HOMOLOGICAL ALGEBRA OF MULTIVALUED ACTION FUNCTIONALS

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ABSTRACT. We outline a cohomological treatment for multivalued (classical) action functionals. We point out that an application of Takens' theorem, after Zuckerman, Deligne and Freed, allows to conclude that multivalued functionals yield globally defined variational equations.

1. INTRODUCTION, MAIN DEFINITIONS AND STATEMENT OF THE RESULTS

1.1. **General motivation.** In recent years there has been a continuously growing interest in the analysis of multivalued action functionals. The precise meaning of "multivalued functional" will be defined below; for now, we can heuristically define an action functional to be "multivalued" if it is given in terms of a collection of Lagrangian densities on the manifold M of parameters (the "space-time") which do *not* glue into a globally defined differential form (of top degree on M).

Amongst the primary motivations to study multivalued actions are the need to incorporate topological terms [13, 11, 3], and the emergence of dynamical fields of new geometric content, such as curvings and connective structures on Gerbes with abelian band, like the B-field in String Theory, see e.g. [12], and more recently, differential cohomology [10]. Multivalued actions can also arise in other geometric contexts, typically when a Lagrangian density, that is, a top-degree form, is produced from a local ansatz, where "local" means that the construction leading to the Lagrangian density is carried out with respect to an explicit choice of an open neighborhood or chart $U \to M$. Such is the case, for example, for the Liouville action constructed in [16] to investigate the Weil-Petersson form on the Teichmüller space of compact Riemann surfaces of genus g, and its chiral "half" used in [2, 1] that yields a variational characterization of the universal projective family. It seems that in all the examples a proper definition of the action usually leads to a generalization of the Lagrangian density as a cocycle in a Čech resolution with respect to a chosen open cover of M. The Čech paradigm is in fact a fundamental one when constructing action functionals. In our work with L.A. Takhtajan (cf. [1]) we stressed its universal nature with respect to the choice of the cover, emphasizing the need (and the possibility) to work with arbitrary coverings. In this way we put the accent on the cocycle itself, rather than on the class it represents: indeed the former is what is usually explicitly computable, given a choice of a covering of M.

In geometric applications, such as in [16] or [1], we need to work with the first and higher variations of the relevant functionals, thus we work firmly within the context of Classical Field Theory [7]. It is obviously of primary importance to ensure that the variational principle associated to these multivalued actions yield well defined variational problems. In other words, we need to ensure that the resulting Euler-Lagrange equations be globally defined on M.

The aim of this note is to point out a simple mechanism by which these (classical) multivalued action functionals yield a globally defined variational equation. Assuming the variations of the relevant dynamical fields glue appropriately on M, we show that Takens' results on the variational bicomplex [15] (see also [7, 17]) force the local variations subordinate to a covering of M to glue into a global one.

1.2. **Multivalued functionals.** The meaning of "multivalued functional" in this context is as follows. Let M be a manifold of dimension n, assumed to be compact for simplicity, and let $\mathcal{U}_M = \{U\}_{i\in I}$ be an open cover. For any p let $\underline{A}^p_M$ be the sheaf of p-forms, with $A^p(M)$ the corresponding module of global sections. (We assume the smooth forms to be $\mathbb{C}$-valued, in general.) Consider the datum of a smooth n-form $\omega_i^{(0)} \in \underline{A}^n_M(U_i)$ for each U_i. Each $\omega_i^{(0)}$ is interpreted as a Lagrangian "density" and if $x_i^1, \dots, x_i^n$ are local coordinates on U_i, we write $\omega_i^{(0)} = L_i dx_i^1 \wedge \cdots \wedge dx_i^n$. We assume the smooth function L_i (the "Lagrangian") depends on a section of some fiber bundle $E \xrightarrow{\pi} M$. (In fact this can be generalized to the situation where we have submersions $E_i \to U_i$ satisfying reasonable descent conditions, see below.) We need to compare two local Lagrangian densities $\omega_i^{(0)}$ and $\omega_j^{(0)}$ on $U_{ij} = U_i \cap U_i$. We consider the following two possibilities:

1. $\omega_i^{(0)} = \omega_j^{(0)}$ for any $i, j \in I$, that is the local lagrangian densities glue to form a globally defined n-form $\omega^{(0)}$ on M. In particular we can define an action functional by integration over M:
$$S = \langle [\omega^{(0)}], [M] \rangle = \int_M \omega^{(0)} .$$
As a result, the corresponding variational principle will be well defined on M.
2. $\omega_j^{(0)} - \omega_i^{(0)} = d\omega_{ij}^{(1)}$, where $\omega_{ij}^{(1)} \in \underline{A}_M^{n-1}(U_{ij})$ is a smooth $(n-1)$-form. We refer to this case as "multivalued", owing to the non-uniqueness of the Lagrangian density.

