SHADOWS OF BLOW-UP ALGEBRAS

PAOLO ALUFFI

ABSTRACT. We study different notions of blow-up of a scheme X along a subscheme Y, depending on the datum of an embedding of X into an ambient scheme. The two extremes in this theory are the ordinary blow-up, corresponding to the identity, and the 'quasi-symmetric blow-up', corresponding to the embedding of X into a nonsingular variety. We prove that this latter blow-up is intrinsic of Y and X, and is universal with respect to the requirement of being embedded as a subscheme of the ordinary blow-up of some ambient space along Y.

We consider these notions in the context of the theory of characteristic classes of singular varieties. We prove that if X is a hypersurface in a nonsingular variety and Y is its 'singularity subscheme', these two extremes embody respectively the *conormal* and *characteristic* cycles of X. Consequently, the first carries the essential information computing Chern-Mather classes, and the second is likewise a carrier for Chern-Schwartz-MacPherson classes. In our approach, these classes are obtained from Segre class-like invariants, in precisely the same way as other intrinsic characteristic classes such as those proposed by Fulton, and by Fulton and Johnson.

We also identify a condition on the singularities of a hypersurface under which the quasi-symmetric blow-up is simply the linear fiber space associated with a coherent sheaf.

1. INTRODUCTION

It is not hard to see that the *conormal* cycle of a hypersurface X of a nonsingular algebraic variety M can be realized as the cycle of the blow-up of X along its singularity subscheme (defined by the partials of an equation defining X). Our guiding question in this paper is, what kind of 'blow-up' realizes similarly the much subtler *characteristic* cycle of a hypersurface? We answer this question, and extract from our construction a unified approach to different characteristic classes associated with a possibly singular hypersurface of a nonsingular variety.

The ordinary blow-up of a scheme X along a subscheme Y—that is, the Proj of the Rees algebra of the ideal sheaf $\mathcal{J}_{Y,X}$ of Y in X—has the remarkable property that it can be recovered from the blow-up of any ambient scheme M along Y, by taking the proper transform of X. As there are other notions of blow-up, obtained by taking the Proj of other 'blow-up algebras' (such as the symmetric algebra of $\mathcal{J}_{Y,X}$), it is natural to ask whether there is a 'largest' blow-up of X along Y that can be embedded in *some* (ordinary) blow-up of an ambient scheme M along Y.

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In the first part of this paper we construct such a blow-up: we define a new *quasi-symmetric algebra* of an ideal $\mathcal{J}_{Y,X}$, and show that it satisfies the universal property summarized above. In fact, we define (Definition 2.5) a quasi-symmetric algebra for every embedding $X \subset M$, then show (Theorem 2.9) that the limit of the corresponding inverse system of algebras equals the quasi-symmetric algebra arising for any nonsingular M (otherwise independently of M). We name the corresponding blow-up the *quasi-symmetric blow-up* of X along Y, $\mathrm{qBl}_Y X$. We also show (Theorem 2.12) that this new blow-up can be obtained by taking a 'principal' transform of X in $\mathrm{Bl}_Y M$, for any nonsingular variety M containing X.

The ordinary Rees blow-up and the new quasi-symmetric blow-up are two extremes in a range. In the second part of the paper we consider the case in which X is a hypersurface in a nonsingular ambient variety M, and we take Y to be its *singularity subscheme.* We find that the two extremes live naturally in the projectivized cotangent bundle of M, and their cycles yield concrete realizations of the *conormal,* resp. *characteristic* cycles of X (Theorems 3.1 and 3.2). As mentioned above, the first of these facts is old fare; the second appears to be new, at least in the form given here. Every quasi-symmetric blow-up in the range should correspond to a Lagrangian cycle in the projectivized cotangent bundle; that is, every embedding of X in another scheme should determine a constructible function on X by this construction. One way to summarize the main results in §3 is by saying that our construction associates the identity $X \overset{=}{\hookrightarrow} X$ with the Euler obstruction of X, and any inclusion $X \subset M$ into a nonsingular variety with the constant function $1\!\!1_X$.

From the point of view of characteristic classes of singular hypersurfaces, this means that 'Rees is to Mather as quasi-symmetric is to Schwartz-MacPherson'. In the third part of the paper we show (Theorem 4.4) how to obtain these classes rather directly from the corresponding blow-up algebras, by a standard intersection-theoretic operation (which is the 'shadow' in the title, Definition 4.1). This set-up gives a unified approach—for hypersurfaces—for the theory of Chern-Mather and Chern-Schwartz-MacPherson classes together with other intrinsic classes defined for singular varieties—notably the classes defined by Fulton and Johnson in [FJ80], and those defined by Fulton in [Ful84], Example 4.2.6.

We also discuss briefly (§3.11) an intriguing condition on the singularities of a hypersurface, under which the quasi-symmetric algebra of the singularity subscheme equals the symmetric algebra; in other words, in this case the characteristic cycle of X is the linear fiber space of the coherent sheaf $\mathcal{J}_{Y,X}$, and the Chern-Schwartz-MacPherson class of X can be computed from the ordinary Segre class of a coherent sheaf. We point out that this condition is automatically verified in several standard situations, and mention an interpretation of the condition in terms of extending vector fields along pieces of a Whitney stratification of the hypersurface.

One should wonder whether an intrinsic realization of the characteristic cycle can be given for more general schemes than hypersurfaces of nonsingular varieties (as we do here). In the end, our attention is directed to a coherent sheaf that is present regardless of whether X is a hypersurface: the cokernel of the dual of the map on differentials determined by the embedding in a nonsingular variety. If X is a hypersurface then a quasi-symmetric algebra can be defined for this sheaf, and our main result shows that this algebra leads to the Chern-Schwartz-MacPherson class of X (Theorem 4.9).

This suggests what the shape of an analogous result for arbitrary schemes might be, but the difficulty in establishing such a general result should not be underestimated. Indeed, the key technical fact allowing us to obtain the result for hypersurfaces in this paper amounts to a specific result relating Fulton-Johnson's classes and Chern-Schwartz-MacPherson classes of hypersurfaces. This relation has now been known for the better part of a decade, and studied intensely from many different viewpoints (cf. [Alu94], [Suw97], [BLSS99], [Yok99], [Alu99a], [Alu99b], [Suw00], [Alu00], [PP01] and the recent [Sch01a] to name a few), yet a generalization to arbitrary schemes has proved exceedingly elusive. A full analog of the results in this paper to arbitrary schemes would amount to a solution of this problem.

Our motivation in pursuing this program is twofold. First, we believe that it would be highly worthwhile to uncover any functoriality feature of classes such as Fulton's or Fulton-Johnson's. Chern-Schwartz-MacPherson's classes owe their existence precisely to their excellent functoriality properties; if such functoriality could be transferred to Segre classes (via formulas such as the ones presented in this article), this would offer a new handle on computing Segre classes, arguably one of the most basic invariants in intersection theory. Second, formulas such as the ones obtained in this paper can be implemented into algorithms running in symbolic computation programs such as `Macaulay2` ([GS]). The only algorithm known to us for such computations ([Alu03]) is woefully slow, and we hope that the approach presented in this paper may lead to substantially improved algorithms.

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2. quasi-Symmetric algebras and blow-ups

2.1. In this section we define and discuss the new blow-up—first in strictly algebraic terms, and next (starting in §2.5) in more geometric ones.

The ordinary blow-up is the Proj of the *Rees algebra* of an ideal, which is a close relative of its symmetric algebra. Our first task is to introduce and study another close relative of the symmetric algebra of an ideal. In fact, in Definition 2.1 we give a whole family of such algebras, depending on the datum of a surjective homomorphism. In Lemma 2.4 we identify conditions under which different homomorphisms lead to the same algebra. In the geometric setting, the family of algebras determines a new notion of blow-up of a scheme X along a subscheme Y, for each embedding of X into an ambient variety M. As a consequence of Lemma 2.4, we can prove (Theorem 2.7) that the new blow-up is independent of the ambient M provided that M is nonsingular.

This canonically determined blow-up is the 'quasi-symmetric blow-up' mentioned in the introduction (Definition 2.6). An explicit computation of the corresponding 'quasi-symmetric' algebra shows (Theorem 2.12) that the quasi-symmetric blow-up can be realized as a residual scheme to the exceptional divisor of the (ordinary) blow-up of the ambient nonsingular variety. This will be the key to one of the main results of the paper (Theorem 3.2), realizing the *characteristic cycle* of a hypersurface in terms of a quasi-symmetric blow-up. In turn, filtering this result through a little intersection theory will yield our applications to characteristic classes (Theorems 4.4 and 4.9),

2.2. Our rings will be Noetherian, commutative, with 1. Homomorphisms of algebras endowed of a natural grading are implicitly understood to preserve the grading.

Let A be a ring, and $\mathfrak{a}$ an ideal of A. Let R be a ring surjecting onto A, and denote by I the inverse image of $\mathfrak{a}$ in R. Note that the symmetric algebra $\mathrm{Sym}_R(I)$ maps to both the Rees algebra $\mathrm{Rees}_R(I) := \oplus_{n\geq 0} I^n$ and (by functoriality of Sym) to $\mathrm{Sym}_A(\mathfrak{a})$.

