COMPUTATION OF THE TEICHMÜLLER DISTANCE BETWEEN ELLIPTIC CURVES

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Abstract. Complex algebraic genus one curves can be uniformized by elliptic integrals. This is both classical and explicit. For any genus one curve C defined by an equation $y^2 = x(x-1)(x-\lambda)$ one can explicitly form a lattice $L = \langle z \mapsto z+\omega_1, z \mapsto z+\omega_2\rangle$ such that $C = \mathbf{C}/L$. One can, furthermore, find the uniformizing projection $\mathbf{C} \to C$ ([2]). In this note that fact is used to find Teichmüller mappings between two given genus one algebraic curves. In fact, for any two elliptic curves given by their defining polynomials it is possible to find all the Teichmüller mappings between them. One can, furthermore, compute the Teichmüller distance between given elliptic curves (in the moduli space).

1. Normal forms for genus one algebraic curves

An algebraic curve of genus one can always be represented as

$$C_\lambda = \{(x, y) \mid y^2 = P(x)\}, \tag{1}$$

where

$$P(x) = x(x-1)(x-\lambda). \tag{2}$$

Observe that such a presentation can be algorithmically found for any genus one curve ([3]). Hence we may, without loss of generality, restrict our considerations to curves given by an equation of the form (1).

Genus one curves C_λ of the form (1) depend on a complex parameter λ, $\lambda \neq 0, 1$. The curves C_{λ_1} and C_{λ_2} are isomorphic if and only if there is a Möbius transformation carrying the unordered set $\{0, 1, \infty, \lambda_1\}$ onto the set $\{0, 1, \infty, \lambda_2\}$. This observation yields the following characterization of isomorphic curves C_{λ_1} and C_{λ_2}.

Let Γ be the group of order 6 generated by the Möbius transformations

$$\gamma_1(z) = \frac{1}{z}, \text{ and } \gamma_2(z) = 1 - z.$$

Then the curves C_{λ_1} and C_{λ_2} are isomorphic if and only if there is an element $\gamma \in \Gamma$ such that $\gamma(\lambda_1) = \lambda_2$.

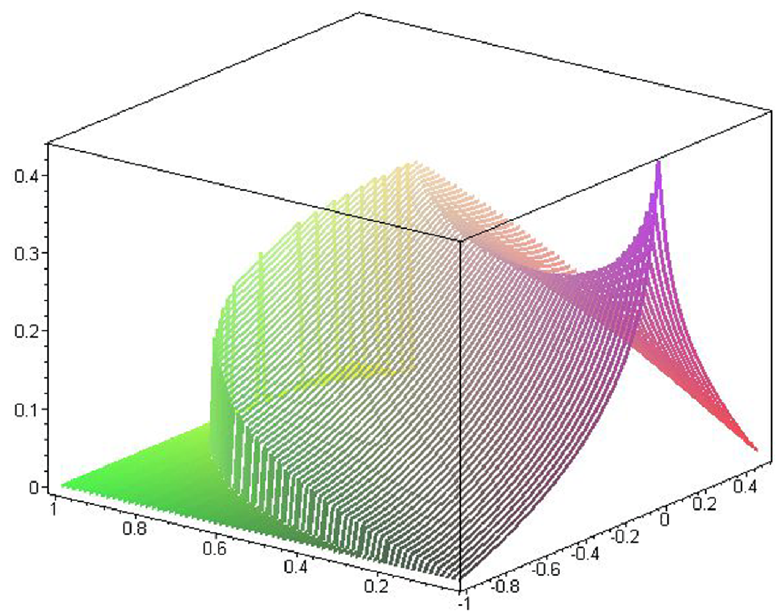

FIGURE 1. Graph of the Teichmüller distance function between the fixed torus $y^2 = x(x-1/2)(x-1)$ and a variable torus $y^2 = x(x-1)(x-\lambda)$.

All this is classical. A fundamental domain for the action of the group Γ is those x's satisfying $\mathrm{Im}\ \lambda \geq 0$, $|\lambda| \leq 1$ and $\mathrm{Re}\ \lambda \leq \frac{1}{2}$. Our graphs are over these domains.