In the second case above, the procedure to construct an action functional is by now standard. We consider the Čech-de Rham complex [4] $\check{C}^q(\mathcal{U}_M, \underline{A}_M^p)$ relative to the cover $\mathcal{U}_M$. If we assume this cover to be good, then we can construct forms $\omega^{(0)}, \omega^{(1)}, \dots, \omega^{(n)}$, with $\omega^{(q)}$ an $(n-q)$-form on a q-fold intersection, from the descent relation

$$(-1)^{n-q} \check{\delta} \omega^{(q)} = d\omega^{(q+1)} , \tag{1.1}$$

where $\check{\delta}$ is the Čech coboundary. We obviously have $d\omega^{(0)} = 0$ for dimensional reasons, and we can close the descent condition at the last step, namely $\check{\delta}\omega^{(n)} = 0$, by invoking the fact that $H^{n+1}(M^n, \mathbb{C}) = 0$. In this way the sequence of forms $\omega^{(0)}, \omega^{(1)}, \dots, \omega^{(n)}$ determines a total cocycle Ω of degree n in the single complex associated to the Čech-de Rham one with total differential $D = d \pm \check{\delta}$, namely we have $D\Omega = 0$. Since the Čech-de Rham complex computes $\mathbb{H}^\bullet(M, \underline{A}_M^\bullet) \cong H^\bullet(M, \mathbb{C})$, the cocycle Ω represents a class of degree n which can then be evaluated against the fundamental class of M. More precisely, Ω can be evaluated against a representative Σ of $[M]$ and the resulting number

$$S = \langle \Omega, \Sigma \rangle \equiv \langle [\Omega], [M] \rangle \tag{1.2}$$

can be taken to be the action functional determined by the collection of local Lagrangian densities $\{\omega_i^{(0)}\}$.

We slightly modify this setup: in many actual examples the descent equations are simply manifestly satisfied up to the last step (so the assumption that the cover is good is not really used), where we may have $\check{\delta}\omega^{(n)} = c \in \mathbb{Z}$, instead of simply $\check{\delta}\omega^{(n)} = 0$. To accommodate this fact it is worth working with Deligne cohomology, as pointed out in [13], namely we replace the de Rham complex $\underline{A}_M^\bullet$ with the augmented one $\mathbb{Z}_M \to \underline{A}_M^\bullet$, where the first arrow is just the inclusion [7]. Thus Ω will be interpreted as a total cocycle of degree $n+1$ corresponding to the sequence $\omega^{(0)}, \omega^{(1)}, \dots, \omega^{(n)}, c$ in the Čech resolution of the Deligne complex above. If indeed the cover is good, the Čech resolution computes the *Deligne cohomology* group $\mathbb{H}^{n+1}(M, \mathbb{Z}_M \to \underline{A}_M^\bullet) \overset{\text{def}}{=} H_{\mathcal{D}}^{n+1}(M, \mathbb{Z}(n+1)) \cong H^n(M, \mathbb{C}^*)$. Therefore the result of (1.2) is to be interpreted as the *exponential* of the action (written additively), rather than the action itself. In any event, the multivalued functionals we have in mind are precisely those Čech cocycles arising from a collections of local lagrangian densities in the manner we have just explained.

1.3. **Main definitions and results.** Let $M, \mathcal{U}_M, \underline{A}_M^\bullet$ as above. Let $\mathbb{Z}_{\mathcal{D}}^\bullet : \mathbb{Z}_M \to \underline{A}_M^0 \to \underline{A}_M^1 \to \cdots$ be the *augmented* de Rham complex—the Deligne complex of length $n+1$. Let $\mathsf{C}^{p,q} = \check{C}^q(\mathcal{U}_M, \mathbb{Z}_{\mathcal{D}}^p)$ the bicomplex with differentials, d and $\check{\delta}$. The associated simple complex will have total differential $D = d + (-1)^p \check{\delta}$ acting on the homogeneous components $\mathsf{C}^{p,\bullet}$. (Note that in degree zero d is just the inclusion $\mathbb{Z}_M$ into the smooth functions $\underline{A}_M^0$.) Let $\{\omega_i^{(0)}\}$ be a collection of lagrangian densities subordinated to the open cover $\mathcal{U}_M$. (In other words, a 0-cochain on $\mathcal{U}_M$ with values in $\underline{A}_M^n$.) For the purpose of this introduction we will assume each $\omega_i^{(0)}$ depends (in a local way, see below) on the restriction to U_i of a section ϕ of a smooth bundle $E \xrightarrow{\pi} M$. (We omit to display this dependency in the notation.)