Definition 2.1. The *quasi-symmetric algebra* $\mathrm{qSym}_{R\to A}(\mathfrak{a})$ is defined by

$$\mathrm{qSym}_{R\to A}(\mathfrak{a}) := \mathrm{Sym}_A(\mathfrak{a}) \otimes_{\mathrm{Sym}_R(I)} \mathrm{Rees}_R(I).$$

A particular case of this notion will be the affine version of our main blow-up algebra, cf. Definition 2.8 below. Note that the algebra corresponding to the identity is the ordinary Rees algebra:

$$\mathrm{qSym}_{A\to A}(\mathfrak{a}) = \mathrm{Rees}_A(\mathfrak{a});$$

thus, the ordinary blow-up can be recovered in terms of the operation studied here. We will be especially interested in the case corresponding to epimorphisms $R \to A$ with R suitably 'nice'; we begin by recording a few properties of the local version of the more general notion.

First of all, the quasi-symmetric algebra is functorial in the sense that any homomorphism of rings $R \to S$ compatible with epimorphisms to A induces an epimorphism

$$\mathrm{qSym}_{R\to A}(\mathfrak{a}) \twoheadrightarrow \mathrm{qSym}_{S\to A}(\mathfrak{a}) \ .$$

Indeed, the homomorphisms $R \to S \to A$ induce the middle row in the diagram

$$\begin{tikzcd} K_R \arrow[r] \arrow[d] & K_S \arrow[d] & \\ \mathrm{Sym}_R(I) \arrow[r] \arrow[d] & \mathrm{Sym}_S(J) \arrow[r] \arrow[d] & \mathrm{Sym}_A(\mathfrak{a}) \\ \mathrm{Rees}_R(I) \arrow[r] & \mathrm{Rees}_S(J) & \end{tikzcd}$$

where J is the inverse image of $\mathfrak{a}$ in S, and K_R, K_S are the kernels of the vertical maps to the Rees algebras. Since $K_R\mathrm{Sym}_A(\mathfrak{a}) \subset K_S\mathrm{Sym}_A(\mathfrak{a})$, there is an induced epimorphism

$$\mathrm{qSym}_{R\to A}(\mathfrak{a}) = \mathrm{Sym}_A(\mathfrak{a})/K_R\mathrm{Sym}_A(\mathfrak{a}) \twoheadrightarrow \mathrm{Sym}_A(\mathfrak{a})/K_S\mathrm{Sym}_A(\mathfrak{a}) = \mathrm{qSym}_{S\to A}(\mathfrak{a}) \ .$$

Pictorially, we have the commutative diagram:

$$\begin{tikzcd} \mathrm{Sym}_R(I) \arrow[r] \arrow[d] & \mathrm{Sym}_S(J) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sym}_A(\mathfrak{a}) \arrow[d] \\ \mathrm{Rees}_R(I) \arrow[r] & \mathrm{Rees}_S(J) \arrow[r] & \mathrm{qSym}_{S\to A}(\mathfrak{a}) \end{tikzcd}$$

where the square on the right is cocartesian by definition. As $\mathrm{qSym}_{R\to A}(\mathfrak{a})$ satisfies a universal property (as a tensor product) there is an induced canonical homomorphism $\mathrm{qSym}_{R\to A}(\mathfrak{a}) \to \mathrm{qSym}_{S\to A}(\mathfrak{a})$.

2.3. The functoriality is the key to most of the following remarks, whose proof is left to the reader.

Lemma 2.2. *Let $R \to A$, $\mathfrak{a}$, I be as above.*

1. *The quasi-symmetric algebra bridges between the Rees algebra and the ordinary symmetric algebra of $\mathfrak{a}$ in A:*

$$\mathrm{Sym}_A(\mathfrak{a}) \twoheadrightarrow \mathrm{qSym}_{R\to A}(\mathfrak{a}) \twoheadrightarrow \mathrm{Rees}_A(\mathfrak{a}) \ .$$

2. *If* $\mathrm{Sym}_R(I) = \mathrm{Rees}_R(I)$, *then* $\mathrm{qSym}_{R\to A}(\mathfrak{a}) = \mathrm{Sym}_A(\mathfrak{a})$.
3. *If $R \to A$ splits, then* $\mathrm{qSym}_{R\to A}(\mathfrak{a}) = \mathrm{Rees}_A(\mathfrak{a})$.
4. *There is an epimorphism* $\mathrm{Rees}_R(I) \twoheadrightarrow \mathrm{qSym}_{R\to A}(\mathfrak{a})$.

Example 2.3. If I is a complete intersection in R, then $\mathrm{qSym}_{R\to A}(\mathfrak{a}) = \mathrm{Sym}_A(\mathfrak{a})$ by Part 2 in Lemma 2.2 (since then the symmetric and Rees algebras of I in R coincide, [Mic64]).

This shows that $\mathrm{qSym}_{R\to A}(\mathfrak{a})$ may depend on R. For example, let $A = \mathbb{C}[x,y]/(xy)$, $\mathfrak{a} = (x,y)$, $R = \mathbb{C}[x,y]$; then

$$\mathrm{qSym}_{R\to A}(\mathfrak{a}) = \mathrm{Sym}_A(\mathfrak{a}) \neq \mathrm{Rees}_A(\mathfrak{a}) = \mathrm{qSym}_{A\to A}(\mathfrak{a}).$$

However, one of the main results of this section (Theorem 2.9) will show that $\mathrm{qSym}_{R\to A}(\mathfrak{a})$ is in fact independent of R provided that R is constrained to be *regular*.

2.4. There are two important cases in which the induced epimorphism is in fact an isomorphism.

Lemma 2.4. *Let $R \to S$ be a ring homomorphism compatible with epimorphisms $R \to A$ and $S \to A$; let $\mathfrak{a}$ be an ideal of A, and let I, J resp. be the inverse images of $\mathfrak{a}$ in R, S. Then the induced epimorphism*

$$\mathrm{qSym}_{R\to A}(\mathfrak{a}) \twoheadrightarrow \mathrm{qSym}_{S\to A}(\mathfrak{a})$$

is an isomorphism if

1. *the homomorphism $R \to S$ splits; or*
2. *S is R-flat, and $J = IS$.*

Proof. In the first situation, if a composition $R \to S \to R$ is the identity we obtain a decomposition of the identity

$$\mathrm{qSym}_R(I) \twoheadrightarrow \mathrm{qSym}_S(J) \twoheadrightarrow \mathrm{qSym}_R(I)$$

implying that both maps are isomorphisms.

In the second situation, since S is flat over R we have $I^m S = S \otimes_R I^m$ for all m. Thus

$$\mathrm{Rees}_S(J) = \mathrm{Rees}_S(IS) = \bigoplus_{m\ge 0} S \otimes_R I^m = S \otimes_R (\oplus_{m\ge 0} I^m) = S \otimes_R \mathrm{Rees}_R(I).$$

On the other hand, and again using flatness,

$$\mathrm{Sym}_S(J) = \mathrm{Sym}_S(IS) = \mathrm{Sym}_S(I \otimes_R S) = S \otimes_R \mathrm{Sym}_R(I),$$

by [Bou74], III §6, Proposition 7. Thus

$$\begin{aligned}
\mathrm{Sym}_S(J) \otimes_{\mathrm{Sym}_R(I)} \mathrm{Rees}_R(I) &= \big(S \otimes_R \mathrm{Sym}_R(I)\big) \otimes_{\mathrm{Sym}_R(I)} \mathrm{Rees}_R(I) \\
&= S \otimes_R \big(\mathrm{Sym}_R(I) \otimes_{\mathrm{Sym}_R(I)} \mathrm{Rees}_R(I)\big) \\
&= S \otimes_R \mathrm{Rees}_R(I) \\
&= \mathrm{Rees}_S(J).
\end{aligned}$$

This shows that the square on the left in the diagram at the end of §2.2 is cocartesian, implying the assertion. □

2.5. We now move to the geometric setting. All our schemes are of finite type over a field k.

Let $Y \subset X \subset M$ be closed embeddings of schemes. We denote by $\mathcal{J}_{Y,X}$, resp. $\mathcal{J}_{Y,M}$ the ideals of Y in X and M, respectively.

Definition 2.5. The *quasi-symmetric algebra* $\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$ is the graded $\mathcal{O}_X$-algebra

$$\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) := \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \otimes_{\mathrm{Sym}_{\mathcal{O}_M}(\mathcal{J}_{Y,M})} \mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}).$$

In other words, $\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$ sheafifies the local construction given by Definition 2.1. Every commutative diagram

$$\begin{tikzcd} & & N \arrow[d, "\pi"] \\ Y \arrow[r, hook] & X \arrow[ur, hook, "j"] \arrow[r, hook, "i"] & M \end{tikzcd}$$

determines an epimorphism

$$\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) \twoheadrightarrow \mathrm{qSym}_{X\subset N}(\mathcal{J}_{Y,X})$$

and we are interested in conditions guaranteeing that this map is an isomorphism.

Lemma 2.6. *The induced epimorphism is an isomorphism if*

1. $N = M \times \mathbb{A}^n$*; or*
2. π *is flat, and* $j(X)$ *is a connected component of* $\pi^{-1}(i(X))$.

Proof. These follow from Lemma 2.4. As the matter can be checked locally, we may assume $M = \operatorname{Spec} R$, $N = \operatorname{Spec} S$, $X = \operatorname{Spec} A$, Y is given by an ideal $\mathfrak{a}$ in A, and we have a commutative diagram

$$\begin{tikzcd} & & S \arrow[dl] \\ A/\mathfrak{a} & A \arrow[l, two heads] & R \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

Denote by K, L resp. the kernels of $R \to A$, $S \to A$ resp.; and by I, J resp. the inverse images of $\mathfrak{a}$ in R, S resp.

