Thermal Expansion Models of Viscous Fluids Based on Limits of Free Energy

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Abstract

Many viscous fluid flows are mechanically incompressible, yet thermally expand and shrink. Various approximations of the compressible Navier-Stokes equations have been proposed to resolve the diverse phenomena that arise in such fluid systems, with a primary goal to remove rapid timescales associated with sound waves. The Boussinesq model for laboratory-scale, buoyancy-driven thermal convection patterns [20] and the anelastic model for atmospheric-scale, density-stratification phenomena [7] are two important examples. Molten polymer and glass flows exhibit thermal expansion where mechanical compressibility is negligible, yet self-consistent approximate models do not exist. With this motivation, we propose a systematic method to derive thermal expansion models based on mechanical incompressibility conditions applied directly on the free energy formulation of the compressible Navier-Stokes system. The method is distinct from other approaches in that the irreversible physics and second law are preserved, whereas the reversible physical mechanisms governed by the gradient and Hessian of the free energy function take special forms imposed by mechanical incompressibility. The four equivalent free-energy formulations of compressible theory lead to four distinct thermal expansion models. Each model is then evaluated from classical thermodynamical perspectives: convexity inequalities of Gibbs on the free energy function which guarantee local well-posedness near the rest state; non-negativity of sound speed, bulk modulus, and specific heat; and linearized stability of thermo-mechanical equilibrium. Two models preserve these fundamental properties. Two others, however, are ill-posed unless an additional degeneracy condition on free energy is imposed; one model then reduces for isothermal flows to the incompressible Navier-Stokes system with constant density, while the other has limited physical applicability. To elucidate this modeling approach, we parametrize the linearized dispersion relation of compressible theory and follow the wave speeds and eigenmodes in the limits which characterize each thermal expansion model.

1 Introduction

For many fluid processes it is observed that mass density (ρ) and pressure (p) are insensitive to changes in each other, whereas both measurably respond to thermal conditions. In this paper, we refer to viscous fluid behavior characterized by these properties as *thermal expansion*. There are several historical approaches to model the effects of thermal expansion with alternative, simplified forms of the fully compressible Navier-Stokes equations, each of which in some way imposes "mechanical incompressibility". The overarching rationale for such models is the decoupling of rapid timescales associated with sound waves which are negligible for the physical features of interest.

The modeling approaches taken, the derivations, and the essential properties which are retained, vary greatly depending on the application and phenomena of interest. Thermal convection experiments on laboratory scales are traditionally modeled by the Boussinesq model [15, 19, 20, 24]. Atmospheric phenomena due to temperature-induced density stratification are generally studied from the anelastic model [7].

Melt spinning of polymer fibers has typically been modeled by *ad hoc* equations in which density-temperature relations are inserted into the incompressible Navier-Stokes equations for constant density [6, 14, 16]; while clearly inconsistent, alternative models have been unavailable. Indeed, the motivation for this study comes from the need for self-consistent thermal expansion models in industrial fiber processing applications, where viscous fluids such as glass and polymers undergo extreme temperature changes through to a solidification transition. Thermal shrinkage can be measured, with no evidence of mechanical compressibility, and yet gravitational effects (buoyancy, background density stratification), as built into the Boussinesq and anelastic models, are negligible.

Quasi-incompressible models for diffuse interfacial mixing between two incompressible liquids [17] are similar in spirit to the models derived here. Lowengrub and Truskinovsky model concentration gradients of the isothermal binary mixture, which introduces weak compressibility. A form similar to Model 3 of this paper for the Gibbs free energy is noted in [17] to characterize isothermal incompressibility. However, their analysis focuses on the isothermal case and the consequences of mixing between two fluids, not on the thermal system for constant concentrations, which is the focus of our study. The study of temperature-dependent Cahn-Hilliard models for mixing of two fluids is possible to derive from the results of this paper combined with those of [17].

All of these approaches are distinguished by approximations applied, in varying levels of sophistication and rigor, directly on the fully compressible equation system. This is done by identifying various length and time scales in the physical process, positing relative orders of the competing physics, and then asserting some small parameter(s) limit. The analysis of Majda and Sethian [18] comes closest to the ideal goal of having an approximation theorem that quantifies closeness of solutions of the fully compressible theory and the limit model; see [1] for an analogous discussion of the anelastic model. Here we pursue a different approach that is based solely on *limits imposed directly on the fully compressible free-energy function.* These limits correspond to simple statements of mechanical incompressibility in each free energy formulation, which only affect the reversible physics, and which make no assumptions or approximations on the irreversible physics or process-dependent scales. The analysis is straightforward, providing explicit thermal expansion models. However, the limits do not necessarily preserve the reversible physical properties of compressible theory: the non-negativity of squared sound speed, bulk modulus, and specific heat at constant volume; the linearized stability of the rest state; or the linearized dispersion relation.

In Section 2 we summarize the four equivalent formulations of the compressible Navier-Stokes equations for viscous fluids, corresponding to the four choices of primitive mechanical-thermal variables: the mechanical variable may be either *density* ρ or *pressure* p, while the thermal variable may be either *absolute temperature* θ or *entropy* η. (Rather than density, most analyses are streamlined by use of the *specific volume*, $\tau = \rho^{-1}$; we will freely interchange τ and ρ in this paper.) For each primitive pair, there corresponds a particular form of the free energy: internal energy $\epsilon(\tau, \eta)$; Helmholtz potential $\psi(\tau, \theta)$; Gibbs potential $\phi(p, \theta)$; and enthalpy $\chi(p, \eta)$. The four potentials are related by Legendre transforms, which allow change of primitive variables and equivalent formulations of the conservation laws for mass, momentum, and energy.

Mechanical incompressibility is a condition between the mechanical variables of compressible theory, pressure p and density ρ. For example, one may impose that pressure is independent of density for constant entropy in the internal energy formulation; the definition of pressure p from the equation of state restates an equivalent degeneracy condition on the internal energy ϵ. Likewise, from the Gibbs free energy formulation, one may impose that density is independent of pressure at fixed temperature; since density is defined from the first variation of Gibbs energy with respect to pressure at fixed temperature, one arrives at a vanishing second variation condition on the free energy. In Section 3 we formulate mechanical incompressibility in this way for each of the four free energy functions, and then derive the corresponding limits of the compressible Navier-Stokes equations. Each choice of primitive variables provides an alternative description of mechanical incompressibility; these four conditions destroy the invertibility of the Legendre transforms, so the limit models are distinct and not recoverable from one another. As we show, these models only share the preservation of shear modes from compressible theory, and otherwise differ dramatically.

The relevant physical properties of the four models require analysis given in Sections 4 & 5. We monitor the Gibbs inequalities on free energy required for well-posedness and linearized stability of the rest state. We then evaluate the limits of sound speed, bulk modulus and specific heat for each model. At this stage, we can immediately reject two models as unphysical. In Section 5 we provide a complete linearized stability analysis of compressible theory and each thermal expansion model. The two unphysical models are seen to have catastrophically unstable rest states, which generalizes the result of Wang [22] based on a short-wave asymptotic analysis of our Model 3 below, previously derived [4] from the Green-Naghdi formalism of internal constraints [11, 12, 13]. (The connection between the theory of internal constraints [11, 12, 13] and the derivations given here is addressed elsewhere [3].) The two physically plausible thermal expansion models are characterized in terms of their near-equilibrium properties. The two unphysical models can be resurrected by an extra degeneracy condition on the free energy, one of which leads to the incompressible Navier-Stokes system for constant density fluids, while the other is physically problematic requiring constant pressure for all processes.

2 Equivalent Formulations for Compressible Newtonian Fluids

The four equivalent formulations of compressible Navier-Stokes equations are summarized only to the extent necessary for the purposes of this paper.

Formulation 1: Specific internal energy $\epsilon(\rho, \eta)$

With density ρ (or specific volume τ) as the mechanical variable and entropy η as the thermal variable, the balance laws are formulated in terms of the internal energy $\epsilon(\rho,\eta)$. Recall the fundamental thermodynamic relation that relates increases in entropy to changes in internal energy and specific volume ($\tau = \frac{1}{\rho}$) ([2]):

$$\theta\dot{\eta} = \dot{\epsilon} - p\frac{\dot{\rho}}{\rho^2} = \dot{\epsilon} + p\dot{\tau}. \tag{1}$$

Conservation of mass, linear and angular momentum are given, respectively, by:

$$\dot{\rho} + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} = 0\ , \qquad \rho\dot{\mathbf{v}} = \boldsymbol{\nabla}\cdot\mathbf{T} + \rho\mathbf{f}\ , \qquad \mathbf{T} = \mathbf{T}^T\ ; \tag{2}$$

conservation of energy, assuming no internal heat production, takes the form

$$\rho\dot{\epsilon} = \mathbf{T} : \mathbf{D} - \boldsymbol{\nabla}\cdot\mathbf{q}\ , \tag{3}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{T}$ is the Cauchy stress, $\mathbf{D}$ is the symmetric part of the velocity gradient, $\mathbf{f}$ represents any body forces, $\mathbf{q}$ is the heat flux, and $\dot{(\ )}$ denotes the convective derivative.

