Topology of Homology Manifolds

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1. Introduction

ANR homology n-manifolds are finite-dimensional absolute neighborhood retracts X with the property that for every $x \in X$, $H_i(X, X - \{x\})$ is 0 for $i \neq n$ and $\mathbb{Z}$ for $i = n$. Topological manifolds are natural examples of such spaces. To obtain nonmanifold examples, we can take a manifold whose boundary consists of a union of integral homology spheres and glue on the cone on each one of the boundary components. The resulting space is not a manifold if the fundamental group of any boundary component is a non-trivial perfect group. It is a consequence of the double suspension theorem of Cannon and Edwards that, as in the examples above, the singularities of polyhedral ANR homology manifolds are isolated. There are, however, many examples of ANR homology manifolds which have no manifold points whatever. See [12] for a good exposition of the relevant theory. The purpose of this paper is to begin a surgical classification of ANR homology manifolds, sometimes referred to in the sequel, simply as *homology manifolds*.

One way to approach this circle of ideas is via the problem of characterizing topological manifolds among ANR homology manifolds. In Cannon's work on the double suspension problem [6], it became clear that in dimensions greater than 4, the right transversality hypothesis is the following (weak) form of general position.

The Disjoint Disks Property (DDP). *For any $\epsilon > 0$ and maps $f, g : D^2 \to X$, there are maps $f', g' : D^2 \to X$ so that $d(f, f') < \epsilon$, $d(g, g') < \epsilon$ and $f'(D^2) \cap g'(D^2) = \emptyset$.*

The following result is an astonishing, powerful extension of the double suspension theorem.

Theorem (Edwards, [12]). *Let X^n, $n \geq 5$, be an ANR homology manifold satisfying the DDP. If $\phi : M \to X$ is a resolution of X, then ϕ is the limit of a sequence of homeomorphisms $h_i : M \to X$.*

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Recall that resolutions are maps $\phi : M \to X$ with the property that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is a homotopy equivalence for every open set $U \subseteq X$, where M is a topological n-manifold. Resolutions are therefore, fine homotopy equivalences that desingularize homology manifolds; they were introduced by Lacher in [22]. The conclusion of Edwards' theorem is that in the presence of the disjoint disks property, *any resolution can be approximated by homeomorphisms*. An important consequence is that a resolvable ANR homology manifold of dimension ≥ 5 is a manifold if and only if it has the DDP.

A natural question then arises: Are all homology manifolds resolvable? In other words, is every homology manifold with the DDP a manifold? In [5], Cannon proposed the following conjecture.

THE RESOLUTION CONJECTURE. *Every ANR homology manifold is resolvable.*

This in turn, by Edwards' theorem, implies the topological manifold characterization conjecture.

THE CHARACTERIZATION CONJECTURE. *Every ANR homology manifold of dimension* ≥ 5 *with the DDP is a topological manifold.*

Early results supporting these conjectures were obtained by Cannon and Bryant-Lacher [7], when the dimension of the singular set of the homology manifold is in the stable range. Quinn made a critical advance with the following resolution theorem.

THEOREM (QUINN, [29]). *There is a locally defined invariant* $\imath(X) \in 1+8\mathbb{Z}$ *for connected ANR homology manifolds* X *which measures the obstruction to resolution.* X *is resolvable, if and only if,* $\imath(X) = 1$.

The local character of Quinn's invariant implies that if X is connected and any open subset of X is a manifold (or even just resolvable), then X is resolvable. Thus, if X is an ANR homology manifold which is a "manifold with singularities" of any sort, then it is resolvable. The existence problem of nonresolvable homology manifolds, however, remained unanswered.

The main result of this paper is a systematic disproof of the *Resolution Conjecture*, one that yields many positive results. Before stating our main theorem we discuss some of its consequences.

Let M^n be a closed, simply connected manifold, $n \geq 6$.

COROLLARY. *Given $\imath \in 1 + 8\mathbb{Z}$, there is a canonical choice of an s-cobordism class of homology manifolds homotopy equivalent to M with resolution obstruction $\imath$.*

Hence, simply connected manifolds have, via the resolution obstruction, homotopy equivalent homology manifold parallels, one (s-cobordism class) for each $\imath \in 1 + 8\mathbb{Z}$.

There is also a relative version of this result for 2-connected manifold pairs, which has yet another corollary regarding the bordism of homology manifolds, answering a question of David Segal.

Let Ω_*^{SH} and Ω_*^{STop} denote the oriented bordism rings of ANR homology manifolds and topological manifolds, respectively.

COROLLARY. *In dimensions ≥ 6, there is an isomorphism $\Omega_*^{SH} \to \Omega_*^{STop}[1+8\mathbb{Z}]$ of graded (oriented) bordism rings, where $\Omega_*^{STop}[1 + 8\mathbb{Z}]$ has the monoid ring structure induced by multiplication in $1 + 8\mathbb{Z}$.*

One shows that every connected homology manifold is cobordant by a simply connected bordism to a simply connected homology manifold (a well-known fact for manifolds), and then uses the correspondence between between manifolds and homology manifolds given by the resolution obstruction. Note that there is no analogue of connected sum; disconnected homology manifolds whose components have different local indices need not be bordant to connected homology manifolds.

The fact that for 2-connected bordisms, the local type can be freely changed is not without parallel in other problems in geometry and topology (see e.g. [19]). When this is the case, it is important to consider $K(\pi, 1)$ spaces. The following observation is somewhat more elementary than the main theorem.

PROPOSITION (COMPARE [16]). *If X is a closed $K(\pi, 1)$ homology manifold, and the Novikov Conjecture holds for the group π, then every ANR homology manifold homotopy equivalent to X has the same local type.*

Thus, it follows from [18] that for groups of nonpositive curvature, the local type is rigid. For example, any ANR homology manifold which is a homotopy torus is resolvable (and in fact, by a torus). This was observed in [16]. Similar considerations give rise to (necessarily nonsimply connected) homology manifolds not homotopy equivalent to manifolds.

The main theorem requires a certain amount of surgery theory to state. Before discussing the result in its general form, we consider a slightly weaker version, suggested by Cappell, that supports our contention that homology manifolds are as natural as manifolds and should be present in any complete theory of topological manifolds.

Recall that classical surgery theory can be phrased as being the study of manifold structures on Poincaré spaces X. We shall consider more generally, ANR homology manifold structures on X. Define the homology manifold structure set of X, $\mathcal{S}^H(X)$, to be the set of equivalence classes of pairs (Y, f), where Y is a homology manifold and $f : Y \to X$ is a (simple) homotopy equivalence. (Y, f) and (Y', f') are equivalent if there exist an s-cobordism $(Z; Y, Y')$ of homology manifolds and a map $F : Z \to X$ extending f and f'. As usual, we shall use $\mathcal{S}(X)$ to denote the manifold structure set of X. For objects with boundary, where the boundary is already given the structure of a (homology) manifold, we will write $\mathcal{S}^H(X, \partial)$ for the structure set relative to the boundary.

Siebenmann observed that $\mathcal{S}(X)$ almost has a four-fold periodicity if X is a manifold. See [25] for a detailed treatment explaining how a $\mathbb{Z}$ factor obstructs genuine periodicity.

COROLLARY. *If X is a manifold, $\mathcal{S}^H(X) \cong \mathcal{S}^H(X \times D^4, \partial)$.*

The right hand side of the formula is a set which consists entirely of structures with manifold representatives, due to the rel ∂ condition. Thus, to get a fully periodic theory of manifolds, we need homology manifolds to fill in for some "missing objects". Note that this formula fails for manifold structure sets, for example, when $X = S^n$.

While surgery theory is usually stated using the language of normal invariants and surgery obstructions (see [3] and [33]), for the purposes of this paper it is more convenient to work with a variant of the more algebraic formulation due to Ranicki [16] that makes use of advances in controlled topology (see also [28] and [35]).

In conventional surgery theory one starts with a degree one normal invariant, which can be viewed as a first approximation to a given Poincaré space by a manifold; it corresponds to a topological reduction of the Spivak normal bundle of X (not all Poincaré spaces have normal invariants, but all ANR homology manifolds do [16]). The surgery obstruction is an element of a Witt group of quadratic forms (or their automorphisms) and measures the obstruction to finding a normal cobordism from this normal map to a (simple) homotopy equivalence.

Ranicki, following an earlier lead of Mischenko, viewed the surgery obstruction as just the algebraic cobordism class of the chain complex (with duality) of the mapping cone of the normal map. In [16], the group of controlled algebraic Poincaré complexes over X was identified with a group isomorphic to the normal invariant group of X. One can now consider the algebraic mapping cone of the duality of a Poincaré space X; it is the assembly of the local mapping cones of X. This is often called the *peripheral complex.* Since X satisfies Poincaré duality, the algebraic mapping cone is (globally) contractible, but its local structure reflects the local geometry of X. The *total surgery obstruction of*

X is the obstruction to algebraically cobording the peripheral complex of X to a locally contractible complex through contractible algebraic complexes. For a homology manifold, the total surgery obstruction vanishes since duality holds locally.

Let X^n, $n \geq 6$, be a (simple) Poincaré complex.

MAIN THEOREM. *There is a homology manifold (simple) homotopy equivalent to X, if and only if, the total surgery obstruction of X vanishes. If this is the case, there is a covariantly functorial 4-periodic exact sequence of abelian groups,*

$$\ldots \to H_{n+1}(X; \mathbb{L}) \to L_{n+1}(\mathbb{Z}\pi_1(X)) \to \mathcal{S}^H(X) \to H_n(X; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(X)),$$

where $\mathbb{L}$ is the simply connected surgery spectrum.

The difference between this spectrum and G/Top (i.e., the 1-connective surgery spectrum [30]) accounts for Quinn's obstruction. There is also a relative version of this theorem which we will not state here, although relative constructions are necessary for the proof of the main theorem. A weaker form of this result was announced in [4].

A word about the organization of the paper. In order to prove the above theorem, we will use results regarding controlled surgery proven in [17]. The idea is to perform constructions of ordinary surgery while keeping careful control on the size of handles. This will be reviewed in §2. In section 3, we reprove Quinn's resolution theorem using this surgery theory. This serves as an introduction to our setup and also shows how the resolution obstruction of the homology manifolds constructed will be detected. We are led to the use of controlled topology since homology manifolds can be characterized as those spaces which are controlled Poincaré complexes over themselves. The self-referential aspect of the solution will necessitate consideration of ϵ-Poincaré complexes realizing the resolution obstruction. Section 4 contains a discussion of these approximate homology manifolds and of other homotopy theoretical aspects of the problem. Section 5 gives a detailed construction of homology manifolds modelled on simply connected closed manifolds with arbitrary resolution obstruction. The main theorem is proved in §6 using similar techniques. In section 7 we apply the theorem to the construction of homology manifolds that are not homotopy equivalent to any closed manifold and in the final section of the paper we discuss refinements of the construction that yield nonresolvable homology manifolds with the disjoint disks property.

2. A REVIEW OF (ϵ, δ)-SURGERY THEORY

This section contains a discussion of the controlled surgery theory we will be using in this paper. Quinn discussed such a theory in lectures given in the late 1970's and early 1980's, but details, except for those in [28], [29], and [35] have never appeared.

