The Grothendieck Group of a Hopf Algebra

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ABSTRACT

Let H be a cosemisimple Hopf algebra over an algebraically closed field k. We show that if H contains a simple subcoalgebra of dimension 4, then H contains either a Hopf subalgebra of dimension 2, 12, or 60, or a simple subcoalgebra of dimension n^2 for each positive integer n. In particular, if H is finite dimensional, then it has even dimension.

Introduction

The theory of cosemisimple Hopf algebras has recently enjoyed considerable success. For example, it was shown in [5] that if H is a finite dimensional cosemisimple Hopf algebra over a field of characteristic 0, then H is semisimple. An account of this and other major results can be found in [6]. Despite this progress, several long-standing questions have remained unanswered. In particular, Kaplansky conjectured [3, Appendix 2] that if a finite dimensional cosemisimple Hopf algebra H over an algebraically closed field contains a simple subcoalgebra of dimension n^2, then n divides the dimension of H.

In this paper we verify this conjecture if H contains a simple subcoalgebra of dimension 4. This follows, via the freeness theorem [7], from our main result below: if H is a cosemisimple Hopf algebra over an algebraically closed field k and contains a simple subcoalgebra of dimension 4, then H contains either a Hopf subalgebra of dimension 2, 12, or 60, or a simple subcoalgebra of dimension n^2 for each positive integer n. Our result does not require any restriction on the characteristic of the underlying field. To achieve this generality we study the Grothendieck group of a Hopf algebra, more precisely, the Grothendieck group of the category of finite dimensional right H-comodules. For completeness we include the definition and elementary properties (compare [2], [1,§16B]). While all results are aimed at helping us prove the main theorem, some are clearly interesting in themselves.

All vector spaces and tensor products will be over a field k. We assume general familiarity with Hopf algebras, as presented in [8].

1. The Grothendieck group of a coalgebra

To give the basic definition of the Grothendieck group that we shall be using, we need to assume only that H is a coalgebra.

Definition. Let H be a coalgebra, and let $\mathcal{M}^H$ be the category of finite dimensional right H-comodules. Let $\mathcal{F}$ be the free abelian group on the symbols (M), one for each isomorphism class of objects in $\mathcal{M}^H$, and $\mathcal{F}_0$ the subgroup of $\mathcal{F}$ generated by all expressions $(M) - (L) - (N)$, where $0 \to L \to M \to N \to 0$ is a short exact sequence of finite dimensional right H- comodules. Then the Grothendieck group of the category $\mathcal{M}^H$ is defined by

$$\mathcal{G}(H) = \frac{\mathcal{F}}{\mathcal{F}_0}.$$

For M in $\mathcal{M}^H$ we write $[M] = (M) + \mathcal{F}_0$ for its image in $\mathcal{G}(H)$. Observe that for M, N in $\mathcal{M}^H$, $[M \oplus N] = [M] + [N]$. Hence, each element x in $\mathcal{G}(H)$ can be written as $x = \sum n_i[M_i] = \sum\limits_{n_i \geq 0} n_i[M_i] - \sum\limits_{n_i<0} (-n_i)[M_i] = [M] - [N]$ for some integers n_i, some M_i in $\mathcal{M}^H$ and some (nonunique) M, N in $\mathcal{M}^H$.

From the definition of $\mathcal{G}(H)$, we have at once:

Proposition 0. *Let T be an abelian group and $h : \mathcal{M}^H \to T$ a map such that $h(M) = h(L) + h(N)$ whenever $0 \to L \to M \to N \to 0$ is a short exact sequence of objects in $\mathcal{M}^H$. Then there exists a unique group homomorphism $g : \mathcal{G}(H) \to T$ satisfying $g([M] - [N]) = h(M) - h(N)$ for every M, N in $\mathcal{M}^H$.* □

Proposition 1. *Let X denote the set of simple subcoalgebras of the coalgebra H. For each $C \in X$, let V_C be a simple right C-comodule. Then $\mathcal{G}(H)$ is a free abelian group with basis $\{[V_C] : C \in X\}$.*

Proof. The (routine) proof of Proposition 1 is analogous to that of [1, Proposition 16.6]. One shows that for each $M \in \mathcal{M}^H$, we have $[M] = \sum r_{_C}(M)[V_C]$, where $r_{_C}(M)$ is the number of composition factors of M which are isomorphic to V_C. □

We shall refer to the $[V_C]$'s as *basic elements* of $\mathcal{G}(H)$, and $\mathcal{X} = \{[V_C] : C \in X\}$ as the *standard basis*. For any $z \in \mathcal{G}(H)$, we write $z = \sum\limits_{x \in \mathcal{X}} m(x, z)x$, and refer to the integer $m(x, z)$ as the *multiplicity* of x in z. If $m(x, z) \neq 0$, we say that x is a *basic component* of z. Extending m to a biadditive function, we define $m(z, z') = \sum\limits_{x \in \mathcal{X}} m(x, z)m(x, z')$, all $z, z' \in \mathcal{G}(H)$.