A *viscous Newtonian fluid* is characterized by the constitutive equations

$$\mathbf{T} = -p\mathbf{I} + \lambda(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{I} + 2\mu\mathbf{D}\ , \qquad \mathbf{q} = -k\boldsymbol{\nabla}\theta\ , \tag{4}$$

where λ and μ are the dilatational and shear viscosities, respectively, k is the thermal conductivity; λ, μ, k are allowed to depend on the primitive thermal and mechanical variables, but not on the velocity field or its gradients. The second law of thermodynamics, in the form of the Clausius-Duhem inequality, implies the constraints:

$$\lambda + \frac{2}{3}\mu \geq 0\ , \qquad \mu \geq 0\ , \qquad k \geq 0\ . \tag{5}$$

Given (1,4), the conservation laws (2,3) determine a closed system for $\rho, \eta, \mathbf{v}$ once the *auxiliary variables*, in this case pressure p and temperature θ, are prescribed. These are provided by the *equations of state* in terms of the gradient of free energy. From (1) and the chain rule,

$$\left(\frac{\partial\epsilon}{\partial\rho} - \frac{p}{\rho^2}\right)\dot{\rho} + \left(\frac{\partial\epsilon}{\partial\eta} - \theta\right)\dot{\eta} = 0\ . \tag{6}$$

Independence of the variations $\dot{\rho}$ and $\dot{\eta}$ yields the prescription for p and θ:

$$p(\rho,\eta) = \rho^2\frac{\partial\epsilon}{\partial\rho} = -\frac{\partial\epsilon}{\partial\tau}, \qquad \theta(\rho,\eta) = \frac{\partial\epsilon}{\partial\eta}. \tag{7}$$

Alternative formulations of compressible Navier-Stokes equation correspond to variable changes in the mechanical or thermal variable, effected by Legendre transforms from internal energy $\epsilon(\rho,\eta)$ to Helmholtz free energy $\psi(\rho,\theta)$, Gibbs free energy $\phi(p,\theta)$, or enthalpy $\chi(p,\eta)$, recalled next.

Formulation 2: Helmholtz free energy $\psi(\rho,\theta)$

The Helmholtz free energy $\psi(\rho,\theta)$ is the Legendre transform of $\epsilon(\rho,\eta)$ with respect to the thermal variable,

$$\psi(\rho,\theta) = \epsilon(\rho,\eta) - \theta\eta. \tag{8}$$

The entropy relation (1) transforms to

$$\dot{\psi} = p\frac{\dot{\rho}}{\rho^2} - \eta\dot{\theta}, \tag{9}$$

while the transformed equations of state (7) yield the auxiliary variables p,η:

$$p(\rho,\theta) = -\frac{\partial\psi}{\partial\tau} = \rho^2\frac{\partial\psi}{\partial\rho}\ , \qquad \eta(\rho,\theta) = -\frac{\partial\psi}{\partial\theta}. \tag{10}$$

The system (2,3) is thereby formulated in terms of $\rho,\theta,\mathbf{v}$.

Formulation 3: Gibbs free energy $\phi(p,\theta)$

The Gibbs free energy $\phi(p,\theta)$ is the Legendre transform of the Helmholtz free energy from specific volume ($\frac{1}{\rho}$) to pressure (p):

$$\phi(p,\theta) = \psi(\rho,\theta) + \frac{p}{\rho}. \tag{11}$$

The entropy relation (1) yields

$$\frac{1}{\rho} = \frac{\partial\phi}{\partial p}\ , \qquad \eta = -\frac{\partial\phi}{\partial\theta}, \tag{12}$$

and the conservation laws (2,3) thereby constitute a closed system for $p,\mathbf{v},\theta$.

Formulation 4: Enthalpy $\chi(p,\eta)$

Enthalpy is the Legendre transform of internal energy from specific volume to pressure,

$$\chi(p,\eta) = \epsilon(\rho,\eta) + \frac{p}{\rho}, \tag{13}$$

which leads to the relations:

$$\frac{1}{\rho} = \frac{\partial\chi}{\partial p}\ , \qquad \theta = \frac{\partial\chi}{\partial\eta}. \tag{14}$$

This formulation then closes in the variables $p,\eta,\mathbf{v}$.

3 Mechanical incompressibility conditions on free energy functions

Formulations 1-4 provide equations of state $(7,10,12,14)_1$, respectively, which define the auxiliary mechanical variable (p in Formulations 1,2; ρ in Formulations 3,4). Mechanical incompressibility states that pressure and density (or specific volume) are insensitive to variations in one another. This notion has different meanings depending on the Formulation:

- *Formulation 1*:

$$\frac{\partial p(\rho,\eta)}{\partial \rho} = \frac{\partial p(\tau,\eta)}{\partial \tau} = 0, \qquad \text{or} \qquad p = p(\eta). \tag{15}$$

- *Formulation 2*:

$$\frac{\partial p(\rho,\theta)}{\partial \rho} = \frac{\partial p(\tau,\theta)}{\partial \tau} = 0, \qquad \text{or} \qquad p = p(\theta). \tag{16}$$

- *Formulation 3*:

$$\frac{\partial \rho(p,\theta)}{\partial p} = 0, \qquad \text{or} \qquad \rho = \tau^{-1} = \rho(\theta). \tag{17}$$

- *Formulation 4*:

$$\frac{\partial \rho(p,\eta)}{\partial p} = 0, \qquad \text{or} \qquad \rho = \tau^{-1} = \rho(\eta). \tag{18}$$

Next, we convert each mechanical incompressibility condition into an equivalent condition on the free energy, simply by using the equation of state $(7,10,12,14)_1$, respectively, for each Formulation. Since specific volume (or pressure) is defined from the first variation with respect to pressure (or specific volume), (15,16,17, 18) are seen as second variation conditions on the respective free energy functions:

- *Formulation 1*:

$$\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial \epsilon(\rho,\eta)}{\partial \rho}) = -\frac{\partial^2 \epsilon(\tau,\eta)}{\partial \tau^2} = 0 \tag{19}$$

- *Formulation 2*:

$$\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial \psi(\rho,\theta)}{\partial \rho}) = -\frac{\partial^2 \psi(\tau,\theta)}{\partial \tau^2} = 0 \tag{20}$$

- *Formulation 3*:

$$\frac{\partial^2 \phi(p,\theta)}{\partial p^2} = 0 \tag{21}$$

- *Formulation 4*:

$$\frac{\partial^2 \chi(\rho,\eta)}{\partial p^2} = 0. \tag{22}$$

In each case, these degeneracy conditions can be integrated twice to deduce *linear dependence of the free energy on the primitive mechanical variable*, where the "slope" and "intercept" functions depend only on the thermal variable. The equations of state $(7{,}10{,}12{,}14)_{1,2}$ are then employed to identify these slope and intercept functions in terms of the auxiliary mechanical and thermal variables, respectively.

Thermal Expansion Model 3

We illustrate the calculation with *Formulation 3*, then state the remaining results. Clearly, (21) implies the Gibbs free energy $\phi(p,\theta)$ is a linear function of p with θ-dependent coefficients,

$$\phi(p,\theta) = a_1(\theta)p + a_2(\theta). \tag{23}$$

The equations of state $(12)_{1,2}$ then determine the arbitrary functions $a_{1,2}(\theta)$ in terms of the auxiliary mechanical (τ) and thermal (η) variables:

$$\bar{\tau}(\theta) = \bar{\rho}^{-1}(\theta) = a_1(\theta), \qquad \eta(p,\theta) = -\frac{\partial \phi}{\partial \theta} = \frac{\bar{\rho}'(\theta)}{\bar{\rho}^2(\theta)}p - a_2'(\theta), \tag{24}$$

where $()'$ denotes ordinary differentiation with respect to the single variable.

Thermal Expansion Model 3 has the following features:

- *Density is a prescribed function of temperature, valid independent of pressure* (eq. $(24)_1$).

- The Gibbs free energy $\phi(\theta,p)$ is a linear function of the pressure p, with slope $\bar{\tau}(\theta) = \frac{1}{\bar{\rho}(\theta)}$. To complete the free energy, the model requires knowledge of one additional temperature-dependent function $a_2(\theta)$, which from (24) specifies the entropy η.

- The model unknowns are θ, p, and $\mathbf{v}$, governed by the following closed system:

– the mass conservation equation $(2)_1$ prescribes an evolution for temperature θ:

$$\bar{\rho}'(\theta)\dot{\theta} + \bar{\rho}(\theta)\nabla\cdot\mathbf{v} = \mathbf{0}; \tag{25}$$

– the momentum equation $(2)_2$ determines the evolution for $\mathbf{v}$; and

– the energy equation, upon using the mass conservation equation to eliminate $\dot{\theta}$, prescribes the evolution for the mechanical variable, p.

Proceeding in the same steps for the other formulations, we derive the remaining models:

Thermal Expansion Model 4

As a consequence of condition (22), enthalpy is a linear function of pressure, requiring knowledge of two functions of entropy. Of these, the slope function presumes knowledge of *density versus entropy, $\hat{\rho}(\eta)$, valid for any pressure*, and

$$\chi(p,\eta) = \frac{1}{\hat{\rho}(\eta)}p + a_4(\eta). \tag{26}$$

The derivative of the scalar function $a_4(\eta)$ determines the temperature:

$$\theta(p,\eta) = \frac{\partial\chi}{\partial\eta} = -\frac{\hat{\rho}'(\eta)}{\hat{\rho}^2(\eta)}p + a_4'(\eta) = \hat{\tau}'(\eta)p + a_4'(\eta). \tag{27}$$

Thermal Expansion Model 4 comprises the following closed system of equations:

- the continuity equation $(2)_1$ prescribes an evolution for entropy η,

$$\hat{\rho}'(\eta)\dot{\eta} + \hat{\rho}(\eta)\nabla\cdot\mathbf{v} = 0; \tag{28}$$

- the momentum equation $(2)_2$ gives the evolution of $\mathbf{v}$, and
- the energy equation (3), upon using (28), determines an elliptic equation for the pressure p. (Model 4 is distinct from Model 3 above and the next two in that the system has a degenerate time-evolution operator; we will expand upon this feature in the subsequent analysis.)