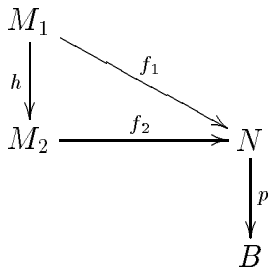
DEFINITION 2.1. If $p : N \to B$ is a map and $\epsilon > 0$ is given:

(i) A map $f : M \to N$ is said to be a $p^{-1}(\epsilon)$-*equivalence* if there exist a map $g : N \to M$ and homotopies $h_t : g \circ f \simeq \mathrm{id}_M$, $k_t : f \circ g \simeq \mathrm{id}_N$ so that $\mathrm{diam}(p \circ f \circ h_t(x)) < \epsilon$ for all $x \in M$ and $\mathrm{diam}(p \circ k_t(x)) < \epsilon$ for all $x \in N$. That is, we require that the tracks of the homotopies have diameter less than ϵ as viewed from B. We will also refer to such an f as an ϵ-*equivalence over* B.

(ii) Maps $f, g : M \to N$ are said to be ϵ-*homotopic over* B if there is a homotopy $h_t : f \simeq g$ so that $\mathrm{diam}(p \circ h_t(x)) < \epsilon$ for all $x \in M$.

DEFINITION 2.2. A proper surjection $p : K \to L$ is a UV^1 *map*, if for every $\epsilon > 0$ and every map $\alpha : P^2 \to L$ of a 2-complex into L with lift $\alpha_0 : P_0 \to K$ defined on a subcomplex P_0, there is a map $\bar{\alpha} : P \to K$ with $\bar{\alpha}|P_0 = \alpha_0$ and $d(p \circ \bar{\alpha}, \alpha) < \epsilon$. This is more-or-less equivalent to saying that p has simply-connected point-inverses. See [23] for details.

DEFINITION 2.3. If N is a manifold and $p : N \to B$ is a UV^1-map, $\mathcal{S}'_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix}$ is the set of equivalence classes of $p^{-1}(\epsilon)$ equivalences $f : M \to N$ where "equivalence" is the relation generated by saying that $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are equivalent if there is a homeomorphism $h : M_1 \to M_2$ so that the diagram



ϵ-homotopy commutes.

THEOREM 2.4 ((ϵ, δ)-SURGERY EXACT SEQUENCE [17]). *If N^n is a compact topological manifold, $n \geq 6$ or $n \geq 5$ when $\partial N = \emptyset$, B is a finite polyhedron, and $p : N \to B$ is UV^1, then there exist an $\epsilon_0 > 0$ and a $T \geq 1$ depending only on n and B so that for every $\epsilon \leq \epsilon_0$ there is a surgery exact sequence*

$$\ldots \longrightarrow H_{n+1}(B; \mathbb{L}) \longrightarrow \mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} \longrightarrow [N, G/Top] \longrightarrow H_n(B; \mathbb{L})$$

where $\mathbb{L}$ is the periodic L-spectrum of the trivial group and

$$\mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} = \mathrm{im} \left(\mathcal{S}'_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} \longrightarrow \mathcal{S}'_{T\epsilon} \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} \right).$$

Moreover, for $\epsilon \leq \epsilon_0$, $\mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} \cong \mathcal{S}_{\epsilon_0} \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix}$.

REMARK 2.5.

(i) If we rewrite $[N, G/Top] = H^0(N; G/Top)$ as $H_n(N; G/Top)$, then the surgery sequence is functorial in the obvious fashion with respect to UV^1 maps $B \to B_1$. The cases $B = \text{pt}$ (for N simply connected) and $B = N$ are particularly instructive. We have a diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{n+1}(N; \mathbb{L}) & \longrightarrow & \mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ N \end{pmatrix} & \longrightarrow & H_n(N; G/Top) & \longrightarrow & H_n(N; \mathbb{L}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & H_{n+1}(B; \mathbb{L}) & \longrightarrow & \mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} & \longrightarrow & H_n(N; G/Top) & \longrightarrow & H_n(B; \mathbb{L}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & H_{n+1}(\text{pt}; \mathbb{L}) & \longrightarrow & \mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ \text{pt} \end{pmatrix} & \longrightarrow & H_n(N; G/Top) & \longrightarrow & H_n(\text{pt}; \mathbb{L})
\end{array}$$

The α-approximation theorem of Chapman-Ferry ([10]) shows that $\mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ N \end{pmatrix}$ is trivial, so the diagram becomes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{n+1}(N; \mathbb{L}) & \longrightarrow & 0 & \longrightarrow & H_n(N; G/Top) & \longrightarrow & H_n(N; \mathbb{L}) \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle p_*} \\
\cdots & \longrightarrow & H_{n+1}(B; \mathbb{L}) & \longrightarrow & \mathcal{S}_\epsilon \begin{pmatrix} N \\ \downarrow \\ B \end{pmatrix} & \longrightarrow & H_n(N; G/Top) & \longrightarrow & H_n(B; \mathbb{L}) \\
 & & \downarrow{\scriptstyle proj} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle proj} \\
\cdots & \longrightarrow & L_{n+1} & \longrightarrow & \mathcal{S}(N) & \longrightarrow & H_n(N; G/Top) & \longrightarrow & L_n
\end{array}$$

where p_* is the induced map on homology, $\mathcal{S}(N)$ is the ordinary structure set of N, proj is the composition $H_n(B; \mathbb{L}) \to H_n(\text{pt}; \mathbb{L}) \cong H_0(pt; L_n) \cong L_n$, and $L_k = \mathbb{Z}, 0, \mathbb{Z}/2\mathbb{Z}, 0$, for $k = 0, 1, 2, 3 \mod(4)$. This computes the map $H_n(N; G/Top) \to H_n(B; \mathbb{L})$ in the surgery exact sequence of a manifold N as being the composition of the map from the connective L-theory to the nonconnective L-theory with the induced map $H_n(N; \mathbb{L}) \to H_n(B; \mathbb{L})$. The reader should be careful hereabouts when working with Poincaré duality spaces rather than manifolds. The map $H_n(X; G/Top) \to H_n(B; \mathbb{L})$ in Theorem 2.8 below is an equivariant map of $H_n(X; G/Top)$-sets with action on $H_n(B; \mathbb{L})$ induced by p_*, rather than a homomorphism. The fact that 0 need not go to 0 gives rise to the resolution obstruction.

(ii) The homology groups $H_n(X; G/Top)$ and $H_n(X; \mathbb{L})$ are easily computed, since away from odd primes these spectra are products of Eilenberg-MacLane spectra and at odd primes, they give real K-theory [24], [32]. This means that in the absence of odd torsion, $H_n(X; \mathbb{L}) \cong \oplus_{p+q=n} H_p(X; L_q)$ and $H_n(X; G/Top) \cong \oplus_{p+q=n, q>0} H_p(X; L_q)$. Induced homomorphisms between such groups are the induced homomorphisms on ordinary homology.

(iii) The (ϵ, δ)-surgery exact sequence is proven in [17] by comparing the ϵ-structure set to the bounded structure set of [16].

There is a corresponding surgery theory at the existence level. In order to state the theorem, we need some definitions.

DEFINITION 2.6. If X and B are finite polyhedra and $p : X \to B$ is a UV^1 map, we say that X is an *(oriented) ϵ-Poincaré complex of formal dimension n* over B if there exist a subdivision of X so that images of simplices have diameter $\ll \epsilon$ in B and a cycle y in the simplicial chains $C_n(X)$ so that $\cap y : C^{\#}(X) \to C_{n-\#}(X)$ is an ϵ-chain homotopy equivalence.

This last means that the morphism $\cap y$ and the chain homotopies have size $< \epsilon$ in the sense that the image of each generator σ only involves generators whose images under p are within ϵ of $p(\sigma) \subset B$. We define Poincaré duality as usual in the unoriented case by using an orientation double cover.

DEFINITION 2.7. Let X be an ϵ-Poincaré duality space of formal dimension n over a metric space B and let ν be a (Top, PL or O) bundle over X. *An ϵ-surgery problem* or *degree one normal map* is a triple (M^n, ϕ, F) where $\phi : M \to X$ is a map from a closed n-manifold M to X such that $\phi_*([M]) = [X]$ and F is a stable trivialization of $\tau_M \oplus \phi^*\nu$. Two problems (M, ϕ, F) and $(\overline{M}, \bar{\phi}, \bar{F})$ are equivalent if there exist an $(n+1)$-dimensional manifold W with $\partial W = M \coprod \overline{M}$, a proper map $\Phi : W \to X$ extending ϕ and $\bar{\phi}$, and a stable trivialization of $\tau_W \oplus \Phi^*\nu$ extending F and $\bar{F}$. Such an equivalence is called a normal bordism. See p. 9 of [33] for further details.

THEOREM 2.8 ((ϵ, δ)-SURGERY EXISTENCE [17]). *If B is a finite complex and $n \geq 5$ is given, then there exist an $\epsilon_0 > 0$ and a $T \geq 1$ so that for every ϵ with $\epsilon_0 > \epsilon > 0$, if* $\begin{array}{ccc} M^n & \xrightarrow{f} & X \\ & & \downarrow p \\ & & B \end{array}$ *is an ϵ-surgery problem with UV^1 control map p, then there is a well-defined obstruction in $H_n(B; \mathbb{L})$ which vanishes if and only if f is normally bordant to a $T\epsilon$-equivalence over B.*

REMARK 2.9.

(i) We will refer to ϵ_0 as the *(n-dimensional) stability constant* for B. T will be called the *(n-dimensional) stability factor* for B.

(ii) The situation is similar in the non-simply connected case, i.e., for control maps that are not UV^1, but it is somewhat more complicated because the surgery groups are not quite homology groups.

3. Quinn's resolution theorem and the obstruction to resolution

To acclimate the reader to our notation and to our setup, we reprove Quinn's resolution theorem and interpret the obstruction to resolutions from our (ϵ, δ) point of view. The argument we give is similar to proofs given by Quinn in lectures and is even more similar to the bounded proof given in [16]. It differs from the proof in [28], [29] in that the use of a full-featured ϵ-surgery theory makes parts of the argument easier to describe.

Theorem 3.1 (Quinn [28], [29]). *Let X be a connected ANR homology n-manifold, $n \geq 5$. Then there is an integral invariant $\imath(X)$ of X such that:*

(i) $\imath(X) \equiv 1 (mod\ 8)$.

(ii) If $U \subset X$ is open, then $\imath(X) = \imath(U)$.

(iii) $\imath(X \times Y) = \imath(X) \times \imath(Y)$.

(iv) $\imath(X) = 1$ if and only if there is a topological manifold M^n and a CE map $f : M^n \to X$.

Proof. By theorem 15.6 of [16], there is a degree one normal map $f : M \to X$. By Theorem 2.8, there is a surgery obstruction $\sigma_f \in H_n(X; \mathbb{L}) \cong [X, G/Top \times \mathbb{Z}]$ which vanishes if and only if f is normally bordant to an δ-equivalence $f_\delta : M_\delta \to X$ for each $\delta > 0$. Changing the map f changes σ_f by an element of $[X, G/Top]$, so we have an invariant in $[X, \mathbb{Z}]$ which measures the component of $G/Top \times \mathbb{Z}$ which is hit by X under σ_f. This is Quinn's invariant $\imath(X)$.