Note that we have bijections between X, $\mathcal{X}$, and the set of isomorphism classes of simple right H-comodules. If C is a simple subcoalgebra of H, V a simple right C-comodule, and $x = [V] \in \mathcal{G}(H)$, we write $V = V_C = V_x$, $C = C_V = C_x$, $x = x_{_V} = x_{_C}$. When appropriate, we suppress double subscripts, writing (for example) V_i and x_i in place of V_{C_i} and $x_{_{C_i}}$.

Let M be a right H-comodule, with comodule structure map $\rho : M \to M \otimes H$. The smallest subcoalgebra C of H such that $\rho(M) \subseteq M \otimes C$ is called the *subcoalgebra associated to* M, and will be denoted $C_{_M}$. If $\{m_j\}$ is a basis of M, and $\rho(m_j) = \sum\limits_i m_i \otimes c_{ij}$ where each $c_{ij} \in H$, then $C_{_M}$ is the span of the c_{ij}'s. Note that the subcoalgebra associated to any subcomodule or quotient comodule of M will be contained in $C_{_M}$.

Proposition 2. *Let H be a coalgebra. For any $M \in \mathcal{M}^H$, and any simple subcoalgebra E of H, we have $m(x_{_E}, [M]) > 0$ if and only if $E \subseteq C_{_M}$.*

Proof. If $m(x_{_E}, [M]) > 0$, then E is associated to a quotient comodule of a subcomodule of M, so $E \subseteq C_{_M}$. We shall show the reverse implication by induction on the dimension of M. If M is zero or simple, the assertion is clear. Otherwise, let N be a nonzero subcomodule of M of lower dimension. By extending a basis of N to a basis of M, it is easy to see that $\rho(M) \subseteq M \otimes C_{_{M/N}} + N \otimes H$. This gives us that $(id \otimes \Delta)\rho(M) = (\rho \otimes id)\rho(M) \subseteq M \otimes H \otimes C_{_{M/N}} + N \otimes C_{_N} \otimes H$, so $\Delta(C_{_M}) \subseteq C_{_N} \otimes H + H \otimes C_{_{M/N}}$. Thus, $C_{_M} \subseteq C_{_N} \bigwedge C_{_{M/N}}$. Since E is simple, $E \subseteq C_{_M}$ implies that $E \subseteq C_{_N}$ or $E \subseteq C_{_{M/N}}$, and we are done by induction. $\square$

For any $z \in \mathcal{G}(H)$, we define the *degree* of z to be $\deg(z) = \sum_{x \in \mathcal{X}} m(x, z) \dim(V_x)$. Note that for any $M \in \mathcal{M}^H$, $\deg([M]) = \dim(M)$.

We next state the definition of intertwining numbers. Some of their properties and their connection to the Grothendieck group are presented in the subsequent Proposition 3.

Definition. Let H be a coalgebra. Let V, W be objects in $\mathcal{M}^H$, the category of finite dimensional right H-comodules. The intertwining number of V and W is $i(V, W) = \dim_k \mathcal{M}^H(V, W)$.

Proposition 3. *Let H be a coalgebra, and let $U_1, U_2, U, V \in \mathcal{M}^H$. Then:*

(1) $i(U_1 \oplus U_2, V) = i(U_1, V) + i(U_2, V)$

$i(V, U_1 \oplus U_2) = i(V, U_1) + i(V, U_2)$

(2) If H is cosemisimple, then $i(U, V) = i(V, U)$

(3) If H is cosemisimple and k is algebraically closed, then if U or V is simple, $i(U, V)$ is

the number of times that the simple one occurs as a composition factor of the other.