Definition 1.1. A *multivalued Lagrangian cocycle* is a cocycle $\Omega = \omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(n)} + c$ of total degree $n+1$ in the total simple complex associated to $\mathsf{C}^{p,q}$. The homogeneous members satisfy the descent condition (1.1) plus the relation $\check{\delta}\omega^{(n)} = c$. If Σ represents the fundamental class $[M]$ of M, the *multivalued action functional* associated to Ω is given by the evaluation (1.2).

For each member U_i of the cover $\mathcal{U}_M$, we consider the variational bicomplex on $\mathcal{S}_i \times U_i$, where $\mathcal{S}_i$ is the restriction to U_i of the space of smooth global sections of E. We have *two* exterior differentials: $\boldsymbol{\delta}$ in the field direction, and d has been already introduced. For simplicity we assume that $\boldsymbol{\delta} d = d\boldsymbol{\delta}$, and include an explicit sign for the total differential $\boldsymbol{\delta} \pm d$.

It follows from Takens' theorem ([15], and also [7]) that the variation $\boldsymbol{\delta}\omega^{(0)}$ can be (uniquely) written as

$$\boldsymbol{\delta}\omega_i^{(0)} = a_i^{(0)} + d\gamma_i^{(0)} \,, \tag{1.3}$$

where the variation $a_i^{(0)}$—the non exact part of $\boldsymbol{\delta}\omega^{(0)}$—is a *source form* of degree $(1, n)$ in the variational bicomplex, and $\gamma_i^{(0)}$ has degree $(1, n-1)$. The source form $a_i^{(0)}$ determines a differential equation which coincides with the classical Euler-Lagrange equations once a coordinate system has been chosen. We call $\mathcal{M}_i$ the zero-locus in $\mathcal{S}_i$ defined by the Euler-Lagrange equation. Following [7], we call $\gamma_i^{(0)}$ the Cartan form.

Then the main result is

Theorem 1.2. *The cochain $\{a_i^{(0)}\}_{i\in I}$ is a 0-cocycle, that is $a_i^{(0)} = a_j^{(0)}$, so it defines a globally defined $(n, 1)$-source form $a^{(0)}$. The variation of the total cocycle $\Omega = \omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(n)} + c$ is solely due to the source form up to a total coboundary, namely we have*

$$\boldsymbol{\delta}\Omega = a^{(0)} + D\Gamma \,, \tag{1.4}$$

where $\Gamma \equiv \sum_{q=0}^{n-1} \gamma^{(q)}$ is a chain of total degree n in the total complex of $\mathsf{C}^{\bullet,\bullet}$ such the last component $\gamma^{(n)} = 0$.

Γ may be called the *global Cartan form* associated to Ω. Notice that Γ is really (up to an index shift) an object in the genuine Čech-de Rham complex.

A number of corollaries are almost immediately available. First, we obviously have

Corollary 1.3. *The zero-loci $\mathcal{M}_i \subset \mathcal{S}_i$ determined by the Euler-Lagrange equations relative to $\{a_i^{(0)}\}_{i\in I}$ glue into a global locus $\mathcal{M}$.*

Remark 1.4. Note that the statement of the corollary is not completely vacuous. In light of the assumptions in 2.5, it means that regardless of the nature of the allowed fields, the loci $\mathcal{M}_i$ will always describe global geometric objects on M.