In the first situation $S = R[u_1, \dots, u_s]$ is a polynomial ring, and the splitting needed in order to apply Lemma 2.4 holds because if K is an ideal of R then any left-inverse of the inclusion $R/K \hookrightarrow (R/K)[u_1, \dots, u_s]$ lifts to a left-inverse of $R \hookrightarrow R[u_1, \dots, u_s]$.

In the second situation, by hypothesis S is flat over R, and there exists an $f \in S$ such that the epimorphism $S \to A$ lifts to an epimorphism $S_f \to A$ from the localization of S at f, with kernel $KS_f = LS_f$. A fortiori $IS_f = JS_f$ is the inverse image of $\mathfrak{a}$ in S_f. As S_f is flat over both S and R, two applications of Part 2 from Lemma 2.4 give the assertion. $\square$

Theorem 2.7. *If* $\pi : N \to M$ *is a smooth map compatible with closed embeddings* $X \subset M$, $X \subset N$, *then for all closed subschemes* $Y \subset X$ *the induced epimorphism*

$$\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) \twoheadrightarrow \mathrm{qSym}_{X\subset N}(\mathcal{J}_{Y,X})$$

is an isomorphism.

Proof. Again the matter can be checked locally, so as π is smooth we may assume that it can be written as a composition

$$N \xrightarrow{\text{étale}} M \times \mathbb{A}^s \longrightarrow M\ ;$$

by Lemma 2.6, Part 1, we may assume that π itself is étale. In this case $\pi^{-1}(X) \to X$ is an étale map with a section; hence the image of X in N must be a connected component of $\pi^{-1}(X)$. As étale maps are flat, Part 2 in Lemma 2.6 concludes the proof. □

Theorem 2.7 shows that the quasi-symmetric algebras of X collect into classes detecting specific 'qualities' of the embeddings $X \subset M$. For example, if $X \subset M$ is a section of a smooth projection $M \to X$ then $\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) = \mathrm{Rees}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ for all closed subschemes $Y \subset X$. In fact, only the features of the embedding $X \subset M$ *near* Y affect the corresponding quasi-symmetric algebra.

2.6. It is time to remove the dependence on the choice of an embedding $X \subset M$. For given $Y \subset X$, the epimorphisms on quasi-symmetric algebras induced by concatenation of embeddings $X \subset M \subset N$ make $\{\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})\}_M$ into an inverse system.

Definition 2.8. Let $Y \subset X$ be a closed embedding of schemes. The *quasi-symmetric algebra* of $\mathcal{J}_{Y,X}$ is defined as the inverse limit

$$\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) := \varprojlim_{M\supset X} \mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}).$$

The *quasi-symmetric blow-up* of X along Y is defined as the Proj of the quasi-symmetric algebra:

$$\mathrm{qBl}_Y X := \mathrm{Proj}(\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})).$$

The quasi-symmetric blow-up carries a tautological line bundle $\mathcal{O}(-1)$, as do the more conventional $\mathrm{sBl}_Y X = \mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}))$ and $\mathrm{Bl}_Y X = \mathrm{Proj}(\mathrm{Rees}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}))$. Also, note that by Lemma 2.2, Part 1, there are closed embeddings

$$\mathrm{Bl}_Y X \subset \mathrm{qBl}_Y X \subset s\mathrm{Bl}_Y X.$$

Theorem 2.7 is the key to the following concrete computation of the 'absolute' quasi-symmetric algebra and blow-up.

Theorem 2.9. *Let $Y \subset X \subset M$ be closed embeddings of schemes, with M nonsingular. Then the canonical epimorphism*

$$\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \twoheadrightarrow \mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$$

is an isomorphism.

Proof. The matter is local. Since locally every scheme is embedded in a nonsingular variety, it suffices to show that if $X \subset M \subset N$ are closed embeddings, with M and N nonsingular varieties, then $\mathrm{qSym}_{X\subset N}(\mathcal{J}_{Y,X}) \to \mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$ is an isomorphism. Factoring the embedding $M \subset N$ through the product, we have the diagram

$$\begin{tikzcd} & M \times N \arrow[dr] & \\ X \arrow[ur, hook] \arrow[r, hook] & M \arrow[u, hook] \arrow[r, hook] & N \end{tikzcd}$$

which induces the commutative diagram of qSym algebras

$$\begin{tikzcd} & \mathrm{qSym}_{X\subset M\times N}(\mathcal{J}_{Y,X}) \arrow[dl, "\sim"'] & \\ \mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) & & \mathrm{qSym}_{X\subset N}(\mathcal{J}_{Y,X}) \arrow[ul, "\sim"'] \arrow[ll, two heads] \end{tikzcd}$$

The diagonal arrow on the left is an isomophism because the diagonal embedding splits; the diagonal arrow on the right is an isomorphism by Theorem 2.7. Thus the horizontal arrow is an isomorphism, as needed. □

2.7. By Theorem 2.9, the inverse system of algebras $\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$ stabilizes at nonsingular ambient varieties M. In fact, by Part 4 in Lemma 2.2 there is a canonical embedding

$$\mathrm{qBl}_Y X \subset \mathrm{Bl}_Y M$$

induced by the surjection $\mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}) \to \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$; the line bundle $\mathcal{O}(-1)$ is the restriction of the line bundle of the exceptional divisor.

Theorem 2.9 implies immediately that $\mathrm{qBl}_Y X$ fulfills the promise made in the introduction.

Corollary 2.10. *The quasi-symmetric blow-up* $\mathrm{qBl}_Y X$ *is the largest subscheme of* $s\mathrm{Bl}_Y X$ *which admits an embedding in* $\mathrm{Bl}_Y M$ *(compatibly with the projection to X) for some scheme M containing X.*

Proof. By Theorem 2.9, the quasi-symmetric blow-up $\mathrm{qBl}_Y X$ can be embedded in $\mathrm{Bl}_Y M$ for any *nonsingular* variety M containing X.

On the other hand, if $\mathcal{S}$ is any quotient algebra of $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ which is also a quotient of $\mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Y,M})$ (for some M) then there is an induced epimorphism $\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}) \to \mathcal{S}$ by the definition of qSym (as a tensor product). Hence we obtain a surjection from $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ to $\mathcal{S}$, showing that $\mathrm{Proj}(\mathcal{S}) \subset \mathrm{qBl}_Y X$, as needed. □

It is natural to ask whether the embedding $\mathrm{qBl}_Y X \subset \mathrm{Bl}_Y M$ (for M a nonsingular variety containing X) can be realized concretely, just as the embedding $\mathrm{Bl}_Y X \subset \mathrm{Bl}_Y M$ can be realized as a 'proper transform'.

Definition 2.11. Let $Y \subset X \subset M$ be closed embeddings of schemes. The *principal transform* of X in the blow-up $\mathrm{Bl}_Y M \overset{\rho}{\to} M$ of M along Y is the residual to the exceptional divisor in $\rho^{-1}(X)$.

Here, 'residual' is taken in the sense of [Ful84], Definition 9.2.1. Explicitly, if $\mathcal{I}$ and $\mathcal{J}$ are respectively the ideals of the exceptional divisor and of $\rho^{-1}(X)$ in $\mathrm{Bl}_Y M$, then since $Y \subset X$ it follows that $\mathcal{J} = \mathcal{I} \cdot \mathcal{K}$ for a uniquely determined ideal $\mathcal{K}$. This ideal $\mathcal{K}$ defines the residual scheme.

The definition of principal transform would appear to depend on M; at any rate, $\rho^{-1}(X)$ certainly depends on M as it contains the exceptional divisor of $\mathrm{Bl}_Y M$. However, the next result claims that the *principal* transform is almost as intrinsic to X, Y as is the *proper* transform.

Theorem 2.12. *Let $Y \subset X \subset M$ be closed embeddings of schemes, with M a nonsingular variety. Then the quasi-symmetric blow-up of X along Y equals the principal transform of X in $\mathrm{Bl}_Y M$.*

This is an easy consequence of the following computation of $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$.

Lemma 2.13. *With notation as in the statement of the theorem,*

$$\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) = \bigoplus_{d\geq 0} \mathcal{J}_{Y,M}^d / \mathcal{J}_{X,M}\mathcal{J}_{Y,M}^{d-1}$$

(where we set $\mathcal{J}_{Y,M}^{-1} = \mathcal{O}_M$).

Proof. By Theorem 2.9 we have $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \cong \mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X})$; we compute the latter. For $d \geq 1$ set up the commutative diagram with exact rows

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & \mathrm{Tors}_d \arrow[r] \arrow[d] & \mathrm{Disc}_d \arrow[r] \arrow[d] & 0 \\
 & J_{X,M} \cdot \mathrm{Sym}^{d-1}\mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d \mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d(\mathcal{J}_{Y,M}/\mathcal{J}_{X,M}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}_{X,M} \cdot \mathcal{J}_{Y,M}^{d-1} \arrow[r] \arrow[d] & \mathcal{J}_{Y,M}^d \arrow[r] \arrow[d] & \mathcal{J}_{Y,M}^d/(\mathcal{J}_{X,M} \cdot \mathcal{J}_{Y,M}^{d-1}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all Sym are over $\mathcal{O}_M$, $J_{X,M}$ denotes the image of $\mathcal{J}_{X,M}$ in $\mathrm{Sym}^1\mathcal{J}_{Y,M} = \mathcal{J}_{Y,M}$, Tors_d is defined to make the central column exact, Disc_d is its image in $\mathrm{Sym}^d(\mathcal{J}_{Y,M}/\mathcal{J}_{X,M})$. A diagram chase shows that the column on the right is exact. This gives

$$\mathcal{O}_M \oplus \bigoplus_{d\geq 1} \mathcal{J}_{Y,M}^d/\mathcal{J}_{X,M}\mathcal{J}_{Y,M}^{d-1} = \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}/\mathcal{J}_{X,M}) \otimes_{\mathrm{Sym}_{\mathcal{O}_M}\mathcal{J}_{Y,M}} \mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}).$$

Tensoring by $\mathcal{O}_X$ only affects the term of degree 0 on the left. On the other hand,

$$\mathcal{O}_X \otimes_{\mathcal{O}_M} \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}/\mathcal{J}_{X,M}) = \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X});$$

thus

$$\bigoplus_{d\geq 0} \mathcal{J}_{Y,M}^d/\mathcal{J}_{X,M}\mathcal{J}_{Y,M}^{d-1} = \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$$

by the associativity of tensor products. □

2.8. For $Y \subset X \subset M$, and M not necessarily nonsingular, we can of course consider a quasi-symmetric blow-up $\mathrm{Proj}(\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}))$.