Property (ii) is clear, since U and X must map to the same component of $G/Top \times \mathbb{Z}$. Properties (i) and (iii) follow from the interpretation of $\imath(X)$ as a difference of signatures. This is discussed in [29], where Quinn defines the invariant by crossing with $\mathbb{C}P^2$ and looking at the signature of the inverse image of $* \times \mathbb{C}P^2$ in $M \times \mathbb{C}P^2$.

Suppose that the invariant is 1. Choose a sequence $\{\epsilon_i\}$ with $\lim_{i \to \infty} \epsilon_i = 0$. The ϵ-structure set of X parameterized over itself is trivial, since we have:

$$H_{n+1}(X; G/Top) \xrightarrow{\cong} H_{n+1}(X; \mathbb{L}) \longrightarrow \mathcal{S}_\epsilon \begin{pmatrix} X \\ \downarrow \\ X \end{pmatrix} \longrightarrow H_n(X; G/Top) \xrightarrow{1-1} H_n(X; \mathbb{L}).$$

The "$\cong$" and the "$1-1$" follow immediately from the Atiyah-Hirzebruch spectral sequence and the fact that $H_*(X, \mathbb{Z}) = 0$ for $* > n$.

Thus, we can choose a sequence $\{\delta_i\}$ so that for each i there is a homeomorphism $M_{\delta_i} \xrightarrow{h_i} M_{\delta_{i+1}}$ with the property that $d(f_{\delta_{i+1}} \circ h_i, f_{\delta_i}) < \epsilon_i$. Consider the maps

$$f_{\delta_i} \circ h_{i-1} \circ \ldots \circ h_1 : M_{\delta_1} \to X.$$

Since

$$d(f_{\delta_{i+1}} \circ h_i \circ \ldots \circ h_1, f_{\delta_i} \circ h_{i-1} \circ \ldots \circ h_1) = d(f_{\delta_i}, f_{\delta_{i+1}} \circ h_i) < \epsilon_i,$$

the sequence converges to a map $f : M_{\delta_1} \to X$, provided that $\sum \epsilon_i < \infty$. Since $f_{\delta_i} \circ h_{i-1} \circ \ldots \circ h_1$ is a δ_i-equivalence, f is CE, giving a resolution of X. ∎

REMARK 3.2.

(i) There is a technical point which should be addressed here – strictly speaking, X is not a polyhedron, so (ϵ, δ)-surgery theory does not apply to maps parameterized over X. Such difficulties are only apparent and are discussed in [26]–[29]. A quick way of avoiding the problem in the present case is to let M be a mapping cylinder neighborhood of X in $\mathbb{R}^\ell$, ℓ large. If Q is the Hilbert cube, there is a homeomorphism $X \times Q \to M \times Q$ whose inverse is as close as we like to $c \times \mathrm{id}$, where $c : M \to X$ is the mapping cylinder projection [9]. Thus there is a CE map $c_1 : X \times Q \to M$ so that $c \circ c_1$ is as close as we like to projection. On the other hand, there is a UV^1 map $c_2 : T^5 \to Q \times T^5$ ([15], [2]) so that composition with projection is as close as we like to id. Composition gives a UV^1 homotopy equivalence

$$X \times T^5 \xrightarrow{id \times c_2} X \times Q \times T^5 \xrightarrow{c_1 \times id} M \times T^5$$

so that composition with $c \times id : M \times T^5 \to X \times T^5$ is as close as we like to the identity. We can consider the structure set $\mathcal{S}_\epsilon \begin{pmatrix} X \times T^5 \\ \downarrow \\ M \times T^5 \end{pmatrix}$, where our UV^1 homotopy equivalence is used as a control map. The argument goes through as before to show that there are manifold structures on $X \times T^5$ which are arbitrarily well controlled over $M \times T^5$, and therefore over $X \times T^5$ via $c \times \mathrm{id}$. This leads as above to a resolution of $X \times T^5$. Passing to a cyclic cover and applying Quinn's End Theorem leads to a resolution of $X \times T^4$ and repeating the destabilization process leads to a resolution of X.

(ii) It is not really necessary to invoke Theorem 15.6 of [16]. We could work with small patches of X over which the Spivak normal bundle is trivial. On the other hand, that is the road to the proof given in [16] that Top reductions exist. Alternatively, we could

work with a controlled version of Ranicki's total surgery obstruction and avoid making explicit assumptions about the Spivak bundle.

4. Some technical preliminaries

This section contains technical lemmas which will be needed in the construction of the counterexamples. The reader is advised to skim this section quickly and then come back to it as needed.

Definition 4.1. We will say that a map $p : L \to B$ between finite polyhedra has the *absolute δ-lifting property* $AL^k(\delta)$ if whenever (P, Q) is a polyhedral pair with $dim(P) \leq k$, $\alpha_0 : Q \to L$ is a map, and $\alpha : P \to B$ is a map with $p \circ \alpha_0 = \alpha | Q$, then there is a map $\bar{\alpha} : P \to L$ extending α_0 with $d(p \circ \bar{\alpha}, \alpha) < \delta$.

Remark 4.2. Note that $p : K \to L$ is UV^1 if and only if it is $AF^2(\delta)$ for all δ.

Here is our first main technical result concerning UV^k maps. Results of this sort are originally due to Bestvina-Walsh-Wilson [2]. This particular formulation appears as theorem 3.1 of [15].

Proposition 4.3. *Let M^n be a compact connected manifold and let B be a connected finite polyhedron. For every $\epsilon > 0$ there is a $\delta > 0$ so that for every $\mu > 0$, if $p : M \to B$ is $AL^{k+1}(\delta)$, $2k+3 \leq n$, then p is ϵ-homotopic to an $AL^{k+1}(\mu)$-map. It follows that there is a $\delta_0 > 0$ so that every $AL^{k+1}(\delta_0)$-map is homotopic to a UV^k map. If $U \subset B$ is open, $p|p^{-1}(U)$ is UV^k, and $C \subset U$ is compact, then we may choose the limiting UV^k map to be equal to p on the inverse image of C.*

The next theorem is a global version of the result above in the sense that it dispenses with the $AL^k(\delta)$ hypothesis and replaces it with the hypothesis that the map have simply connected homotopy fiber. Theorems of this type first appeared in Bestvina's thesis [2].

Theorem 4.4. *If $f : (M^n, \partial M) \to K$ is a map from a compact manifold to a polyhedron, $n \geq 5$, and the homotopy fiber of f is simply connected, then f is homotopic to a UV^1 map. If $f|\partial M$ is UV^1, f is homotopic rel ∂ to a UV^1 map.*

We will also need some results concerning controlled Poincaré duality. In his thesis, Spivak proves that a polyhedron X is a Poincaré duality space of formal dimension k, if and only if, when $N(X)$ is a regular neighborhood of X in $\mathbb{R}^{n+k}$, n large, the inclusion $\partial N \to N$ has homotopy fiber S^{k-1} [31]. This says that $(N, \partial N)$ looks like a tubular neighborhood of X from the viewpoint of homotopy theory. If X is a Poincaré duality space, a cycle which links the top class of X generates the homology of the fiber. Conversely, if $S^{k-1} \to \partial N \to$

X is nullhomotopic, we have a cocycle in $(N, \partial N)$ corresponding to (D^k, S^{k-1}) whose dual in $H_n(X)$ is the top class of X. The next proposition is an estimated version of this result.

PROPOSITION 4.5. *Given n and B, there is an $\epsilon_0 > 0$ and $T > 0$ such that if $0 < \epsilon \leq \epsilon_0$ and X is an ϵ-Poincaré duality space of topological (not formal) dimension $\leq n$ over B with UV^1 control map $p : X \to B$, then for every abstract regular neighborhood N of X in which X has codimension at least 3, the restriction of the regular neighborhood projection to ∂N has the $T\epsilon$-lifting property.*

Proof. Let N be a regular neighborhood of X with $\dim N = n + m$, $m \geq 3$, and with regular neighborhood retraction $\gamma : N \to X$. The composition $\gamma \circ p : N \to B$ provides a control map for N.

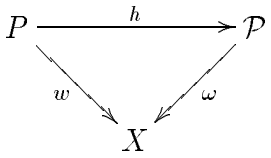
Assume $N \supseteq X$ are triangulated so that γ is simplicial and $\mathrm{diam}(\gamma \circ p)(\sigma) \ll \epsilon$ for each simplex σ in N. By hypothesis, there is an n-cycle y in $C_n(X)$ such that $\cap y : C^\sharp(X) \to C_{n-\sharp}(X)$ is an ϵ-chain homotopy equivalence over B. Given $\delta > 0$ the inclusion induced maps $i^\sharp : C^\sharp(N) \to C^\sharp(X)$ and $i_\sharp : C_\sharp(X) \to C_\sharp(N)$ can be chosen to be δ-chain homotopy equivalences over X by taking a sufficiently fine triangulation of N. Similarly, there is a fundamental cycle $z \in C_n(N, \partial N)$ giving a δ-chain homotopy equivalence $\cap z : C^\sharp(N, \partial N) \to C_{n+m-\sharp}(N)$ over X. Consider the following diagram

$$\begin{array}{ccccc} C^{m+q}(N, \partial N) & \xrightarrow{\cap z} & C_{n-q}(N) & \xleftarrow{i_\sharp} & C_{n-q}(X) \\ & \nwarrow^{G_q} & \uparrow{\scriptstyle \cap i_\sharp(y)} & & \uparrow{\scriptstyle \cap y} \\ & & C^q(N) & \xrightarrow{i^\sharp} & C^q(X) \end{array}$$

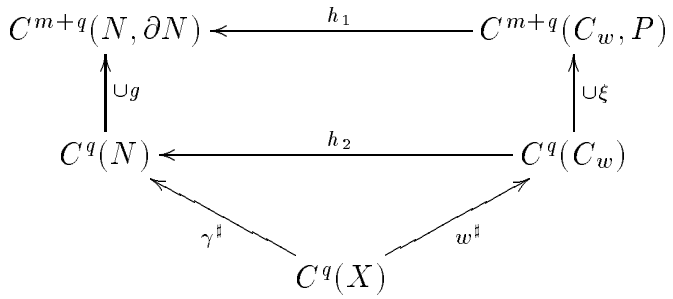
Since $\cap y$ is an ϵ-chain homotopy equivalence over B, we can make δ sufficiently small so that $\cap i_\sharp(y)$ is an ϵ-equivalence. We get G_q by composing $\cap i_\sharp(y)$ with a chain homotopy inverse to $\cap z$. Hence, we may assume that G_q is a ϵ-chain homotopy equivalence over B and that the left triangle is ϵ-chain homotopy commutative. By [31], we may take $G_q = \cup g$, where $g = G_0(1)$, since for each $c \in C^q(N)$, $(c \cup G_0(1)) \cap z = c \cap (G_0(1) \cap z)$, which is ϵ-chain homotopy equivalent to $c \cap (1 \cap i_\sharp(y)) = c \cap i_\sharp(y)$.

