INCLUSION-EXCLUSION AND SEGRE CLASSES, II

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Abstract. Considerations based on the known relation between different characteristic classes for singular hypersufaces suggest that a form of the 'inclusion-exclusion' principle may hold for Segre classes. We formulate and prove such a principle for a notion closely related to Segre classes. This is used to provide a simple computation of the classes introduced in [Alu], in certain special (but representative) cases.

1. Introduction

Recent work on relations between the Chern-Schwartz-MacPherson class of a hypersurface X in a nonsingular ambient variety M and the class of its virtual tangent bundle has revealed a tight connection between the former and the *Segre class* of the Jacobian subscheme of X (cf. [Alu94] and [Alu99], §1.1). A strong motivation to pursue this connection is the important rôle played by Segre classes in intersection theory—the hope is that functoriality properties of Segre classes (such as those that may be inherited via the connection with Chern-Schwartz-MacPherson's classes) would lead to new computational tools for Segre classes, or at least point in the right direction to look for such tools. In this paper we discuss an 'inclusion-exclusion principle' for Segre classes, inspired by this connection.

This article is a counterpoint to [Alu], where we have *imposed* an inclusion-exclusion principle on a Segre class-type notion. The resulting *SM-Segre class* satisfies a number of remarkable properties, which must be a reflection of unknown and potentially useful properties of ordinary Segre classes. This leads us in this article to search for other instances where an inclusion-exclusion principle may be at work in the theory of Segre classes.

By *inclusion-exclusion* we refer to the familiar counting principle according to which the number of elements in the intersection of a family of finite sets may be computed by adding the cardinalities of the sets, subtracting the cardinalities of their pairwise unions, adding back the cardinalities of triple unions, etc. An analog of this principle is trivially satisfied by the topological Euler characteristic, and the simplest form of the functoriality property of the Chern-Schwartz-MacPherson is an expression of the same principle. This is the observation leading to the definition of SM-Segre classes in [Alu].

We can now abstract away from Chern-Schwartz-MacPherson classes for a moment, and look for other situations where Segre classes express a behavior reminiscent of inclusion-exclusion. Our candidate in this paper is proposed in §2. We show that with a suitable definition of the *union Segre class,* which unfortunately is *not* quite the Segre class of the union, a straightforward inclusion-exclusion formula computes the (conventional) Segre class of the intersection Y of closed subschemes $X_1,\dots,X_n$ in an ambient irreducible scheme M. In fact we give a rather broad statement (Theorem 2.5), which specializes to inclusion-exclusion but encompasses a substantially more general situation; indeed, some information can be obtained as soon as Y is contained in the intersection of the X_i, provided (maybe surprisingly) that *enough* X_i are considered.

After the fact, we go back to Chern-Schwartz-MacPherson classes: in our main application of Theorem 2.5, we observe that the union Segre class in fact equals the SM-Segre class studied in [Alu], in a particularly well-behaved class of examples (Theorem 3.5, Corollary 3.6). This recovers immediately the main result of [Alu99] in a very particular, but representative case (cf. Example 3.7). Our hope, and our main motivation in this paper, is that applying Theorem 3.5 to similarly representative situations may suggest how to extend [Alu99] to a wider class of varieties, e.g., complete intersections. However, this issue is not further explored here.

We also include in §3 comments meant to illustrate the inclusion-exclusion principle in action as the number of loci X_i increases, and a few explicit examples.

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2. Union Segre classes and the inclusion-exclusion formula

We work in a fixed ambient variety M.

As we are aiming for an inclusion-exclusion formula, we first recall the familiar counting analog for finite sets. Assume a set Y is the intersection of a finite number of finite sets X_i:

$$Y = X_1 \cap \dots \cap X_r \quad .$$

Let $X_{i_1\dots i_s}$ denote the union $X_{i_1}\cup\dots\cup X_{i_s}$; then the cardinality of Y can be computed in terms of the cardinalities of these unions by the formula

$$\#Y = \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1<\dots<i_s}\#X_{i_1\dots i_s} \quad ,$$

as is proved immediately by induction on r.