Thermal Expansion Model 1

The model variables are ρ, η and $\mathbf{v}$. With condition (19), the internal energy ϵ is a linear function of specific volume τ with slope and intercept functions prescribed functions of entropy η:

$$\epsilon(\tau,\eta) = -\hat{p}(\eta)\tau + a_6(\eta); \tag{29}$$

the slope function requires knowledge of *pressure as a function of entropy, $\hat{p}(\eta)$, which must hold independent of density.* Temperature is specified from the internal energy:

$$\theta = \frac{\partial \epsilon}{\partial \eta} = -\frac{\hat{p}'(\eta)}{\rho} + a_6'(\eta). \tag{30}$$

The closed system of equations for Thermal Expansion Model 1 is:

- the mass conservation equation $(2)_1$ prescribes an evolution for density ρ or specific volume,
- the momentum equation $(2)_2$ determines the evolution for $\mathbf{v}$, and
- the energy equation, after using mass conservation, provides an evolution equation for entropy η.

Thermal Expansion Model 2

The model variables are ρ, θ and $\mathbf{v}$. With condition (20) the Helmholtz potential $\psi(\rho, \theta)$ is a linear function of specific volume τ, requiring two specified functions of temperature, one of which consists of knowledge of *pressure versus temperature, $\bar{p}(\theta)$, which holds independent of density*:

$$\psi(\tau, \theta) = -\tau \bar{p}(\theta) + a_8(\theta). \tag{31}$$

The intercept function $a_8(\theta)$ prescribes the entropy:

$$\eta(\tau, \theta) = -\frac{\partial \psi}{\partial \theta} = \bar{p}'(\theta)\tau - a_8'(\theta). \tag{32}$$

The closed system of equations is:

- the mass conservation equation $(2)_1$ prescribes an evolution for ρ,
- the momentum equation $(2)_2$ determines the evolution for $\mathbf{v}$, and
- the energy equation, upon using the mass equation, provides an evolution for θ.

4 Thermal Expansion Models: Physical Properties

We now determine various physical properties associated with each model. The first approach is to monitor inequalities due to Gibbs that arise from different statements of local stability of the rest state. These inequalities are equivalently transformed into conditions on sound speed, bulk modulus, and specific heat, showing their limits for each model.

4.1 Gibbs convexity conditions on free energy potentials

Gibbs provided three formulations of stability: convexity of the free energy potential; non-negativity of sound speed, bulk modulus, and specific heat; and linearized stability of thermo-mechanical equilibrium. While it is difficult to find precise statements of equivalence among these features from Gibbs' collected works, clearly he understood these concepts in the language of the time. We shall state these properties of the compressible Navier-Stokes equations in each of the four Formulations, and then monitor their limits in each Thermal Expansion Model 1-4. In the formulas to follow, subscripts denote partial differentiation with respect to that variable.

Convexity conditions on free energy potentials

$$\text{Formulation 1:} \quad \epsilon_{\tau\tau} \geq 0\ , \qquad \epsilon_{\eta\eta} \geq 0\ , \qquad \epsilon_{\tau\tau}\epsilon_{\eta\eta} - \epsilon_{\tau\eta}^2 \geq 0. \tag{33}$$

$$\text{Formulation 2:} \quad \psi_{\tau\tau} \geq 0, \qquad \psi_{\theta\theta} \leq 0, \qquad \psi_{\tau\tau}\psi_{\theta\theta} - \psi_{\tau\theta}^2 \leq 0 \tag{34}$$

$$\text{Formulation 3:} \quad \phi_{pp} \leq 0, \qquad \phi_{\theta\theta} \leq 0, \qquad \phi_{pp}\phi_{\theta\theta} - \phi_{p\theta}^2 \geq 0; \tag{35}$$

$$\text{Formulation 4:} \quad \chi_{pp} \leq 0, \qquad \chi_{\eta\eta} \geq 0, \qquad \chi_{\eta\eta}\chi_{pp} - \chi_{p\eta}^2 \leq 0. \tag{36}$$

We now simply monitor the limits of these convexity inequalities for Thermal Expansion Models 1-4 by application of the degeneracy conditions (19,20,21,22). Note that in each case only two of the inequalities need be monitored since, for instance, inequalities $(33)_1$ and $(33)_3$ imply inequality $(33)_2$.

- Thermal Expansion Model 1 **does not in general preserve** the convexity of internal energy $\epsilon(\tau, \eta)$ unless an additional degeneracy constraint $\epsilon_{\tau\eta} = 0$ is imposed.

The Hessian determinant condition $(33)_3$, by (19), becomes $-\epsilon_{\tau\eta}^2 \geq 0$, which can only be satisfied if we further impose $\epsilon_{\tau\eta} = 0$ (or equivalently $\epsilon_{\rho\eta} = 0$). From equations of state (7) and the special form (29) of internal energy in Model 1, *the condition that local convexity of the free energy function is preserved therefore requires the pressure to be identically constant* for all flows, and the temperature to depend only on entropy (i.e., not on density):

$$p = p_0, \qquad \theta = a_6'(\eta), \qquad \epsilon = -p_0\tau + a_6(\eta); \tag{37}$$

inequality $(33)_2$ in the further restricted system becomes $a_6''(\eta) \geq 0$.

To summarize, Model 1 is unphysical unless the additional constraint $\epsilon_{\rho\eta} = 0$ on free energy is imposed, and in that case the model is restricted to flows where one can maintain constant pressure; such flows appear to be of little physical interest. The properties of sound speed, bulk modulus, and specific heat, and linearized behavior, are given below for both the general Model 1 system and the further restricted system.

- Thermal Expansion Model 2 **preserves** convexity of the Helmholtz free energy $\psi(\tau, \theta)$.

With (20), condition $(34)_1$ is satisfied with equality, whereas $(34)_3$ is automatically satisfied.

- Thermal Expansion Model 3 **does not preserve** convexity of the Gibbs free energy $\phi(p, \theta)$ unless an additional constraint, $\phi_{p\theta} = 0$, is imposed.

The Hessian inequality $(35)_3$, by (21), becomes $-\phi_{p\theta}^2 \geq 0$, which *can only be satisfied if we impose another degeneracy condition*, $\phi_{p\theta} = 0$. From equations of state (12) and the special form(23) of the Gibbs energy, this second condition *constrains the specific volume, or density, to be identically constant independent of temperature and pressure*, and the Gibbs free energy reduces to $\phi = \tau_0 p + a_2(\theta)$, where, because of inequality $(35)_3$, $a_2(\theta)'' \leq 0$. The entropy is also forced to be independent of pressure, given by $-a_2'(\theta)$. For isothermal flows, the doubly degenerate limit of the Gibbs free energy formulation of mechanical incompressibility leads to the incompressible Navier-Stokes equations.

We show below that there are unphysical sound speed and specific heat in Model 3 if the second degeneracy condition is not imposed.

- Thermal Expansion Model 4 ($\rho = \rho(\eta)$) **preserves** convexity of the enthalpy $\chi(p, \eta)$, analogous to Model 2.

With (22), condition $(36)_1$ is satisfied with equality, whereas $(36)_3$ is automatically satisfied.

4.2 Analysis of sound speed, bulk modulus, and specific heat

We now use Table 1 in the Appendix to evaluate the degeneracy conditions on free energy in terms of the fundamental inequalities on sound speed c, bulk modulus κ, and specific heat C_V.

- Model 1 for the general case where $\epsilon_{\rho\eta} \neq 0$: $c^2 \to 0^+, \kappa \to \bar{\kappa} < 0, C_V \to \bar{C}_V > 0$.

- Degenerate limit of Model 1 imposing $\epsilon_{\rho\eta} = 0$: $c^2 = \kappa = 0$, $C_V = \bar{C}_V > 0$.

- Model 2: $\kappa \to 0^+, C_V \to \bar{C}_V > 0, c^2 \to \bar{c}^2 \geq 0$.

- Model 3 for the general case where $\phi_{p\theta} \neq 0$: $C_V \to -\infty, \kappa \to +\infty$ with their ratio remaining finite, $c^2 \to \bar{c}^2 < 0$.

- Degenerate limit of Model 3 imposing $\phi_{\rho\theta} = 0$: $c^2 \to +\infty, \kappa \to +\infty, C_V \geq 0$.

- Model 4: $C_V \to 0^+, c^2 \to +\infty$ with their product remaining finite, $\kappa \to \bar{\kappa} > 0$.

Consistent with the Gibbs convexity inequalities, we find that Thermal Expansion Models 2 and 4 yield physically consistent limits of these reversible fluid properties, whereas Models 1 and 3 lead to unphysical limits of bulk modulus (Model 1), sound speed (Model 3) or specific heat (Model 3) unless an extra degeneracy condition is imposed. As noted earlier, in isothermal flows the degenerate limit of Model 3 corresponds to the incompressible Navier-Stokes system for constant density fluids. The limit of Model 1 is of limited applicability since it requires constant pressure flows. We remark that Model 4 is the only model that retains temperature dependence and removes sound waves; this feature will be further elaborated upon next.

5 Linearization of the Rest State

We now proceed to a detailed analysis of the linearized stability of thermo-mechanical equilibrium for the full compressible Navier-Stokes equations, and each of these model systems.

