Seifert-Takens-Singhof fillings of 3-manifolds*

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1 Seifert-Takens-Singhof Fillings

The purpose of this paper is to give a short description of some results obtained recently about Seifert-Takens-Singhof fillings (STS-fillings) of 3-manifolds and to show how these decompositions are related to other classical concepts used in the study of manifolds.

The motivation for defining these *STS-fillings* comes from Seifert fiber spaces, which can be decomposed into solid tori T_i such that $T_i \cap T_j = \partial T_i \cap \partial T_j$ is empty or consists of essential annuli (for $i \neq j$). However, the pieces for a STS-filling need not be solid tori and the manifolds admitting STS-fillings need not be Seifert fiber spaces. The concept of a filling was introduced by Takens [14] in his investigation of the minimal number of critical points of a function on a compact manifold and was later used by Singhof [13] as a central tool in his study of minimal coverings of manifolds with balls. Following these ideas of Seifert, Takens and Singhof we make the following more general definition.

Let $\mathcal{V}$ be a family of compact, connected 3-manifolds with non-empty boundary. We say that a compact, connected 3-manifold M has a $(\mathcal{V}, \mathrm{N})$-*STS-filling* if M admits a decomposition into $V_1, \ldots, V_{\mathrm{N}}$, where each V_i is

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homeomorphic to some manifold of $\mathcal{V}$ and such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ for $i \neq j$. Furthermore if a point p in $int\,(M)$ meets k different $V_{i_1}, \ldots, V_{i_k}$ for some $2 \leq k$, then we require that a neighborhood of p is as in Fig 1 (here

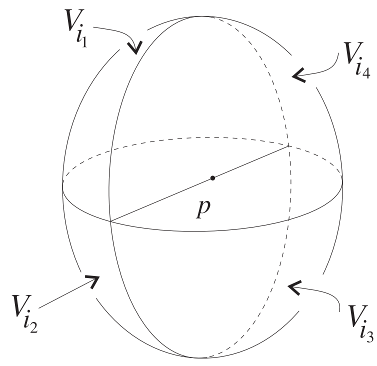

Figure 1:

$k = 4$). In this case we say that p *has order* k.

Note that the set of points of order 2 constitute the interior of compact surfaces whose boundary components are empty or consist of simple closed curves such that all points in a boundary curve C are of the same order $k \geq 3$. Such a curve C is called an *intersection curve of order* k.

Furthermore, if $\partial M \neq \emptyset$ and $p \in \partial M$ meets k different $V_{i_1}, \ldots, V_{i_k}$ for some $k \geq 2$, then a neigborhood of p is required to be as in Fig 2 (here

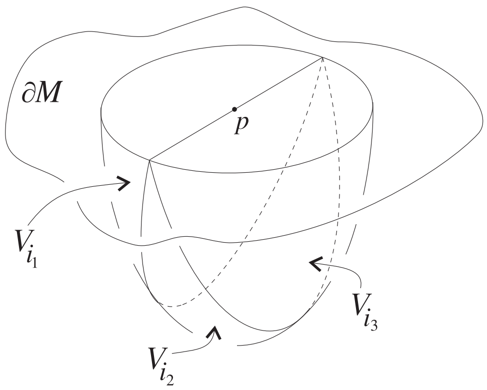

Figure 2:

$k = 3$). Again in this case we say that p has order k and note that the set of points on ∂M of order $k \geq 2$ constitute simple closed curves. All points in one of these simple closed curves C are of the same order $k \geq 2$ and we call such a curve an intersection curve of order k, observe that each of these $C \subset \partial M$ is also a boundary curve of (at least) one surface formed by interior points of M of order 2.

We note that if M admits a $(\mathcal{V}, \textsc{n})$-STS-filling $M = \bigcup_{j=1}^{k} V_{i_j}$ where $V_{i_j} \in \mathcal{V}$ and V_{i_j} is *orientable;* $j = 1, \ldots, k$ then *we can always assume that the order of an intersection curve* C *is three (resp. two) if* $C \in M - \partial M$ *(resp.* $C \in \partial M$*).* This can be obtained in the following way: first, it is easy to see (e.g. [8]) that a regular neighboorhood of C, $N(C)$, is a solid tori. Now, perform an isotopy in $N(C)$ as Figure 3(a) (resp. Figure 3(b)) shows when