Using the same notations in the algebro–geometric situation, assume Y and the X_i are proper subschemes of the ambient M; then we can aim to a similar formula for the Segre class $s(Y,M)$:

$$s(Y,M) = \sum_{s=1}^{r}(-1)^{s-1}\sum_{i_1<\dots<i_s} s(Y;X_{i_1},\dots,X_{i_s};M) \quad ; \tag{1}$$

such an equality could be sought, for example, in the Chow group of the union $\cup X_i$ (and we will be omitting evident push-forward notations). The question we explore is how to define the class

$$s(Y;X_{i_1},\dots,X_{i_s};M)$$

so that such a formula may hold.

Remark 2.1. Counter to the first possible guess, this should not be the class of the 'union': for example, take $Y = p$, a point in $M = \mathbb{P}^2$, and X_1, X_2 two lines meeting at the point; then

$$s(X_1, \mathbb{P}^2) + s(X_2, \mathbb{P}^2) - s(X_1 \cup X_2, \mathbb{P}^2) = 2[p]$$

rather than $[p]$.

On the basis of a few such simple examples, one may reach the conclusion that we have no right to expect a formula such as (1) to hold. The surprise is that on the contrary there *is* a straightforward way to define a Segre–like class $s(Y; X_{i_1}, \dots, X_{i_s}; M)$ so that inclusion–exclusion works; this class is defined on $X_{i_1} \cup \dots \cup X_{i_s}$, as it should, but it is not (defined as) the Segre class of a subscheme in M (unless $s = 1$, cf. Proposition 2.4 below).

In fact, the class will depend on the specific choice of $X_{i_1}, \dots, X_{i_s}$ and on Y, and not only on the scheme $X_{i_1} \cup \dots \cup X_{i_s}$; this seems an undesirable feature, but we don't see any way around it at this moment. On the other hand, we will see in §3 that the class is effectively computable in several representative situations.

We will give the definition for proper closed subschemes $X_1, \dots, X_s$ of M, all containing a closed subscheme Y in fact this requirement could be dropped, but at the expense of any geometric interpretation of the classes arising in the process. Let $\pi : \widetilde{M} \to M$ be a proper birational morphism such that Y and all the X_i pull–back to Cartier divisors $\overline{Y}, \overline{X}_1, \dots, \overline{X}_s$ in $\widetilde{M}$; we say then that π is a 'resolving' morphism. By construction, all the $\overline{X}_i$ contain $\overline{Y}$ as a component, so that we can write

$$\overline{X}_i = \overline{Y} + R_i$$

for well–defined residual divisors R_i in $\widetilde{M}$.

Definition 2.2. With notations as above, we define the *union Segre class* of $X_1, \dots, X_s$ in M, w.r.t. Y, to be the class

$$s(Y; X_1, \dots, X_s; M) = \pi_* \frac{[R_1 + \dots + R_s + \overline{Y}]}{1 + R_1 + \dots + R_s + \overline{Y}}$$

in $A_*(X_1 \cup \dots \cup X_s)$.

Of course a resolving morphism π exists (blow-up everything in sight). However we must prove that the result is independent of the choice of π.

Lemma 2.3. *The definition of $s(Y; X_1, \dots, X_s; M)$ is independent of the chosen resolving morphism π.*

Proof. If $\pi : \widetilde{M} \to M$, $\pi' : \widetilde{M}' \to M$ are two proper birational morphisms for which Y and the X_i pull-back to Cartier divisors, then so is $(\widetilde{M} \times_M \widetilde{M}')^\sim \to M$, where $(\widetilde{M} \times_M \widetilde{M}')^\sim$ is the component of the fiber product dominating M. So we may assume $\widetilde{M}'$ dominates $\widetilde{M}$:

$$\begin{tikzcd} \widetilde{M}' \arrow[r, "p"'] \arrow[rr, bend left, "\pi'"] & \widetilde{M} \arrow[r, "\pi"'] & M \end{tikzcd} \quad .$$

Denoting by a $'$ the corresponding subschemes in $\widetilde{M}'$, we have then

$$\overline{Y}' = p^*\overline{Y} \quad , \quad \overline{X}_i' = p^*\overline{X}_i \quad , \quad R_i' = p^* R_i \quad ,$$

and the stated independence follows immediately. □

Before proving our inclusion-exclusion formula, we comment on relations between the class we introduced and ordinary Segre classes.

