Journal of Knot Theory and Its Ramifications

SEIFERT UNIONS AND SPACES OF GRAPHS IN S^3

J. C. GOMEZ-LARRAÑAGA
Centro de Investigación en Matemáticas
A. P. 402, Guanajuato 36000, Gto. México.

WOLFGANG H. HEIL
Department of Mathematics, Florida State University
Tallahassee, FL 32306, USA.

ABSTRACT

Let Γ be a non-splittable graph in S^3, not a tree. It is shown that if the space $M = \overline{S^3 - N(\Gamma)}$ of Γ is a Seifert union of solid tori then M is homeomorphic to the space of a tree of Burde-Murasugi links.

Keywords: Burde-Murasugi links, graphs, Seifert unions, Seifert-webs.

1. Introduction

Let l be a link in S^3. In [2] and [1], Burde-Zieschang and Burde-Murasugi showed that if the space $\overline{S^3 - N(l)}$ of l in S^3 is a Seifert fiber space then each component of l can be chosen to be a fiber of some Seifert fibration of S^3, including the singular fibration and its unknotted circle. This singular fibration of S^3 is the trivial fibration of an unknotted solid torus is S^3 that does not extend to a fibration of S^3. A component of the Hopf link is a fiber of this fibration; the other component of the Hopf link is the unknotted circle that consists of the singular points not included in the fibration. We will call such links Burde-Murasugi links. It is clear that any Seifert fiber space M with $\partial M \neq \phi$ can be obtained by gluing together solid tori along a mutually disjoint collection of essential annuli in their boundaries. Of course, in this case, if M is orientable, then ∂M is a finite set of tori.

In order to consider spaces of complements of graphs in S^3 with boundary components surfaces of arbitrary genus we consider Seifert unions. A manifold is a

Seifert union if it is obtained from any number of (punctured) solid tori and (punctured) balls by gluing them together along any family of mutually disjoint surfaces (not necessarily essential) in their boundaries (e.g. see fig. 1 for an example of a Seifert union of two solid tori).

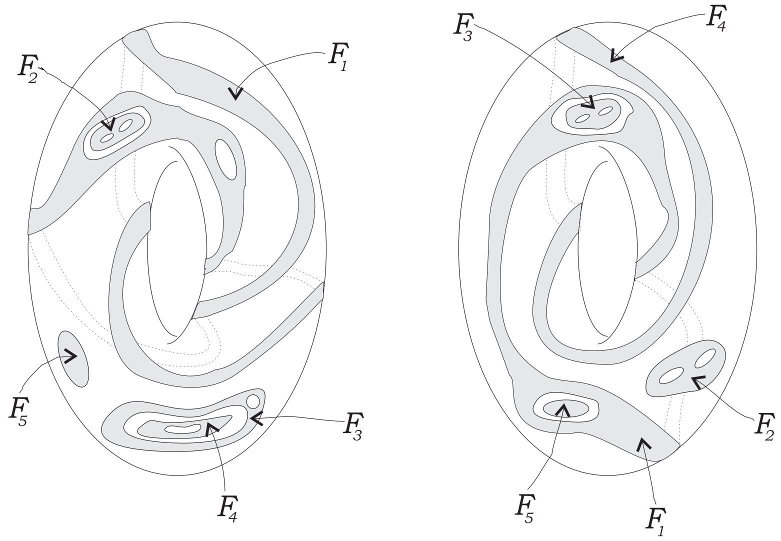

Fig. 1.

The Seifert unions were classified in [3]. Let Γ be a non-splittable graph in S^3. In this paper we show that if the space $M = \overline{S^3 - N(\Gamma)}$ of Γ is a Seifert union, not a ball, then M is homeomorphic to the space of a tree of Burde-Murasugi links as defined in section 1. A basic reference for Seifert fiber spaces is [4].

2. Trees of Burde-Murasugi Links

Let l be a link in S^3 which is not the unlink of more than one component. We say that l is a *Burde-Murasugi link* if the components of l are components of a torus link on an unknotted torus T^2 in S^3, possibly together with curves α and β as in fig. 2.

In other words, see [1], each component of l can be chosen to be a fiber of some Seifert fibration of S^3, including the singular fibration and its unknotted circle.

A *graph* in S^3 is $\left(S^3, \Gamma\right)$, where Γ is an embedding of a compact graph in S^3. As usual, the *space* of a graph Γ in S^3 is $\overline{S^3 - N(\Gamma)}$ for a regular neighborhood $N(\Gamma)$ of Γ. A graph Γ in S^3 is *non-splittable* if the space of Γ is irreducible.