Example 2.14. Let $Y \subset X \subset M$, with $M = X \times \mathbb{P}^1$ and X embedded as $X \times \{\infty\}$. Then the corresponding quasi-symmetric blow-up equals the ordinary blow-up $\mathrm{Bl}_Y X$.

This follows (for example) from Theorem 2.7 and the fact that the quasi-symmetric algebra corresponding to the identity is the Rees algebra, cf. the comments immediately following the proof of Theoremm 2.7.

In this case the blow-up of the ambient space M along Y is the 'deformation to the normal cone' of [Ful84]. The statement of Theorem 2.12 corresponds to the description of the fiber over ∞ of the deformation, cf. p. 87 of loc. cit.

In general, the analog of Theorem 2.12 realizes any such quasi-symmetric blow-up as the principal transform of X in the (ordinary) blow-up of M along Y. We observe that every Proj of a quasi-symmetric algebra is contained in the quasi-symmetric blow-up of Definition 2.8, since any M is contained locally in a nonsingular variety. In any case, this more general notion will not be used in the rest of this paper.

Some of the intuition regarding the quasi-symmetric blow-up of Definition 2.8 is captured by the following simple examples.

Example 2.15. Let $X =$ a pair of distinct lines in $M = \mathbb{A}^2$, and $Y =$ the point of intersection.

Consider the blow-up $\rho : \mathrm{Bl}_Y M \to M$; the exceptional divisor is a $\mathbb{P}^1$. The ordinary $\mathrm{Bl}_Y X$ is the proper transform of X, and it consists of two disjoint $\mathbb{A}^1$. The inverse image $\rho^{-1}(X)$ consists (as a divisor) of $\mathrm{Bl}_Y X$ plus the exceptional divisor *with multiplicity 2.* By Theorem 2.12, the quasi-symmetric blow-up $\mathrm{qBl}_Y X$ sits between these two schemes: it consists of $\mathrm{Bl}_Y X$ plus the exceptional divisor *with multiplicity 1.*

Example 2.16. By the same token, if X consists of the union of m distinct lines through a point Y in a plane, then $\mathrm{qBl}_Y X$ consists of m disjoint lines union a $(m-1)$-multiple $\mathbb{P}^1$ intersecting each of them at a point.

Example 2.17. The picture is drastically different if the lines are not coplanar. For example, let X be the union of the coordinate axes in $\mathbb{A}^3$, and let Y be the origin. Then $\mathrm{qBl}_Y X$ consists of three disjoint lines, union a *plane* $\mathbb{P}^2$ intersecting each of them at a point.

This is again checked by using Theorem 2.12. Using coordinates (x, y, z) in $M = \mathbb{A}^3$, the ideals of Y and X are (x, y, z), (xy, xz, yz) respectively. We can choose coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ in an affine chart of the blow-up of $\mathbb{A}^3$ at the origin so that the blow-up map ρ is given by

$$\begin{cases} x = \tilde{x}\tilde{z} \\ y = \tilde{y}\tilde{z} \\ z = \tilde{z} \end{cases},$$

with $(\tilde{z})$ the ideal of the exceptional divisor. The ideal of $\rho^{-1}(X)$ in this chart is

$$(\tilde{x}\tilde{y}\tilde{z}^2, \tilde{x}\tilde{z}^2, \tilde{y}\tilde{z}^2) = (\tilde{x}\tilde{z}^2, \tilde{y}\tilde{z}^2).$$

By Theorem 2.12, $\mathrm{qBl}_Y X$ is the residual to the exceptional divisor in $\mathrm{qBl}_Y M$; hence it has ideal

$$(\tilde{x}\tilde{z}, \tilde{y}\tilde{z}) = (\tilde{x}, \tilde{y}) \cap (\tilde{z})$$

in this chart: that is, $\mathrm{qBl}_Y M$ meets this chart in the proper transform of the line $x = y = 0$, union the exceptional divisor. The proper transforms of the other lines are contained in the other charts covering the blow-up.

Example 2.17 shows that $q\mathrm{Bl}_Y X$ may have components of higher dimension than X. Contrast this situation with Example 2.16: the quasi-symmetric blow-up of the union of three *coplanar* lines has dimension 1 (it consists of three disjoint lines union a double line connecting them). More generally:

Corollary 2.18. *If X can locally be embedded as a hypersurface in a nonsingular irreducible variety, then for every $Y \subset X$ the quasi-symmetric blow-up $\mathrm{qBl}_Y X$ is equidimensional.*

Proof. This follows from Theorem 2.12, which shows that in this case $\mathrm{qBl}_Y X$ is a Cartier divisor in $\mathrm{Bl}_Y M$. □

Hypersurfaces of nonsingular varieties will be our main concern in the rest of the paper.

3. The conormal and characteristic cycles of a hypersurface

3.1. We now move from the generalities in §2 to our application to the theory of Chern classes of singular varieties. In this section we will deal with the theory at the level of Lagrangian cycles in the cotangent bundle of an ambient nonsingular variety; in the next section we will extract the information more closely pertaining to characteristic classes.

Our main objective in this section is to show that the notion introduced in §2 gives a concrete realization of the characteristic cycle of a hypersurface X in a nonsingular ambient variety M. In a nutshell, the *characteristic* cycle of X is the cycle of the *quasi-symmetric blow-up* of X along its singularity subscheme. This fact should be appreciated in conjunction with the (straightforward) observation that the *conormal* cycle of X is the cycle of its ordinary blow-up along the same subscheme.

Realizing the characteristic cycle allows us to give a direct computation of the Chern-Schwartz-MacPherson classes of a hypersurface, following the same philosophy behind other characteristic classes (specifically the classes introduced in [FJ80] and [Ful84], Example 4.2.6). This requires a certain care in handling the appropriate tautological line bundles; we work this out in §4.

After the preliminary work done in §2, the main result in this section follows easily from the existing literature on characteristic classes for singular hypersurfaces.

In this section we also identify a condition under which the quasi-symmetric blow-up needed here equals the symmetric blow-up. In this situation, the Chern-Schwartz-MacPherson class of the hypersurface can be efficiently expressed in terms of the Chern class of a certain coherent sheaf defined on it.

3.2. We work over an algebraically closed field of characteristic 0. Throughout the rest of the paper M will denote a nonsingular irreducible algebraic variety, and X will be the zero-scheme of a nonzero section F of a line bundle $\mathcal{L}$ on M; we will say that X is a *hypersurface* for short. For convenience we will implicitly assume that X is *reduced,* although this is not an essential requirement (cf. §3.12).

The singularity locus of X has an interesting, possibly nonreduced scheme structure. We will denote by Y this *singularity subscheme* of X (see §3.8 for the precise definition).

We begin by recalling several well-established notions, for the benefit of the non-expert and in order to establish notation. The informed and impatient reader can safely skip to §3.6.

3.3. A *constructible function* on a variety V is a finite linear combination

$$\sum n_W \, 1\!\!1_W$$

where the summation ranges over (closed, irreducible) subvarieties $W \subset V$, $n_W \in \mathbb{Z}$, and $1\!\!1_W$ denotes the function that is the constant 1 on W, and 0 outside of W. We denote by $C(V)$ the group of constructible functions on V. If $f : V_1 \to V_2$ is a proper map, a push-forward $C(f) : C(V_1) \to C(V_2)$ is defined by setting, for W a subvariety of V_1 and $p \in V_2$,

$$C(f)(1\!\!1_W)(p) = \chi(f^{-1}(p) \cap W),$$

and extending by linearity. Here χ denotes the topological Euler characteristic when working over $\mathbb{C}$; see [Ken90], §3, for the extension of the theory to arbitrary algebraically closed field of characteristic 0.

With this push-forward, the assignment

$$\mathcal{C} : V \mapsto C(V)$$

yields a covariant functor from algebraic varieties to abelian groups.

3.4. A fundamental result of MacPherson ([Mac74] and [Ken90]) compares this functor to the functor

$$\mathcal{A} : V \mapsto A(V)$$

assigning to a variety its Chow group: there exists a natural transformation

$$c_* \quad : \quad \mathcal{C} \rightsquigarrow \mathcal{A}$$

such that, for V a *nonsingular* variety, the induced group homomorphism

$$C(V) \to A(V)$$

maps $1\!\!1_V$ to the total Chern class of the tangent bundle of V:

$$1\!\!1_V \mapsto c(TV) \cap [V].$$

For arbitrarily singular V, one may then define a (total) 'Chern class' in the Chow group of V, by setting

$$c_{\mathrm{SM}}(V) := c_*(1\!\!1_V);$$

thus $c_{\mathrm{SM}}(V) = c(TV) \cap [V]$ if V is nonsingular. Brasselet and Schwartz later discovered that this class defined by MacPherson is in fact Alexander dual to a class previously defined by Schwartz ([Sch65a], [Sch65b]; and [BS81]); nowadays, $c_{\mathrm{SM}}(V)$ is commonly named the *Chern-Schwartz-MacPherson class* of V.