Let $\omega : \mathcal{P} \to X$ denote the path fibration associated with the map $\gamma|\partial N : \partial N \to X$. Here, $\mathcal{P} = \{(x, \lambda) \in \partial N \times X^I : \gamma(x) = \lambda(0)\}$ and $\omega(x, \lambda) = \lambda(1)$. The inclusion $j : \partial N \to \mathcal{P}$ given by $j(x) = (x, \lambda_x)$, where λ_x is the constant path at x, is a homotopy equivalence. Let $\mathcal{C}_\omega$ denote the mapping cylinder of ω with mapping cylinder retraction $\overline{\omega} : \mathcal{C}_\omega \to X$. Since N is the mapping cylinder of $\gamma : \partial N \to X$, the inclusion $\partial N \subseteq \mathcal{P}$ extends to an inclusion $j : (N, \partial N) \to (\mathcal{C}_\omega, \mathcal{P})$ such that $\omega \circ j = \gamma|\partial N$ and $\overline{\omega} \circ j = \gamma$. By [31], ω is a spherical fibration, since $m \geq 3$. Thus, given $\mu > 0$, one can take a fine triangulation of

X and inductively construct, as in the proof of theorem 5.3 of [15], a finite CW complex P and a spherical block bundle $w : P \to X$, for which there is a homotopy equivalence $h : P \to \mathcal{P}$ with homotopy inverse $h' : \mathcal{P} \to P$, such that the diagram



is μ-homotopy commutative. In particular, $w : P \to X$ will have the μ-homotopy lifting property. One can construct a Thom cocycle $\xi \in C^m(C_w, P)$ such that $\cup\xi : C^q(C_w) \to C^{m+q}(C_w, P)$ is a μ-chain homotopy equivalence over X. Assume that $h' \circ j$ is a cellular map. Consider the diagram



where the maps h_1 and h_2 are induced by $h' \circ j$. The definition of ξ guarantees that the upper square is ϵ-chain homotopy commutative. The maps $\gamma^\sharp$, $\omega^\sharp$, $\cup\xi$, and h_2 are controlled chain homotopy equivalences over X, hence over B, and the map $\cup g$ is an ϵ-chain homotopy equivalence over B. Hence, h_1 is an ϵ-chain homotopy equivalence over B. Comparing the short exact sequences of chain complexes for the pairs $(N, \partial N)$ and (C_w, P), one sees that the map $h' \circ j : \partial N \to P$, which we may assume is an inclusion, induces an ϵ-chain homotopy equivalence $h_3 : C^\sharp(P) \to C^\sharp(\partial N)$. Consequently, the pair $(P, \partial N)$ is homologically (ϵ, k)-connected for all $k \geq 0$ (see [26]).

Pick $\epsilon_0 > 0$ and a $T > 0$ such that if $0 < \epsilon \leq \epsilon_0$, then any two maps of a space into B within ϵ of each other are $T\epsilon$-homotopic, and any subset of B of diameter $< \epsilon$ can be contracted in a subset of diameter $< T\epsilon$. The constants ϵ_0 and T can be obtained, for example, from a fixed regular neighborhood $\pi : W \to B$, using the fact that the regular neighborhood projection is Lipschitz. For such a choice of ϵ, the pair $(P, \partial N)$ is $(T\epsilon, 1)$-connected, since all control maps are UV^1. We may assume that w is UV^1 by Proposition 4.3. By the controlled Hurewicz theorem [1], it follows that $(P, \partial N)$ is $(T\epsilon, k)$-connected for all k so that $h' \circ j : \partial N \to P$ is a $T\epsilon$-homotopy equivalence over B. Therefore, $\gamma|\partial N : \partial N \to X$ has the $T\epsilon$-homotopy lifting property over B. ■

If we paste together two Poincaré pairs using a homotopy equivalence on the boundary, we get a Poincaré duality space. The following is an ϵ-controlled analogue of that result.

PROPOSITION 4.6. *Given n and B there is an $\epsilon_0 > 0$ and $T > 0$ such that if $0 < \epsilon \leq \epsilon_0$, $(M_1, \partial M_1)$ and $(M_2, \partial M_2)$ are orientable manifolds, $p_1 : M_1 \to B$ and $p_2 : M_2 \to B$ are UV^1 maps, and $h : \partial M_1 \to \partial M_2$ is an orientation preserving ϵ-equivalence over B (this includes $d(p_1, p_2 \circ h) < \epsilon$), then $M_1 \cup_h M_2$ is a $T\epsilon$-Poincaré duality space over B.*

Proof. As in the proof of Proposition 4.5, there is an $\epsilon_0 > 0$ and a $T > 0$ such that if $0 < \epsilon \leq \epsilon_0$, then any two maps of a space into B that are within ϵ of each other are $T\epsilon$-homotopic. Given $0 < \epsilon \leq \epsilon_0$, suppose that $(M_1, \partial M_1)$, $(M_2, \partial M_2)$, and $h : \partial M_1 \to \partial M_2$ are given, where h is an ϵ-equivalence over B. Assume M_1 and M_2 are equipped with triangulations such that $h : \partial M_1 \to \partial M_2$ is simplicial. Let C_h be the simplicial mapping cylinder of h. We can take C_h to be a subcomplex of $(\partial M_1)' * (\partial M_2)'$, the join of first barycentric subdivisions of ∂M_1 and ∂M_2; C_h is homeomorphic to the topological mapping cylinder of h [11]. The maps $p_i : \partial M_i \to B$ extend to a map $p_0 : C_h \to B$ with respect to which ∂M_1 is a $T\epsilon$-deformation retract of C_h. Hence, the chain complexes $C_\sharp(M_1, \partial M_1)$ ($C^\sharp(M_1, \partial M_1)$, resp.) and $C_\sharp^{(\sharp)}(M_1 \cup C_h, \partial M_2)$ ($C^\sharp(M_1 \cup C_h, \partial M_2)$, resp.) are $T\epsilon$-chain homotopy equivalent.

Let $X_1 = M_1 \cup_{\partial M_1} C_h$, $X_2 = M_2$, $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2 = \partial M_2$, and $p : X \to B$ be the union of the p_i's, $i = 0, 1, 2$. The fundamental cycles $y_i \in C_n(M_i, \partial M_i)$ give rise to δ-chain homotopy equivalences $\cap y_i : C^\sharp(M_i, \partial M_i) \to C_{n-\sharp}(M_i)$ and $\cap y_i : C^\sharp(M_i) \to C_{n-\sharp}(M_i, \partial M_i)$, where δ can be made arbitrarily small by refining the triangulations of M_1 and M_2. Let $\overline{y}_1 \in C_n(X_1, X_0)$ be the fundamental cycle corresponding to y_1 under the $T\epsilon$-chain equivalence $C_\sharp(M_1, \partial M_1) \to C_\sharp(X_1, X_0)$. Choose orientations of $\overline{y}_1$ and y_2 so that $\partial \overline{y}_1 = -\partial y_2 \in C_{n-1}(X_0)$. Then $y = j_1(\overline{y}_1) + j_2(y_2) \in C_n(X)$ is a fundamental cycle for X, where j_i is the composition $C_\sharp(X_i, X_0) \to C_\sharp(X_i) \to C_\sharp(X)$. Write $C^\sharp(X) = C^\sharp(X_1) \oplus C^\sharp(X_2, X_0)$ and $C_{n-\sharp}(X) = C_{n-\sharp}(X_1, X_0) \oplus C_{n-\sharp}(X_2)$. Then, $\cap y = \cap(j_1(\overline{y}_1) + j_2(y_2)) : C^\sharp(X_1) \oplus C^\sharp(X_2, X_0) \to C_{n-\sharp}(X_1, X_0) \oplus C_{n-\sharp}(X_2)$ is easily seen to be a $T\epsilon$-chain homotopy equivalence. ∎

A map between simply connected Poincaré spaces which is an equivalence through the middle dimension is a homotopy equivalence. The following is an ϵ version of that (see [17]).

PROPOSITION 4.7. *Given n and B there is a $T > 0$ so that if $p_1 : X_1 \to B$ and $p_2 : X_2 \to B$ are ϵ-Poincaré spaces over B of the same formal dimension and topological dimension $\leq n$ with UV^1 control maps, and $f : X_1 \to X_2$ is a map with $d(p_2 \circ f, p_1) < \epsilon$ such that*

the algebraic mapping cone of f is ϵ-acyclic through the middle dimension, then f is a $T\epsilon$-equivalence.

We will also need an estimated version of the classical homotopy extension theorem.

THEOREM 4.8 (ESTIMATED HOMOTOPY EXTENSION THEOREM). *If X is a metric space, $f : X \to Z$ and $p : Z \to B$ are maps, $A \subset X$ is closed and Z is an ANR, and $F_t : A \to Z$ is an ϵ-homotopy over B starting at $F_0 = f$, then there is an ϵ- homotopy $\bar{F}_t : X \to Z$ extending F_t. Moreover, we can take $\bar{F}_t = f$ outside of any neighborhood of $A \times I$.*

Proof. The proof is the usual one. The map F extends to a map $X \times 0 \cup A \times I \to Z$. Since Z is an ANR, this map extends to a neighborhood U of $X \times 0 \cup A \times I \to Z$ in $X \times I$. We construct a map $r : X \times I \to U$ by pushing down lines $\{x\} \times I$ to the graph of a function. Composing with the extension of F gives $\bar{F}$. ∎

REMARK 4.9. A similar argument works if A is an ANR and Z is arbitrary.

The next proposition shows that if $r : V \to X$ is a retraction and X is controlled homotopy equivalent to Y, then there is a retraction $s : V \to Y$ which is ϵ close to r. This will be used in showing that the limit space we construct is an ANR.

PROPOSITION 4.10. *Suppose that X and Y are finite polyhedra, V is a regular neighborhood of X with $\dim V \geq 2 \dim Y + 1$, $p : V \to B$ is a map, $r : V \to X$ is a retraction and $f : Y \to X$ is an ϵ-equivalence over B. Then we can choose an embedding $i : Y \to V$ so that there is a retraction $s : V \to i(Y)$ with $d(p \circ r, p \circ s) < 2\epsilon$.*

Proof. Since $r \circ f = f$, given $\mu > 0$, we can choose an embedding $i : Y \to V$ with $r \circ i$ μ-homotopic to f. If $g : X \to Y$ is an ϵ-inverse for f, then we can choose μ small enough that $g \circ r \circ i$ is ϵ-homotopic to id_Y, which implies that $g \circ r|Y$ is ϵ-homotopic to id_Y. By the estimated homotopy extension theorem, there is a retraction $s : V \to i(Y)$ with $d(p \circ r, p \circ s) < 2\epsilon$. ∎

5. THE CONSTRUCTION OF SIMPLY CONNECTED EXAMPLES

Let M^n be a simply connected closed manifold of dimension $n \geq 6$. We begin the construction of a nonresolvable ANR homology manifold homotopy equivalent to M. We start by choosing a sequence $\{\delta_i, i \geq 0\}$, with $\delta_i > 0$ and $\sum \delta_i < \infty$.

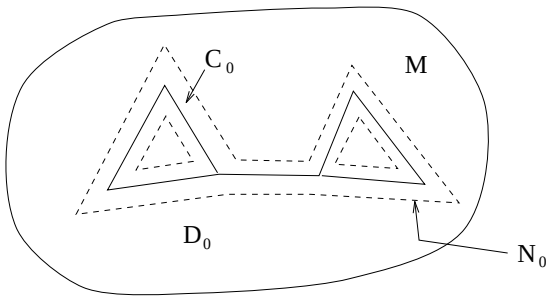
STEP I.