Proof. (1) is clear from the corresponding properties of $\mathcal{M}^H(U,V)$. Concerning (2), it is clear that for arbitrary H, $i(U,V) = i(V,U)$ if both U and V are simple H-comodules. Thus, if U,V are direct sums of simple H-comodules, as we are assuming, $i(U,V) = i(V,U)$ follows using (1). Regarding (3), we observe that if U,V are simple and k algebraically closed, then $i(U,V)$ is 1 if U is isomorphic to V and 0 otherwise. Therefore (3) follows from (2). □

2. Bialgebras and subcoalgebras

If we assume that H is a bialgebra, $\mathcal{G}(H)$ becomes a ring. Recall that if H is a bialgebra and M,N are right H-comodules, then $M \otimes N$ is a right H-comodule via $\rho(m \otimes n) = \sum m_0 \otimes n_0 \otimes m_1 n_1$ for all $m \in M$, $n \in N$.

Proposition 4. *Let H be a bialgebra. Then the Grothendieck group $\mathcal{G}(H)$ is a ring, with multiplication given by $[M][N] = [M \otimes N]$ for each $M,N \in \mathcal{M}^H$.*

Proof. The proof is as in [1, p. 407]. □

We shall denote $[k1]$, the unit of $\mathcal{G}(H)$, as 1.

Note that for $x,y \in \mathcal{X}$ we have $\deg(xy) = \deg([V_x \otimes V_y]) = \dim(V_x \otimes V_y) = \dim(V_x)\dim(V_y) = \deg(x)\deg(y)$. It follows that $\deg(zz') = \deg(z)\deg(z')$ for all $z,z' \in \mathcal{G}(H)$. Thus we have the formula:

$$\deg(z)\deg(z') = \sum_{x \in \mathcal{X}} m(x,zz')\deg(x), \text{ for all } z,z' \in \mathcal{G}(H).$$

If H is a bialgebra, and hence $\mathcal{G}(H)$ is a ring, we are particularly interested in how products can be expressed in terms of our standard basis. Proposition 5 gives a partial answer, and an application.

Proposition 5. *Let H be a bialgebra.*

(1) Let $M, N \in \mathcal{M}^H$, and let E be a simple subcoalgebra of H. Then $m(x_E, [M][N]) > 0$ if and only if $E \subseteq C_M C_N$.

(2) Let C, D, E be subcoalgebras of H, with E simple. Then $E \subseteq CD$ iff there exist simple subcoalgebras F of C and G of D such that $m(x_E, x_F x_G) > 0$.

Proof. (1). It is easy to see that $C_{M\otimes N} = C_M C_N$, so (1) follows from Proposition 2.

(2). It suffices to prove this in the case in which C and D are finite dimensional. Applying (1), we have that $E \subseteq CD$ iff $m(x_E, [C][D]) > 0$. Now $m(x_E, [C][D]) > 0$ iff there exist basic components x_F of $[C]$, x_G of $[D]$ with $m(x_E, x_F x_G) > 0$. By Proposition 2, x_F and x_G are basic components of $[C]$ and $[D]$ respectively iff $F \subseteq C$ and $G \subseteq D$, and we are done. □

Let H be a bialgebra. Let us say that a *standard subring* of $\mathcal{G}(H)$ is a subring of $\mathcal{G}(H)$ which is spanned as an abelian group by a subset of the standard basis of $\mathcal{G}(H)$.

Theorem 6. *Let H be a bialgebra. There is a 1-1 correspondence between standard subrings of $\mathcal{G}(H)$ and subbialgebras of H generated as algebras by their simple subcoalgebras, given by: the subbialgebra B generated by its simple subcoalgebras corresponds to the standard subring spanned by* $\{x_C : C$ is a simple subcoalgebra of $B\}$.

Proof. Let B be a subbialgebra of H. The functorial ring homomorphism $\mathcal{G}(B) \to \mathcal{G}(H)$ is an isomorphism onto the standard subring spanned by $\{x_C : C$ is a simple subcoalgebra of $B\}$. Since two subbialgebras of H correspond to the same standard subring of $\mathcal{G}(H)$ iff they contain the same simple subcoalgebras, there is a 1-1 correspondence between subbialgebras of H generated by their simple subcoalgebras and certain standard subrings of $\mathcal{G}(H)$.