Proposition 2.4. 1. *For $r = 1$: $s(Y; X; M) = s(X, M)$ is the Segre class of X in M (in particular, it is independent of Y).*

2. *If $Y = \emptyset$, then $s(Y; X_1, \dots, X_s; M) = s(X_1 \cup \dots \cup X_s, M)$, where $X_1 \cup \dots \cup X_s$ denotes the subscheme with ideal given by the product of the ideals of $X_1, \dots, X_s$.*
3. *$s(Y; X_1, \dots, X_s; M)$ is a birational invariant: if $f : M' \to M$ is a proper birational morphism, and $Y' = f^{-1}(Y)$, $X_i' = f^{-1}(X_i)$ (scheme-theoretically) then*

$$f_* s(Y'; X_1', \dots, X_s'; M') = s(Y; X_1, \dots, X_s; M) \quad .$$

Proof. The first two statements follow immediately from the definition. For the third: if $\pi' : \widetilde{M}' \to M'$ is a resolving morphism for Y' and $X_1', \dots, X_s'$, then $\pi' \circ f$ is resolving for Y and $X_1, \dots, X_s$, and the result follows easily. □

The second statement of Proposition 2.4 is a property shared with conventional Segre classes, cf. Proposition 4.2 in [Ful84]; it can be useful for concrete computations.

Here is the promised inclusion-exclusion principle:

Theorem 2.5 (Inclusion–Exclusion formula)**.** *With notations as above, let $Y \subset X_1 \cap \dots \cap X_r$, and assume $R_1 \cdot \dots \cdot R_r = 0$. Then*

$$s(Y, M) = \sum_{s=1}^{r} (-1)^{s-1} \sum_{i_1 < \dots < i_s} s(Y; X_{i_1}, \dots, X_{i_s}; M)$$

in $A_*(X_1 \cup \dots \cup X_r)$.

Remark 2.6. The condition on the R_i seems hard to check, but it is automatically verified in at least two situations:

1. if Y in fact *equals* $X_1 \cap \dots \cap X_r$ (scheme–theoretically); and
2. if $r > \dim M$.

The first case is the one which best agrees with the set–theoretic analogy; the second says (maybe surprisingly) that we could, if we want, choose *all X_i to be the same;* so long as they contain Y, and we are choosing enough of them, then the formula must hold. We will illustrate this point in the beginning of §3.

Proof. By Proposition 2.4, we can replace M by a birational variety obtained by a resolving morphism. Thus we may assume that Y and the X_i are Cartier divisors, and we are reduced to proving that

$$\frac{[Y]}{1+Y} - \sum_{s=1}^{r}(-1)^{s-1} \sum_{i_1<\cdots<i_s} \frac{[R_{i_1}+\cdots+R_{i_s}+Y]}{1+R_{i_1}+\cdots+R_{i_s}+Y}$$

is zero under the stated conditions; that is, that

$$\sum_{s=0}^{r}(-1)^{s} \sum_{i_1<\cdots<i_s} \frac{[R_{i_1}+\cdots+R_{i_s}+Y]}{1+R_{i_1}+\cdots+R_{i_s}+Y}$$

is 0 if $R_1\cdot\cdots\cdot R_r=0$. But for this it is enough to show that this last expression, as a formal power series in the R_i's and Y, is divisible by $R_1\cdot\cdots\cdot R_r$; and in turn for this it is enough to show that it is divisible by R_r; and finally, for this it suffices to show that the expression vanishes when R_r is set to 0. Rewriting the expression by separating the part which contains R_r transforms it into

$$\sum_{s=0}^{r}(-1)^{s+1} \sum_{i_1<\cdots<i_s<r} \frac{[R_{i_1}+\cdots+R_{i_s}+R_r+Y]}{1+R_{i_1}+\cdots+R_{i_s}+R_r+Y}$$

plus

$$\sum_{s=0}^{r}(-1)^{s} \sum_{i_1<\cdots<i_s<r} \frac{[R_{i_1}+\cdots+R_{i_s}+Y]}{1+R_{i_1}+\cdots+R_{i_s}+Y} \quad .$$