2. Uniformization of elliptic curves

Without loss of generality we may restrict our consideration to elliptic curves of the type (1) with the parameter λ in the fundamental domain illustrated by Figure (1). Then the elliptic integrals

$$\omega_1 = \int_0^\lambda \frac{1}{\sqrt{P(x)}} dx$$

and

$$\omega_2 = \int_1^\lambda \frac{1}{\sqrt{P(x)}} dx$$

define the lattice $\Lambda = \langle z \to z+1, z \to z+\tau \rangle$, $\tau = \omega_2/\omega_1$ for which

$$C = \mathbf{C}/\Lambda.$$

This is classical ([2, Theorem 1, p. 42]).

The Weierstrass $\mathcal{P}$ function provides a way to compute a defining equation for a torus $\mathbf{C}/\Lambda$ ([2, 1.11]). That brings us back from the

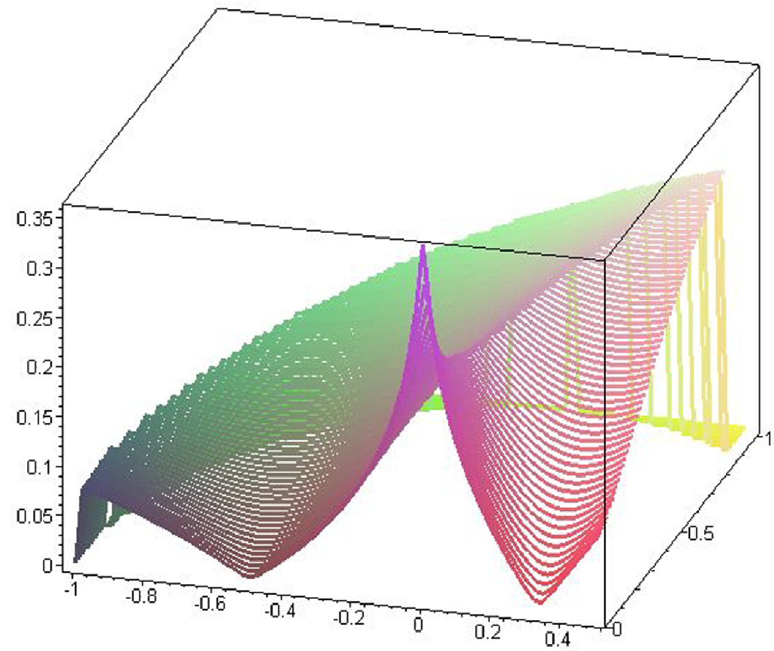

FIGURE 2. Graph of the Teichmüller distance function between the fixed torus $y^2 = x(x-1)(x+1/2)$ and a variable torus $y^2 = x(x-1)(x-\lambda)$.

category of Riemann surfaces to that of algebraic curves, and gives a way to check the accuracy of the computations. The method of uniformization of elliptic curves by elliptic integrals is extremely accurate.

Observe that the correspondence between the lattice parameter τ and the parameter λ in the defining equation (1) of the corresponding algebraic curve of genus one is the classical λ function (see [1, 7.3.5]).

3. QUASICONFORMAL MAPPINGS BETWEEN ALGEBRAIC CURVES OF GENUS ONE

Let

$$C_{\lambda_j} = \{(x,y) \mid y^2 = x(x-1)(x-\lambda_j)\},\ j = 1,2,$$

be two elliptic curves. Consider the problem of finding explicit quasiconformal mappings between the curves C_{λ_1} and C_{λ_2}. This can rather easily be solved using the above uniformization of the curves C_{λ_j}.