Let A be a standard subring of $\mathcal{G}(H)$, which is spanned (as an abelian group) by the subset $\mathcal{Y}$ of $\mathcal{X}$. Let D be the subcoalgebra of H spanned by the simple subcoalgebras C_y, $y \in \mathcal{Y}$. By repeated use of Proposition 5(2), we see that the simple subcoalgebras of D^2, D^3, etc., are all contained in D. Thus the subbialgebra generated by D has coradical D and corresponds to A, as required. $\square$

For any subset $\mathcal{Y}_1$ of $\mathcal{X}$, the smallest standard subring of $\mathcal{G}(H)$ containing $\mathcal{Y}_1$ is called the standard subring *generated* by $\mathcal{Y}_1$. It corresponds to the subbialgebra generated as an algebra by $\{C_y : y \in \mathcal{Y}_1\}$. Arguing as above, we see that a basis for the standard subring generated by $\mathcal{Y}_1$ is given by $\{1\}$ together with the union of the subsets $\mathcal{Y}_n$, $n \geq 1$ of $\mathcal{X}$, where for each $n \geq 1$, we set $\mathcal{Y}_{n+1} = \{x \in \mathcal{X} : m(x, zy) > 0 \text{ for some } z \in \mathcal{Y}_n, y \in \mathcal{Y}_1\}$.

3. Hopf algebras and " * "

When H is a Hopf algebra, the dual of a right H-comodule is a right H-comodule in a natural way. Let $U \in \mathcal{M}^H$, with its comodule structure map $\rho : U \to U \otimes H$. Let $\{u_j\}$ be a basis of U, and write $\rho(u_j) = \sum_i u_i \otimes a_{ij}$ where each $a_{ij} \in H$. We have that U^* is a left comodule via $\rho^t : U^* \to H \otimes U^*$ given by $\rho^t(u_i^*) = \sum_j a_{ij} \otimes u_j^*$, where $\{u_i^*\}$ is the dual basis. Furthermore, if H is a Hopf algebra with antipode S, then U^* becomes a right H-comodule via $u_i^* \to \sum_j u_j^* \otimes S(a_{ij})$ all i. Note that $C_{U^*} = S(C_U)$. If U is a simple right H-comodule associated to the simple subcoalgebra C of H and the antipode of H is injective, then U^* is a simple right H-comodule associated to $S(C)$. Thus, in this situation, U^{**} is isomorphic to U whenever $S^2(C) = C$.

For any vector spaces V and W with V being finite dimensional, we usually identify $\mathrm{Hom}_k(V, W)$ with $W \otimes V^*$ via $(w \otimes v^*)(v) = \langle v^*, v\rangle w$. Thus if $\{v_1, \ldots, v_n\}$ is a basis of V and $\{v_1^* \cdots v_n^*\}$ the dual basis of V^*, we write $f \in \mathrm{Hom}_k(V, W)$ as $f = \sum_i f(v_i) \otimes v_i^*$.

Proposition 7. *Let H be a Hopf algebra, and $U, V, W \in \mathcal{M}^H$. Let $f \in \mathrm{Hom}_k(U, V)$. Then:*

(1) $f \in \mathcal{M}^H(U, V)$ if and only if $f^* \in {}^H\mathcal{M}(V^*, U^*)$, i.e., $f^* : V^* \to U^*$ is a left H-comodule map.

(2) If $f \in \mathcal{M}^H(U, V)$, then $f^* \in \mathcal{M}^H(V^*, U^*)$. If $f^* \in \mathcal{M}^H(V^*, U^*)$ and the antipode S of H is bijective, then $f \in \mathcal{M}^H(U, V)$. Thus $i(U, V) \leq i(V^*, U^*)$, and $i(U, V) = i(V^*, U^*)$ if S is bijective.

(3) The natural map $\psi : \mathrm{Hom}_k(U, V \otimes W) \to \mathrm{Hom}_k(V^*, W \otimes U^*)$ given by $\psi(v \otimes w \otimes u^*) = w \otimes u^* \otimes v$ sends $\mathcal{M}^H(U, V \otimes W)$ to $\mathcal{M}^H(V^*, W \otimes U^*)$, identifying these spaces if the antipode S of H is bijective.

Proof. (1). Recall that the comodule structures of U^* and U are related in such a way that for all $u^* \in U$, $u \in U$ we have $\sum u^*_{-1}\langle u^*_0, u\rangle = \sum \langle u^*, u_0\rangle u_1$. Write $f = \sum v \otimes u^*$. For $u \in U$, we have $\rho(f(u)) = \sum v_0 \otimes v_1 \langle u^*, u\rangle$, and $(f \otimes id)\rho(u) = \sum f(u_0) \otimes u_1 = \sum v \langle u^*, u_0\rangle \otimes u_1 = \sum v \otimes u^*_{-1}\langle u^*_0, u\rangle$, so $f \in \mathcal{M}^H(U, V)$ iff $\sum v_0 \otimes v_1 \otimes u^* = \sum v \otimes u^*_{-1} \otimes u^*_0$. We have $f^* = \sum u^* \otimes v$, and a similar calculation shows that $f^* \in {}^H\mathcal{M}(V^*, U^*)$ is equivalent to the same condition.