Setting $R_r=0$ in the first summands makes it equal and opposite in sign to the second; so the sum is 0 when $R_r=0$, as needed. □

Without hypotheses on the residual R_i (via a resolving morphism π) and if $r\le n=\dim M$, it is not hard to still say something rather precise regarding the right-hand-side of our inclusion-exclusion formula. As an illustration, consider the case $r=n$; then the proof of the theorem shows that the difference between the inclusion–exclusion formula and the Segre class of Y is a multiple of $\pi_*(R_1\cdot\cdots\cdot R_n)$. This multiple can be evaluated easily.

Proposition 2.7. *With notations as above, let $Y\subset X_1\cap\cdots\cap X_n$, with $n=\dim M$; then*

$$\sum_{s=1}^{r}(-1)^{s-1} \sum_{i_1<\cdots<i_s} s(Y;X_{i_1},\ldots,X_{i_s};M) = s(Y,M)+n!\,\pi_*(R_1\cdot\cdots\cdot R_n)$$

in $A_*(X_1\cup\cdots\cup X_r)$.

Proof. Again replace M by a resolving variety. The coefficient of $R_1\cdot\cdots\cdot R_n$ can be computed after assuming $Y=0$ and all R_i are equal to a fixed class R.

With these positions the left-hand-side becomes

$$\sum_{s=1}^{n}(-1)^{s-1}\binom{n}{s}\frac{s[R]}{1+sR} \quad ;$$

the coefficient of R^n in this expression is

$$\sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}s^n \quad .$$

So the result is implied by the combinatorial identity which follows. ☐

Lemma 2.8.

$$\sum_{s=0}^{n}(-1)^{n-s}\binom{n}{s}s^n = n!$$

This is easily verified: n^n counts the number of n–digit strings from n objects; $-n\cdot(n-1)^n$ takes away those which only use $n-1$ objects; but those that used $n-2$ or less have been taken away twice, so $+\binom{n}{2}\cdot(n-2)^n$ evens out the count; except that those that used $n-3$ or less have been counted too many times; etc. etc. The conclusion is that the formula in the lemma computes the number of strings of length n using exactly n objects, that is, $n!$.

As a side remark, we note that the argument in the proof of the theorem shows, by the same token, that

$$\sum_{s=0}^{n}(-1)^{r-s}\binom{n}{s}s^r = 0 \quad \text{for } 0\le r<n \quad ;$$

which is also immediate from the counting argument just provided, since there are *no* strings of length n using fewer than n objects.

3. Applications and examples

There are two main situations in which Theorem 2.5 applies (cf. Remark 2.6): when the schemes X_i cut out Y, and when there simply are enough of them.

To illustrate the second situation, let

$$s(Y;X^{(r)};M) := s(Y;\overbrace{X,\dots,X}^{r};M)$$

where X is any scheme containing Y. The classes

$$\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s(Y;X^{(s)};M)$$

may be seen as 'successive approximations' of $s(Y,M)$: indeed, by Proposition 2.4, part 1., and Theorem 2.5, this class equals $s(X,M)$ for $r=1$, and $s(Y,M)$ for $r\gg 0$.

