1985), 223–238.

[2] S. Drury, *Restrictions of Fourier transforms to curves*, Ann. Inst. Fourier (Grenoble) **35** (1985), 117–123.

[3] ———, *Degenerate curves and harmonic analysis*, Math. Proc. Cambridge Phil. Soc. **108** (1990), 89–96.

[4] S. Drury and B. Marshall, *Fourier restriction theorems for curves with affine and Euclidean arclengths*, Math. Proc. Cambridge Phil. Soc. **97** (1985), 111–125.

[5] ———, *Fourier restriction theorems for degenerate curves*, Math. Proc. Cambridge Phil. Soc. **101** (1987), 541–553.

[6] D. Oberlin, *Convolution estimates for some measures on curves*, Proc. Amer. Math. Soc **99** (1987), 56–60.

[7] G. Polya and G. Szegö, *Problems and theorems in analysis*, Springer-Verlag, 1976.

[8] E. Prestini, *A restriction theorem for space curves*, Proc. Amer. Math. Soc **70** (1978), 8–10.