Since (2) is a special case of (3) with $W = k1$, it remains only to prove (3). Let $f = \sum v \otimes w \otimes u^* \in \mathrm{Hom}_k(U, V \otimes W)$. By the calculation in (1), $f \in \mathcal{M}^H(U, V \otimes W)$ if and only if $\sum v_0 \otimes w_0 \otimes v_1 w_1 \otimes u^* = \sum v \otimes w \otimes u^*_{-1} \otimes u^*_0$, and $\psi(f) \in \mathcal{M}^H(V^*, W \otimes U^*)$ if and only if $\sum w_0 \otimes u^*_0 \otimes w_1 S(u^*_{-1}) \otimes v = \sum w \otimes u^* \otimes S(v_1) \otimes v_0$. Now $f \in \mathcal{M}^H(U, V \otimes W)$ implies $\sum w \otimes u^*_0 \otimes S(u^*_{-1}) \otimes v = \sum w_0 \otimes u^* \otimes S(v_1 w_1) \otimes v_0$, so $\sum w_0 \otimes u^*_0 \otimes w_1 S(u^*_{-1}) \otimes v = \sum w_0 \otimes u^* \otimes w_1 S(w_2) S(v_1) \otimes v_0 = \sum w \otimes u^* \otimes S(v_1) \otimes v_0$, as required. On the other hand, if $\psi(f) \in \mathcal{M}^H(V^*, W \otimes U^*)$ and S is bijective, then $\sum v_0 \otimes w \otimes v_1 \otimes u^* = \sum v \otimes w_0 \otimes u^*_{-1} S^{-1}(w_1) \otimes u^*_0$, and hence $\sum v_0 \otimes w_0 \otimes v_1 w_1 \otimes u^* = \sum v \otimes w_0 \otimes u^*_{-1} S^{-1}(w_2) w_1 \otimes u^*_0 = \sum v \otimes w \otimes u^*_{-1} \otimes u^*_0$, showing $f \in \mathcal{M}^H(U, V \otimes W)$. $\square$

Proposition 8. *Let H be a Hopf algebra. Then the map $^*:\mathcal{G}(H)\to\mathcal{G}(H)$ given by $[M]^*=[M^*]$ is a group homomorphism and a ring anti-homomorphism. If H is cosemisimple, then " $*$ " is an involution.*

Proof. If $0\to L\to M\to N\to 0$ is a short exact sequence in $\mathcal{M}^H$, then by Proposition 7(2) $0\leftarrow L^*\leftarrow M^*\leftarrow N^*\leftarrow 0$ is also a short exact sequence in $\mathcal{M}^H$. By Proposition 0 the assignment " $*$ " determines a group homomorphism. Furthermore, it is easily checked that the natural map $\varphi: M^*\otimes N^*\to(N\otimes M)^*$ given by $\varphi(m^*\otimes n^*)(n\otimes m)=\langle m^*,m\rangle\langle n^*,n\rangle$ is an isomorphism of right H-comodules. Hence $([N][M])^*=[N\otimes M]^*=[(N\otimes M)^*]=[M^*\otimes N^*]=[M^*][N^*]=[M]^*[N]^*$ as claimed. When H is cosemisimple , we have $S^2(C)=C$ for every simple subcoalgebra C [4, Theorem 3.3], so $(x_C)^{**}=x_C$ follows from the discussion preceding Proposition 7. □

Remark. Let $M\in\mathcal{M}^H$. Since $C_{M^*}=S(C_M)$, we have by Proposition 2 that the basic components of $[M]^*$ are given by the simple subcoalgebras of $S(C_M)$. Now let B be a subbialgebra of H, and view $\mathcal{G}(B)$ as a subring of $\mathcal{G}(H)$. It is clear that if B is a Hopf subalgebra of H, then $\mathcal{G}(B)^*\subseteq\mathcal{G}(B)$. Conversely, suppose that $\mathcal{G}(B)^*\subseteq\mathcal{G}(B)$. If the antipode S of H is injective, we obtain that $S(\mathrm{corad}(B))\subseteq B$. This implies that $S(B)\subseteq B$; we shall make use of this implication only in the case in which H is cosemisimple, for which it is obvious.