Example 3.1. Let $M = \mathbb{P}^5$, $X = \mathbb{P}^4 \subset M$, and $Y =$ a quadric surface in a $\mathbb{P}^3$ contained in X. The successive approximations of $s(Y, M)$ are (after push-forward to $M = \mathbb{P}^5$):

$$\begin{array}{rrrrrr}
s(X, M) = & [\mathbb{P}^4] - & [\mathbb{P}^3] & +[\mathbb{P}^2] & -[\mathbb{P}^1] & +[\mathbb{P}^0] \\
s(Y; X^{(2)}; M) = & & 2[\mathbb{P}^3] & -4[\mathbb{P}^2] & +6[\mathbb{P}^1] & -8[\mathbb{P}^0] \\
s(Y; X^{(3)}; M) = & & & -4[\mathbb{P}^2] & +4[\mathbb{P}^1] & -8[\mathbb{P}^0] \\
s(Y; X^{(4)}; M) = & & & 2[\mathbb{P}^2] & +16[\mathbb{P}^1] & +22[\mathbb{P}^0] \\
s(Y; X^{(5)}; M) = & & & 2[\mathbb{P}^2] & -8[\mathbb{P}^1] & -98[\mathbb{P}^0] \\
s(Y; X^{(6)}; M) = & & & 2[\mathbb{P}^2] & -8[\mathbb{P}^1] & +22[\mathbb{P}^0]
\end{array}$$

and they stabilize from this point on, having reached $s(Y, M)$.

Since successive approximations must eventually agree, we obtain a relation recursively computing $s(Y; X^{(r)}; M)$ for large r. Explicitly:

$$s(Y; X^{(r+1)}; M) = \sum_{s=1}^{r} (-1)^{r-s} \binom{r}{s-1} s(Y; X^{(s)}; M) \quad \text{for } r > \dim M.$$

More generally, Theorem 2.5 could be used to obtain a recursion to compute the class $s(Y; X_1, \dots, X_r; M)$ for large r, in terms of classes involving fewer of the X_i's.

For $Y = \emptyset$ and all X_i equal to a fixed hypersurface X, the recursion says the following. Let M be a nonsingular variety, and let $\mathcal{L}$ be a line bundle on M. Denote by $c(rX) \in A_*M$ the total Chern class of a nonsingular section of $\mathcal{L}^{\otimes r}$ (if any exists). Then

- For $r > \dim M$:
$$c(rX) = \sum_{s=1}^{r-1} (-1)^{r-s-1} \binom{r}{s} c(sX)$$
- For $r = \dim M$:
$$c(rX) = \sum_{s=1}^{r-1} (-1)^{r-s-1} \binom{r}{s} c(sX) - (\dim M)!\, c_1(\mathcal{L}^\vee)^{\dim M} \cap [M]$$

The first formula is immediate from the above considerations and Proposition 2.4, part 2; the second one follows likewise by using Proposition 2.7.

These facts are essentially obvious independently of the results in this paper (exercise), but may deserve to be better known then they seem to be. For instance, this recursive behavior applies to the *Euler characteristic* of nonsingular sections of higher and higher powers of a line-bundle.

Example 3.2. The simplest possible example of this phenomenon has to be the case of curves in $\mathbb{P}^2$, where it can of course be immediately verified from the genus formula. Even in this case, however, the recursive recipe seems rather pretty: taking $\mathcal{L} = \mathcal{O}(1)$, the successive Euler characteristics of degree r curves, for $r = 1, 2, 3, \dots$ are

$$\mathbf{2}, \mathbf{2}, \mathbf{0}, -\mathbf{4}, -\mathbf{10}, -\mathbf{18}, -\mathbf{28}, -\mathbf{40}, \dots \quad ;$$

in this case, the observation amounts to the fact that

$$\begin{gathered}\binom{2}{1}\cdot\mathbf{2}-2!\cdot 1=\mathbf{2}\\ -\binom{3}{1}\cdot 2+\binom{3}{2}\cdot\mathbf{2}=\mathbf{0}\\ \binom{4}{1}\cdot 2-\binom{4}{2}\cdot 2+\binom{4}{3}\cdot\mathbf{0}=-\mathbf{4}\\ -\binom{5}{1}\cdot 2+\binom{5}{2}\cdot 2-\binom{5}{3}\cdot 0+\binom{5}{4}\cdot(-\mathbf{4})=-\mathbf{10}\\ \binom{6}{1}\cdot 2-\binom{6}{2}\cdot 2+\binom{6}{3}\cdot 0-\binom{6}{4}\cdot(-4)+\binom{6}{5}\cdot(-\mathbf{10})=-\mathbf{18}\end{gathered}$$

etc.