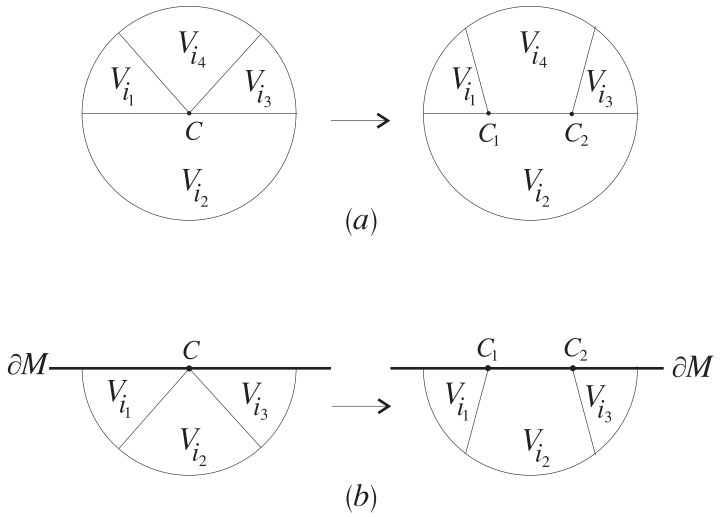

Figure 3:

$C \subset M - \partial M$ (resp. $C \subset \partial M$). Note that we may increase the number of intersections curves after the isotopy but the order of the new curves is the required. Also, note that the family $\{V_{i_j}\}_{j=1}^{k}$ is not changed.

As typical examples (cf. Corollary 1 and 2 below) consider the case when $\mathcal{V}$ consists of one ball or one solid torus. Note that if $\mathcal{V}$ is the family of (orientable or non-orientable) handlebodies then a $(\mathcal{V}, 2)$-STS filling of a closed, connected 3-manifold M is precisely a Heegaard splitting of M. It was pointed out in [4] that if $\mathcal{V}$ is the family of *orientable* handlebodies, then every closed, connected (non-orientable) 3-manifold M has a $(\mathcal{V}, 3)$-STS-filling. A classification of all closed, connected 3-manifolds admitting a $(\mathcal{V}, 3)$-STS filling where $\mathcal{V}$ consists of a 3-ball, a solid torus, a solid Klein bottle, and $P^2 \times I$, was obtained in [6].

On the other hand note that the existence of a Heegaard splitting shows that every closed, connected 3-manifold admits a decomposition into four 3-balls meeting only along their boundaries. However the local intersection pattern is different from that of a STS-filling since one is forced to accept points p in M with a local intersection pattern as in Fig 4, where the B_i are balls.

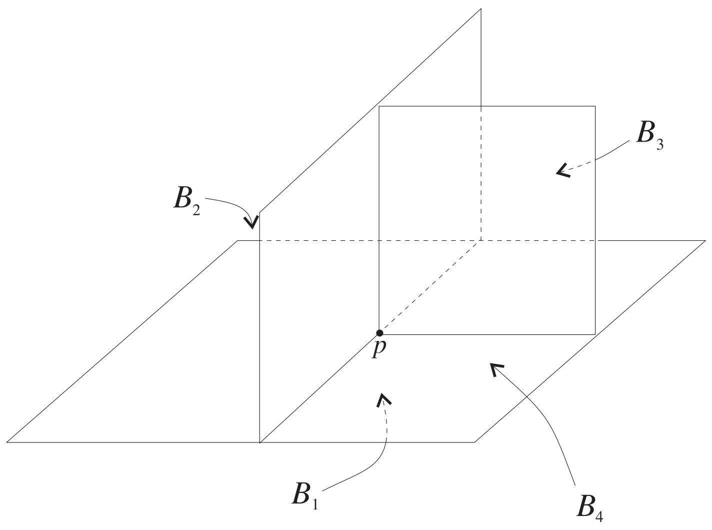

Figure 4:

2 Some Results About STS-fillings

Now let $\mathcal{T}$ be the family of punctured balls and punctured solid tori (any number of punctures) and let M be a closed, connected 3-manifold. We show in [8] that if M has a $(\mathcal{T}, \textsc{n})$-STS-filling, then M is a connected sum of Seifert fiber spaces (for any $\textsc{n} \geq 2$). More precisely, let $\hat{S}\,(m)$ denote any closed Seifert fiber space (orientable or non-orientable) with arbitrary orbit surface and at most m exceptional fibers, and different from a lens space. Let L or L_i denote a lens space, possibly S^3, but different from $S^2 \times S^1$. Let $\mathcal{B}$ denote the family of connected sums of S^3 and S^2-bundles over S^1 (with arbitrarily many factors). For a 3-manifold M and collections $\mathcal{A}_1$ and $\mathcal{A}_2$ of 3-manifolds we denote by $M\#\mathcal{A}_1$, resp. $\mathcal{A}_1\#\mathcal{A}_2$, the collection of manifolds that are connected sums of M with a manifold from $\mathcal{A}_1$, resp. manifolds that are a connected sum $A_1 \# A_2$ for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Then in [8] we prove

Theorem 1 *If M admits a $(\mathcal{T}, \text{N}+\text{M})$-STS filling with N punctured solid tori and M punctured balls then M is of the form*

$$M = \hat{S}(m_1) \# \cdots \# \hat{S}(m_k) \# L_1 \# \cdots \# L_r \# \mathcal{B}.$$

Furthermore each $m_i \geq 3$ and $\sum_{i=1}^{k} m_i + r \leq \text{N}$.

As a corollary we have the following results:

Corollary 1 *Let $\mathcal{V} = \{B^3\}$, B^3 a 3-ball. If M has a $(\mathcal{V}, \text{N})$-STS-filling then $M \in B$.*

Corollary 2 *Let $\mathcal{V} = \{T\}$, T a solid torus. If M has a $(\mathcal{V}, 3)$-STS-filling, then M is of the form $L \# \mathcal{B}$, $L_1 \# L_2 \# \mathcal{B}$, $L_1 \# L_2 \# L_3 \# \mathcal{B}$, or $\hat{S}(3) \# \mathcal{B}$.*

To obtain a result similar to Theorem 1 for manifolds with non-empty boundary we let $S(m)$ denote a Seifert fiber space (orientable or non-orientable) with arbitrary orbit surface and at most m exceptional fibers, and with *non empty boundary.*

A *Seifert-web* $W(m_1, \ldots, m_k)$ is a connected manifold obtained from the disjoint union $S(m_1) \cup \cdots \cup S(m_k)$ by attaching 1-handles (either to the same or distinct $S(m_i)$'s). The $S(m_i)$'s are called the *nodes* of the Seifert-web.

We also use the following notations

- $\hat{\mathcal{S}}$ is the collection of all connected sums of closed Seifert fiber spaces,
- $\mathcal{W}$ is the collection of all connected sums of Seifert-webs,
- $\mathcal{L}$ is the collection of 3-manifolds that are connected sums of lens spaces L_i,
- $\mathcal{F}$ is the collection of all 3-manifolds of the form $\mathcal{B} \# H_1 \# \cdots \# H_k \# H$, where $H_1, \ldots, H_k$ are handlebodies (orientable or non-orientable, *not a ball*), H is a punctured ball (with arbitrary many punctures) and k is any non-negative integer.

As before let $\mathcal{T}$ be the family of punctured solid tori and punctured balls (with any number of punctures). A manifold M is a *Seifert union of* $\mathcal{T}$ if M admits a $(\mathcal{T}, \text{N})$-STS-filling with no points p of order $k \geq 3$. In [7], Theorem 8 we proved

Theorem 2 *A Seifert union of punctured solid tori and punctured balls is in $\mathcal{F}\#\mathcal{L}\#\mathcal{W}$. Conversely, every manifold M of $\mathcal{F}\#\mathcal{L}\#\mathcal{W}$ with $\partial M \neq \emptyset$ can be realized as a Seifert union of tori and balls (without punctures).*

Note in Theorem 2 that if $M \in \mathcal{F}\#\mathcal{L}\#\mathcal{W}$, and $\partial M = \emptyset$, then M is a connected sum of S^2-bundles over S^1 and lens spaces, and is therefore clearly a Seifert union of punctured solid tori and punctured balls.

We now obtain the following generalization of Theorem 1.

Theorem 3 *Let M be a compact, connected 3-manifold (orientable or non-orientable, closed or with non-empty boundary). If M has a $(\mathcal{T}, \text{N})$-STS-filling then M is in $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$.*

Conversely, every manifold M of $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$ has a $(\mathcal{T}, \text{N})$-STS-filling.