Furthermore, in ref. [17] Zuckerman introduces the variational differential of the Cartan form and calls it the *universal conserved current*. Thus for each $i \in I$ we would consider the local universal conserved current $\theta_i^{(0)} = \boldsymbol{\delta}\gamma_i^{(0)}$, which is a form of bidegree $(2, n)$ in the variational complex. Elementary manipulations show that the main property of the Cartan form is that

$$\boldsymbol{\delta}\theta_i^{(0)} = 0 \,, \qquad d\theta_i^{(0)} = -\boldsymbol{\delta} a_i^{(0)} \,, \tag{1.5}$$

It follows from (1.5) that the restriction of the universal current to $\mathcal{M}_i$ is a *closed* $(2, n)$-form (in the variational bicomplex). To complete the picture, we consider the global current $\Theta \stackrel{\text{def}}{=} \boldsymbol{\delta}\Gamma$ and similarly to ref. [17] we have:

Proposition 1.5. *The global current Θ satisfies*

$$\boldsymbol{\delta}\Theta = 0 \,, \qquad D\Theta = -\boldsymbol{\delta} a^{(0)} \tag{1.6}$$

so it is a conserved current. The restriction to $\mathcal{M}$ is closed with respect to $\boldsymbol{\delta} + D$.

In analogy with the variation being a homogeneous element in the Čech resolution of $\mathbb{Z}^{\bullet}_{\mathcal{D}}$, thanks to Theorem 1.2, the proposition shows the D-differential (and obviously the total differential) of the global current Θ is also a homogeneous object of pure degree $(n + 1, 0)$.

1.4. **Organization.** This note is organized as follows. In the first part of sect. 2 we collect some notation and some notions we need about Deligne complexes and variational bicomplexes. In subsect. 2.4 we make more precise assumptions on the allowed objects in the variational process (in particular relaxing the conditions stated in subsect. 1.3) and we state a lemma on the gluing properties of the resulting local variational complexes. Once this is done, the proofs of Thm. 1.2 and Prop. 1.5 reduce to a homological manipulation of various differential complexes. They are presented in some detail in sect. 3. Finally, we draw some conclusions and look at possible future directions in sect. 4.

2. Setup

We keep the assumptions on M, $\mathcal{U}_M$, $\underline{A}^{\bullet}_M$ made in sect. 1. Also, we use the notation U_{ij} for $U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$, and so on.

2.1. **Double and triple complexes.** If $\mathsf{C}^{\bullet,\bullet}$ is any double complex with *commuting* differentials d_1 and d_2 we denote by $\mathsf{C}^\bullet$ or by $\mathrm{Tot}^\bullet\,\mathsf{C}$ the associated total simple complex, with $\mathsf{C}^k = \oplus_p \mathsf{C}^{p,k-p}$, and total differential $d(x) = d_1(x) + (-1)^p d_2(x)$, for $x \in \mathsf{C}^{p,k-p}$.

Unfortunately (or more interestingly), we will have to consider also *triple* complexes. If $\mathsf{C}^{p,q,r}$ is such a tricomplex with differentials d_1, d_2, d_3, then the associated total complex has a total differential d equal to $d = d_1 + (-1)^p d_2 + (-1)^{p+q} d_3$ when acting on the homogeneous component of (triple) degree (p,q,r).

2.2. **Deligne complexes.** The use of Deligne cohomology is by now fairly common, so we just introduce the notation. A brief introduction can be found in [7], a more thorough treatment can be found in refs.[9, 6]. Recall that M has dimension n. The smooth Deligne complex of length p is the complex of sheaves

$$\mathbb{Z}^\bullet_{\mathcal{D}} : \mathbb{Z}_M \xrightarrow{\imath} \underline{A}_M \xrightarrow{d} \underline{A}^1_M \xrightarrow{d} \dots \xrightarrow{d} \underline{A}^{p-1}_M$$

$\mathbb{Z}$ is placed in degree zero and the degree of each term $\underline{A}^r_M$ in $\mathbb{Z}^\bullet_{\mathcal{D}}$ is $r+1$. The first differential is just the inclusion $\imath$ of $\mathbb{Z}$ in $\underline{A}_M$, while d is the usual de Rham differential. The complex is truncated to zero after degree p.[1] The *smooth Deligne cohomology groups* of M — denoted by $H^q_{\mathcal{D}}(M,\mathbb{Z})$ — are the hypercohomology groups $\mathbb{H}^q(M,\mathbb{Z}^\bullet_{\mathcal{D}})$. In practice, if the cover $\mathcal{U}_M$ is sufficiently fine (as we assume here) these groups can be calculated using Čech cohomology from the double complex $\mathsf{C}^{p,q} = \check{C}^q(\mathcal{U}_M, \mathbb{Z}^p_{\mathcal{D}})$, equipped with the differentials d and $\check{\delta}$ and the total differential $D = d + (-1)^r \check{\delta}$.