3.5. In MacPherson's approach, the natural transformation c_* is defined directly by requiring that

$$c_*(\mathrm{Eu}_V) = c_{\mathrm{Ma}}(V)$$

for all varieties V. Here $c_{\mathrm{Ma}}(V)$ stands for the *Chern-Mather* class of V, and Eu_V is the *local Euler obstruction,* a measure of the singularities of V. Both these notions were defined in [Mac74], and have been the subject of intense study since; again we refer the reader to [Ken90] for a very readable treatment and for the extension of the theory to arbitrary algebraically closed fields of characteristic 0.

A different approach to the definition of c_* emerged in the work of Sabbah ([Sab85], [Ken90], and [PP01], §1). The natural transformation c_* can be obtained (up to taking a harmless dual) as the composite of two transformations

$$\mathcal{C} \rightsquigarrow \mathcal{L} \rightsquigarrow \mathcal{A},$$

where $\mathcal{L}$ denotes the functor assigning to a variety V the group $L(V)$ of *Lagrangian cycles* over V, with a suitably defined push-forward. If $V \subset M$ is an embedding of V into a nonsingular variety M, the Lagrangian cycles over V are the Lagrangian cycles in the restriction $\mathbb{P}(T^*M)|_V$ of the projectivized cotangent bundle of M. As is well known ([Ken90], Lemma 3), every Lagrangian subvariety over V is in fact the projective conormal space $\mathbb{P}(T^*_W M)$ of a closed subvariety $W \subset V$. Hence $L(V)$ is the free abelian group on the set of subvarieties of V; the realization of $L(V)$ as a group of cycles in $\mathbb{P}(T^*M)|_V$, for some nonsingular M containing V, yields a good notion of push-forward of elements of $L(V)$ (see p. 2829-31 in [Ken90] for details).

The second step $\mathcal{L} \rightsquigarrow \mathcal{A}$ in the above decomposition can be expressed in terms of standard intersection theory, and will be recalled in §4.3. The first step, $\mathcal{C} \rightsquigarrow \mathcal{L}$, is considerably subtler. It is determined by the requirement that, for all (closed, irreducible) subvarieties $W \subset V$, the local Euler obstruction of W correspond (up to sign) to the *conormal cycle* of W in M:

$$(-1)^{\dim W} \operatorname{Eu}_W \mapsto [\mathbb{P}(T^*_W M)]$$

For every constructible function $\varphi \in C(V)$ we obtain then a *characteristic cycle*

$$\operatorname{Ch}(\varphi) \in L(V).$$

The cycle $\operatorname{Ch}(1\!\!1_V)$ (realized as above, that is, in terms of an embedding $V \subset M$) is called the *characteristic cycle of V (in M).*

3.6. Summarizing, there are two important cycles associated to a variety V in the projectivized cotangent bundle $\mathbb{P}(T^*M) = \operatorname{Proj}(\operatorname{Sym}_M((\Omega^1_M)^\vee))$ (here and elsewhere, $^\vee$ denotes 'dual' in the ordinary sense of locally free sheaves, as in [Har77], p. 123) of any nonsingular variety M in which V is embedded:

- the conormal cycle $[\mathbb{P}(T^*_V M)]$, corresponding (up to sign) to the local Euler obstruction of V, and to the Chern-Mather class of V; and
- the characteristic cycle $\operatorname{Ch}(V)$ of V, likewise corresponding to the constant function $1\!\!1_V$ and to the Chern-Schwartz-MacPherson class of V.

Explicitly realizing $\operatorname{Ch}(V)$ 'from the definition' requires finding subvarieties W of V and integers e_W such that $1\!\!1_V = \sum_W e_W \operatorname{Eu}_W$. This information is extremely subtle. 'Index formulas' (cf. [BDK81]) provide an approach to extracting this information, but we do not know of any computationally effective method to implement such formulas.

Our goal here is the construction of a scheme whose cycle is the characteristic cycle of a hypersurface X of a nonsingular variety M. In principle this construction can be performed by symbolic computation programs such as `Macaulay2`. An entirely analogous realization of the conormal cycle is more readily available, and will be recalled in a moment.

The theory recalled above applies to varieties, and in particular requires V to be *reduced.* Because of this, we will assume that our hypersurfaces are reduced in what follows (but see 3.12 below).

3.7. According to the framework recalled above, the conormal and characteristic cycles arise as cycles in the (projectivized) cotangent bundle of an ambient nonsingular variety. It is our opinion that these cycles have a right to exist freely, independent of an ambient variety; but we will wait until §4 to fully make this point. For the time being we will house the cycles in the usual place, which amounts to finding an appropriate ambient for the blow-ups considered in §2.

The section F of $\mathcal{L}$ defining X determines a section s of the bundle $\mathcal{P}^1_M\mathcal{L}$ of principal parts of $\mathcal{L}$:

$$s : \mathcal{O}_M \to \mathcal{P}^1_M\mathcal{L};$$

(see [EGA], 16.7, for the definition of $\mathcal{P}^1$; we recommend the appendix of [Per95] for a thorough but concise treatment). We let Y denote the zero-scheme of s in M, and we call Y the 'singularity subscheme' of X. Composing s with the projection to $\mathcal{L}$ recovers F:

$$\begin{tikzcd} \mathcal{O}_M \arrow[r, "s"'] \arrow[rr, bend left, "F"] & \mathcal{P}^1_M\mathcal{L} \arrow[r, two heads] & \mathcal{L} \end{tikzcd} ;$$

hence s induces a section of $\Omega^1_M \otimes \mathcal{L}$ on X, which is natural to name dF:

$$dF : \mathcal{O}_X \longrightarrow (\Omega^1_M \otimes \mathcal{L})|_X;$$

the subscheme Y is the zero-scheme of dF on X. It is easily checked that, locally, dF is given by the partial derivatives of F with respect to a set of local parameters for M; hence Y is supported on the singular locus of X, justifying its name. Locally, we can write (abusing notation):

$$\mathcal{J}_{Y,M} = \left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)$$

for the ideal or Y in M. We will write (F) for the ideal of X in M, as this is given by the vanishing of the section F of $\mathcal{L}$.

Dualizing $s : \mathcal{O}_M \to \mathcal{P}^1_M\mathcal{L}$ we get an epimorphism

$$(\mathcal{P}^1_M\mathcal{L})^\vee \twoheadrightarrow \mathcal{J}_{Y,M}$$

and from this, Lemma 2.2, and Theorem 2.9 the epimorphisms

$$\mathrm{Sym}_{\mathcal{O}_M}((\mathcal{P}^1_M\mathcal{L})^\vee) \twoheadrightarrow \mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Y,M}) \twoheadrightarrow \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) .$$

Since $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ is an $\mathcal{O}_X$-module, tensoring by $\mathcal{O}_X$ gives an epimorphism

$$\mathrm{Sym}_{\mathcal{O}_M}((\mathcal{P}^1_M\mathcal{L})^\vee|_X) \twoheadrightarrow \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) .$$

Finally, composing with $\mathcal{L}^\vee \hookrightarrow (\mathcal{P}^1_M\mathcal{L})^\vee$ gives the zero-map over X, showing that there is a surjection

$$\mathrm{Sym}((\Omega^1_M \otimes \mathcal{L})^\vee|_X) \twoheadrightarrow \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) .$$

Since $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ dominates all quasi-symmetric algebras of $\mathcal{J}_{Y,X}$, and in particular the Rees algebra, this shows (taking Proj) that there are closed embeddings

$$\mathrm{Bl}_Y X \subset \mathrm{qBl}_Y X \subset \mathrm{Proj}(\mathrm{Sym}((\Omega^1_M \otimes \mathcal{L})^\vee|_X)) = \mathbb{P}(T^*M \otimes \mathcal{L}|_X) \cong \mathbb{P}(T^*M|_X).$$

3.8. The following statement is only one step away from the definitions, but it is excellent preparation for the main result of the section, Theorem 3.2 below.

Theorem 3.1. *The conormal cycle* $[\mathbb{P}(T^*_X M)]$ *of* X *in* M *equals*

$$[\mathrm{Bl}_Y X] = [\mathrm{Proj}(\mathrm{qSym}_{X\subset X}(\mathcal{J}_{Y,X}))].$$

Proof. Recall that we are assuming that X is reduced. The conormal space $T^*_X M$ of X in M is the closure in T^*M of the kernels of the projection

$$(T^*M)_x \longrightarrow (T^*X)_x \longrightarrow 0$$

over nonsingular points x of X. In other words, the projectivized conormal space of X is the closure of the image of the section

$$X^{\mathrm{reg}} \longrightarrow \mathbb{P}(T^*M)|_X = \mathbb{P}(T^*M \otimes \mathcal{L})|_X$$

induced on the set X^{reg} of regular points of X by the section dF determined above. Chasing the morphisms collected above shows that this is precisely how $\mathrm{Bl}_Y X$ is embedded in $\mathbb{P}(T^*M|_X)$ over regular points of X. Hence $\mathrm{Bl}_Y X$ and the projectivized conormal space agree over regular points of X, and it follows that they agree everywhere, as needed. Finally, recall that the quasi-symmetric algebra corresponding to the identity $X \overset{id}{\hookrightarrow} X$ equals the Rees algebra. □

3.9. The next result is our main application of the construction developed in §2; it does for the characteristic cycle precisely what Theorem 3.1 does for the conormal cycle.