Let

$$\omega_1^j = \int_0^{\lambda_j} \frac{1}{\sqrt{x(x-1)(x-\lambda_j)}}dx$$

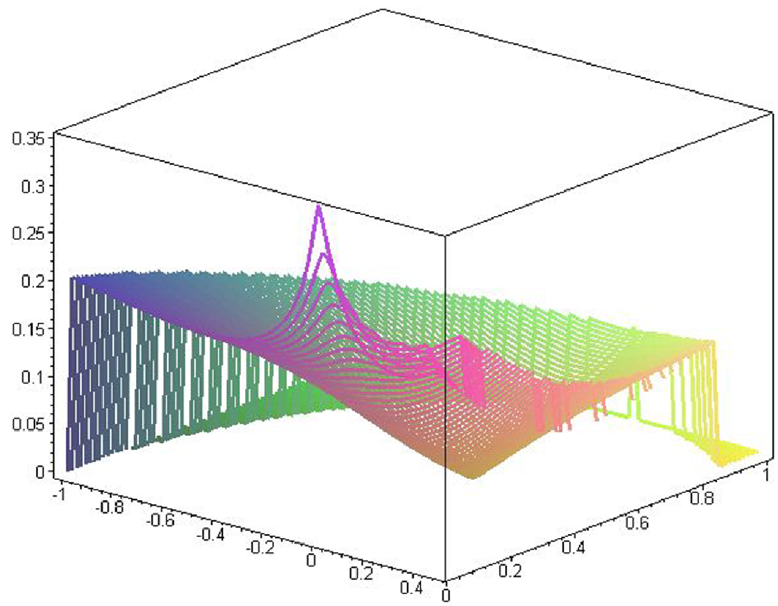

FIGURE 3. Graph of the Teichmüller distance function between the fixed torus $y^2 = x(x-1)(x-(1/4+i/2))$ and a variable torus.

and

$$\omega_2^j = \int_1^{\lambda_j} \frac{1}{\sqrt{x(x-1)(x-\lambda_j)}}dx,\ j=1,2,$$

denote corresponding elliptic integrals, and let

$$\Lambda^j = \langle z \to z+1, z \to z+\tau_j \rangle,\ \tau_j = \frac{\omega_2^j}{\omega_1^j},$$

be the corresponding lattices.

The problem of finding quasiconformal mappings between the algebraic curves C^1 and C^2 becomes simply the problem of finding quasiconformal mappings F which map a fundamental parallelogram of the lattice Λ^1 onto that of the lattice Λ^2. Best such mappings are affine stretchings.

The fundamental parallelograms for the lattices Λ^j can be readily obtained from the bases used for the lattices. Affine stretchings, mapping one such fundamental parallelogram onto another one, have the smallest maximal dilatation in their homotopy classes. Replacing this fundamental parallelogram with another one changes the homotopy class of the corresponding stretching.

To find quasiconformal mappings with the smallest maximal dilatation (among *all* qc mappings between the tori), one has to use a standard basis as described below.

To find a Teichmüller mapping, i.e., a mapping induced by affine stretching as described above, between two given tori C_{λ_1} and C_{λ_2} observe that, without loss of generality, we may assume that

$$\mathrm{Im}\ \frac{\omega_2^j}{\omega_1^j} = \mathrm{Im}\ \tau_j > 0,\ j = 1, 2,$$

when considering the lattices

$$\Lambda^j = \langle z \to z + 1, z \to z + \tau_j\rangle,\ j = 1, 2.$$

The affine mapping $F(z) = Az + B\bar{z}$, with

$$A = \frac{\tau_2 - \bar{\tau}_1}{\tau_1 - \bar{\tau}_1} \text{ and } B = \frac{\tau_1 - \tau_2}{\tau_1 - \bar{\tau}_1} \tag{3}$$

takes Λ^1 to Λ^2. Moreover the quasiconformal mapping F has the minimal dilatation among all quasiconformal mappings that take the lattice points of Λ^1 to those of Λ^2. The complex dilatation of F is

$$\mu = \frac{\partial F/\partial \bar{z}}{\partial F/\partial z} = \frac{B}{A} = \frac{\tau_1 - \tau_2}{\tau_2 - \bar{\tau}_1}. \tag{4}$$

The dilatation of the quasiconformal mapping F is then

$$K = \frac{1 + |\mu|}{1 - |\mu|} \tag{5}$$

and the Teichmüller distance between the two complex structures is

$$T(\tau_1, \tau_2) = \frac{1}{2} \log K. \tag{6}$$

Hence we can explicitly write down Teichmüller mappings between any given genus one curves. In the above construction, the integration paths for the elliptic integrals ω_k^j can be chosen in many ways. Different choices lead to different bases for the lattices. Suitable normalization will ensure that the mapping constructed above is the best possible quasiconformal mapping between the curves.

4. Maple code for computing Teichmüller mappings and distances between elliptic curves

4.1. Computation of the lattice.

Input: parameter λ defining the genus one algebraic curve C given by the equation (2).

Output: Standardized number τ such that $\mathrm{Im}\tau > 0$ and $C = \mathbf{C}/\langle z \mapsto z + 1,\ z \mapsto z + \tau\rangle$.

We describe the algorithms that appear in the pseudocode below. Given λ from (1) the `EllipticK` program in `MAPLE` calculates a corresponding basis for the lattice. After dividing by a basis element we obtain an equivalent basis of the form $(1, \tau)$. In the event that $\mathrm{Im}\tau < 0$ we replace τ with $-\tau$.