Proof. Suppose M has a $(\mathcal{T}, \text{N})$-STS-filling. If M is closed then Theorem 1 shows that $M \in \mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$. So assume that $\partial M \neq \emptyset$.

As we pointed out before, we may assume that the order of an intersection curve $C \subset M - \partial M$ is three and is two if $C \subset \partial M$. Also, a regular neighborhood of an intersection curve $C \subset M - \partial M$ of order $= 3$ is a solid torus. Let R be the manifold obtained from M by removing the interiors of regular neighborhoods of all intersection curves of order $= 3$. Then R is a Seifert union of manifolds from $\mathcal{T}$. By Theorem 2, R is of the form

$$R = \mathcal{B}\#L_1\# \cdots \#L_r\#H_1\# \cdots \#H_s\#H\#W_1\# \cdots \#W_t$$

where the H_i's are handlebodies and the W_j's are Seifert-webs. Now M is obtained from R by attaching solid tori to R along tori. If a solid torus is attached to a H_i then this factor of R is changed to a lens space factor. In [10] it was shown that attaching a solid torus to a Seifert fiber space results in a Seifert fiber space, a connected sum of lens spaces or S^2-bundles over S^1, or handlebodies of genus 1. (The proof given in [10] for orientable Seifert fiber spaces with orientable orbit space works also for non-orientable Seifert fiber spaces with orientable or non-orientable orbit space). From this it is easy to see that attaching solid tori to a W_k results in a manifold in $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$. Hence $M \in \mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$.

The converse is proved similarly to the proof of Theorem 8 (b) in [7]. ∎

We now consider the case that a compact, connected 3-manifold M which admits an STS-filling as in Theorem 3 embeds in S^3. If M is irreducible then the only possibilities are that M is an orientable handlebody, B^3, S^3, or a Seifert-web. The Seifert-webs that embed in S^3 are classified in [9] and can be described as follows.

A *tree* of links $l_1, \ldots, l_n$ is a graph Γ in S^3 obtained from a finite tree T that is embedded in S^3 with vertices $v_1, \ldots, v_m$ lying in disjoint balls $B_1, \ldots, B_m$ by replacing each vertex v_i in B_i with the link l_i in B_i and replacing each edge e_k of T from v_i to v_j by an arc α_k in S^3 that joins a component of l_i to a component of l_j. A *Burde-Murasugi link* l is a link in S^3, not the unlink of more than one component, whose components are components of a torus link on an unknotted torus T^2, possibly together with curves α and β as in the Fig. 5. As we know from [2] and [1], these Burde-

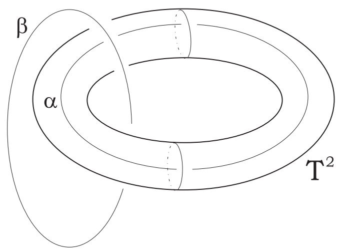

Figure 5:

Murasugi links are precisely those for which the space of the link is a Seifert fiber space. Here, as usual, the space of a graph Γ in S^3 is $\overline{S^3 - N(\Gamma)}$ for a regular neighborhood of Γ.

In [9] we show that an irreducible 3-manifold M with $\partial M \neq \emptyset$ that admits a SFS filling and embeds in S^3 is either a ball or homeomorphic to the space of a tree of Burde-Murasugi links. From this we have

Theorem 4 *If a compact, connected* 3*-manifold* M *has a* $(\mathcal{T}, \textsc{n})$*-STS filling and embeds in* S^3 *then* $M = M_1 \# \cdots \# M_k$*, where each* M_i *is either* S^3 *or a ball or homeomorphic to the space of a tree of Burde-Murasugi links.*

We now would like to compile a table that gives an overview of the STS-fillings of closed, connected 3-manifolds with orientable handlebodies (see Table 1). For this table let H_n denote the orientable handlebody of genus n, $n \geq 1$, and H_0 denote a ball. Let W_n be a core of H_n, i.e. W_n is a wedge of n circles for $n \geq 1$, and W_0 is a point.