For the length $n+1$ complex used to describe the multivalued functionals there is in fact no truncation, so it is an augmented de Rham complex $\mathbb{Z} \to \underline{A}^\bullet_M$, as noted before. In this case $H^{n+1}_{\mathcal{D}}(M,\mathbb{Z}) \cong H^n(M,\mathbb{C}^*)$. (See also [8].) However, in some case the dynamical fields themselves are cocycles in $\mathsf{C}^{\bullet,\bullet}$ for some appropriate length, so the general formalism may be needed.

2.3. **Jets and the variational bicomplex.** We need to recall a bit of notation concerning jet bundles and the variational complexes of local forms. An in-depth account can be found in [14]. We follow the approach in [17, 7].

For any manifold U of dimension n, let $\pi : E \to U$ be a smooth fibration, with the manifold of smooth sections $\mathcal{S} = \Gamma(U;E)$. (Later on we will consider the case where $U \subset M$ is an open set.) Let $ev : \mathcal{S} \times U \to E$ be the evaluation map. By taking the infinite jet of $ev(\phi,m) = \phi(m)$ at $m \in U$ it extends to $Ev : \mathcal{S} \times U \to JE$, where JE is the infinite jet bundle of $E \to U$. A tangent vector ξ to $\mathcal{S}$ at ϕ (a "variation") is a section of the vector bundle $\phi^{-1}(TE/U)$, where TE/U is the vertical bundle of the fibration $E \to U$. By taking the infinite jet $j(\phi)$ of ϕ we obtain a section of the vertical bundle $j(\phi)^{-1}(JE/U)$. The vertical bundle sequence of JE splits; in particular the fiber of supplementary (horizontal) bundle at $m \in U$ is the image of $T_m M$ under $j_m(\phi)_*$. By duality, there is a corresponding splitting of the complex of differential forms on JE into vertical and horizontal components.

The complex of differential forms on $\mathcal{S} \times U$ splits according to the product structure into components of bidegree (p,q), and the exterior differential splits accordingly as $d_{\mathcal{S}\times U} = \boldsymbol{\delta} + (-1)^p d$. In the complex of smooth differential forms on $\mathcal{S} \times U$ we are only interested in the subcomplex obtained as the inverse image under Ev^* of the complex of differential forms on JE, namely the complex of *local* forms. These are the forms whose dependency on $\phi \in \mathcal{S}$ and tangent vectors ξ_i factors (locally, in general) through some finite jet of ϕ and ξ_i at $m \in U$. The local forms inherit a splitting induced by the one on JE via Ev^*. It is compatible with the one induced by product structure of $\mathcal{S} \times U$. We denote by $A^{p,q}_{\mathrm{loc}}(\mathcal{S} \times U)$ the homogeneous component of degree (p,q).

In [15] (see also [7] for a different proof) Takens proves

Theorem 2.1 (Takens). *For $p \geq 1$ the complex $(A^{p,\bullet}_{\mathrm{loc}}(\mathcal{S}) \times U, d)$ is exact except in top degree $|\bullet| = n+1$.*

(Recall that we are shifting the form degrees by 1.) A consequence of Thm. 2.1 is that in degree $n+1$ $A^{p,n+1}_{\mathrm{loc}}(\mathcal{S} \times U)$ is the direct sum of the image of $A^{p,n}_{\mathrm{loc}}(\mathcal{S} \times U)$ under d and a non-exact component consisting of *source forms:* these are $(p, n+1)$-forms in the variational bicomplex whose dependency on the variation factors through the 0-jet.

2.4. **Homology of dynamical fields and gluing of variational complexes.** The definition 1.1 of multivalued action functional was given under the assumption that the components of Ω be local forms depending on the restriction of a section of E over M to the various open sets of $\mathcal{U}_M$. This is one of the main examples. If $E \to M$ is a smooth fibration, a global section ϕ is then a collection $\{\phi_i\}_{i\in I}$ of sections of $E|_{U_i}$ such that $\phi_i = \phi_j$ over U_{ij}. All the statements, however, remain valid in a more general context. Rather than just sections of global fibrations, we can allow more general objects with more relaxed gluing properties.

[1] We do not use the "Algebraic Geometers' twist" $\mathbb{Z}(p) = (2\pi i)^p \mathbb{Z}$, here.