Next τ is adjusted so that $|\tau| \geq 1$ and $\frac{1}{2} \leq \mathrm{Re}\tau \leq \frac{1}{2}$. The tau's in this region correspond to elliptic curves except for identification of the boundary under the map $z \mapsto -\bar{z}$. We first adjust the $\mathrm{Re}\tau$, effectively by applying a power of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ to the basis, so that $|\mathrm{Re}\tau| \leq \frac{1}{2}$. If $|\tau| \geq 1$, then the algorithm stops. Otherwise, τ is replaced by $-\frac{1}{\tau}$ (the matrix $(0, -1, 1, 0) \in SL_2(\mathbb{Z})$) and looped back so that $|\mathrm{Re}\tau| \leq \frac{1}{2}$. This repeats until convergence.

Pseudocode for the program. The following code uses Maple's built-in special function `EllipticK`.

$\omega_1 \leftarrow \left(\lambda \to 4\texttt{EllipticK}(1/\sqrt{\lambda})/\sqrt{\lambda}\right)$
$\omega_2 \leftarrow \left(\lambda \to 4\texttt{EllipticK}(\sqrt{\lambda})/\sqrt{\lambda}\right)$
$G \leftarrow \omega_1(\lambda)$
$H \leftarrow \omega_2(\lambda)$
$\tau \leftarrow G/H$
if $\mathrm{Im}(\tau) < 0$ **then**
    $\tau \leftarrow -\tau$
end if
while $\frac{1}{2} < \mathrm{Re}(\tau)$ **or** $\mathrm{Re}(\tau) < -\frac{1}{2}$ **or** $\mathrm{Re}(\tau)^2 + \mathrm{Im}(\tau)^2 < 1$ **do**
    $M \leftarrow \mathrm{Re}(\tau)$
    if $\frac{1}{2} < |M|$ **then**
        $M \leftarrow M - \lfloor \mathrm{Re}(\tau) + \frac{1}{2} \rfloor$
    end if
    $\tau \leftarrow M + \sqrt{-1}\mathrm{Im}(\tau)$
    if $\mathrm{Re}(\tau)^2 + \mathrm{Im}(\tau)^2 < 1$ **then**
        $\tau \leftarrow -1/\tau$
    end if
end do
output τ

The above program computes first elliptic integrals defining a basis for the lattice corresponding to the genus one curve C given by the parameter λ.

Next the program standardizes this lattice.

4.2. **Computation of the Teichmüller distance.** The pseudohyperbolic distance $\rho(z_1, z_2)$ between the points z_1 and z_2 in the unit disk is given by the formula

$$\rho(z_1, z_2) = \left| \frac{z_1 - z_2}{z_1 - \overline{z_2}} \right|. \tag{7}$$

In the following code we use that formula to compute the Teichmüller distance between tori. Here the inputs are complex numbers τ_1 and τ_2 with $|\tau_i| \geq 1$ and $-\frac{1}{2} \leq \operatorname{Re} \tau_i \leq \frac{1}{2}$, $i = 1, 2$. These are obtained as outputs from the previous program. the code calculates the pseudohyperbolic distance between t hese points and nearby equivalent points and chooses the minimum.

Input: Complex numbers τ_1 and τ_2, $\operatorname{Im}(\tau_j) > 0$, defining tori $T_j = \mathbf{C}/\langle z \mapsto z+1, z \mapsto z + \tau_j \rangle$, $j = 1, 2$.

Output: The Teichmüller distance between the tori T_j in the moduli space of genus one algebraic curves.

$M \leftarrow \operatorname{Re}(\tau_1) - \operatorname{Re}(\tau_2)$
if $\frac{1}{2} < M$ **then**
  $\tau_2 \leftarrow \tau_2 + 1$
end if
if $M < -\frac{1}{2}$ **then**
  $\tau_2 \leftarrow \tau_2 - 1$
end if
$\rho \leftarrow \left((z_1, z_2) \mapsto \left| \frac{z_1 - z_2}{z_1 - \overline{z_2}} \right| \right)$
$k_1 \leftarrow \rho(\tau_1, \tau_2)$
$k_2 \leftarrow \rho(\tau_1, -1/\tau_2)$
$k \leftarrow \min(k_1, k_2)$
output $\frac{1}{2} \log \frac{1+k}{1-k}$

References

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