Theorem 9. *Let H be a Hopf algebra.*

(1) If the antipode of H is injective, then $m(x,y)=m(x^*,y^*)$ for all $x,y\in\mathcal{G}(H)$.

(2) If H is cosemisimple and k is algebraically closed, then $m(x,yz)=m(y^*,zx^*)=m(y,xz^*)$ for all $x,y,z\in\mathcal{G}(H)$.

Proof. (1). It suffices to show this when x and y are basic. From our observations preceding Proposition 7, we have that when S is injective, " * "yields an injective mapping from the set of basic elements of $\mathcal{G}(H)$ to itself, and (1) follows.

(2). It suffices to establish the first equality, for then the second follows by (1) and Proposition 8. Since m and multiplication are $\mathbf{Z}$-bilinear, we may assume that x, y, and z are basic. Say $x = [U]$, $y = [V]$, $z = [W]$, for simple H- comodules U, V, W. Then $m(x, yz) = i(U, V \otimes W)$ and $m(y^*, zx^*) = i(V^*, W \otimes U^*)$ by Proposition 3(3), and we are done by Proposition 7(3). □

4. Main results

The basic elements of $\mathcal{G}(H)$ of degree 1 are of the form $[kg]$, where $g \in G(H)$, the set of grouplike elements of H. We shall write $[kg]$ simply as g. The facts that we shall need about these elements of $\mathcal{G}(H)$ are contained in the following theorem, the first statement of which is a generalization of the orthogonality relations for characters [4,Theorem 2.7.].

Theorem 10. *Let H be a cosemisimple Hopf algebra over an algebraically closed field. Let x, y be basic elements of $\mathcal{G}(H)$.*

(1) For each grouplike element g of H, we have

$$m(g, xy) = \begin{cases} 1, & \text{if } y = x^*g \\ 0, & \text{otherwise} \end{cases}$$

In particular, $m(g, xy) = 0$ if $\deg(x) \neq \deg(y)$.

(2) Let C be the simple subcoalgebra of H associated to x. Then for any grouplike element g of H, $m(g, xx^*) > 0$ iff $m(g, xx^*) = 1$ iff $gx = x$ iff $gC = C$ iff $g \in CS(C)$. The set of such grouplike elements forms a group, of order at most $(\deg x)^2$.

(3) Suppose $xx^* = 1+y$. Then $g \mapsto gx$ implements a 1-1 correspondence between elements of the group $\{g \in G(H) : gy = y\}$ and the set of basic components of yx which have the same degree as x. Each such component of yx occurs with multiplicity 1.

Proof. (1). We have from Theorem 9 that $m(g, xy) = m(x^*, yg^{-1})$. Note that yg^{-1} is basic (in fact, $C_{yg^{-1}} = C_y g^{-1}$). Thus, $m(x^*, yg^{-1}) = 0$ unless $x^* = yg^{-1}$, in which case $m(x^*, yg^{-1}) = 1$.

(2). We have by Proposition 5(2) that $g \in CS(C)$ iff $m(g, xx^*) > 0$. By (1), this occurs iff $m(g, xx^*) = 1$, iff $x^* = x^* g$. This last condition is equivalent to $x = g^{-1}x$, which is in turn equivalent to $gx = x$. The basic elements gx, x of $\mathcal{G}(H)$ are equal iff their associated simple subcoalgebras gC, C are equal. It is clear that the grouplikes g satisfying (say) $gx = x$ form a group. If g is in the group, then g is a basic constituent of xx^*, so the bound on the order of the group follows from the formula $(\deg x)^2 = \deg(x)\deg(x^*) = \sum_{x' \in \mathcal{X}} m(x', xx^*) deg(x')$ as given after Proposition 4.

(3). We have $1 = m(y, xx^*) = m(x, yx)$. If $gy = y$, then we calculate $m(gx, yx) = m(x^*, (gx)^* y) = m(x^*, x^* g^{-1} y) = m(x^*, x^* y) = m(x, yx) = 1$. Since we assume that $m(g, xx^*) > 0$ iff $g = 1$, note that if $g_1 x = g_2 x$ for $g_1, g_2 \in G(H)$, then $g_1^{-1} g_2 x = x$, so $g_1 = g_2$ by (2). Now suppose that z is basic, z occurs in yx, and $\deg(z) = \deg(x)$. We have $1 \le m(z, yx) = m(y, zx^*)$. Since $\deg(y) + 1 = \deg(xx^*) = \deg(zx^*)$, we must have $zx^* = y + g$ for some grouplike g. Then (1) yields $x^* = z^* g$, so $z = gx$, and $gy = y$ follows from $g + gy = gxx^* = zx^* = y + g$. □

Theorem 11. *Let H be a cosemisimple Hopf algebra over an algebraically closed field k. Suppose that H has a simple subcoalgebra C of dimension 4. Then (at least) one of the following conditions holds:*

(i) H has a grouplike element g of order 2 such that $gC = C$.