Theorem 3.2. *The characteristic cycle* $[\mathrm{Ch}(X)]$ *of* X *in* M *equals*

$$(-1)^{\dim X}[\mathrm{qBl}_Y X] = (-1)^{\dim X}[\mathrm{Proj}(\mathrm{qSym}_{X\subset M}(\mathcal{J}_{Y,X}))].$$

The annoying sign is due to established (thus unavoidable) conventions, and reflects the fact that the Lagrangian point of view is best suited to build a *co*tangent theory of characteristic classes.

Modulo the work done in §2, the statement is an easy consequence of results in the literature on characteristic classes for singular varieties.

Proof. By Theorem 2.12, $[\mathrm{qBl}_Y M]$ equals the principal transform of X in $\mathrm{Bl}_Y M$, so the claim is that the latter computes $\mathrm{Ch}(X)$, with due attention to the sign. Over $\mathbb{C}$, this statement is Corollary 2.4 in [PP01]; for arbitrary algebraically closed fields of characteristic 0, it can be obtained from Claim 2.1 in [Alu00]. □

3.10. We will now identify a technical condition under which the algebra $\mathrm{qSym}_X(Y)$ is nothing but the symmetric algebra of $\mathcal{J}_{Y,X}$. As a consequence of Theorem 3.2, the characteristic cycle of hypersurfaces satisfying this condition is (up to sign) the cycle of the *symmetric* blow-up of their singularity subschemes. This both simplifies matters computationally (since packages such as `Macaulay2` have built-in functions for computing symmetric algebras) and is philosphically intriguing: in this case, the characteristic cycle is realized as the 'linear fiber space' (*Linearer Faserraum*, cf. [Fis67]) corresponding to the ideal sheaf $\mathcal{J}_{Y,X}$. While the fibers of the characteristic cycle are always linear, we do not know if every characteristic cycle can be realized as a linear fiber space.

As above, F denotes the section of the line bundle $\mathcal{L}$ on M whose zero-scheme is the hypersurface X. For the purpose of this discussion, a *homogeneous, degree d differential operator satisfied by F* is a local section of $\mathrm{Sym}^d(\mathcal{P}^1_M\mathcal{L})^\vee$ mapping to 0 in $\mathcal{J}^d_{Y,M}$ via the map induced by the surjection $(\mathcal{P}^1_M\mathcal{L})^\vee \to \mathcal{J}_{Y,M}$ whose existence we pointed out in §3.7. In terms of local parameters $x_1, \dots, x_n$ on M, this object is nothing but a homogeneous polynomial

$$P(T_0, \dots, T_n)$$

with coefficients (local) functions on M, such that

$$P\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \equiv 0;$$

we will express the condition in this slightly imprecise but more vivid language, leaving to the reader the task of translating it into a global, coordinate-free formulation.

The simplest way to manufacture homogeneous differential operators of degree d satisfied by F is as a sum

$$P = P_0 \cdot T_0 + P_1 \cdot T_1 + \cdots + P_n \cdot T_n,$$

where the P_i are homogeneous polynomials of degree $d-1$ in $T_0, \dots, T_n$, and

$$P_0 \cdot F + P_1 \cdot \frac{\partial F}{\partial x_1} + \cdots + P_n \cdot \frac{\partial F}{\partial x_n} = 0.$$

We say that such operators are *trivially* satisfied by F. The *$\times$-condition* on X is a softening of this requirement, on operators of sufficiently high degree satisfied by F.

Definition 3.3. A hypersurface X in M *satisfies the $\times$-condition* if there exists a d_0 such that every homogeneous differential operator P of degree $d \geq d_0$ and satisfied by F can be written as

$$P = P_0 \cdot T_0 + P_1 \cdot T_1 + \cdots + P_n \cdot T_n,$$

where the P_i are homogeneous polynomials of degree $d-1$ in $T_0, \dots, T_n$, and

$$P_1 \cdot \frac{\partial F}{\partial x_1} + \cdots + P_n \cdot \frac{\partial F}{\partial x_n} \in (F).$$

Proposition 3.4. *A hypersurface X satisfies the $\times$-condition if and only if*

$$\mathrm{Sym}^d_X(Y) \cong \mathrm{qSym}^d_X(Y)$$

for $d \gg 0$.

Proof. In the hypersurface case, we can complete the diagram in the proof of Theorem 2.12 so that all rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_0 \cdot \mathrm{Tors}_{d-1} \arrow[r] \arrow[d] & \mathrm{Tors}_d \arrow[r] \arrow[d] & \mathrm{Disc}_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_0 \cdot \mathrm{Sym}^{d-1}\mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d\mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d(\mathcal{J}_{Y,M}/(F)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \cdot \mathcal{J}^{d-1}_{Y,M} \arrow[r] \arrow[d] & \mathcal{J}^d_{Y,M} \arrow[r] \arrow[d] & \mathcal{J}^d_{Y,M}/(F)\mathcal{J}^{d-1}_{Y,M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(the leftmost column is exact as F is a non-zero-divisor, and it follows that the top row is exact). We have to verify that $\mathrm{Disc}_d = 0$ for $d \gg 0$ if and only if X satisfies the $\times$-condition.

Now (cf. for example [Vas94], Chapter 2) Tors_d can be described as the space of degree-d homogeneous operators satisfied by F, modulo those trivially satisfied by F. Hence

$$\mathrm{Disc}_d = \mathrm{Tors}_d / T_0 \cdot \mathrm{Tors}_{d-1}$$

is 0 if and only if every degree-d homogeneous operator satisfied by F is equivalent to a multiple of T_0 modulo trivial ones. That is, if and only if for every homogeneous P of degree d such that

$$P\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \equiv 0$$

there exists a Q, homogeneous of degree $d-1$ and such that

$$P - T_0 \cdot Q = P_0 \cdot T_0 + P_1 \cdot T_1 + \cdots + P_n \cdot T_n$$

with

$$P_0 \cdot F + P_1 \cdot \frac{\partial F}{\partial x_1} + \cdots + P_n \cdot \frac{\partial F}{\partial x_n} = 0.$$

It is straightforward to verify that this latter condition is satisfied for $d \gg 0$ if and only if X satisfies the $\times$-condition. $\square$

Corollary 3.5. *If X satisfies the $\times$-condition, then the characteristic cycle of X in M equals $(-1)^{\dim X}[\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}))]$.*

Proof. Immediate consequence of Theorem 3.2: if X satisfies the $\times$-condition, then by Proposition 3.4 the algebras $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ and $\mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$ are isomorphic in high degree, so they have the same Proj. □

3.11. **The $\times$-files.** The '$\times$' in *$\times$-condition* has been chosen as it reminds us of the prototypical singularities satisfying it: the conic $xy = 0$ in the plane is (i) a hypersurface with a nonsingular singularity subscheme; (ii) a hypersurface with quasi-homogeneous isolated singularities; and (iii) a divisor with normal crossing divisor. Each of these classes of hypersurfaces satisfies the $\times$-condition. In fact, as the interested reader may verify, in each of these cases the embedding of the singularity subscheme in the ambient space is 'linear'.

Recall ([Kee93]) that an embedding of schemes $S \subset T$ is *linear* if the Rees algebra and the symmetric algebra of the ideal of S in T are isomorphic; it is *weakly linear* if the Rees algebra and the symmetric algebra are isomorphic in high degree, that is, if $\mathrm{Proj}(\mathrm{Sym}_T(S))$ is isomorphic to the (Rees) blow-up of T along S. These conditions have been studied extensively, see for example [Mic64], [Hun80], [Val80].

Proposition 3.6. *Let X be a hypersurface in a nonsingular variety M, with singularity subscheme Y. If the embedding of Y in M is weakly linear, then X satisfies the $\times$-condition.*

Proof. With the notation in the proof of Proposition 3.4, the embedding of Y in M is weakly linear if and only if $\mathrm{Tors}_d = 0$ for $d \gg 0$. This implies $\mathrm{Disc}_d = 0$ for $d \gg 0$, which is equivalent to the $\times$-condition, as observed in that proof. □

For example, this implies immediately the $\times$-condition for the first case listed above: if Y is nonsingular, then its embedding in M is regular, hence linear, hence weakly-linear. However, we should remind the reader that the requirement that the singularity *subscheme* of a hypersurface be nonsingular is very strong; substantially stronger, for example, than the requirement that the singularity *locus* be nonsingular. Some constraints on this situation are studied in [Alu95], §3. Hypersurfaces whose singularity subscheme is nonsingular are in particular *nice* in the sense of [AB03].

Example 3.7. The plane curve $x^4 + x^3y^2 + y^6 = 0$ has an isolated singularity at the origin; the embedding of its singularity subscheme in the plane is not linear.

This is checked by explicit calculations, which we performed with `Macaulay2`.