5.1 Linearized analysis for the compressible theory

Without loss of generality, we assume Formulation 1 in order to recall this classical calculation. The structure of the linearized operators will be useful for analysis of the Thermal Expansion Models. Consider a constant thermo-mechanical equilibrium:

$$\rho = \rho_0, \qquad \eta = \eta_0, \qquad \mathbf{v} = \mathbf{0}, \tag{38}$$

and allow infinitesimal disturbances of each primitive variable (starred quantities),

$$\rho = \rho_0 + \rho^\star, \qquad \eta = \eta_0 + \eta^\star, \qquad \mathbf{v} = \mathbf{v}^\star. \tag{39}$$

We posit standard Fourier eigenfunctions for each primitive variable of the form

$$F^\star(\mathbf{x}, t) = \tilde{F} \exp\left(is\mathbf{n} \cdot \mathbf{x} - \sigma t\right) \;, \tag{40}$$

where the wave vector $s\mathbf{n}$ has norm s and arbitrary but constant direction $\mathbf{n}$. We note that p_0, θ_0 are determined from equations of state (7) evaluated at ρ_0, η_0. Similarly, linearized disturbances $p^\star$, $\theta^\star$ are deduced from $\rho^\star$, $\eta^\star$ by linearization of the equations of state (7). The representation (40) converts the linearization of the compressible system (1)–(4) into the following algebraic system in Fourier space:

$$(\mathbf{B}\sigma + \mathbf{A})\tilde{\mathbf{f}} = \mathbf{0}, \tag{41}$$

where

$$\tilde{\mathbf{f}} \;=\; (\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{\eta}) \tag{42}$$

is constant and nonzero, $\mathbf{B}$ is the 5×5 identity matrix, and

$$\mathbf{A} = \begin{pmatrix} -\frac{s^2\mu_0}{\rho_0}\mathbf{I} - \frac{s^2(\mu_0+\lambda_0)}{\rho_0}\mathbf{n}\otimes\mathbf{n} & -i\frac{s}{\rho_0}(\rho_0^2\epsilon_{\rho\rho} + 2\rho_0\epsilon_\rho)\mathbf{n} & -is\rho_0\epsilon_{\rho\eta}\mathbf{n} \\ -is\rho_0\mathbf{n} & 0 & 0 \\ 0 & -\frac{s^2 k_0}{\rho_0\theta_0}\epsilon_{\rho\eta} & -\frac{s^2 k_0}{\rho_0\theta_0}\epsilon_{\eta\eta} \end{pmatrix} \tag{43}$$

with

$$\mu_0 = \mu(\theta_0), \qquad \lambda_0 = \lambda(\theta_0), \qquad k_0 = k(\theta_0). \tag{44}$$

The linearized dispersion relation consists of the branches $\sigma_j(s)$ of

$$\det(\mathbf{B}\sigma + \mathbf{A}) = 0, \tag{45}$$

which has five roots since $\mathbf{A}$ generically has rank 5. The rest state is **stable** if $\mathrm{Re}\,(\sigma_j) \geq 0$ for all σ_j at all wavenumbers s, **asymptotically stable** if $\mathrm{Re}\,(\sigma_j) > 0$ for all σ_j at all wavenumbers s, and **unstable** if $\mathrm{Re}\,(\sigma_j) < 0$ for some root σ_j at some s.

In Appendix A we detail the structure of the matrix $\mathbf{A}$ for compressible theory and the four Thermal Expansion Models. This structure allows an *explicit factorization of the dispersion relation into quadratic and cubic factors*, determination of the corresponding eigenfunctions, and control over the limits that correspond to each approximate model. The *compressible dispersion equation (45) factors* into

$$\left(\sigma - \frac{\mu_0}{\rho_0}s^2\right)^2 \left[\sigma - \frac{\lambda_0 + 2\mu_0}{\rho_0}s^2 + \frac{s^2\left(\sigma(\rho_0^2\epsilon_{\rho\rho}^{(0)} + 2\rho_0\epsilon_\rho^{(0)}) - \dfrac{k_0}{\rho_0\theta_0}s^2\Delta_0\right)}{\sigma\left(\sigma - \epsilon_{\eta\eta}^{(0)}\dfrac{k_0}{\rho_0\theta_0}s^2\right)}\right] = 0\ , \tag{46}$$

where

$$\Delta_0 = \rho_0^2\left[\left(\epsilon_{\rho\rho}^{(0)} + 2\frac{\epsilon_\rho^{(0)}}{\rho_0}\right)\epsilon_{\eta\eta}^{(0)} - \left(\epsilon_{\rho\eta}^{(0)}\right)^2\right], \tag{47}$$

and superscript (0) denotes that the function is evaluated at the rest state. The quadratic factor corresponds to the **double shear wave branch**, $\sigma_{1,2} = \frac{\mu_0}{\rho_0} s^2$, **which is independent of the form of the free energy**.

If one assumes the inequalities (5) on viscosities and thermal conductivity, then the Gibbs convexity inequalities are equivalent to stability of the rest state. This is standard knowledge, but we give the proof in Appendix C in which we develop explicit formulas for comparison with the limit models.

5.2 Linearized stability of the Thermal Expansion Models

The linearization of Model 1 takes the form (41) where $\mathbf{B}$ is the 5×5 identity, and

$$\tilde{\mathbf{f}} = (\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{\eta}), \quad \mathbf{A} = \begin{pmatrix} -\frac{s^2\mu_0}{\rho_0}\mathbf{I} - \frac{s^2(\mu_0+\lambda_0)}{\rho_0}\mathbf{n}\otimes\mathbf{n} & 0 & -is\rho_0\epsilon_{\rho\eta}^{(0)}\mathbf{n} \\ -is\rho_0\mathbf{n} & 0 & 0 \\ 0 & -\frac{s^2k_0}{\rho_0\theta_0}\epsilon_{\rho\eta}^{(0)} & -\frac{s^2k_0}{\rho_0\theta_0}\epsilon_{\eta\eta}^{(0)} \end{pmatrix}. \tag{48}$$

The dispersion equation is fifth order,

$$\begin{gathered}\left(\sigma - \frac{s^2\mu_0}{\rho_0}\right)^2\Big[\sigma^3 - s^2\Big(\frac{2\mu_0+\lambda_0}{\rho_0} + \frac{k_0}{\rho_0\theta_0}\epsilon_{\eta\eta}^{(0)}\Big)\sigma^2 + \frac{s^4k_0(2\mu_0+\lambda_0)}{\rho_0^2\theta_0}\epsilon_{\eta\eta}^{(0)}\sigma \\ +\frac{s^4k_0}{\rho_0\theta_0}\epsilon_{\rho\eta}^{(0)2}\Big] = 0.\end{gathered} \tag{49}$$

One of the roots of the cubic equation must be negative for $k_0 > 0$ since

$$\sigma_3\sigma_4\sigma_5 = -\frac{s^4k_0{\epsilon_{\rho\eta}^{(0)}}^2}{\rho_0\theta_0}. \tag{50}$$

Thus, the rest state is *unstable* provided $\epsilon_{\rho\eta} \neq 0$. A short wavelength analysis shows the growth rate is catastrophic, proportional to wave number s as $s \to \infty$.

The degenerate limit of Model 1, with the additional degeneracy constraint $\epsilon_{\rho\eta} = 0$ imposed, yields $\sigma_5 = 0$ while

$$\begin{aligned}\sigma_{3,4} = \tfrac{1}{2}[s^2(\tfrac{2\mu_0+\lambda_0}{\rho_0} + \tfrac{k_0}{\rho_0\theta_0}\epsilon_{\eta\eta}^{(0)}) \pm ((s^2(\tfrac{2\mu_0+\lambda_0}{\rho_0} + \tfrac{k_0}{\rho_0\theta_0}\epsilon_{\eta\eta}^{(0)}))^2 - \\ 4\tfrac{s^4k_0(2\mu_0+\lambda_0)}{\rho_0^2\theta_0}\epsilon_{\eta\eta}^{(0)})^{\frac{1}{2}}] > 0.\end{aligned} \tag{51}$$

So, the rest state becomes neutrally stable with a one-dimensional center-manifold, consistent with the analysis of the previous section. Again, the constant pressure constraint relegates this model to a novelty.

The linearized form of Model 2 has the properties:

$$\tilde{\mathbf{f}} = (\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{\theta}), \quad \mathbf{B} = \begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \psi_{\rho\theta}^{(0)} & \psi_{\theta\theta}^{(0)} \end{pmatrix},$$
$$\mathbf{A} = \begin{pmatrix} -\frac{s^2\mu_0}{\rho_0}\mathbf{I} - \frac{s^2(\mu_0+\lambda_0)}{\rho_0}\mathbf{n}\otimes\mathbf{n} & 0 & -is\rho_0\psi_{\rho\theta}^{(0)}\mathbf{n} \\ -is\rho_0\mathbf{n} & 0 & 0 \\ 0 & 0 & \frac{s^2k_0}{\rho_0\theta_0} \end{pmatrix}. \tag{52}$$

$\mathbf{B}$ is nonsingular provided $\psi_{\theta\theta}^{(0)} \neq 0$, so there are five dispersion branches given explicitly by $\sigma_{1,2} = \pm s^2\frac{\mu_0}{\rho_0}$, $\sigma_3 = 0$, and $\sigma_{4,5}$ are roots of the quadratic

$$\sigma^2 - \left(s^2\frac{2\mu_0+\lambda_0}{\rho_0} - \frac{s^2k_0}{\rho_0\theta_0\psi_{\theta\theta}^{(0)}}\right)\sigma - \frac{s^2}{\psi_{\theta\theta}^{(0)}}\left(\frac{s^2k_0(2\mu_0+\lambda_0)}{\rho_0^2\theta_0} + \rho_0^2\psi_{\rho\theta}^{(0)2}\right) = 0. \tag{53}$$

The neutral branch $\sigma_3 = 0$ has a pure density eigenmode,

$$\tilde{\mathbf{f}}_3 = (\mathbf{0}, 1, 0)^T. \tag{54}$$

The real parts of $\sigma_{4,5}$ from (53) are non-negative since $\psi_{\theta\theta}^{(0)} \leq 0$. Hence, the rest state of Model 2 is *stable.*