By the above considerations, the same structure governs all sequences of Euler characteristics of successive powers of a line bundle over every variety; if the dimension of M is n, the structure is visible from the n-th row of Pascal's triangle onward. The same applies to the terms in all dimensions of the classes $c(rX)$.

A more substantial application of Theorem 2.5 is to the computation of *SM-Segre classes,* that is (equivalently) of Chern-Schwartz-MacPherson classes. The reader is addressed to [Alu] and references therein for more thorough discussions of these notions. As a quick reminder, the Chern-Schwartz-MacPherson classes are classes defined in the Chow group of a possibly singular variety V, agreeing with $c(TV)\cap[V]$ when V is nonsingular, and fitting a functorial prescription (originally envisioned by Deligne and Grothendieck; see [Ken90] for a nice treatment). There has been substantial activity in recent years, comparing these classes to other classes generalizing the classes of the tangent bundle. These other classes are typically defined in terms of a Segre class; thus it is natural to ask whether the Chern-Schwartz-MacPherson classes admit a description in terms of Segre classes.

This is indeed the case for hypersurfaces ([Alu99]): the difference between Chern-Schwartz-MacPherson classes and the classes of the virtual tangent bundle of a hypersurface X can be measured effectively in terms of the Segre classes of the *singularity subscheme* of X. This difference has been named *Milnor class,* and studied in a different context; see [PP01] for the hypersurface case, and [Sch] for the state of the art on this point of view (now embracing a substantially more general class of varieties).

This otherwise substantial progress does not seem to yield a general expression for the Chern-Schwartz-MacPherson class in terms of Segre classes. Theorem 3.5 below is a (small) step in this direction.

Let M be a *nonsingular* ambient variety.

Definition 3.3. We say that a reduced subscheme $X\subset M$ is *almost nonsingular* if its irreducible components X_i are nonsingular, and $X_i\cap X_j=Y$ (as schemes) for all $i\neq j$, where Y is a nonsingular variety independent of i, j.

Example 3.4. The simplest example of an almost nonsingular variety is the transversal union of two nonsingular hypersurfaces in M. Another typical example would be the union of any number of lines through a point in a projective space, or more generally the union of a collection of general subspaces of dimension $\leq\frac{n}{2}$ containing a fixed subspace Y in $\mathbb{P}^n$.

In particular, note that *almost nonsingular* varieties are *not* necessarily pure-dimensional, Cohen-Macaulay, complete intersections ...

Theorem 3.5. *Let $X \subset M$ be almost nonsingular, let $X_1, \dots, X_r$ be its components, and let Y be the common pairwise intersection of the components of X. Then*

$$c_{\mathrm{SM}}(X) = c(TM) \cap s(Y; X_1, \dots, X_r; M) \quad .$$

Proof. Induction on the number r of components of X. If $r = 1$, then X is nonsingular, so $c_{\mathrm{SM}}(X) = c(TX) \cap [X] = c(TM) \cap s(X, M)$; so the formula holds, by Proposition 2.4, part 1.