$\diagdown$ N n	H_0	H_1	H_2	$\cdots$	H_n	$\cdots$	Σ
2	S^3	L	$\mathcal{H}_2$	$\cdots$	$\mathcal{H}_n$	$\cdots$	$\mathcal{M}_0$
3	$\mathcal{B}$ Cor. 1	Cor. 2					$\mathcal{M}$
$\vdots$	$\vdots$	$\vdots$					$\vdots$
N	$\mathcal{B}$ Cor. 1	Theo. 1					$\mathcal{M}$
$\vdots$	$\vdots$	$\vdots$					$\vdots$
Σ	$\mathcal{B}$	$\hat{\mathcal{S}}$					$\mathcal{M}$

Table 1: $(\{H_n, \text{N}\})$-STS-fillings

In the Table 1, $\mathcal{M}$ (resp. $\mathcal{M}_0$) denote the family of all closed, connected (resp. closed, connected orientable) 3-manifolds. Also $\mathcal{H}_n$ denote the family of all closed, connected orientable 3-manifolds of Heegaard genus less than equal to n. We note that all manifolds in the i^{th} row and j^{th} column are contained in the $(i+1)$st row and j^{th} column and also in the i^{th} row and $(j+1)$st column, so that each row is contained in its succeeding row and each column in its succeeding column. Observe that the union of all manifolds in the 1^{st} row (for N$=2$) is $\mathcal{M}_0$. Using the observation that every non-orientable 3-manifold is a union of three orientable handlebodies [4], it is not hard to see that the union of all manifolds in row 2 consists of $\mathcal{M}$.

Observe that the union all manifolds in the 1^{st} column (for $n = 0$) are all the manifolds in $\mathcal{B}$ and the union of all manifolds in the 2^{nd} column is $\hat{\mathcal{S}}$. It is an open question what the union of the manifolds in the m^{th} column is, when $m \geq 3$ (i.e. $n \geq 2$).

3 STS-fillings and Some Classical Invariants

Following Clapp and Puppe [3] we define the W_n-category of M as follows. A subset $A \subset M$ is *W_n-categorical* if the inclusion map $i : A \longrightarrow M$ factors homotopically through W_n. The *W_n-category of M*, denoted by $cat_n(M)$, is the smallest number of sets, open and W_n-categorical, needed to cover M.

(This is denoted by $\{W_n\}$-cat (M) in [3]).

Clearly $\operatorname{cat}_0(M) = \operatorname{cat}(M)$, where $\operatorname{cat}(M)$ denote the Lusternik-Schnirelmann category of M, i.e. the smallest number of sets, open and contractible in M needed to cover M. Sometime $\operatorname{cat}_1(M)$ is called the *round category of* M (cf. [12]).

By $C_n(M)$ we denote the minimal number of sets homeomorphic to $\overset{\circ}{H}_n$, needed to cover M, e.g. $C_0(M)$ is the minimal number of open balls needed to cover M (cf. [11]).

We now consider invariants associated to smooth functions $f : M \longrightarrow \mathbb{R}^1$. The number of critical points of f is denoted by $\mu_M(f)$ and $F_0(M) = \min_f \{\mu_M(f)\}$. Following Khimshiashvilli and Siersma [12] we say that f is a *round function* if the critical set $C(f)$ is a smooth link in M; the components of $C(f)$ are called the *critical loops* of f. In [12], it is showed that round function exist on closed, connected orientable 3-manifolds. The *round complexity*, $F_1(M)$, is the minimal number of critical loops, taken over all round functions on M.

In Table 2, we show the results that are known about the invariants F_0, F_1, C_0 and cat_0. It is clear that from Table 1 several interesting questions can be proposed about C_1 and cat_1. In a coming joint work with F. González-Acuña, we will explore some of these questions.

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N \ n	W_0	W_1
	$F_0(M) \leq$ N; [14]	$F_1(M) \leq$ N, M orientable; [12]
2	S^3	L
3	$\mathcal{B}$	M is orientable and admits a STS-filling with three solid tori
4	$\mathcal{M}$	$\mathcal{M}_0$
	$C_0(M) \leq$ N; [11]	$C_1(M) \leq$ N
2	S^3	
3	$\mathcal{B}$	
4	$\mathcal{M}$	$\mathcal{M}$
	$\mathrm{cat}_0(M) \leq$ N; [5]	$\mathrm{cat}_0(M) \leq$ N
2	$\pi_1(M) = 0 \Leftrightarrow M \simeq S^3$	
3	$\pi_1(M)$ is a free group $\Leftrightarrow M \simeq M' \in \mathcal{B}$	
4	$\mathcal{M}$	$\mathcal{M}$

Table 2:

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