Thermal Expansion Model 3 has a linearized representation (41) with

$$\tilde{\mathbf{f}} = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\theta}), \quad \mathbf{B} = \begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & \rho_0^2\phi_{p\theta}^{(0)} \\ 0 & \phi_{p\theta}^{(0)} & \phi_{\theta\theta}^{(0)} \end{pmatrix},$$
$$\mathbf{A} = \begin{pmatrix} -\frac{s^2\mu_0}{\rho_0}\mathbf{I} - \frac{s^2(\mu_0+\lambda_0)}{\rho_0}\mathbf{n}\otimes\mathbf{n} & -i\frac{s}{\rho_0}\mathbf{n} & 0 \\ is\rho_0\mathbf{n} & 0 & 0 \\ 0 & 0 & \frac{s^2k_0}{\rho_0\theta_0} \end{pmatrix}; \tag{55}$$

$\mathbf{B}$ is nonsingular (except in the degenerate case $\phi_{p\theta}^{(0)} = 0$, corresponding to $\rho =$ constant). In Model 3 there are five dispersion branches, given by the roots of

$$\left(\sigma - \frac{s^2\mu_0}{\rho_0}\right)^2\left(\sigma^3 - s^2\frac{2\mu_0+\lambda_0}{\rho_0}\sigma^2 + s^2\frac{\phi_{\theta\theta}^{(0)}}{\rho_0^2\phi_{p\theta}^{(0)2}}\sigma + \frac{s^4k_0}{\rho_0^3\theta_0\phi_{p\theta}^{(0)2}}\right) = 0, \tag{56}$$

where $\phi_{p\theta}^{(0)}$ is assumed nonzero. The quadratic factor is from compressible shear waves which we noted earlier are not affected by any of the mechanical incompressibility conditions.

The cubic factor, by virtue of the constant term, implies

$$\sigma_3\sigma_4\sigma_5 = -\frac{k_0s^4}{\rho_0^3\theta_0\phi_{p\theta}^{(0)2}}, \tag{57}$$

and since k_0 must be nonnegative by the second law, at least one of the decay rates $\sigma_3, \sigma_4, \sigma_5$ is real and negative, leading to *instability.* The real and negative root was analyzed previously

[22] and shown to be proportional to $\sqrt{\frac{k_0}{\lambda_0+2\mu_0}}s$ for $s \gg 1$, corresponding to catastrophic growth.

The eigenmode corresponding to the unstable growth rate is given by $\tilde{\mathbf{f}} = (\mathbf{n}, \tilde{p}, \tilde{\theta})$ with

$$\begin{pmatrix} \tilde{p} \\ \tilde{\theta} \end{pmatrix} = \frac{is}{\sigma\rho_0\phi_{p\theta}^{(0)2}} \begin{pmatrix} \phi_{\theta\theta}^{(0)} + \frac{s^2 k_0}{\rho_0\theta_0\sigma} \\ -\phi_{p\theta}^{(0)} \end{pmatrix}. \tag{58}$$

For the degenerate limit of Model 3, imposing $\phi_{p\theta}^{(0)} = 0$, the dispersion equation reduces to

$$\left(\sigma - \frac{s^2\mu_0}{\rho_0}\right)^2 \left(\phi_{\theta\theta}^{(0)}\sigma + \frac{s^2 k_0}{\rho_0\theta_0}\right) = 0. \tag{59}$$

Among $\sigma_{3,4,5}$, two dispersion branches are lost due to the degeneracy of the eigenvalue problem (expanded upon more below) and the remaining non-shear dispersion branch is given by

$$\sigma_3 = -\frac{s^2}{\rho_0\theta_0\phi_{\theta\theta}^{(0)}} > 0. \tag{60}$$

This result is consistent with the constant density, isothermal, incompressible Navier-Stokes equations, which is what the degenerate Model 3 reduces to.

The linearized system for Model 4 is given by:

$$\begin{gathered} \tilde{\mathbf{f}} = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{\eta}), \quad \mathbf{B} = \begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & 0 & \rho_0^2\chi_{p\eta}^{(0)} \\ 0 & 0 & -1 \end{pmatrix}, \\ \mathbf{A} = \begin{pmatrix} -\frac{s^2\mu_0}{\rho_0}\mathbf{I} - \frac{s^2(\mu_0+\lambda_0)}{\rho_0}\mathbf{n}\otimes\mathbf{n} & -i\frac{s}{\rho_0}\mathbf{n} & 0 \\ is\rho_0\mathbf{n} & 0 & 0 \\ 0 & \frac{s^2 k_0}{\rho_0\theta_0}\chi_{p\eta}^{(0)} & \frac{s^2 k_0}{\rho_0\theta_0}\chi_{\eta\eta}^{(0)} \end{pmatrix}. \end{gathered} \tag{61}$$

B is *singular with rank four*, so the dispersion relation is fourth order:

$$\left(\sigma - \frac{s^2\mu_0}{\rho_0}\right)^2 \left(\sigma^2 - (s^2\frac{2\mu_0+\lambda_0}{\rho_0} + \frac{\theta_0}{\rho_0 k_0 \chi_{p\eta}^{(0)2}})\sigma + s^2\frac{\chi_{\eta\eta}^{(0)}}{\rho_0^2\chi_{p\eta}^{2(0)}}\right) = 0. \tag{62}$$

The shear waves remain intact, and the remaining two branches have generically positive real parts. The loss of one linearized wavefunction is a result of vanishing specific heat and infinite sound speed, with a positive bulk modulus κ; this is the linearized form of the property of Model 4 whereby the mechanical variable (p in this case) is prescribed by an elliptic pde instead of an evolution equation. The rest state for Model 4 is therefore *asymptotically stable.*

Comparison of Linearized Dispersion Relations

Returning to the dispersion relation for compressible theory, (46) can be written from the relations in Appendix B *completely in terms of all reversible and irreversible parameters*:

$$\left(\sigma - \frac{\mu_0}{\rho_0}s^2\right)^2 \left[\sigma - \frac{\lambda_0 + 2\mu_0}{\rho_0}s^2 + \frac{s^2\left(\sigma c_0^2 - \frac{k_0\kappa_0}{\rho_0^2 C_V^{(0)}}s^2\right)}{\sigma\left(\sigma - \frac{k_0}{\rho_0 C_V^{(0)}}s^2\right)}\right] = 0\ . \tag{63}$$

Near thermo-mechanical equilibrium, we want to characterize experimental regimes of compressible flow in terms of dimensionless ratios in order to pass to the limits identified for the four thermal expansion models. To do so, we scale the linearized growthrate σ and wavenumber s with respect to the dispersion relation of the shear waves, which we have shown is invariant for all four limits. This yields a *non-dimensionalized compressible dispersion relation*:

$$\left(\Sigma - S^2\right)^2 \left[\Sigma - DS^2 + \frac{S^2\left(Br\Sigma - GS^2\right)}{\Sigma\left(\Sigma - \frac{1}{Pr}S^2\right)}\right] = 0\ , \tag{64}$$

where

$$\sigma = \frac{k_0\theta_0\rho_0}{\mu_0^2}\Sigma, \qquad s = \sqrt{\frac{k_0\theta_0}{\mu_0^3}}\rho_0 S, \tag{65}$$

with dimensionless numbers

$$Br = \frac{c_0^2\mu_0}{k_0\theta_0}, \qquad G = \frac{\kappa_0}{\theta_0 C_V^{(0)}\rho_0} \qquad \frac{1}{Pr} = \frac{k_0}{\mu_0 C_V^{(0)}}, \qquad D = \frac{\lambda_0 + 2\mu_0}{\mu_0}; \tag{66}$$

The viscosities λ_0, μ_0 and conductivity k_0 are fixed, so that D is fixed. The remaining three parameters are:

- The Brinkman number Br is a normalized squared sound speed.
- The Prandtl number Pr is a normalized specific heat at constant volume.
- G measures bulk modulus relative to specific heat.
- GPr is a normalized bulk modulus; $\frac{Br}{G}$ is a normalized specific heat.

Stability is equivalent to Pr, Br, G being non-negative.

Dispersion equations for Models 1-4 are recovered from (64) in the following limits:

- Model 1 without the additional degeneracy constraint: $Br \to 0^+, G < 0$, and $Pr > 0$.
- Degenerate Model 1: $Br \to 0^+, G \to 0^+$, with a finite limiting finite ratio $\frac{Br}{G} > 0$, and $Pr > 0$.
- Model 2: $G \to 0^+, Br \geq 0$, and $Pr > 0$.
- Model 3 without the additional degeneracy constraint: $Br < 0$ and $G < 0$ finite and $Pr \to -\infty$.
- Degenerate Model 3: $Br \to +\infty, G \to +\infty$, with a finite limiting finite ratio $\frac{G}{Br} > 0$, and $Pr > 0$.
- Model 4: $Br \to +\infty, Pr \to 0^+$, and $G \to +\infty$ with $BrPr > 0$ and $PrG > 0$ finite.

We now visualize the linearized dispersion behavior as parametrized by Br, Pr, G, with $D = 3$ throughout. Figure 1 depicts *a sequence of physically consistent compressible parameter regimes* (Pr, Br, G non-negative) that preserve the Gibbs inequalities on free energy. Figure 1a conveys a generic parameter set of $O(1)$ values: $Br = 2, G = 1, Pr = 1$. The next three Figures 1b,c,d *tend toward the limit that characterizes Thermal Expansion Model 4*, which shows emergence of salient features:

- one wave speed tends toward the vertical, a consequence of $G \to +\infty$ (equivalently $C_V \to 0$), since the product of the three non-shear dimensionless wavespeeds is G;
- the dotted curve is the double shear-wave branch which is insensitive to Br, Pr, G;
- one wave speed appears to converge to a finite, positive value in the longwave $S = 0$ limit; indeed, analysis of (64) reveals one branch has longwave limiting value $PrBr$, which takes the value 1 for parameters in the figure.