For $r > 1$, observe that since c_{SM} itself satifies inclusion-exclusion (as a particular instance of functoriality) then

$$c_{\mathrm{SM}}(Y) = \sum_{s=1}^{r-1} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} c_{\mathrm{SM}}(X_{i_1} \cup \cdots \cup X_{i_s}) + (-1)^{r-1} c_{\mathrm{SM}}(X) \quad .$$

Capping by $c(TM)^{-1}$, and using the induction hypothesis:

$$s(Y, M) = \sum_{s=1}^{r-1} (-1)^{s-1} \sum_{i_1 < \cdots < i_s} s(Y; X_{i_1}, \dots, X_{i_s}; M) + (-1)^{r-1} c(TM)^{-1} \cap c_{\mathrm{SM}}(X)$$

The result follows by comparing this formula with Theorem 2.5. □

Using the definition of *SM-Segre class* $s^\circ(X, M)$ introduced in [Alu]:

Corollary 3.6. *With notations as in Theorem 3.5,*

$$s^\circ(X, M) = s(Y; X_1, \dots, X_r; M) \quad .$$

Proof. This now follows from Theorem 3.5 and [Alu], Theorem 3.1. □

Corollary 3.6 gives a suprisingly straightforward computation of $s^\circ(X, M)$ in the particular case of almost nonsingular varieties. The definition of $s^\circ(X, M)$ given in [Alu] is a certain combination of the Segre classes of hypersurfaces cutting out X in M and of their singularity subschemes; it would seem very difficult to have any control over it in that form. The expression in terms of a union Segre class is by contrast very direct, and matches well the intuition (from the case of a hypersurface, cf. [Alu94], §1) that the right class should be obtained by 'removing' a copy of the singularity subscheme.

Example 3.7. This is perhaps the main punch-line in the whole paper, so we expand on this point. As mentioned above, the simplest example of an almost nonsingular variety is the union X of two transversal, nonsingular hypersurfaces X_1, X_2. If (F_1), (F_2) are (local) ideals for X_1, X_2, then X has ideal $(F_1 F_2)$, and hence singularity subscheme Y with ideal $(F_1 F_2, F_1\, dF_2 + F_2\, dF_1)$. As X_1 and X_2 are transversal, dF_1 and dF_2 are linearly independent at every point of Y; hence the ideal of Y is simply (F_1, F_2): that is, $Y = X_1 \cap X_2$.

Now using Theorem 3.5 we get

$$c_{\mathrm{SM}}(X) = c(TM) \cap s(Y; X_1, X_2; M) \quad .$$

Blowing up along Y, we can write $s(Y;X_1,X_2;M)$ as the push-forward of

$$\frac{[X_1+X_2-E]}{1+X_1+X_2-E} \quad ,$$

where E denotes the exceptional divisor and we use X for the inverse image of its namesake in M; in this sense we 'remove one copy of Y from X'. Simple manipulations (using the notational device from [Alu99], §1.4) give

$$\frac{[X-E]}{1+X-E}=\frac{[X]}{1+X}+\frac{1}{1+X}\cdot\frac{-[E]}{1+X-E}=\frac{[X]}{1+X}+\frac{1}{1+X}\left(\left(\frac{[E]}{1+E}\right)^{\vee}\otimes\mathcal{O}(X)\right),$$

and finally pushing forward we get

$$s(Y;X_1,X_2;M)=s(X,M)+c(\mathcal{O}(X))^{-1}\cap\left(s(Y,M)^{\vee}\otimes\mathcal{O}(X)\right) \quad .$$

This is precisely the formula giving $s(X\setminus Y,M)$ (cf. Lemma I.3 in [Alu99]). That is, although the case of 'almost nonsingular varieties' is extremely special, it is representative enough to capture precisely the correct definition for the class studied in full generality in [Alu99].

We hope that studying union Segre classes for similarly representative cases in higher codimension may suggest extensions of the result of [Alu99].

Conversely, the remarkable properties of $s^\circ(X,M)$ (see [Alu], Theorem 2.3) tell us something about the union Segre class, at least in the case of almost nonsingular varieties. For instance:

Corollary 3.8. *If X is almost nonsingular, with components X_i, and $Y=X_i\cap X_j$ for $i\neq j$, and $X\subset M$, $X\subset M'$ are embeddings of X in nonsingular varieties, then*

$$s(Y;X_1,\dots,X_r;M)=c(TM)^{-1}c(TM')\cap s(Y;X_1,\dots,X_r;M') \quad .$$

Proof. By Theorem 3.5, both classes $c(TM)\cap s(Y;X_1,\dots,X_r;M)$ and $c(TM')\cap s(Y;X_1,\dots,X_r;M')$ compute $c_{\mathrm{SM}}(X_1\cup\cdots\cup X_r)$. □

Such observations indicate that union Segre classes are better behaved than one may expect. For example, they must be less sensitive to scheme structure than ordinary Segre classes.