(ii) H has a Hopf subalgebra of dimension 24, which contains a grouplike element g of order 2 such that $gC \neq C$.

(iii) H has a Hopf subalgebra of dimension 12 or 60.

(iv) H has a family $\{C_n : n \ge 1\}$ of simple subcoalgebras, with $\dim C_n = n^2$ for each n,

such that for all $n \geq 2$,

$$C_n S(C_2) = C_{n-1} + C_{n+1} \text{ if } n \text{ is even, and}$$

$$C_n C_2 = C_{n-1} + C_{n+1} \text{ if } n \text{ is odd.}$$

Proof. We assume that (i), (ii), and (iii) do not hold, and show (iv). We must show that there exists a family $\{x_i : i \geq 1\}$ of basic elements of $\mathcal{G}(H)$, with $\deg x_i = i$ for all i, such that for all $i \geq 2$, $x_i u = x_{i-1} + x_{i+1}$, where $u = x_2^*$ if i is even and $u = x_2$ if i is odd.

We take $x_1 = 1$, $x_2 = x_C$. By Theorem 10(2), the basic constituents of $x_2 x_2^*$ of degree 1 form a group of order ≤ 4, and thus of order 1, 2, or 4. Since (i) does not hold, the group must be trivial, so we conclude that $x_2 x_2^* = 1 + x_3$ for some basic $x_3 \in \mathcal{G}(H)$ of degree 3, corresponding to a simple subcoalgebra C_3 of H of dimension 9. (We note for future reference that $x_2 x_2^* = 1 + x_3$ implies that $x_3^* = x_3$.)

We now suppose that $i \geq 2$, and that we have basic x_i, x_{i-1}, x_{i+1} of degrees $i, i-1, i+1$ respectively such that $x_i v = x_{i-1} + x_{i+1}$, where $v = x_2^*$ if i is even, $v = x_2$ if i is odd. We have by Theorem 9(2) that $1 = m(x_{i+1}, x_i v) = m(x_i, x_{i+1} v^*)$, so $x_{i+1} v^* = x_i + z$, where $z \in \mathcal{G}(H)$ has degree $i+2$, and x_i is not a component of z. If we can show that z is basic, then we can set $x_{i+2} = z$, and we will be done by induction.

Suppose that x'_r is a basic component of z of degree r. We have $1 \leq m(x'_r, x_{i+1} v^*) = m(x_{i+1}, x'_r v)$, so $2r \geq i+1$.

Case 1: Suppose $i = 2j$ is even, $j \geq 1$. Then $2r \geq 2j+2$, so $r \geq j+1$. Since z has degree $2j+2$, z can fail to be basic only if z has 2 basic components x'_r, x''_r, each of degree $r = j+1$. From $1 \leq m(x_{i+1}, x'_r v)$ we obtain $x'_{j+1} x_2^* = x_{2j+1} + g$ for some grouplike g of H. By Theorem 10(1), we have $j = 1$, $x'_2 = g x_2$, and thus $g x_3 = x_3$. Note that if w is a basic component of $x_3 x_2$ of degree 2, then $m(w, x_3 x_2) = m(x_3, w x_2^*)$ shows that w occurs only once, so $x_3 x_2$ must have 3 distinct basic components of degree 2. By Theorem 10(3), $\{g \in G(H) : g x_3 = x_3\}$ is a group of order 3, so the grouplike g above for which $g x_2 = x'_2$

is of order 3, and we have $x_2'' = g^2x_2$, $x_3x_2 = x_2 + gx_2 + g^2x_2$. Multiplying on the right by x_2^*, we obtain $x_3^2 = 1 + g + g^2 + 2x_3$. We have $x_3^* = x_3$, so $(x_3g)^* = g^{-1}x_3^* = g^{-1}x_3 = x_3$ shows $x_3g = x_3$. It now follows from Theorem 6 and the remark after Proposition 8 that $k1 + kg + kg^2 + C_3$ is a Hopf subalgebra of H of dimension 12. Since we are assuming that none exists, the discussion of case 1 is complete.