Let ϵ_0 be the stability constant for M and T_0 the stability factor. Given $\eta_0 > 0$, our goal in this step is to produce $p_0 : X_0 \to M$, where:
(a_0) p_0 is UV^1.

(b_0) X_0 is η_o-Poincaré over M.

(c_0) p_0 is a homotopy equivalence.

Choose $\mu_0 \ll \epsilon_0$. Take a triangulation of M with mesh $< \mu_0$ and consider a regular neighborhood C_0 of the 2-skeleton.



Let N_0 be the boundary of the regular neighborhood C_0 and D_0 be the closure of the complement of C_0 in M. By Proposition 4.3, there is a UV^1 map $q_0 : M \to M$ close to the identity that restricts to UV^1 maps on N_0, C_0 and D_0. The restriction of q_0 to N_0 will be denoted $\overline{q}_0 : N_0 \to M$. Let

$$\sigma \in H_n(M; \mathbb{L}) \cong H_n(M; L_0) \times H_n(M; G/Top)$$

be a nonzero element of $H_n(M; L_0) \cong \mathbb{Z}$.

According to the controlled analog of Wall's realization theorem (theorem 5.8 of [33]) applied to the controlled surgery target $N_0 \times I \to M$ obtained by composing $\overline{q}_0$ with the projection, for each $\mu > 0$, there is a cobordism $(V; N_0, N_0')$ over M and a degree one normal map $F_\sigma : V \to N_0 \times I$ so that

(i) $F_\sigma | N_0 = id$

(ii) $f_\sigma = F_\sigma | N_0'$ is a μ-equivalence over M.

(iii) The controlled surgery obstruction of F_σ rel ∂ over M is σ.

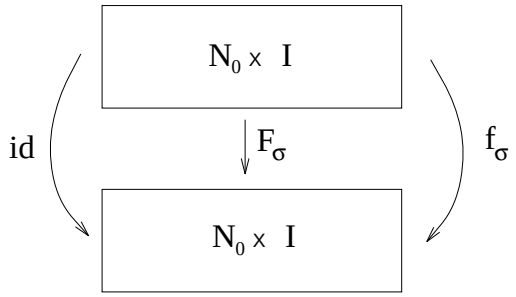
By Proposition 4.3, we can take f_σ to be a UV^1 map. The map $f_\sigma : N_0 \to N_0'$ obtained at the upper end of the normal cobordism represents a μ-controlled structure on N_0. We compare the controlled and uncontrolled surgery exact sequences to analyze f_σ as an uncontrolled structure:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & [\Sigma N_0, G/Top] & \longrightarrow & H_n(M; \mathbb{L}) & \longrightarrow & \mathcal{S}_\mu \begin{pmatrix} N_0 \\ \downarrow \\ M \end{pmatrix} & \longrightarrow & [N_0, G/Top] \\
 & \| & & \downarrow \mathcal{A} & & \downarrow & & \| \\
\cdots \longrightarrow & [\Sigma N_0, G/Top] & \longrightarrow & L_n(e) & \longrightarrow & \mathcal{S}(N_0) & \longrightarrow & [N_0, G/Top].
\end{array}$$

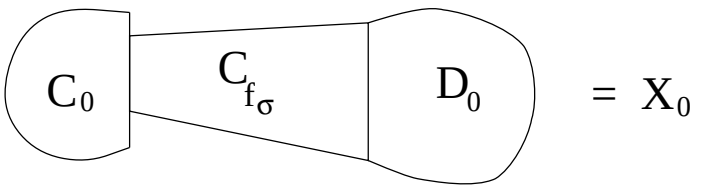
By theorem 15.8 of [16], the vertical map $\mathcal{A} : H_n(M; \mathbb{L}) \to L_n(e)$ is the surgery assembly map. Since M is simply connected, $\mathcal{A}$ can be factored as

$$H_n(M; \mathbb{L}) \cong H_n(M; L_0) \times [M, G/Top] \xrightarrow{\pi} [M, G/Top] \to L_n(e),$$

where π denotes projection. This implies that $\sigma \in \ker \mathcal{A}$ and therefore, $f_\sigma : N_0' \to N_0$ is trivial as an uncontrolled structure on N_0. By the h-cobordism theorem we can assume that $V = N_0 \times I$. By Proposition 4.4, we can also assume that F_σ is UV^1 after a homotopy relative to the boundary.



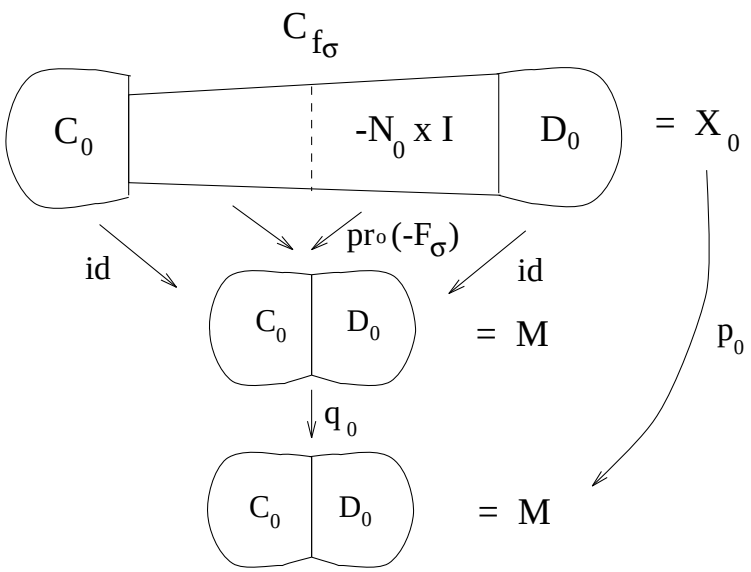
Choose such $f_\sigma : N_0 \to N_0$, for $\mu = \mu_0$. Form X_0 by cutting M open along N_0 and pasting the pieces back together using f_σ, where the copy of N_0 in D_0 is viewed as the domain of f_σ. X_0 is homeomorphic to the space obtained by splitting M along N_0 and pasting in a copy of the mapping cylinder C_{f_σ}.



We extend to X_0 the map that coincides with q_0 on $D_0 \cup C_0 \subseteq X_0$, to obtain a UV^1 homotopy equivalence $p_0 : X_0 \to M$, by collapsing the bottom half of C_{f_σ} to N_0 and defining p_0 to be the composition

$$N_0 \times I \xrightarrow{F_\sigma} N_0 \times I \to N_0 \xrightarrow{\overline{q}_0} M$$

on the upper half which is identified to $N_0 \times I$ along mapping cylinder lines.



By Proposition 4.6, if μ_0 is small enough so that $T\mu_0 < \eta_0$, then X_0 is an η_0-Poincaré duality space over M. This completes Step I.

STEP II.

Embed M into $\mathbb{R}^L$, L large, and approximate p_0 by an embedding $\iota_0 : X_0 \to \mathbb{R}^L$. From now on we identify X_0 with $\iota_0(X_0)$. This fixes a metric on X_0. Let W_0 be a regular neighborhood of X_0 in $\mathbb{R}^L$ with regular neighborhood collapse $r_0 : W_0 \to X_0$. Let $0 < \zeta_0 < \delta_0$ be such that L-dimensional (ζ_0, h)-cobordisms over M admit δ_0-product structures[1] [26], and let $\epsilon_1 > 0$ be the stability constant for X_0 and T_1 the stability factor. Our goal in this step is to show that if η_0 was chosen sufficiently small, then for any $\eta_1 > 0$ we can construct X_1 and $p_1 : X_1 \to X_0$ such that:

(a_1) p_1 is UV^1.

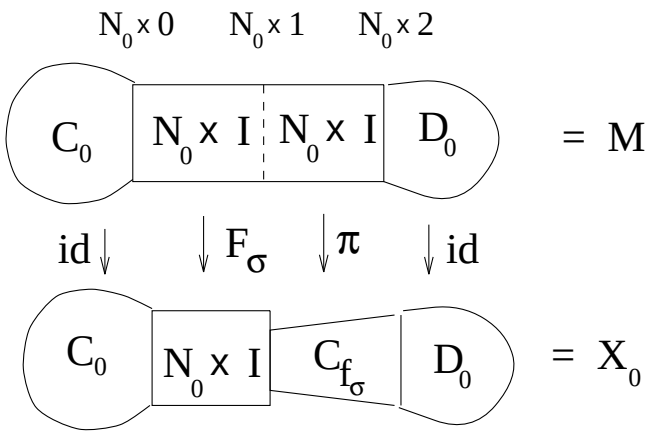
(b_1) X_1 is η_1-Poincaré over X_0.

(c_1) $p_1 : X_1 \to X_0$ is a ζ_0-equivalence over M.

(d_1) There exist an embedding $X_1 \to W_0$ and a retraction $r_1 : W_0 \to X_1$ such that $d(r_0, r_1) < \zeta_0$.

Let $f_0 : M \to X_0$ be the degree one normal map indicated in the picture below.

[1] Recall that the sequence $\{\delta_i\}$ was chosen at the very beginning.



Here, π denotes the obvious quotient map. Since f_0 induces a homeomorphism on the complement of $N_0 \times [1,2]$ in M, its controlled surgery obstruction over M is $\sigma \in H_n(M; \mathbb{L})$. By Proposition 4.4, after a homotopy, we can assume that f_0 is a UV^1 map.

Given an arbitrary $\mu_1 > 0$, triangulate M so that the image of the triangulation under f_0 has mesh $\ll \mu_1$. Let C_1 be a regular neighborhood of the 2-skeleton in this triangulation and let the closure of the complement of C_1 be D_1. Set $N_1 = C_1 \cap D_1$ and let $q_1 : M \to X_0$ be a UV^1 map close to f_0 that restricts to UV^1 maps on C_1, D_1 and N_1. The restriction of q_1 to N_1 will be denoted $\overline{q}_1 : N_1 \to X_0$.

Since $p_0 : X_0 \to M$ is UV^1, $(p_0)_* : H_n(X_0; \mathbb{L}) \to H_n(M; \mathbb{L})$ identifies the surgery obstruction groups so that we can think of σ as an element of $H_n(X_0; \mathbb{L})$. This isomorphism maps the component $H_n(X_0; L_0) \subseteq H_n(X_0; \mathbb{L})$ onto the corresponding component $H_n(M; L_0)$ of $H_n(M; \mathbb{L})$. As in Step I, realize $\sigma \in H_n(X_0; \mathbb{L})$ by a controlled surgery problem

$$\begin{array}{ccc} N_1 \times I & \longrightarrow & N_1 \times I \\ & & \downarrow \\ & & X_0 \end{array}$$

Cut M open along N_1 and paste in a copy of the mapping cylinder of the UV^1 controlled equivalence obtained at the upper end of this normal problem, to obtain a singular space X_1' which satisfies η_1-Poincaré duality over X_0, if μ_1 is small enough. The surgery obstruction of $X_1' \to X_0$ over M is 0, so we can do surgery to get a $T_0\eta_0$-controlled (over M) equivalence $p_1'' : X_1 \to X_0$. The homotopy equivalence is this large because X_0 is an η_0-Poincaré duality space over M.