Example 2.2. Let us consider the case of cocycles of degree p in the Čech resolution of some Deligne complex of the (same) length p, namely an object of the form

$$\Phi = \phi^{(0)} + \cdots + \phi^{(p-1)} + c\,,$$

with $\phi^{(j)} \in \check{C}^{p-j}(\mathbb{Z}^j_{\mathcal{D}})$. (And $c = \phi^{(p)} \in \mathbb{Z}$.) This includes the case of connections in line bundles and curving structures on gerbes with abelian band [12]. The main dynamical field will be the collection $\{\phi^{(0)}_i\}_{i\in I}$ while the other members determine the gluing law, namely

$$\phi^{(0)}_j - \phi^{(0)}_i = \pm d\phi^{(1)}_{ij}\,, \quad \phi^{(1)}_{ij} - \phi^{(1)}_{ik} + \phi^{(1)}_{ij} = \pm d\phi^{(2)}_{ijk}\,, \quad \ldots$$

and so on according to the cocycle condition and the relevant sign rules. In a case like this, we will demand that the *variations glue,* namely that $\boldsymbol{\delta}\phi^{(0)}_j = \boldsymbol{\delta}\phi^{(0)}_i$, which intuitively amounts to say that the gluing objects $\phi^{(1)}, \phi^{(2)}, \ldots$ are spectators from the point of view of the dynamics.

Other examples we want to consider include the following.

Example 2.3. The category of connections on a principal G-bundle over M for a non-abelian group G with Lie algebra $\mathfrak{g}$. The dynamical field is a 0-cochain $\{A_i\}_{i\in I}$ with values in $\underline{A}^1_M \otimes \mathfrak{g}$ with the gluing law:

$$A_j - \mathrm{ad}(g^{-1}_{ij})(A_i) = g^{-1}_{ij}dg_{ij}$$

for a 1-cocycle $\{g_{ij}\}$ with values in G.

Example 2.4. (G, X)-structures: consider an action $G \times X \to X$ and fibrations $X \to M$ and $G \to M$, with corresponding sheaves of sections $\underline{X}_M$ and $\underline{G}_M$. The dynamical field is a collection of local sections x_i over U_i of $\underline{X}_M$, with elements $g_{ij} \in \underline{G}_M(U_{ij})$ acting as gluing morphisms.

With these examples as main motivation, we make the following

Assumption-Definition 2.5. *Let $\mathcal{U}_M$ be a covering of M, and let $\underline{E}_M$ a sheaf over M with an appropriate structure, for example the sheaf of sections of smooth fibration $E \to M$. Let $\{\phi_i \in \underline{E}_M(U_i)\}_{i\in I}$ be a collection of sections. Assume that either:*

1. *$\underline{E}_M$ can be realized as the highest degree object in a complex of abelian groups;*

$$\cdots \longrightarrow \underline{E}^{-2}_M \longrightarrow \underline{E}^{-1}_M \longrightarrow \underline{E}^0_M \equiv \underline{E}_M$$

   and $\{\phi_i\}_{i\in I}$ completes to a cocycle of the appropriate length;

2. *or $\underline{E}_M$ can be realized as the zero level of a truncated simplicial object of length one:*

$$\underline{E}^{-1}_M \rightrightarrows \underline{E}^0_M \equiv \underline{E}_M\,.$$

   $\underline{E}^{-1}_M$ acts on $\underline{E}_M$ by isomorphisms. In this case we assume $\{\phi_i\}_{i\in I}$ is the object part of an appropriate decomposition of $\underline{E}^{\bullet}_M$, namely there are isomorphisms ψ_{ij} over U_{ij} such that $\phi_i = \psi_{ij}(\phi_j)$, where the ψ_{ij} do not *necessarily satisfy a cocycle condition, see* [5] *for more details.*

In both cases, we assume that the relevant sheaves of jets satisfy the descent condition, namely elements of $\underline{E}^{-1}_M(U_{ij})$ induce isomorphisms $j\psi_{ij} : J\underline{E}_M|_{U_{ij}} \to J\underline{E}_M|_{U_{ij}}$ satisfying the usual compatibility condition over U_{ijk}.