Example 3.9. The *ordinary* Segre class of the union X of three distinct lines L_1, L_2, L_3 through a point p in $\mathbb{P}^3$ depends on the position of the lines.

If the three lines are *not* coplanar, then a direct computation shows that $s(X, \mathbb{P}^3) = [X] - 10[\mathbb{P}^0]$; if the lines *are* coplanar, then $s(X, \mathbb{P}^3) = [X] - 12[\mathbb{P}^0]$. Here we are taking the reduced structure on the union, and in a sense this is the problem: as one of the lines moves to become coplanar with the others, the flat limit acquires an embedded point at the origin; in other words, the reduced union of coplanar lines is not a limit of the union of configurations of noncoplanar ones.

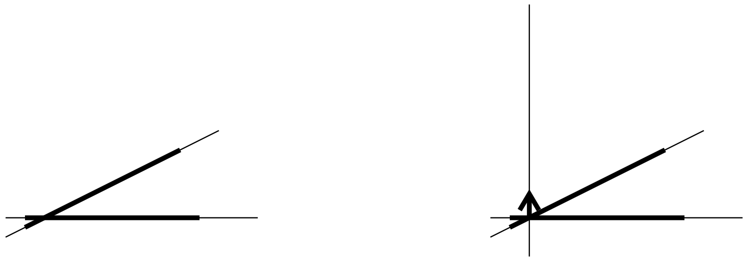
The union Segre class does not detect this difference. Using Definition 2.2, or Theorem 3.5, one computes that

$$s(p; L_1, L_2, L_3; \mathbb{P}^3) = [X] - 12[\mathbb{P}^0]$$

regardless of the directions of the lines, so long as these remain distinct.

On the other hand, properties such as Corollary 3.8 do depend on rather subtle information, in that they do not extend blindly to all union Segre classes.

Example 3.10. Consider the union X of two distinct lines L_1, L_2 through a point, with ideal given by the *product* of the ideals of the components.



This scheme depends on the ambient variety: in $\mathbb{P}^2$ it is reduced, in $\mathbb{P}^3$ it acquires an embedded component supported at the intersection point. Computing the ordinary Segre classes, and applying Proposition 2.4, part 2., gives

$$s(\emptyset; L_1, L_2; \mathbb{P}^2) = [X] - 4[\mathbb{P}^0] = s(\emptyset; L_1, L_2; \mathbb{P}^3) \quad ;$$

thus

$$s(\emptyset; L_1, L_2; \mathbb{P}^2) \neq c(T\mathbb{P}^2)^{-1} c(T\mathbb{P}^3) \cap s(\emptyset; L_1, L_2; \mathbb{P}^3) = [X] - 6[\mathbb{P}^0] \quad .$$

However, by Corollary 3.8 we must have

$$s(p; L_1, L_2; \mathbb{P}^2) = c(T\mathbb{P}^2)^{-1} c(T\mathbb{P}^3) \cap s(p; L_1, L_2; \mathbb{P}^3) \quad ,$$

where p is the intersection point. Using Definition 2.2, the reader can check 'by hand' that $s(p; L_1, L_2; \mathbb{P}^2) = [X] - 3[\mathbb{P}^0]$, $s(p; L_1, L_2; \mathbb{P}^3) = [X] - 5[\mathbb{P}^0]$, in agreement with this formula.

Summarizing, the inclusion-exclusion principle draws a connection between the union Segre classes considered here and the SM-Segre classes considered in [Alu]. In view of applications to the study of Chern-Schwartz-MacPherson and Milnor classes, it would be very interesting to establish the precise circumstances under which union and SM-Segre classes agree.

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