Constructing p_1'' involves surgery on a singular space, but this is not difficult in our case. The spheres we need can be moved off of the 2-dimensional spine of C_1 and pushed away from the singular set by small moves. The bundle data tells us how to thicken the handles in the manifold part of X_1, so the problem is the same as a controlled manifold surgery

problem. It is important to note that F_σ preserves the bundle data so the map $X_1' \to X_0$ is a degree one normal map between Poincaré spaces respecting the given reductions of the Spivak normal bundles. This kind of Poincaré surgery goes back to Lowell Jones [20].

After surgery, we approximate the resulting homotopy equivalence by a UV^1 map $p_1 : X_1 \to X_0$, which is a $2T_0\eta_0$-equivalence. The existence of the embedding and the retraction follow immediately from Proposition 4.10. Choosing η_0 small allows us to verify conditions (c_1) and (d_1).

STEP III.

Steps I and II differ in that p_1 is a controlled homotopy equivalence, while p_0 is not. We continue the construction as in Step II, starting with a degree one normal map $f_i : M \to X_i$ with surgery obstruction $\sigma \in H_n(X_{i-1}; \mathbb{L})$. Let $0 < \zeta_i < \delta_i$ be such that L-dimensional (ζ_i, h)-cobordisms over X_{i-1} admit δ_i-product structures. If ϵ_i and T_i are the stability constant and stability factor for X_{i-1}, we choose η_i so that for any $\eta_{i+1} > 0$ we can construct X_{i+1} and $p_{i+1} : X_{i+1} \to X_i$ so that:

(a_i) p_{i+1} is UV^1.

(b_i) X_{i+1} is η_{i+1}-Poincaré over X_i.

(c_i) p_{i+1} is a ζ_i-equivalence over X_{i-1}.

(d_i) There is an embedding $X_{i+1} \to W_i \subset W_0$ and a retraction $r_{i+1} : W_0 \to X_{i+1}$ so that $d(r_i, r_{i+1}) < \zeta_i$. Here, W_i is a (very thin) regular neighborhood of X_i.

STEP IV.

Let $X = \cap_{i=1}^{\infty} W_i$. Taking the limit of the r_i's gives a retraction $r : W_0 \to X$; this shows that X is as ANR. X is homotopy equivalent to X_i, because for i large, we can retract a straight line homotopy from r_i to r into both X and X_i. We now refine the maps $\{r_i\}$ to retractions $\rho_i : W_0 \to X_i$ in order to argue that X is a homology manifold.

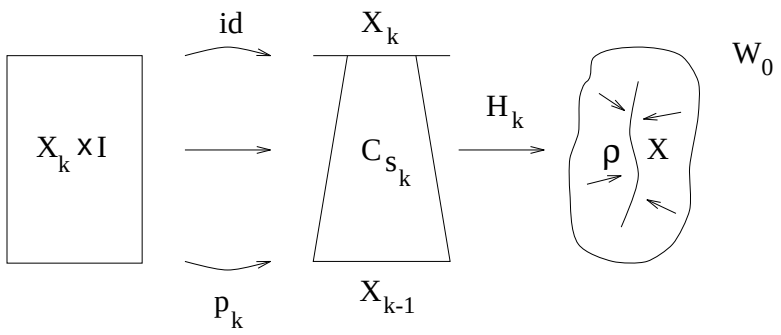
Let W_i be a small regular neighborhood of X_i. $W_{i-1} \setminus \mathrm{int}(W_i)$ is a thin h-cobordism with respect to the control map $r_i : W_0 \to X_i$. Deforming $W_0 \setminus \mathrm{int}(W_i)$ to ∂W_i along (thin) product structures and composing with a regular neighborhood collapse $W_i \to X_i$ induces a retraction $\rho_i : W_0 \to X_i$. In the limit, we obtain a new retraction $\rho : W_0 \to X$. By Proposition 4.5, the restriction of r_i to $\partial(W_i)$ has the $T\delta_{i-1}$-lifting property. Hence, $\rho_i|\partial W_0$ has the $T\delta_{i-1}$-lifting property and the restriction of $\rho : W_0 \to X$ to $\partial(W_0)$ is an approximate fibration. This shows that X is a homology manifold by [13].

Next, we show that the resolution obstruction of X is σ. It suffices to exhibit a degree one normal map $f : M \to X$ whose surgery obstruction over X is σ, since as pointed out earlier, $[X, G/Top]$ acts trivially on the $H_n(X; L_0) \cong \mathbb{Z}$ summand of $H_n(X; \mathbb{L})$ that gives rise to Quinn's obstruction. We assume that X is a polyhedron, since by Remark 3.2 (i)

we can do so after crossing with T^5 and replacing the control map $id : X \times T^5 \to X \times T^5$ by the UV^1 composition $X \times T^5 \to X \times Q \times T^5 \to W_0 \times T^5$, where Q denotes the Hilbert cube. Let

$$\begin{array}{ccc} M & \xrightarrow{f_k} & X_k \\ & & \downarrow p_k \\ & & X_{k-1} \end{array}$$

be the surgery problem with surgery obstruction $\sigma \in H_n(X_{k-1}; \mathbb{L})$ obtained in Step III of the construction. We show that for k large enough, the composition $M \xrightarrow{f_k} X_k \hookrightarrow W_0 \xrightarrow{\rho} X$ is a degree one normal map with surgery obstruction $\sigma \in H_n(X; \mathbb{L})$. After a small deformation, we can assume that the restriction of ρ to X_{k-1} is a small UV^1 homotopy equivalence, provided that k is large enough. This implies that $\rho_* : H_n(X_{k-1}; \mathbb{L}) \to H_n(X; \mathbb{L})$ is an isomorphism, and therefore $f_k : M \to X_k$ has surgery obstruction $\sigma \in H_n(X; \mathbb{L})$ with respect to the control map $X_k \xrightarrow{p_k} X_{k-1} \xrightarrow{\rho} X$. Let $s_k : X_{k-1} \to X_k$ be a fine homotopy inverse to $p_k : X_k \to X_{k-1}$ over X. The inclusions $X_{k-1} \subseteq W_0$ and $X_k \subseteq W_0$ extend to a map $H_k : C_{s_k} \to W_0$ defined on the mapping cylinder of s_k, that sends mapping cylinder lines to straight lines in W_0 connecting points $x \in X_{k-1}$ to their images $s_k(x) \in X_k$. Composing H_k with the retraction ρ induces a homotopy between the control maps $X_k \xrightarrow{p_k} X_{k-1} \xrightarrow{\rho} X$ and $X_k \xrightarrow{\rho} X$ as indicated in the picture.



For k large, we can assume that all maps are UV^1 after small homotopies. This shows that the surgery obstruction of the normal map $f_k : M \to X_k$ with respect to the control map $X_k \xrightarrow{\rho} X$ is also $\sigma \in H_n(X; \mathbb{L})$. Since $\rho : X_k \to X$ can be assumed to be a UV^1 controlled equivalence, it follows that the surgery problems

$$\begin{array}{ccc} M & \xrightarrow{f_k} & X_k \\ & & \downarrow \rho \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\rho \circ f_k} & X \\ & & \downarrow id \\ & & X \end{array}$$

have the same obstruction over X. This concludes the construction.

6. The general case

Let Z^n, $n \geq 6$, be a (simple) Poincaré complex.

THEOREM 6.1. *There is a homology manifold (simple) homotopy equivalent to Z, if and only if, the total surgery obstruction of Z vanishes. If this is the case, there is a covariantly functorial 4-periodic exact sequence of abelian groups*

$$\ldots \to H_{n+1}(Z; \mathbb{L}) \to L_{n+1}(\mathbb{Z}\pi_1(Z)) \to \mathcal{S}^H(Z) \to H_n(Z; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(Z)),$$

where $\mathbb{L}$ is the simply connected surgery spectrum.

Proof. Consider Ranicki's algebraic surgery sequence

$$\ldots \to H_n(Z; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(Z)) \to \mathcal{S}_{n-1}(Z) \to H_{n-1}(Z; \mathbb{L}) \to \ldots.$$

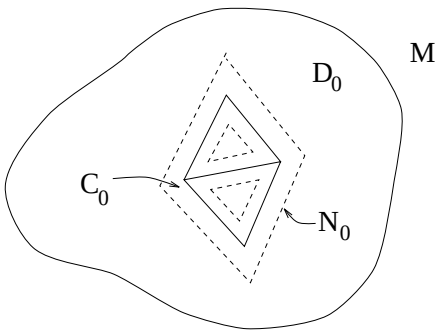
A Poincaré duality space Z has a total surgery obstruction $\mathcal{O}(Z) \in \mathcal{S}_{n-1}(Z)$ with the property that the image of $\mathcal{O}(Z)$ in $H_{n-1}(Z; \mathbb{L})$ is the obstruction to lifting the Spivak fibration to Top. When this vanishes, $\mathcal{O}(Z)$ is the image of $\sigma(f) \in L_n(\mathbb{Z}\pi_1(Z))$, where $f : M \to Z$ is any degree one normal map. If Z is an ANR homology manifold and $f : M \to Z$ is a degree one normal map as promised by [16], the obstruction $\sigma_c(f)$ to doing surgery to an ϵ-equivalence with respect to the control map $Z \xrightarrow{id} Z$ lies in the controlled Wall group $H_n(Z; \mathbb{L})$. By naturality, the ordinary surgery obstruction $\sigma(f)$ is the image of $\sigma_c(f)$ in $L_n(\mathbb{Z}\pi_1(Z))$. Since $\mathcal{O}(Z)$ is the image of $\sigma(f)$, $\mathcal{O}(Z) = 0$. When Z is homotopy equivalent to a homology manifold, the result follows from the functoriality of the surgery sequence.

Conversely, suppose that Z is a Poincaré duality space with $\mathcal{O}(Z) = 0$. If $f : M \to Z$ is a degree one normal map, the surgery obstruction $\sigma(f)$ lies in the image of the controlled surgery obstruction group $H_n(Z; \mathbb{L})$. Choose $\sigma \in H_n(Z; \mathbb{L})$ so that the image of σ is $\sigma(f)$ under the natural map $H_n(Z; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(Z))$.

Let $\{\delta_i, i \geq 0\}$ be a sequence with $\delta_i > 0$ and $\sum \delta_i < \infty$.

STEP I.

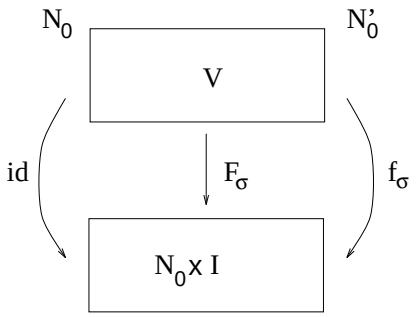
We may assume that $f : M \to Z$ is connected up to the middle dimension. By Proposition 4.3, we may take f to be a UV^1 map. Let ϵ_0 be the stability constant for Z and T_0 the stability factor. Take a triangulation of M with mesh $< \mu_0$, where $\mu_0 \ll \epsilon_0$, and consider the regular neighborhood C_0 of the 2-skeleton.