Figures 2a-d depict dispersion branches at limits of (Br, Pr, G) corresponding to Thermal Expansion Models 1-4, respectively. *Figures 2b,d show results for Models 2 and 4, which preserve the Gibbs inequalities and all physical properties nearby thermomechanical equilibrium.* Model 2 has four decaying modes in each wavenumber S, and one identically zero branch corresponding to a neutrally stable eigenfunction. This property arises because the product of the three non-shear linearized rates, $\Sigma_3\Sigma_4\Sigma_5 = G$, and $G = 0$ for Model 2 (since bulk modulus κ vanishes.) The remaining two non-shear branches are qualitatively similar to compressible theory. Physically, a concern for this model nearby equilibrium data is the failure to damp eigenmodes associated with finite bulk modulus in the intermediate and short wavelengths, and the fact that sound waves remain with finite speed.

Model 4, Figure 2d, has the limiting characteristics developed in Figure 1, with only 4 dispersion branches in the limit, all exhibiting decay, and one branch limiting as $S \to 0$ to the finite value $PrBr$. The slowest decaying mode associated with finite bulk modulus appears to be well approximated; the finite wavespeed associated with specific heat has become infinite with a corresponding degenerate time evolution in the model system.

Note from Figures 2a,c that Models 1 and 3 have one unstable (negative) growth rate branch that can be shown by a shortwave analysis $S \to \infty$ to be unbounded and therefore Hadamard unstable. Thus, not only is the rest state unstable, both model systems have one linearized wavefunction that exhibits catastrophic instability. These unphysical phenomena are consistent with the corresponding unphysical limits of sound speed, bulk modulus, and specific heat.

The degenerate limits of Models 1 and 3 which preserve the Gibbs inequalities and local stability of equilibria have dispersion relations graphed in Figure 3. The degenerate Model 1 requires $G = 0$, so like Model 2 has one neutral dispersion branch instead of the unphysical, unbounded growth branch of Figure 2a. Thus the extra degeneracy condition on the free energy arrests the unstable branch by enforcing zero growth, or neutral stability, in all wavenumbers.

The degenerate Model 3 has infinite sound speed (instead of the previous finite "imaginary" sound speed) and infinite bulk modulus, with finite positive specific heat (instead of negatively infinite specific heat). The degenerate model then has two infinite wavespeeds, since Br and G approach ∞ as in Model 4, but furthermore $BrPr$ and GPr approach ∞. *The three surviving dispersion branches for the degenerate Model 3 are exactly those of the incompressible Navier-Stokes equations for constant density and temperature.*

6 Conclusion

Four models for thermal expansion of viscous fluids have been derived by imposing the condition of mechanical incompressibility on the free energy of the compressible theory. Each degeneracy condition on the free energy is evaluated from equivalent classical thermodynamical perspectives: convexity inequalities of Gibbs on the free energy function; inequalities on sound speed, bulk modulus, and specific heat; and linearized stability of thermo-mechanical equilibrium.

Two models that impose density is independent of pressure at constant temperature, and pressure is independent of density at constant entropy, respectively, are unphysical, corresponding to unphysical limits of sound speed, specific heat, or bulk modulus, and shown also to yield catastrophic instability of the rest state.

The two other models, in which density is independent of pressure at constant entropy, and pressure is independent of density at constant temperature, respectively, are physically plausible, and pose as candidates for applications to thermal expansion flows. Their relationship to compressible theory has been made explicit, which can guide appropriate use of the models at least in the near-equilibrium regime. Further analysis and study of these models for applied problems remains.

We emphasize the approach of this paper is distinct from standard perturbation theory derivations of the Boussinesq, anelastic, or other “weakly compressible” models. In particular, this class of models corresponds to a specific form of the free energy, namely that it must be linear in the mechanical variable (specific volume τ or pressure p). Thus the approximation is posited through the form of the free energy, with no conditions placed on the irreversible fluid properties (viscosities or thermal conductivity) or length, time, and temperature scales of the process. In a sense, we have modularized the limits of the reversible physics and taken care of them independently of the irreversible physics and specifics of the process. Presumably, one can now take these two plausible models, especially Model 4 (density slaved to entropy) which has removed the effects of sound waves, and further address specific phenomena by introducing additional assumptions on viscosities, thermal conductivity, and imposed length and time scales.

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A Linearized equations: basic structure

For compressible theory and Models 1-4, the linearized equations can be written as a 5×5 linear system,

$$(\mathbf{B}\sigma + \mathbf{A})\tilde{\mathbf{f}} = \mathbf{0}, \tag{67}$$

where $\mathbf{A}, \mathbf{B}$, and $\tilde{\mathbf{f}}$ are identified in the body. In each case the 5×5 matrix $\mathbf{B}\sigma + \mathbf{A}$ has the structure

$$\begin{aligned} &\mathbf{B}\sigma + \mathbf{A} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \\ &C_{11} = a_0 \mathbf{I} + b_0 \mathbf{n} \otimes \mathbf{n}, \qquad C_{12} = \mathbf{n} \otimes (a_1, a_2), \qquad C_{21} = (a_3, 0)^T \otimes \mathbf{n}, \end{aligned} \tag{68}$$

where $a_0 = \left(\sigma - \frac{\mu_0}{\rho_0} s^2\right)$, the scalars b_0, a_1, a_2, and a_3 and 2×2 matrix C_{22} are functions of σ, and $\mathbf{n}$ is the unit wave vector in the perturbation (40). If C_{22} is nonsingular, the eigenvalue problem (67) is equivalent to

$$\begin{pmatrix} C_{11} - C_{12} C_{22}^{-1} C_{21} & 0 \\ C_{21} & C_{22} \end{pmatrix} \tilde{\mathbf{f}} = \mathbf{0}. \tag{69}$$

Thus,

$$\begin{aligned} &\det(\mathbf{B}\sigma + \mathbf{A}) = \det(C_{11} - C_{12} C_{22}^{-1} C_{21}) \det(C_{22}), \\ &C_{11} - C_{12} C_{22}^{-1} C_{21} = a_0 \mathbf{I} + (b_0 - (a_1, a_2) C_{22}^{-1} \left(a_3, 0)^T\right) \mathbf{n} \otimes \mathbf{n}. \end{aligned} \tag{70}$$

With this structure the determinant can easily be calculated as

$$\begin{aligned} \det(\mathbf{B}\sigma + \mathbf{A}) &= \det(C_{11} - C_{12} C_{22}^{-1} C_{21}) \det(C_{22}) \\ &= a_0^2 \left(a_0 + b_0 - (a_1, a_2) C_{22}^{-1} (a_3, 0)^T\right) \det(C_{22}). \end{aligned} \tag{71}$$

The eigenvalues of (68) are then given by the solutions of the equations

$$a_0^2 = 0, \qquad (a_0 + b_0 - (a_1, a_2) C_{22}^{-1} (a_3, 0)^T) \det(C_{22}) = 0. \tag{72}$$

The two eigenvectors corresponding to the zeros $\sigma_{4,5} = \frac{\mu_0}{\rho_0} s^2$ of $(72)_1$ are $\tilde{\mathbf{f}}_{4,5} = (\mathbf{n}_i, \mathbf{0})^T, i = 1, 2$, where $\mathbf{n}_{1,2}$ are two mutually orthonormal vectors perpendicular to $\mathbf{n}$; these *two shear waves* are independent of free energy convexity conditions (i.e., independent of sound speed, bulk modulus, and specific heat), and therefore are *preserved in all four thermal expansion models.*

The remaining three linear waves are characterized by the zeros $\sigma_{1,2,3}$ of $(72)_2$, where the corresponding eigenvectors are $\tilde{\mathbf{f}}_{1,2,3} = (\mathbf{n}, \mathbf{g}_{1,2,3})^T$, with

$$C_{22}(\sigma_j)\mathbf{g} = -(a_3(\sigma_j), 0)^T. \tag{73}$$

B Relations between material properties and thermodynamic potentials

The potentials $\epsilon(\rho, \eta), \psi(\rho, \theta), \phi(p, \theta), \chi(p, \eta)$ obey the following relations (subscripts denote partial differentiation with respect to the respective variables)

$$\epsilon_{\rho\rho} = \psi_{\rho\rho} - \frac{\psi_{\rho\theta}^2}{\psi_{\theta\theta}} = -\frac{2p}{\rho^3} - \frac{1}{\rho^4 \chi_{pp}}, \qquad \epsilon_{\eta\eta} = -\frac{1}{\psi_{\theta\theta}} = \chi_{\eta\eta} - \frac{\chi_{p\eta}^2}{\chi_{pp}} = -\frac{\phi_{pp}}{\phi_{pp}\phi_{\theta\theta} - \phi_{p\theta}^2},$$

$$\epsilon_{\rho\eta} = -\frac{\psi_{\rho\theta}}{\psi_{\theta\theta}} = -\frac{\chi_{p\eta}}{\rho^2\chi_{pp}} = \frac{1}{\rho^2}\left(\frac{\phi_{p\theta}}{\phi_{pp}\phi_{\theta\theta} - \phi_{p\theta}^2}\right) ,$$
$$\chi_{pp} = \phi_{pp} - \frac{\phi_{p\theta}^2}{\phi_{\theta\theta}} , \qquad \chi_{p\eta} = -\frac{\phi_{p\theta}}{\phi_{\theta\theta}} , \qquad \chi_{\eta\eta} = -\frac{1}{\phi_{\theta\theta}} ,$$
$$\psi_{\rho\rho} = -\frac{2p}{\rho^3} - \frac{1}{\rho^4\phi_{pp}} , \qquad \psi_{\rho\theta} = -\frac{\phi_{p\theta}}{\rho^2\phi_{pp}} , \qquad \psi_{\theta\theta} = \phi_{\theta\theta} - \frac{\phi_{p\theta}^2}{\phi_{pp}}. \tag{74}$$

Sound speed c and isothermal bulk modulus κ are defined by

$$c^2 = \frac{\partial p(\rho,\eta)}{\partial\rho} , \qquad \kappa = \rho\frac{\partial\breve{p}(\rho,\theta)}{\partial\rho} , \tag{75}$$

where p is alternately considered as a function of ρ, η and ρ, θ. If one expresses the internal energy, normally a function of ρ and η, as a function of ρ and θ,

$$\breve{\epsilon}(\rho,\theta) = \psi(\rho,\theta) - \theta\frac{\partial\psi}{\partial\theta}(\rho,\theta) , \tag{76}$$

then specific heat at constant volume C_V is defined by

$$C_V = \frac{\partial\breve{\epsilon}(\rho,\theta)}{\partial\theta} . \tag{77}$$

Table 1 provides the explicit expressions for sound speed c^2, bulk modulus κ, and specific heat at constant volume C_V in terms of each free energy formulation.