It can also be shown that if the $\times$-condition implies that every vector tangent at a point x to a stratum in a Whitney stratification of X extends to fiberwise linear functions on $\mathcal{P}^1_M\mathcal{O}(X)$, tangent to nearby 'level hypersurfaces'. In this sense, the $\times$-condition may be viewed as a strong regularity requirement on extensions of tangent vectors near strata of a Whitney stratification of X. It is known that tangent vectors to strata of a Whitney stratification of X are suitably 'close' to the tangent spaces of nearby level hypersurfaces (the so-called *w_f-condition of Thom*). This suggests that the techniques in [Par93] or [BMM94] may be apt to characterizing hypersurfaces satisfying the $\times$-condition.

3.12. For simplicity we have assumed that the hypersurface X is reduced in the preceding subsections. It should be noted, however, that the quasi-symmetric blow-up is defined, and determines a cycle in $\mathbb{P}(T^*M)$, regardless of whether X is reduced or not. The arguments given above can be traced in this case, and show that this cycle is nothing but the characteristic cycle of the support X_{red}. This rather remarkable fact implies that simply setting $c_{\text{SM}}(X) := c_{\text{SM}}(X_{\text{red}})$ leads to a consistent theory of Chern-Schwartz-MacPherson classes, at least when X is a hypersurface.

We leave the details to the interested reader (cf. §2.1 in [Alu99a]).

4. Shadows of blow-up algebras

4.1. Theorems 3.1 and 3.2 give intrinsic constructions of the two key cycles associated with X. We would like to deal with the corresponding schemes $\text{Bl}_Y X$, $\text{qBl}_Y X$ as stand-alone entities, and determine precisely what type of information they carry in relation with the ambient nonsingular variety M.

With this in mind, we first discuss the transformation $\mathcal{L} \rightsquigarrow \mathcal{A}$ mentioned in §3.5, which produces the Chern-Mather, resp. Chern-Schwartz-MacPherson classes from the conormal, resp. characteristic cycle; then we separate the rôle of the ambient variety in this computation from that of the blow-ups themselves, and find that the blow-ups carry 'normal data' regarding the embedding $X \subset M$. This point of view unifies the computation of the Chern-Mather and Chern-Schwartz-MacPherson classes with the approach yielding the classes defined by Fulton and Fulton-Johnson ([Ful84], [FJ80], and cf. §4.6 below).

4.2. If $\mathcal{E}$ is a locally free sheaf of rank $e+1$ on a scheme S, there is a precise structure theorem for the Chow group of the projective bundle

$$\mathbb{P}(\mathcal{E}) := \text{Proj}(\text{Sym}\mathcal{E}^\vee) \xrightarrow{\epsilon} S$$

([Ful84], §3.3): every class $C \in A_r\mathbb{P}(\mathcal{E})$ can be written uniquely as

$$C = \sum_{j=0}^{e} c_1(\mathcal{O}(1))^j \cap \epsilon^*(C_{r-e+j})$$

where $\mathcal{O}(1)$ denotes the tautological line bundle on $\mathbb{P}(\mathcal{E})$, and $C_{r-e+j} \in A_{r-e+j}S$.

Therefore, knowledge of C is equivalent to knowledge of the collection of $e+1$ classes $C_{r-e}, \dots, C_r$ on S.

Definition 4.1. We say that the class $C_{r-e} + \cdots + C_r \in AS$ is the *shadow* of the class C.

As its real world namesake, the shadow neglects some of the information carried by the object that casts it. For example, $c_1(\mathcal{O}(1))^j \cdot [\mathbb{P}(\mathcal{E})]$ has shadow $[S]$ for all $j = 0, \dots, e$. However, a pure-dimensional class C can be reconstructed from its shadow if its dimension is known, as follows immediately from the structure theorem recalled above.

It will be convenient to have a direct way to obtain the shadow of a given class.

Lemma 4.2. *The shadow of C is the class*

$$c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right)$$

Proof. Writing C as above, we have

$$\begin{aligned} c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right) &= c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap \sum_{j=0}^{e} c_1(\mathcal{O}(1))^j \cap \epsilon^*(C_{r-e+j})\right) \\ &= \sum_{j=0}^{e} c(\mathcal{E}) \cap \epsilon_* \left(\sum_{k\geq j} c_1(\mathcal{O}(1))^k \cap \epsilon^*(C_{r-e+j})\right). \end{aligned}$$

Since $c_1(\mathcal{O}(1))^k \cap \epsilon^*\alpha = 0$ for $0 \leq k < e$ and any $\alpha \in A_*S$, this says

$$c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right) = \sum_{j=0}^{e} c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap \epsilon^*(C_{r-e+j})\right).$$

Finally, this equals $\sum_{j=0}^{e} C_{r-e+j}$ by [Ful84], Example 3.3.3. □

4.3. As recalled in §3.5, MacPherson's natural transformation c_* can be expressed by a two-step procedure: ($\mathcal{C} \rightsquigarrow \mathcal{L}$) taking the characteristic cycle $\mathrm{Ch}(\varphi)$ of a constructibile function φ, and ($\mathcal{L} \rightsquigarrow \mathcal{A}$) extracting a rational equivalence class from the characteristic cycle. As the natural habitat of Lagrangian cycles is the projectivized *co*tangent bundle $\mathbb{P}(T^*M)$, we find it convenient to arrange things so as to obtain a class $\check{c}_*(\varphi)$ differing from $c_*(\varphi)$ by the sign of the components of odd dimension:

$$\{\check{c}_*(\varphi)\}_r = (-1)^r \{c_*(\varphi)\}_r$$

in dimension r. For example,

$$\check{c}_*(1\!\!1_M) = (-1)^{\dim M} c(T^*M) \cap [M]$$

for the nonsingular ambient M.

Lemma 4.3. *The class $\check{c}_*(\varphi)$ is the shadow of the characteristic cycle* $\mathrm{Ch}(\varphi)$.

Proof. This is formula (12) on p. 67 of [PP01], filtered through Lemma 4.2. As observed in [PP01], this is in agreement with [Mac74]. □

The statement of Lemma 4.3, while implicit in the existing literature, is mysteriously absent in this explicit form relating the transformation $\mathcal{L} \rightsquigarrow \mathcal{A}$ to the structure theorem of the Chow group of projective bundles. This interpretation streamlines the proof that $\mathcal{L} \rightsquigarrow$

$\mathcal{A}$ is a natural transformation; Schürmann has independently made the same observation [Sch01b].

4.4. We are ready to justify the title of this article. Denote by $\check{c}_{\mathrm{Ma}}(X)$, $\check{c}_{\mathrm{SM}}(X)$ resp. the classes obtained by changing the sign of the components of odd dimension in $c_{\mathrm{Ma}}(X)$, $c_{\mathrm{SM}}(X)$.

Theorem 4.4. *Let X be a hypersurface of a nonsingular variety M, and let Y be its singularity subscheme. Then*

- *the shadow of $[\mathrm{Bl}_Y X]$ is $(-1)^{\dim X}\check{c}_{\mathrm{Ma}}(X)$;*
- *the shadow of $[\mathrm{qBl}_Y X]$ is $(-1)^{\dim X}\check{c}_{\mathrm{SM}}(X)$.*

Proof. This now follows from Theorems 3.1 and 3.2, and Lemma 4.3. □

4.5. The next step in our program consists of carefully distinguishing the rôle of the ambient space and of the blow-ups in the statement of Theorem 4.4. There is an interesting twist to this story, which highlights the need for a subtle change of perspective.

We have so far focused on the ideal $\mathcal{J}_{Y,X}$ as the most natural source of information concerning the singularities of X; and indeed we have defined our main notions in §2 starting from the data of an ideal sheaf in $\mathcal{O}_X$. We are now going to shift the attention to a different coherent sheaf, defined for any subscheme X of a nonsingular variety M; it will be easy to relate this sheaf to $\mathcal{J}_{Y,X}$ when X is a hypersurface, and this will naturally extend quasi-symmetric blow-up algebras to this coherent sheaf. To summarize what we will find, this new algebras agree locally with the algebras obtained for $\mathcal{J}_{Y,X}$; in fact, their Proj will be isomorphic as schemes to the quasi-symmetric blow-ups of $\mathcal{J}_{Y,X}$. But the *algebras* carry more information than the schemes: the grading determines a line bundle on the blow-ups, and this information will turn out to be essential.

The new blow-up algebras will thus determine a Segre-class type of invariant, and we will show that using this invariant yields the Mather and Schwartz-MacPherson classes in essentially the same way as ordinary Segre classes of coherent sheaves, resp. of cones lead to Fulton-Johnson, resp. Fulton classes.

4.6. Here is a quick reminder concerning these latter two classes, in order to clarify the context underlining our motivation.

If Z is any scheme embedded in a nonsingular variety M (of dimension $> \dim Z$ for convenience), there are several ways to obtain 'normal data' relating to the embedding. For example, such data is carried by the conormal sheaf $\mathcal{N}_Z M = \mathcal{J}_{Z,M} \otimes_{\mathcal{O}_M} \mathcal{O}_Z$, and can be effectively encoded in the *Segre class* of this coherent sheaf, defined by

$$s(\mathcal{N}_Z M) := p_* \sum c(\mathcal{O}(1))^i \cap [\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{N}_Z M))]$$

where p is the structure morphism on Proj.

Definition 4.5. The *Fulton-Johnson* class of Z is the class

$$c(TM) \cap s(\mathcal{N}_Z M)$$

in the Chow group of Z.

It can be shown that this class is independent of the ambient variety M, and agrees with the total Chern class of the tangent bundle of Z when Z is nonsingular (cf. [FJ80] or [Ful84], Example 4.2.6 (c)).