A *tree T of links* $l_1, \ldots, l_n$ is a graph Γ in S^3 obtained from a finite tree T as follows:

Embed T in S^3 such that the vertices $v_1, \ldots, v_m$ lie in disjoint balls $B_1, \ldots, B_m$.

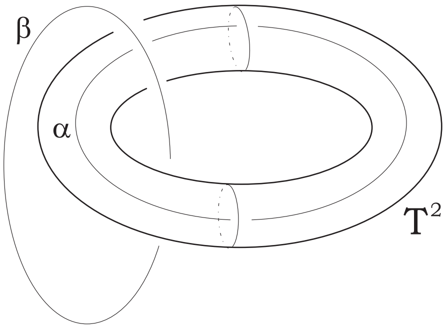

Fig. 2.

Replace each vertex v_i in B_i with the link l_i in B_i and replace each edge e_k of T from v_i to v_j by an arc α_k in S^3, that joints a component of l_i to a component of l_j. (The arcs are mutually disjoint and $\alpha_k \cap \cup l_\mu = \alpha_k \cap (l_i \cup l_j) = \partial\alpha_k$).

Note that a tree of nonsplittable links $l_1, \ldots, l_n$ is an nonsplittable graph. Since the spaces of Burde-Murasugi links are Seifert fiber spaces it follows that the space M of a tree T of Burde-Murasugi links is a disk sum (sometimes called boundary connected sum) of Seifert fiber spaces, called the *nodes* of M.

3. Seifert Unions and Spaces of Graphs

It is not true that if the space of a graph Γ is a Seifert union then Γ is a tree of Burde-Murasugi links, e.g. [3]. However we show

Theorem. *If the space M of a nonsplittable graph Γ in S^3 is a Seifert union, then either Γ is a tree or M is homeomophic to the space of a tree of Burde-Murasugi links.*

As illustration see fig. 3 for a tree of Burde-Murasugi links of ten components. Note that in any tree of Burde-Murasugi links, each component is a tree of torus knots.

Proof. In [3] it is shown that any Seifert union is a connected sum of S^2-bundles over S^1, lens spaces, a punctured ball and Seifert-webs. Here a Seifert-web $W(S_1, \ldots, S_n)$ is a connected manifold obtained from Seifert fiber spaces S_i with $\partial S_i \neq \phi$ by attaching a finite number of 1-handles.

In the case at hand, $\partial M \neq \phi$ and M is irreducible, hence M is either a ball or a Seifert-web. If M is a ball then Γ is a tree. So assume M is a Seifert-web $W(S_1, \ldots, S_n) \subset S^3$. Deleting all the 1-handles we see that each S_i is a Seifert fiber space embedded in S^3 and hence by [1], S_i is the space of a Burde-Murasugi link.

In the Seifert-web $W(S_1, \ldots, S_n)$, no 1-handle is attached to different boundary components of the same S_i. Otherwise we obtain a simple closed curve by connecting the endpoints of the core of the 1-handle by an arc in S_i that intersects a boundary