Table 1: c^2, κ, C_V as functions of the potentials

	c^2	κ	C_V
ϵ	$2\rho\epsilon_\rho + \rho^2\epsilon_{\rho\rho} = \tau^2\epsilon_{\tau\tau}$	$2\rho^2\epsilon_\rho + \rho^3\epsilon_{\rho\rho} - \rho^3\frac{\epsilon_{\rho\eta}^2}{\epsilon_{\eta\eta}}$	$\frac{\theta}{\epsilon_{\eta\eta}}$
ψ	$2\rho\psi_\rho + \rho^2\psi_{\rho\rho} - \rho^2\frac{\psi_{\rho\theta}^2}{\psi_{\theta\theta}}$	$2\rho^2\psi_\rho + \rho^3\psi_{\rho\rho}$	$-\theta\psi_{\theta\theta}$
ϕ	$-\frac{\phi_{\theta\theta}}{\rho^2(\phi_{pp}\phi_{\theta\theta} - \phi_{p\theta}^2)}$	$-\frac{1}{\rho\phi_{pp}}$	-$\theta\left(\frac{\phi_{pp}\phi_{\theta\theta} - \phi_{p\theta}^2}{\phi_{pp}}\right)$
χ	$-\frac{1}{\rho^2\chi_{pp}}$	$-\frac{1}{\rho(\chi_{pp}\chi_{\eta\eta} - \chi_{p\eta}^2)}$	$\theta\left(\frac{\chi_{pp}}{\chi_{pp}\chi_{\eta\eta} - \chi_{p\eta}^2}\right)$

C Equivalence Between Convexity and Linearized Stability

A repeated root of dispersion relation (46) is

$$\sigma_4 = \sigma_5 = s^2\mu_0/\rho_0. \tag{78}$$

Therefore $\sigma_4 = \sigma_5 \geq 0$ is equivalent to $\mu_0 \geq 0$. The other three dispersion branches satisfy

$$\begin{aligned}\sigma^3 \quad & -s^2\left(\frac{\lambda_0+2\mu_0}{\rho_0}+\epsilon_{\eta\eta}^{(0)}\frac{k_0}{\rho_0\theta_0}\right)\sigma^2 \\ & +s^2\left(\rho_0^2\epsilon_{\rho\rho}^{(0)}+2\rho_0\epsilon_{\rho}^{(0)}+\frac{\lambda_0+2\mu_0}{\rho_0}\epsilon_{\eta\eta}^{(0)}\frac{k_0}{\rho_0\theta_0}s^2\right)\sigma-\frac{k_0}{\rho_0\theta_0}s^4\Delta_0=0.\end{aligned} \tag{79}$$

Since the coefficients of this cubic equation are real, either all three roots are real or one is real and the other pair consists of complex conjugates. We label the roots of (79) by $\sigma_1, \sigma_2, \sigma_3$, where the constant term shows

$$\sigma_1\sigma_2\sigma_3 = \frac{k_0}{\rho_0\theta_0}s^4\Delta_0. \tag{80}$$

Since $k_0 \geq 0$, stability conditions on $\sigma_1, \sigma_2, \sigma_3$ (nonnegative real parts) imply $\Delta_0 \geq 0$, which is inequality $(33)_3$.

We now use stability of the rest state for the special perturbations

$$\tilde{\mathbf{f}} \;=\; (\mathbf{0}, 0, \tilde{\eta}). \tag{81}$$

Then (41) reduces to

$$\left(\sigma - \frac{s^2 k_0}{\rho_0\theta_0}\epsilon_{\eta\eta}\right)\tilde{\eta} = 0.$$

We deduce $(33)_2$, and the result is established.

We now prove that the Gibbs inequalities (33), together with (5), imply stability of the rest state: Recall that either all three roots $\sigma_1, \sigma_2, \sigma_3$ of the cubic equation (79) are real or one is real and the other pair consists of complex conjugates. Since $k_0 \geq 0$, inequality $(33)_3$ implies $\Delta_0 \geq 0$, and from eq. (80) at least one of the roots, say σ_1, must be real and nonnegative. We rewrite (79) in the form

$$\sigma = \frac{\sigma^2 s^2\left(\frac{\lambda_0+2\mu_0}{\rho_0}+\epsilon_{\eta\eta}^{(0)}\frac{k_0}{\rho_0\theta_0}\right)+\frac{k_0}{\rho_0\theta_0}s^4\Delta_0}{\sigma^2+s^2\left(\rho_0^2\epsilon_{\rho\rho}^{(0)}+2\rho_0\epsilon_{\rho}^{(0)}+\frac{\lambda_0+2\mu_0}{\rho_0}\epsilon_{\eta\eta}^{(0)}\frac{k_0}{\rho_0\theta_0}s^2\right)}. \tag{82}$$

The coefficients in the right hand side are all positive because of the convexity of ϵ and (5). Hence if the remaining two roots σ_2 and σ_3 are real they must be nonnegative, and if they are complex conjugates then

$$\mathrm{Re}\,(\sigma_2) = \mathrm{Re}\,(\sigma_3) = \frac{|\sigma_2^2 + C|^2}{2D}\ , \tag{83}$$

where

$$C = s^2 \left(\rho_0^2 \epsilon_{\rho\rho}^{(0)} + 2\rho_0 \epsilon_\rho^{(0)} + \frac{\lambda_0 + 2\mu_0}{\rho_0} \epsilon_{\eta\eta}^{(0)} \frac{k_0}{\rho_0 \theta_0} s^2 \right),$$

$$\begin{aligned} D &= s^4 (\lambda_0 + 2\mu_0)(\rho_0 \epsilon_{\rho\rho}^{(0)} + 2\epsilon_\rho^{(0)}) + (\lambda_0 + 2\mu_0)^2 \epsilon_{\eta\eta}^{(0)} \frac{k_0}{\rho_0^3 \theta_0} s^6 \\ &\quad + (\lambda_0 + 2\mu_0)(\epsilon_{\eta\eta}^{(0)})^2 \frac{k_0^2}{\rho_0^3 \theta_0^2} s^6 + (\epsilon_{\rho\eta}^{(0)})^2 \frac{\rho_0 k_0}{\theta_0} s^4 , \end{aligned} \tag{84}$$

and so $\mathrm{Re}\,(\sigma_2) = \mathrm{Re}\,(\sigma_3)$ is nonnegative, and the rest state is stable. We therefore have equivalences between free energy convexity inequalities (33), (35), (34), or (36), and inequalities on the square of sound speed, $c^2 \geq 0$, specific heat at constant volume, $C_V \geq 0$, and isothermal bulk modulus, $\kappa \geq 0$.