Case 2: Suppose $i = 2j + 1$ is odd, $j \geq 1$. We have $r \geq j + 1$, and the degree of z is $2j + 3$. We must rule out the case in which z has basic components x'_{j+1}, x'_{j+2}. From $1 \leq m(x_{i+1}, x'_r v)$ we obtain $x'_{j+1}x_2 = x_{2j+2}$, and $x'_{j+2}x_2 = x_{2j+2} + x_2''$ for some basic $x_2'' \in \mathcal{G}(H)$. We have $1 = m(x_2'', x'_{j+2}x_2) = m(x'_{j+2}, x_2''x_2^*)$, which may occur only for $j = 1$ or $j = 2$.

We first consider the case $j = 1$. We have $x_4x_2^* = x_3 + x_2' + x_3'$ and $x_3'x_2 = x_4 + x_2''$. From $1 = m(x_2'', x_3'x_2) = m(x_3', x_2''x_2^*)$, we get $x_2''x_2^* = x_3' + g$ for some grouplike g. Then $g^{-1}x_2''x_2^* = g^{-1}x_3' + 1$ shows $g^{-1}x_2'' = x_2$ and $g^{-1}x_3' = x_3$. Now $g \neq 1$, since $g = 1$ gives $x_3' = x_3$, but (in general) x_i is not a component of z. We have $g^{-1}x_4 = g^{-1}x_3'x_2 - g^{-1}x_2'' = x_3x_2 - x_2 = x_4$, so $gx_4 = x_4$. Then from $x_4x_2^* = x_3 + x_2' + x_3'$ we get that $gx_2' = x_2'$ and $g^2x_3 = x_3$. Since x_2 is the only component of x_2x_3 of degree 2, we have $g^2 = 1$ by Theorem 10(3). Recall $x_2'x_2 = x_4$ from above. Substituting for x_4 in $x_4x_2^* = x_3 + x_2' + x_3'$ yields $x_2'x_3 = x_3 + x_3'$. By multiplying $x_3x_2 = x_2 + x_4$ on the right by x_2^*, we deduce that $x_3^2 = 1 + x_2' + x_3 + x_3'$. Multiplying on the left by g, we obtain $x_3'x_3 = g + x_2' + x_3 + x_3'$. It now follows, as discussed after Theorem 6, that the standard subring generated by $\{x_3\}$ is $\mathbf{Z}1 + \mathbf{Z}g + \mathbf{Z}x_2' + \mathbf{Z}x_3 + \mathbf{Z}x_3'$. Since $x_3^* = x_3$, the corresponding subbialgebra, which is of dimension 24, is a Hopf subalgebra, and we have arrived at a situation that we have assumed does not hold.

Thus it remains to rule out the case $j = 2$. We have $x_6x_2^* = x_5 + x_3' + x_4'$, $x_3'x_2 = x_6$ (so $x_3' \neq x_3$), $x_4'x_2 = x_6 + x_2''$. Note that $1 = m(x_2'', x_4'x_2) = m(x_4', x_2''x_2^*)$ yields $x_4' = x_2''x_2^*$. By multiplying on the right by x_2^* and using $x_2x_2^* = 1 + x_3$, we find that $x_3x_2 = x_2 + x_4$ implies $x_3^2 = 1 + x_3 + x_5$, $x_3'x_2 = x_6$ implies $x_3'x_3 = x_5 + x_4'$, $x_4'x_2 = x_6 + x_2''$ implies $x_4'x_3 = x_5 + x_3' + x_4'$, and $x_5x_2 = x_4 + x_6$ implies $x_5x_3 = x_3 + x_5 + x_3' + x_4'$.

Thus the standard subring generated by $\{x_3\}$ is $\mathbf{Z}1 + \mathbf{Z}x_3 + \mathbf{Z}x'_3 + \mathbf{Z}x'_4 + \mathbf{Z}x_5$. Since $x_3^* = x_3$, the corresponding subbialgebra, which has dimension 60, is a Hopf algebra. But this is contrary to our assumptions, so we are done. $\square$

By the freeness theorem in [7], we have as an immediate consequence in the finite dimensional case:

Corollary 12. *Let H be a finite dimensional cosemisimple Hopf algebra over an algebraically closed field k. Suppose H contains a simple subcoalgebra of dimension 4. Then 2 divides the dimension of H.*

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