A different way to access normal data amounts to taking a Rees point of view rather than a Sym point of view. Replacing

$$\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{N}_Z M) = \mathrm{Sym}_{\mathcal{O}_M}(\mathcal{J}_{Z,M}) \otimes_{\mathcal{O}_M} \mathcal{O}_Z$$

by

$$\mathrm{Rees}_{\mathcal{O}_M}(\mathcal{J}_{Z,M}) \otimes_{\mathcal{O}_M} \mathcal{O}_Z$$

defines the *normal cone* of Z in M, whose Segre class (again defined by pushing forward powers of the first Chern class of $\mathcal{O}(1)$) is properly called the *Segre class of Z in M*, $s(Z, M)$.

Applying the same principle as above leads to the following notion.

Definition 4.6. The *Fulton* class of Z is the class

$$c(TM) \cap s(Z, M)$$

in the Chow group of Z.

Again, it can be shown that this class is independent of the ambient nonsingular variety M (cf. [Ful84], Example 4.2.6), and agrees with the total Chern class of the tangent bundle of Z when Z is nonsingular.

The formulas in Definitions 4.5 and 4.6 should be compared with the formulas for the Chern-Mather and Chern-Schwartz-MacPherson classes that we will obtain in Theorem 4.9.

4.7. How else can one extract normal data from an embedding $Z \subset M$ of a scheme in a nonsingular variety? Again we assume that $\dim M > \dim Z$. There is a surjection

$$\Omega^1_M|_Z \longrightarrow \Omega^1_Z \longrightarrow 0,$$

from which we obtain the exact sequence

$$0 \longrightarrow \mathcal{H}om(\Omega^1_Z, \mathcal{O}_Z) \longrightarrow \mathcal{H}om(\Omega^1_M|_Z, \mathcal{O}_Z) \longrightarrow \mathcal{T}_Z M \longrightarrow 0,$$

defining the coherent sheaf $\mathcal{T}_Z M$ on Z. If Z is nonsingular then $\mathcal{T}_Z M$ is locally free, and in fact it is the sheaf of sections of the normal bundle of Z in M.

Now our idea consists of following the same guiding principle which rules in §4.6, but employing Segre classes obtained from quasi-symmetric algebras associated with $\mathcal{T}_Z M$.

As things stand now, we only have defined such objects for ideals, and this limits the scope of our aim. However, in the case we have considered in §3 and in Theorem 4.4 the day is saved by a special form taken by $\mathcal{T}_Z M$.

Lemma 4.7. *If $Z = X$ is a hypersurface in a nonsingular variety M, with line bundle $\mathcal{L}$ and singularity subscheme Y, then $\mathcal{T}_X M = \mathcal{J}_{Y,X} \otimes_{\mathcal{O}_X} \mathcal{L}$.*

Proof. Let $\mathcal{J} = \mathcal{J}_{X,M}$ denote the ideal of X in M. Taking $\mathcal{H}om(-, \mathcal{O}_X)$ in the exact sequence of differential gives the exact sequence

$$0 \longrightarrow \mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om(\Omega^1_M|_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_X).$$

A local computation determines the image of the rightmost map as the subsheaf of $\mathcal{O}_X$-morphisms $\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{O}_X$ factoring through $\mathcal{J}_{Y,X}$. In other words, if X is a hypersurface in M then

$$\mathcal{T}_X M = \mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{J}_{Y,X}) = \mathcal{L}|_X \otimes_{\mathcal{O}_X} \mathcal{J}_{Y,X},$$

as claimed. □

By virtue of Lemma 4.7 we can make sense of quasi-symmetric algebras of $\mathcal{T}_X M$ if X is a hypersurface in M. The two extremes in the range of quasi-symmetric algebras are the following two definitions:

$$\mathrm{qSym}_{X\subset X}(\mathcal{T}_X M) := \mathrm{Rees}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \otimes_{\mathcal{O}_X} \mathcal{L}$$
$$\mathrm{qSym}_{X\subset M}(\mathcal{T}_X M) := \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X}) \otimes_{\mathcal{O}_X} \mathcal{L}$$

and the corresponding Segre class-like notions:

$$\check{s}_{\mathrm{Ma}}(X, M) := p_* \sum c(\mathcal{O}(1))^i \cap [\mathrm{Proj}(\mathrm{qSym}_{X\subset X}(\mathcal{T}_X M))]$$
$$\check{s}_{\mathrm{SM}}(X, M) := p_* \sum c(\mathcal{O}(1))^i \cap [\mathrm{Proj}(\mathrm{qSym}_{X\subset M}(\mathcal{T}_X M))]$$

where p denotes the projection from the corresponding Proj, and $\mathcal{O}(1)$ is the tautological line bundle. We remark that the two Proj equal $\mathrm{Bl}_Y X$, $\mathrm{qBl}_Y X$ as schemes—only the tautological bundles are affected upon tensoring by $\mathcal{L}$.

We have defined a 'checked' notion of Segre class in view of artificially taking a dual that brings us back to the *tangent* world. So we set

$$s_{\mathrm{Ma}}(X, M) := (-1)^{\dim X} \sum_{r\geq 0} (-1)^r \check{s}_{\mathrm{Ma}}(X, M)_r$$
$$s_{\mathrm{SM}}(X, M) := (-1)^{\dim X} \sum_{r\geq 0} (-1)^r \check{s}_{\mathrm{SM}}(X, M)_r$$

where subscripts mark dimensions; that is, we change the sign of components in the checked Segre classes of every other *co*dimension in X.

Example 4.8. If X is a nonsingular hypersurface, then all notions of Segre class coincide: $s(\mathcal{N}_X M) = s(X, M) = s_{\mathrm{Ma}}(X, M) = s_{\mathrm{SM}}(X, M) = c(\mathcal{L})^{-1} \cap [X]$. If X may be singular, but satisfies the $\times$-condition (see §3.10), then $s_{\mathrm{SM}}(X, M) = s(\mathcal{J}_{Y,X} \otimes \mathcal{L})$.

4.8. Summarizing, we have extracted normal data from our hypersurface X in M by defining a coherent sheaf $\mathcal{T}_X M$ in a rather simple-minded way from the exact sequence of differentials of X; adapting to $\mathcal{T}_X M$ the construction of §2; and defining from the resulting blow-up algebra a notion of Segre class. These classes achieve precisely what we set out to do, that is, they yield the Chern-Mather and Chern-Schwartz-MacPherson classes by the same method behind the classes of Fulton and Fulton-Johnson (cf. Definitions 4.5 and 4.6). That is:

Theorem 4.9. *Let X be a hypersurface in a nonsingular variety M. Then*

- $c_{\mathrm{Ma}}(X) = c(TM) \cap s_{\mathrm{Ma}}(X, M)$;
- $c_{\mathrm{SM}}(X) = c(TM) \cap s_{\mathrm{SM}}(X, M)$.

Proof. We will give the argument for the second equality; the first is treated similarly.

Tensoring by $\mathcal{L}$ the epimorphism

$$\mathrm{Sym}((\Omega^1_M \otimes \mathcal{L})^\vee|_X) \twoheadrightarrow \mathrm{qSym}_{\mathcal{O}_X}(\mathcal{J}_{Y,X})$$

from §3.7 we obtain

$$\mathrm{Sym}((\Omega^1_M)^\vee|_X) \twoheadrightarrow \mathrm{qSym}_{X\subset M}(\mathcal{T}_X M),$$

inducing the embedding

$$\mathrm{qBl}_Y X \hookrightarrow \mathbb{P}(T^* M)$$

realizing the characteristic cycle of X (by Theorem 3.2), and showing that the restriction of $\mathcal{O}(-1)$ to $\mathrm{qBl}_Y X$ is the universal bundle $\mathcal{O}(-1)$ of $\mathrm{Proj}(\mathrm{qSym}_{X\subset M}(\mathcal{T}_X M))$. By Lemma 4.2, the shadow of the blow-up algebra $\mathrm{qBl}_Y X$ is computed by

$$c(T^*M) \cap (c(\mathcal{O}(-1))^{-1} \cap [\mathrm{qBl}_Y X]) = c(T^*M) \cap \check{s}_{\mathrm{SM}}(X, M).$$

This equals $(-1)^{\dim X} \check{c}_{\mathrm{SM}}(X)$, by Theorem 4.4. The equality for $c_{\mathrm{SM}}(X)$ follows by changing the sign of the components of every other codimension. □

4.9. At this point it is only too natural to pose the problem of defining quasi-symmetric algebras for coherent sheaves so as to validate Theorem 4.9 for more general schemes X, following the same strategy (that is, by obtaining Segre classes from the quasi-symmetric algebras of $\mathcal{T}_X M$). The advantage in formulas such as those in Theorem 4.9 is not only theoretical: these formulas can be implemented in procedures for symbolic computation programs such as `Macaulay2`. At present a routine is implemented that computes Chern-Schwartz-MacPherson classes of projective schemes ([Alu03]), exploiting the hypersurface case in order to compute classes in the general case, by a computationally expensive 'inclusion-exclusion' procedure.

An upgrade of Theorem 4.9 to more general schemes would bring about a drastic improvement in the speed of such routines.

Regarding a possible definition of quasi-symmetric algebras for coherent sheaves, this would presumably pivot on a good notion of Rees algebra of a module; such notions have been introduced and studied by several authors—for example Micali, [Mic64]. Even in the simpler case of ideals treated here, it would be quite interesting to relate our construction with the ideals defined by Micali in loc. cit., interpolating between the symmetric and the Rees algebras.

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Mathematics Department, Florida State University, Tallahassee, Florida 32306, USA

E-mail address: aluffi@math.fsu.edu