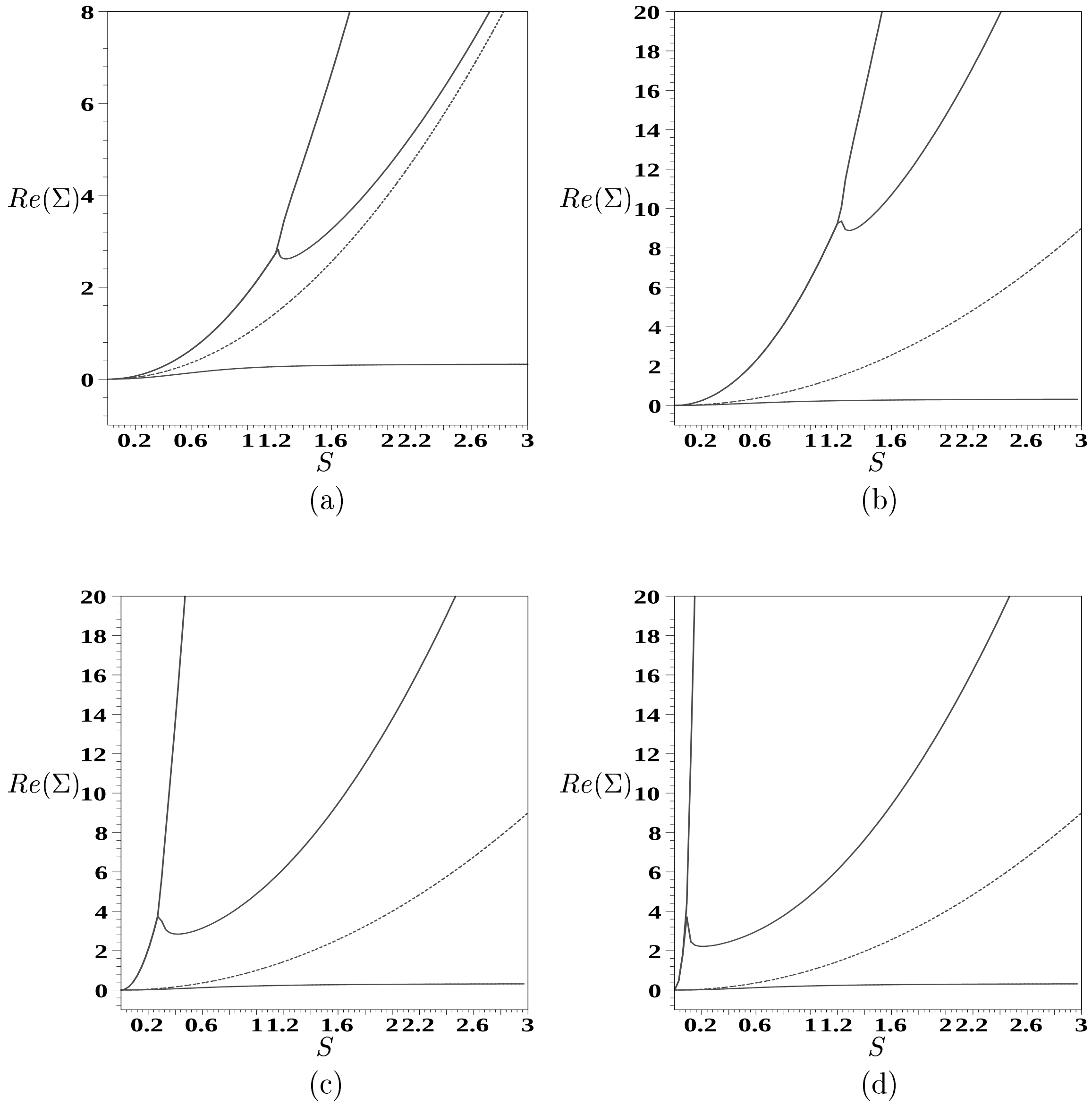

Figure 1: Dimensionless decay rate $Re(\Sigma)$ as functions of S in the compressible theory. The dashed curves correspond to the growth rate associated to the shear wave and the solid curves correspond to the other modes. $D = 3$. (a). $Br = 2, G = 1, Pr = 1$. (b). $Br = 20, G = 10, Pr = 0.1$. (c). $Br = 200, G = 100, Pr = 0.01$. (d). The compressible theory with near limiting values. $Br = 2000, G = 1000, Pr = 0.001$. One of the growthrate curve approaches the vertical axis as $Br, G \to +\infty$ and $Pr \to 0$in the process of limiting to Model 4. A slowly varying growthrate exists shown as the lowest curve.

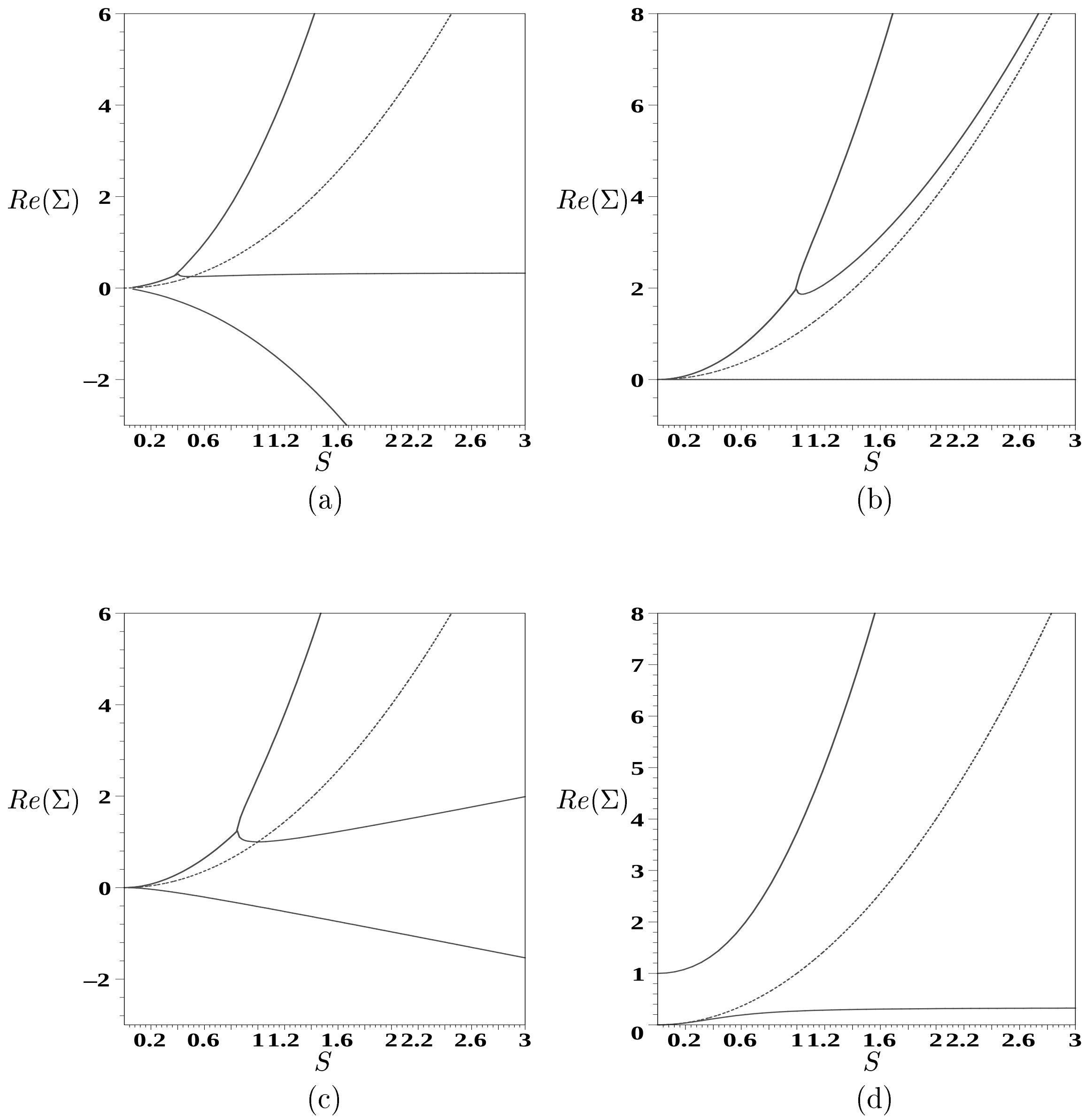

Figure 2: Dimensionless decay rates $Re(\Sigma)$ as functions of dimensionless wave number S. The dashed curves correspond to the growth rate associated to the shear wave. $D = 3$. (a). Model 1: $Br = 0, G = -1, Pr = -1$. There exists a negative $Re(\Sigma)$ so that the rest state is unstable. (b). Model 2: $Br = 1, Pr = 1, G = 0$. (c). Model 3: $Br = 1, G = -1, Pr \to -\infty$. There exists a negative $Re(\Sigma)$ so that the rest state is unstable. (d). Model 4: $BrPr = 1, GPr = 1, Br, G \to +\infty, Pr = 0$, where one of the growthrate curve is "lost".

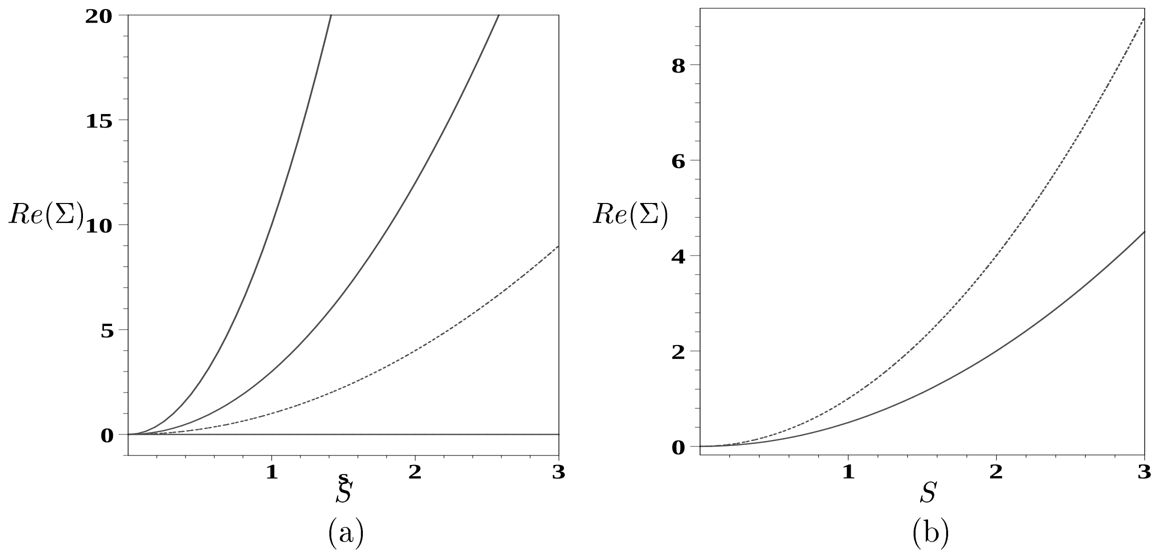

Figure 3: Dimensionless decay rates $Re(\Sigma)$ as functions of dimensionless wave number S of the two doubly degenerate limits. The dashed curves correspond to the growth rate associated to the shear wave. $D = 3$. (a). Model 1: five growthrates; $\frac{Br}{G} = 2, Pr = 0.1$, where one of the growthrate is $\Sigma = 0$. (b). Model 3: three growth rates; $\frac{Br}{G} = 2$. Two growth rates are lost due to the doubly degeneracy.