As a consequence we have a "gluing lemma" for the variational bicomplexes above the various members of the cover $\mathcal{U}_M$. Indeed, let $A^{p,q}_{\mathrm{loc}}(\mathcal{S}_i \times U_i)$ be the variational bicomplex determined by $\underline{E}_M|_{U_i}$.[2] Restriction to U_{ij} and the action of ψ_{ij} determine maps $A^{p,q}_{\mathrm{loc}}(\mathcal{S}_j \times U_{ij}) \to A^{p,q}_{\mathrm{loc}}(\mathcal{S}_i \times U_{ij})$. Then we have

Lemma 2.6 (Gluing Lemma)**.** *For $p \geq 1$ the variational complexes $A^{p,\bullet}_{\mathrm{loc}}(\mathcal{S}_i \times U_i)$ descend to a global object on M. Also, $\boldsymbol{\delta}\check{\delta} = \check{\delta}\boldsymbol{\delta}$.*

Proof. Essentially immediate. First of all, in general it is easy to verify that an isomorphism of $\underline{E}$ into $\underline{F}$ covering the identity "prolongs" [14] to an isomorphisms between $J\underline{E}$ and $J\underline{F}$ that preserves the splittings in the respective complexes of differential forms. Thus, given the isomorphism ψ_{ij} over U_{ij}, the induced $j\psi_{ij}$ preserves the splitting in the complex of differential forms over $JE|_{U_{ij}}$, and this is consistent thanks to the descent condition assumption in 2.5. □

[2] Here $\mathcal{S}_i$ and $\underline{E}_M|_{U_i}$ are really two names for the same object.

An immediate corollary is that given a cocycle Φ of the type specified in 2.5, the variation $\boldsymbol{\delta}\Phi$ is a well defined global object on M.

3. Proofs

We will (implicitly) work with the triple complex $\mathsf{C}^{p,q,r} = \check{C}^r(\mathcal{A}^{p,q})$, where p is the variational degree, q the Deligne complex one, and r is the Čech degree. The respective differentials are $\boldsymbol{\delta}$, d, and $\check{\delta}$, assumed to commute with one another. The relevant associated total differentials are constructed according to the rules spelled out in subsect. 2.1. So, for example, we have $D = d + (-1)^q\check{\delta}$, and $\triangle = \boldsymbol{\delta} + (-1)^p d + (-1)^{p+q}\check{\delta}$.

3.1. **Proof of Theorem 1.2.** Let $\Omega = \sum_{r=0}^{n} \omega^{(r)} + c$ be a Lagrangian cocycle, so that $D\Omega = 0$, the latter relation being equivalent to (1.1). Assume that on U_i the variation of the Lagrangian densities $\omega_i^{(0)}$ be given by eqn. (1.3). From the component of highest form degree in (1.1) we have on $U_{ij} = U_i \cap U_j$,

$$\omega_j^{(0)} - \omega_i^{(0)} = (-1)^n d\omega_{ij}^{(1)} ,$$

and taking the variation

$$\boldsymbol{\delta}\omega_j^{(0)} - \boldsymbol{\delta}\omega_i^{(0)} = (-1)^n d\boldsymbol{\delta}\omega_{ij}^{(1)} .$$

By the uniqueness of the decomposition $source + exact$ implied by Takens' theorem [15], plus the gluing lemma 2.6 above, we must have

$$a_j^{(0)} = a_i^{(0)} , \quad \text{and} \quad d\gamma_j^{(0)} - d\gamma_i^{(0)} = (-1)^n d\boldsymbol{\delta}\omega_{ij}^{(1)} . \tag{3.1}$$

The second relation is an equality between differentials of $(n-1)$-forms (i.e. of degree n in the Deligne complex). But now the variational complex is exact, so we obtain

$$\boldsymbol{\delta}\omega^{(1)} = (-1)^n\check{\delta}\gamma^{(0)} + d\gamma^{(1)} ,$$

for a $\gamma^{(1)}$ of degree $(1, n-1, 1)$.

We continue by recursion. For $1 \leq r \leq n-1$, assume

$$\boldsymbol{\delta}\omega^{(r)} = (-1)^{n-r+1}\check{\delta}\gamma^{(r-1)} + d\gamma^{(r)} , \tag{3.2}$$

then apply $\boldsymbol{\delta}$ to

$$d\omega^{(r+1)} = (-1)^{n-r}\check{\delta}\omega^{(r)} .$$

We have

$$d\boldsymbol{\delta}\omega^{(r+1)} = (-1)^{n-r}\check{\delta}\boldsymbol{\delta}\omega^{(r)} = (-1)^{n-r}d\boldsymbol{\delta}\gamma^{(r)} ,$$

having used (3.2), and by Takens, again, we obtain

$$\boldsymbol{\delta}\omega^{(r+1)} = (-1)^{n-r}\check{\delta}\gamma^{(r)} + d\gamma^{(r+1)} ,$$

for a $\gamma^{(r+1)}$ of degree $(1, n-r-1, r+1)$, as wanted.