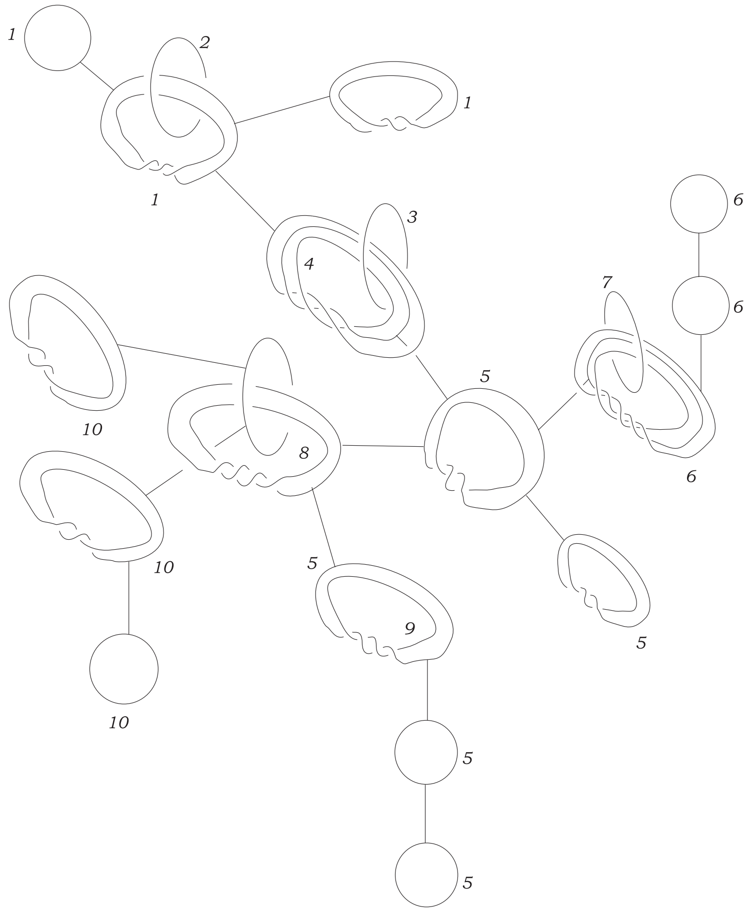

Fig. 3.

surface of S_i transversely in one point. This is impossible since every closed surface in S^3 is separating.

By the same argument there are no two 1-handles joining different boundary components of one S_i to one or more boundary components of some S_j, $j \neq i$.

If there are $k \geq 2$ 1-handles joining one boundary component of S_i to one boundary component of S_j then we slide the attaching disks of the first $(k-1)$ 1-handles across the last 1-handle to either S_i or S_j.

Thinking of the S_i's as vertices and the 1-handles as edges, suppose there is a simple directed cycle $\alpha_0\alpha_1\cdots\alpha_{k-1}$ where the 1-handle α_p joins S_{i_p} to $S_{i_{p+1}}$, where p is mod k.

Then the attaching disks for the 1-handles α_p and α_{p+1} in $S_{i_{p+1}}$ must lie in the same boundary component of $S_{i_{p+1}}$ (otherwise we can construct a simple closed curve intersecting a boundary component exactly once). In this case slide the attaching disk of the 1-handle α_0 in S_{i_1} across the 1-handles $\alpha_1, \ldots, \alpha_{k-1}$ into the boundary component of S_{i_0} containing the other attaching disk of α_0 (see fig. 4).

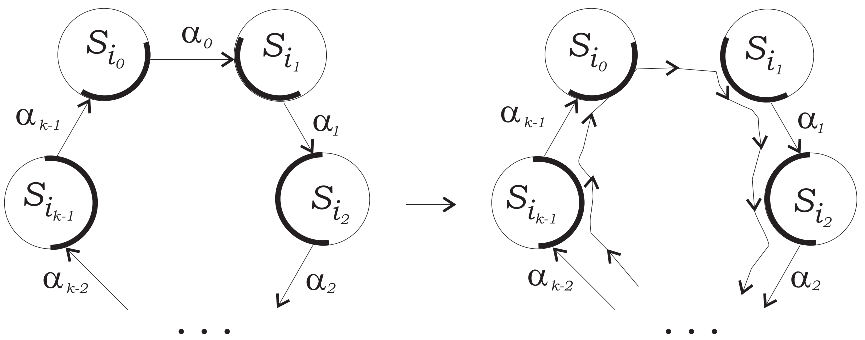

Fig. 4.

Finally, each 1-handle with attaching disks to the same boundary component of S_i can be represented as a disk sum of S_i with a solid torus (see fig. 5). Considering

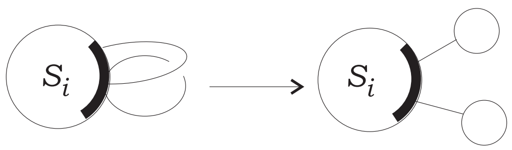

Fig. 5.

the solid torus as the space of a Burde-Murasugi link (the unknot) we see that M is the space of a graph Γ', which is a tree of Burde-Murasugi links $\square$.

A linear tree is a tree where each vertex has degree at most 2. It is now easy to see the following

Corollary. *If the closure M of the complement of a handlebody of genus $g \geq 1$ in S^3 is a Seifert union then M is homeomorphic to the space of a linear tree of torus knots.*

Acknowledgements

We would like to thank Hernán González Aguilar for his help drawing the pictures. This work was partially supported by CONACyT grant 28490-E and NSF grant INT-9602950.

References

[1] G. Burde and K. Murasugi, *Links and Seifert Fiber Spaces*, Duke Math. J. **37** (1970), 89–93.

[2] G. Burde and H. Zieschang, *Eine Kennzeichnung der Torus Knoten*, Math. Ann. **167** (1966), 169–176.

[3] J. C. Gómez-Larrañaga and W. Heil, *Seifert Unions of Solid Tori*, to appear in Math. Zeit..

[4] H.Seifert, *Toplogie 3-dimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147-238.(English translation in H. Seifert and W. Threlfall, *A Textbook of Topology* (1980), Academic Press).