As before, we call the boundary of the regular neighborhood N_0 and let D_0 be the closure of the complement of C_0 in M. Let $q_0 : N_0 \to M$ be a UV^1 map close to the identity such that its restrictions to C_0, D_0 and N_0 are also UV^1. We denote the restriction of q_0 to N_0 by $\overline{q}_0 : N_0 \to M$. Consider the surgery exact sequence

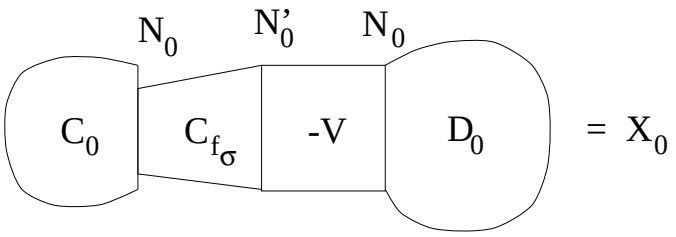
$$\ldots \to H_n(Z; \mathbb{L}) \to \mathcal{S}_\mu \begin{pmatrix} N_0 \\ \downarrow \\ Z \end{pmatrix} \to [N_0, G/Top].$$

By Wall realization, for $\mu = \mu_0$, there is a cobordism $(V; N_0, N_0')$ and a degree one normal map $F_\sigma : (V; N_0, N_0') \to (N_0 \times I; N_0 \times \{0\}, N_0 \times \{1\})$ realizing σ.



We can assume that $F_\sigma | N_0 = id$ and that $f_\sigma = F_\sigma | N_0'$ is a UV^1 μ_0-equivalence over Z. Doing surgery below middle dimension, by Proposition 4.4 we can also assume that F_σ is UV^1. Unlike the simply connected case, however, V is not necessarily a product since the uncontrolled surgery obstruction of F_σ, which coincides with $\sigma(f)$, may be nontrivial.

Let X_0' be the space obtained by splitting M along N_0 and pasting in a copy of V together with a copy of the mapping cylinder of f_σ.



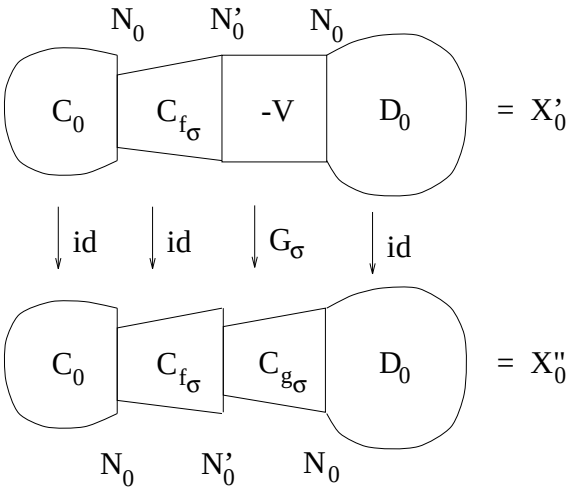
Our next goal is to construct a degree one UV^1 map $X_0' \to Z$. Let $g_\sigma : N_0 \to N_0'$ be a UV^1 μ_0-controlled homotopy inverse to f_σ and let $G'_\sigma : V \to N_0' \times I$ be the composition $G'_\sigma = (g_\sigma \times id) \circ F_\sigma$. Using the controlled homotopy extension theorem and Proposition 4.3, we can construct G_σ homotopic to G'_σ so that:

(i) $G_\sigma | N_0' = id$.

(ii) $G_\sigma | N_0 = g_\sigma$.

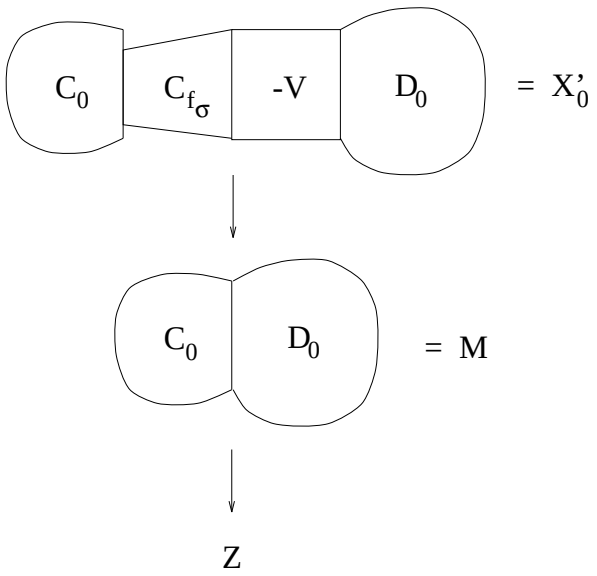
(iii) G_σ is UV^1.

Form the space X_0'' and define a map $X_0' \to X_0''$ as pictured below.



We obtain a UV^1 map $X_0' \to M$ by constructing a UV^1 map $c : C_{g_\sigma} \cup_{N_0'} C_{f_\sigma} \to N_0$ which is the identity on the two ends and forming the quotient space $X_0'' \cup_c N_0 = M$. The map c is constructed by collapsing C_{f_σ} and the bottom half of C_{g_σ} to N_0 and using Proposition 4.3 to extend to a UV^1 map over the top half of C_{g_σ} relatively to the ends. The composition $X_0' \to X_0'' \xrightarrow{c} M \xrightarrow{f} Z$ gives a degree one UV^1 map $p_0' : X_0' \to Z$ with respect to which X_0' is a $T\mu_o$-Poincaré duality space. If μ_0 is small enough, we have that

$T\mu_0 < \eta_0$.



The ordinary surgery obstruction of P_0' is $\sigma + (-\sigma) = 0$, so we can do surgery on X_0' to obtain an η_0-controlled Poincaré space X_0 with respect to a UV^1 homotopy equivalence $p_0 : X_0 \to Z$. This involves surgery on a singular space as discussed in the last section.

The rest of the construction proceeds essentially as in the simply connected case – starting with a degree one normal map $f_0 : M_0 \to X_0$, we construct an η_1-Poincaré space X_1 over X_0 with respect to a ζ_0-equivalence to X_0 over Z, an η_2-Poincaré space X_2 over X_1 with a ζ_1-equivalence to X_1 over X_0, et cetera. We embed the X_i's in a large Euclidean space R^L and take the limit. If the ζ_i's are such that L-dimensional (ζ_i, h)-cobordisms over X_{i-1} admit δ_i-product structures, we obtain an ANR homology manifold X homotopy equivalent to Z. The only difference is that, as in Step I, in the cut-paste construction we insert a copy of the mapping cylinder of the gluing map together with a copy of (the negative of) the cobordism obtained from the surgery obstruction realization theorem. We need this variant of the construction since, as observed earlier, the image of σ under the forget control map may be nontrivial. This concludes the proof of the first assertion of the theorem.

Our argument up to this point establishes the following bounded analog of the existence result (see [16]for a discussion of bounded surgery theory).

THEOREM 6.2. *Let X be a bounded Poincaré complex controlled with respect to a UV^1 map $X \to Z$. If the bounded total surgery obstruction vanishes, there is a homology*

manifold W bounded simple homotopy equivalent to X over Z.

REMARK 6.3. One could, of course, relax the UV^1 condition and prove controlled rather than bounded versions of this result.

Now we can complete the proof of exactness of the ordinary surgery exact sequence (and, by the "same" argument, the controlled and bounded surgery exact sequences). Suppose $\phi : X_1 \to X_2$ are simple homotopy equivalent homology manifolds. Consider

$$\mathcal{X} = O^{\geq 1}(X_1) \cup M(\phi) \cup O^{\geq 1}(X_2),$$

where $O^{\geq 1}(X)$ denotes the part of the open cone on X which lies outside of the unit ball. An easy calculation identifies the total surgery obstruction of this Poincaré duality space with the structure on X_2 represented by ϕ. If this obstruction vanishes, we can glue copies of X_1 and X_2 onto the ends of the resulting ANR homology manifold, obtaining an ANR homology manifold s-cobordism connecting X_1 and X_2. ∎

7. Homology manifolds not homotopy equivalent to any manifold

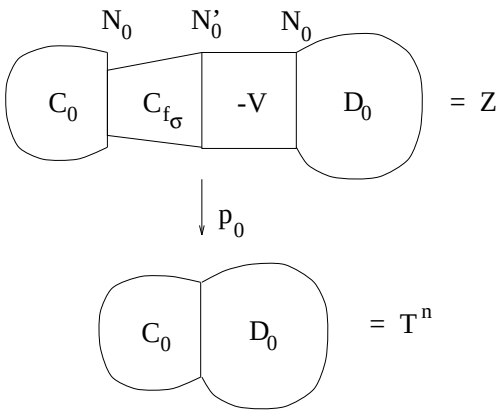
In this section, we apply Theorem 6.1 to the construction of homology manifolds that are not homotopy equivalent to any closed manifold.

Take a very fine triangulation of T^n, $n \geq 6$, and consider the regular neighborhood C_0 of the 2-skeleton. As before, let N_0 be the boundary of C_0 and let D_0 be the closure of the complement of C_0 in T^n. Let $p_0' : N_0 \to T^n$ be a UV^1 map close to the inclusion and

$$\sigma \in H_n(T^n; \mathbb{L}) \cong \bigoplus_{p+q=n} H_p(T^n; L_q)$$

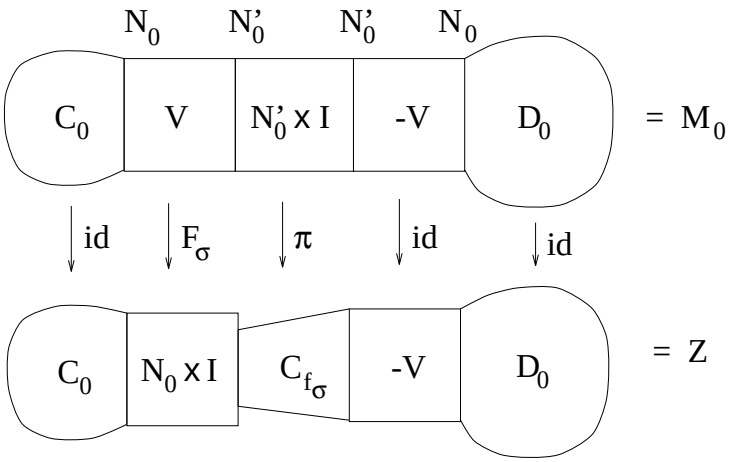
be a non-zero element of $H_n(T^n; L_0) \cong \mathbb{Z}$. As before, by Wall realization, there is a cobordism $(V; N_0', N_0)$ and a degree one normal map $F_\sigma : V \to N_0 \times I$ realizing σ, such that $F_\sigma | N_0 = id$ and $f_\sigma = F_\sigma | N_0' : N_0' \to N_0$ is a fine homotopy equivalence over T^n.

Form the Poincaré complex Z by pasting $C_{f_\sigma} \cup_{N_0'} (-V)$ into T^n, and construct a UV^1 control map $p_0 : Z \to T^n$ as in the previous section.



We show that Z is a Poincaré space whose total surgery obstruction $\mathcal{O}(Z)$ vanishes and that Z is not homotopy equivalent to any closed manifold. By Theorem 6.1, there is a homology manifold homotopy equivalent to Z.