The last step we need to check is the relation (1.1) for $r = n-1$. We have

$$d\omega^{(n)} = -\check{\delta}\omega^{(n-1)} ,$$

and applying $\boldsymbol{\delta}$ to both sides we get

$$d\boldsymbol{\delta}\omega^{(n)} = -\check{\delta}\boldsymbol{\delta}\omega^{(n-1)} = -d\check{\delta}\gamma^{(n-1)} ,$$

having used (3.2) for $r = n-1$. Notice that the latter is an equation for differentials of *functions.* A final application of the variational complex acyclicity property yields

$$\boldsymbol{\delta}\omega^{(n)} = \check{\delta}\gamma^{(n-1)} , \tag{3.3}$$

without constant terms, and 1.2 is proved.

3.2. **Proofs for the universal current.** The universal current calculation in proposition. 1.5 is a formal manipulation of differentials. Let $\Theta = \boldsymbol{\delta}\Gamma$ so that $\Theta = \sum_{r=0}^{n-1}\theta^{(r)}$, where of course $\theta^{(r)} = \boldsymbol{\delta}\gamma^{(r)}$. First, we obviously have $\boldsymbol{\delta}\theta^{(r)} = 0$, and, second:

$$d\theta^{(0)} = \boldsymbol{\delta} d\gamma^{(0)} = -\boldsymbol{\delta} a^{(0)}\,, \tag{3.4}$$

having used (1.3) once again. It follows from (3.4) that $d\theta^{(0)}$ is in fact a well defined form on M. For the other terms, we have

$$\begin{aligned} d\theta^{(r)} &= d\boldsymbol{\delta}\gamma^{(r)} = \boldsymbol{\delta} d\gamma^{(r)} \\ &= \boldsymbol{\delta}\bigl(\boldsymbol{\delta}\omega^{(r)} - (-1)^{n-r+1}\check{\delta}\gamma^{(r-1)}\bigr) \end{aligned}$$

having used (3.2)

$$= -(-1)^{n-r+1}\check{\delta}\theta^{(r-1)}\,.$$

It follows that

$$D\Theta = \sum_{r=0}^{n-1}\bigl(d\theta^{(r)} + (-1)^{n-r}\check{\delta}\theta^{(r)}\bigr) = -d\theta^{(0)} = \boldsymbol{\delta} a^{(0)}\,,$$

thanks to the last relation, which proves the proposition.

In fact, it is easy to convince oneself that all the previous manipulations amount to the following simple calculation:

$$D\Theta = \boldsymbol{\delta} D\Gamma = \boldsymbol{\delta}(\boldsymbol{\delta}\Omega - a^{(0)}) = -\boldsymbol{\delta} a^{(0)}\,.$$

Also it follows that $\triangle\Theta = -\boldsymbol{\delta} a^{(0)}$ so Θ is indeed globally closed on $\mathcal{M}$.

4. Conclusions and outlook

We have shown that under certain conditions (specified in 2.5) action functionals that appear to be ill-defined under coordinate changes nevertheless yield a well defined variational equation. Besides the more historical and consolidated example of actions containing topological terms, a compelling motivation is provided by action functionals (of not necessary topological flavor) arising from geometric structures, such as [16, 1]. In fact, the analysis performed in [1] was one of our main motivations is writing this note.

One limitation of the present approach lies precisely in the assumption 2.5 used above. As an example, in [1] the allowed variations were just those that did not change the complex structure, namely the vertical variation along the Earle-Eells fibration over the Teichmüller space. But in a geometric situation, like the one provided by a functional depending on a complex structure, for example, one would like to do exactly what 2.5 forbids: performing a full variation whereby the local isomorphisms themselves become dynamical fields. It is rather easy to formulate counterexamples to lemma 2.6 and Thm 1.2 in this more general framework, whereby the source forms can be shown *not* to glue. One is led to conjecture that a form of Thm 1.2 should be valid with a cocycle a of length equal to the number of members in the complexes in assumption 2.5 that are allowed to have dynamical fields. We hope to return to this question in a future publication.

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