Let M_0 be the manifold obtained by splitting T^n along N_0 and pasting $V \cup_{N_0'} (-V)$ between the two parts. Consider the normal map $f_0 : M_0 \to Z$ indicated below.



The surgery obstruction of $f_0 : M_0 \to Z$ over T^n is $\sigma \in H_n(T^n; \mathbb{L})$. Since the ordinary surgery obstruction of f_0 is the image of σ under the natural map $H_n(T^n; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(T^n))$, it follows from Ranicki's algebraic surgery sequence that $\mathcal{O}(Z) = 0$, as discussed in the proof of Theorem 6.1.

Any other degree one normal map $f : M \to Z$, differs from f_0 by the action of $H_n(Z; G/Top)$. Hence, the controlled surgery obstruction of f has σ as its $H_n(T^n; L_0)$

component since choosing a different normal map changes the surgery obstruction by the action of $H_n(Z; G/Top)$ on $H_n(T^n; \mathbb{L})$ induced by $(p_0)_*$, and this does not affect the $H_n(T^n; L_0)$ component.

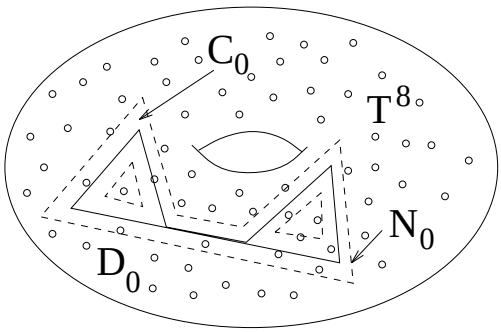
The forgetful map from the controlled surgery obstruction group $H_n(T^n; \mathbb{L})$ to the uncontrolled surgery obstruction group $L_n(\mathbb{Z}(\mathbb{Z}^n))$ is an isomorphism (this appears disguised as the map $[T^n \times D^4, \partial; G/Top] \to L_{n+4}(\mathbb{Z}(\mathbb{Z}^n))$). Thus, the *uncontrolled* surgery obstruction of any degree one normal map $f : M \to Z$ is also nonzero. This implies that Z does not have the homotopy type of any closed topological manifold.

REMARK 7.1. The same construction works if we replace T^n by any closed manifold M^n for which the summand $H_n(M; L_0)$ of $H_n(M; \mathbb{L})$ injects in $L_n(\mathbb{Z}\pi_1(M))$ under the assembly map $\mathcal{A} : H_n(M; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(M))$. Hence, manifolds for which the Novikov conjecture holds give rise to homology manifolds not homotopy equivalent to manifolds.

Using techniques from [8], we can vary the construction to obtain a $\mathbb{Z}$-homology T^n with fundamental group $\mathbb{Z}^n$ which is not homotopy equivalent to a manifold. Since $\mathcal{S}(M) \cong \mathcal{S}(N)$ if there is a map $M \to N$ which is a π_1-isomorphism and an integral homology equivalence, this gives a counterexample to a "homology" analog of the well-known conjecture that every Poincaré duality group is the fundamental group of an aspherical manifold. See [34].

REMARK 7.2. Here is a more concrete construction of the cobordism V used above. We begin with the Milnor plumbing, which gives us a degree one normal map $f : M^8 \to D^8$ which is a homeomorphism on the boundary. Take the product withn T^8. The result is a 16-dimensional surgery problem which cannot be solved rel boundary since it has nontrivial normal invariant which is detected by the signature of the inverse image of the D^8 and there are no nontrivial structures on $T^n \times D^k$ rel ∂. This problem is invariant under passage to finite covers of the torus.

Do surgery up to middle dimension. Call the resulting problem $(M_1^{16}, \partial) \to (D^8 \times T^8, \partial)$. The remaining obstruction is a quadratic form on the 8-dimensional homology of M_1. Pass to a large finite cover of T^8. We obtain a "geometric quadratic form" over T^8, that is, a quadratic form for which the basis elements are associated to points in T^8 and for which basis elements only interact with nearby basis elements.



The cobordism V used in the first stage of the construction is obtained by pushing the geometric quadratic form off of the 2-skeleton and out of C_0 by general position and then doing a "geometric" Wall realization of this form over T^8 starting with $N_0 \times [0,1]$. Notice that we have shifted dimensions. The original form was on 8-dimensional homology, while the construction of V uses the "same" form on 4-dimensional homology.

8. Final comments and conjectures

A. Using a mild refinement of the arguments, we can arrange that the homology manifolds we construct have the disjoint disks property.

Definition 8.1. Given $\epsilon > \delta > 0$, we say that a space X has the (ϵ, δ)-DDP if for each pair of maps $f, g : D^2 \to X$ there is a pair of maps $\overline{f}, \overline{g} : D^2 \to X$ so that $d(\overline{f}(D^2), \overline{g}(D^2)) > \delta$, $d(f, \overline{f}) < \epsilon$ and $d(g, \overline{g}) < \epsilon$.

Proposition 8.2. *If M^n is a PL manifold, $n \geq 5$, then for every $\epsilon > 0$ there is a $\delta > 0$ such that M has the (ϵ, δ)-DDP.*

Proof. Choose a fine triangulation of M so that maps $D^2 \to M$ can be displaced into the 2-skeleton or the dual $(n-3)$-skeleton by ϵ-moves. The distance between the 2-skeleton and the dual $(n-3)$-skeleton gives the desired δ. ∎

Remark 8.3. Using [21], one can show that Proposition 8.2 is also valid for topological manifolds.

Proposition 8.4. *Let $\{X_i\}_{i=1}^{\infty}$ and X be subsets of $\mathbb{R}^L$ and $r_i : X_i \to X$, $s_i : X \to X_i$ be maps so that $\lim_{i\to\infty} \sup\{d(x, r_i(x)), x \in X_i\} = 0$ and $\lim_{i\to\infty} \sup\{d(x, s_i(x)), x \in X\} = 0$. If X_i has the (ϵ, δ)-DDP for each i, then X has the $(2\epsilon, \delta/2)$-DDP.*

Proof. Choose i large enough so that $d(x, r_i(x)) < \delta/4$ and $d(x, s_i(x)) < \delta/4$. Given $f, g : D^2 \to X$, consider the compositions $s_i \circ f$, $s_i \circ g : D^2 \to X_i$. By hypothesis, there

are maps f', $g' : D^2 \to X_i$ such that $d(s_i \circ f, f') < \epsilon$, $d(s_i \circ g, g') < \epsilon$ and $d(s_i \circ f(D^2), s_i \circ g(D^2)) > \delta$. Then, $r_i \circ f'$, $r_i \circ g' : D^2 \to X$ are maps with $d(r_i \circ f'(D^2), r_i \circ g`(D^2)) > \delta/2$, $d(f, r_i \circ f') < \epsilon + \delta < 2\epsilon$ and $d(g, r_i \circ g') < 2\epsilon$. ■

If X has the (ϵ_i, δ_i)-DDP for some sequence $\epsilon_i \to 0$, then X has the DDP. This suggests a method for manufacturing ANR homology manifolds with the DDP. In step II of §5, we start with a map $f_0 : M \to X_0$. Let $\overline{\epsilon}_0 > 0$ be given. After surgery below the middle dimension and a homotopy, we can assume that this map is UV^1. Choose an embedding f_0' so that $f_0 = r_0 \circ f_0'$ (add some extra Euclidean factors, if necessary, and push up to the graph) and so that f_0' is η-close to f_0, where $\eta \ll \eta_0 T_0$. Here, $r_0 : W_0 \to X_0$ is a regular neighborhood collapse. We may assume that W_0 was chosen so that $d(r_0, id) < \overline{\epsilon}_0/4$. By Propostion 8.2, there is a $\overline{\delta}_0$ so that $f_0'(M)$ has the $(\overline{\epsilon}_0/2, \overline{\delta}_0)$-DDP. By simplicial approximation, any sufficiently fine 2-skeleton of M will also have the $(\overline{\epsilon}_0/2, \overline{\delta}_0)$-DDP.

Continue the construction of X_1, taking C_1 to be a regular neighborhood of a fine 2-skeleton of this embedded copy of M and taking the map $p_1|C_1$ to be a UV^1 approximation to the restriction of $r_0 \circ f_0$. As before, we extend this to a UV^1 map $p_1 : X_1 \to X_0$. Using the same graphing trick, we can arrange that $p_1 = r_0|X_1$. It follows that X_1 has the $(\overline{\epsilon}_0, \overline{\delta}_0)$-DDP. If $f, g : D^2 \to X_1$ are maps, then there are lifts $\overline{r_0 \circ f}$ and $\overline{r_0 \circ g}$ of $r_0 \circ f$ and $r_0 \circ g$ into C_1 which are $\overline{\epsilon}_0/2$-close to f and g. These can be pushed $\overline{\delta}_0$ units apart using a move of no more than $\overline{\epsilon}_0/2$ units.

If the η_i's are small enough, Proposition 8.4 guarantees that the ANR homology manifold X constructed by our process will have the $(2\overline{\epsilon}_0, \overline{\delta}_0/2)$-DDP. Choosing $\overline{\epsilon}_1 \ll \overline{\epsilon}_0$, doing surgery below the middle dimension to produce a UV^1 degree one normal map, and repeating the process at the next stage of the construction guarantees that the limit space will have the $(2\overline{\epsilon}_1, \overline{\delta}_1/2)$-DDP, as well. Iterating the process for all n produces an ANR homology manifold X with the DDP. Note that the various approximations in this process reinforce each other – choosing a triangulation of finer mesh leads to a smaller Poincaré duality constant η_1 while allowing a smaller homotopy from $r_0 \circ f_0'$ to a UV^1 map and preserving the (ϵ, δ)-DDP.

B. To what extent do these new spaces resemble manifolds? We made a number of conjectures in [4]. Here is a unified version of those conjectures.

UNIFIED CONJECTURE. *There exist spaces $\mathbb{R}^4_k$, $k \in \mathbb{Z}$, so that every connected ANR homology manifold X_n, $n \geq 5$, with the DDP and $\iota(X) = 8k + 1$ is locally homeomorphic to $\mathbb{R}^4_k \times \mathbb{R}^{n-4}$. These ANR homology manifolds are classified up to homeomorphism by Ranicki's algebraic surgery theory. In particular, we conjecture that high-dimensional*

ANR homology manifolds with the DDP are homogeneous, that the s-cobordism theorem holds for ANR homology manifolds with the DDP and that such homology manifolds are classified up to homeomorphism by a surgery exact sequence

$$\ldots \to H_{n+1}(X; \mathbb{L}) \to L_{n+1}(\mathbb{Z}\pi_1(X)) \to \mathcal{S}^H(X) \to H_n(X; \mathbb{L}) \to L_n(\mathbb{Z}\pi_1(X)).$$

As "evidence" for the conjecture, we point out that we could have performed the construction in §5 using any element of the controlled surgery obstruction group $H_n(M; \mathbb{L})$. In case the $H_n(M; L_0)$-component of this obstruction is zero, the construction yields a resolvable ANR homology manifold with the DDP, i.e., a topological manifold. Our conjecture says that the case when the $H_n(M; L_0)$-component vanishes is not anomalous and that equally canonical results can be achieved in all cases.

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